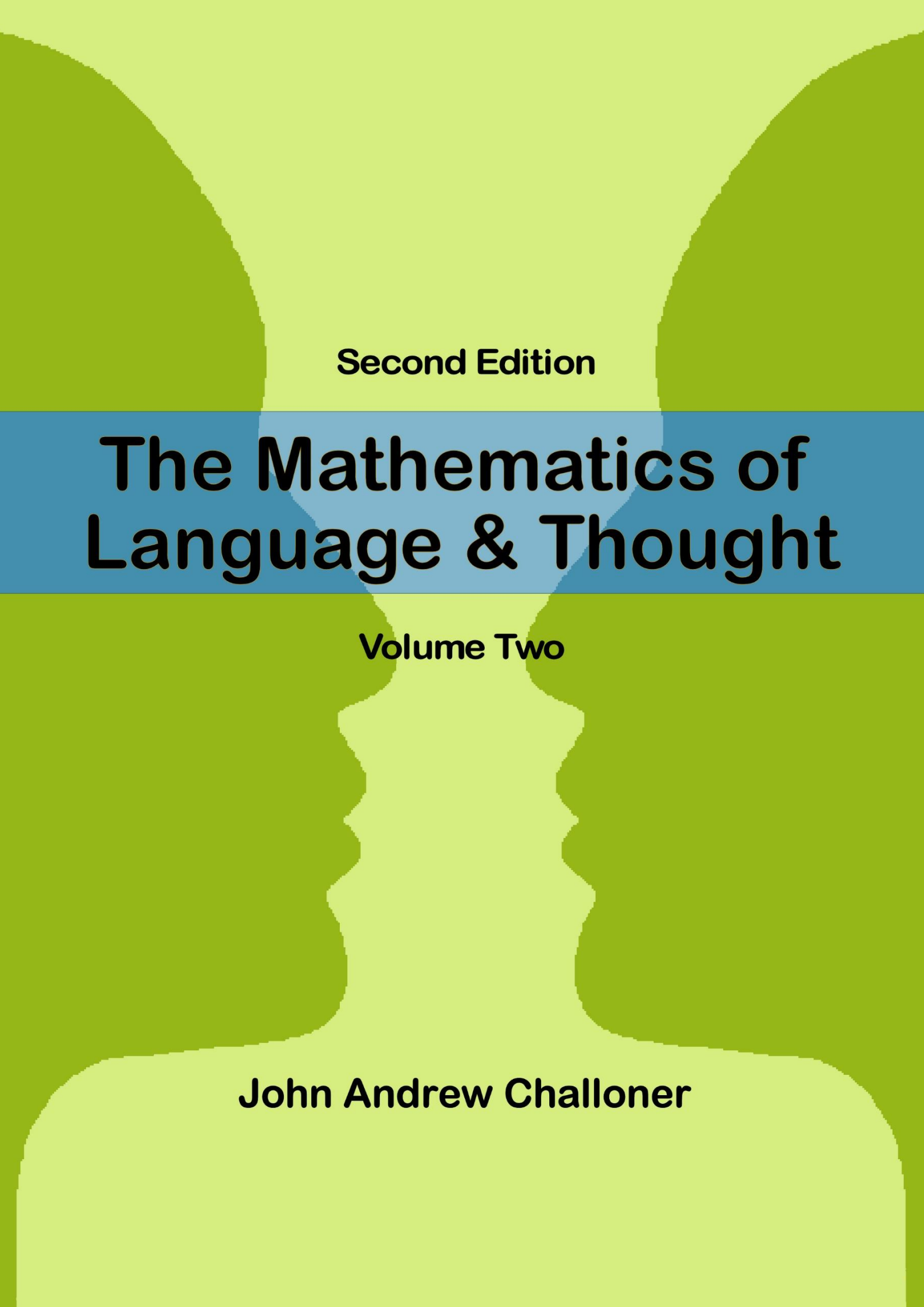


The book cover features a background of green and yellow. A horizontal blue band runs across the middle. The title is centered in the blue band. Above and below the band, the text 'Second Edition' and 'Volume Two' are centered. At the bottom, the author's name is centered. The background also features large, stylized green shapes that resemble the profiles of two faces facing each other.

Second Edition

The Mathematics of Language & Thought

Volume Two

John Andrew Challoner

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*This book is dedicated to my wife, Indy, for her understanding, patience, and support
and for still being there when I had finished it.*

Preface

In this second volume, I provide detailed proofs of the theorems of Symbolic Reasoning from its definitions and axioms. Since the first edition, I have made several revisions and have introduced new definitions, axioms, and theorems.

Each has its own unique reference number. However, they are not as orderly as I would wish. Had it been a simple matter of presenting definitions, axioms, and theorems already formed in my mind, then a neat, sequential numbering system could have been applied. However, in practice, they came and went as the theory developed. This left gaps which were, subsequently, reused for newer theorems. Having used a word processor, it would have been relatively easy for me to renumber the references and produce a more orderly system. However, I have not done so for several reasons. Firstly, the introduction of new definitions, axioms, and theorems as Symbolic Reasoning develops will quickly reintroduce disorder. Secondly, I am now familiar with many of the references used over the last 23 years, and renumbering would make navigation more difficult. The same is, of course, true of anyone who wishes to learn Symbolic Reasoning and I would, therefore, discourage any such renumbering. Thirdly, the system allows developers to identify their work. For example, (A_{B01}) might be used to reference a new axiom introduced by a Dr. Brown. Last, but by no means least, retention of the original references has an honesty about it that I prefer.

John Andrew Challoner, BSc(Eng), CEng, MICE, Brighton, UK, April, 2022.

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Chapter 1 Proofs of Membership Theorems – Part 1

1.1 Introduction

Proofs of the Membership Theorems now follow over the next three chapters. These proofs are necessary in support of Symbolic Reasoning. Once proven it is unnecessary to repeat the process and the theorems can then be used in any other proof or in the manipulation of statements.

It is suggested that the reader reviews only some of the proofs to gain an understanding of the techniques, but treats the remainder of these chapters as reference material.

With a very few exceptions, these theorem proofs are ordered so that:

- (1) each proof uses only theorems which have previously been proven; and
- (2) each theorem is only used in the proof of theorems which follow.

Where this has not been possible an explanation is given.

In this way circularity is avoided.

1.2 Proofs Using The Axioms and Definitions of Membership Theory Only

A. The Universality of Axioms and Definitions.

All proofs use axioms and definitions as their basis. All axioms and definitions are universal, i.e., they are true in all circumstances, and their truth set is the same as the universal set. Any theorem proven from a conjunction of axioms and definitions is also universal, i.e., if a theorem can be proven true in all circumstances, then it is true in all circumstances.

Thus, where (A_n) is any axiom, (D_n) any definition, and (T_n) any theorem:

$$(A_n) = E, \text{ and } (D_n) = E, \text{ and } (T_n) = E \quad (T_{M001}) \text{ Universality of Axioms Definitions and Theorems}$$

This can be shown by applying the Order Axiom to any axiom, definition or theorem as shown in the following example:

Example Proof of (T_{M001})

$$\begin{array}{ll} a \subseteq E & (T_{M744})(\text{Universal Set Theorem}) \\ (a \subseteq E) = E & (A_{MX16})(\text{Order Axiom}) \end{array}$$

Two implications can be drawn from this theorem. Firstly, an axiom, definition or theorem may replace the universal set in an equation.

Example

$$\begin{array}{ll} a \subseteq b.E & (S_1)(\text{Eqn1}) \\ (a = a) = E & (T_{M001})(\text{Universality of } (D_{M00}) \text{ Definition of } =)(\text{Eqn2}) \\ (a = E) \subseteq (b = E)(a = a) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{array}$$

Note however that the order of the other terms must be adjusted to match. Hence, for example, a becomes $a = E$.

Secondly, whenever an axiom or definition is encountered in an equation it may be replaced by the universal set. This can be used as a technique for proving or disproving the equation.

B. Other Proofs

$(a = b)(a \subset b) = \emptyset$	(T_{M037}) Conjunction of Binary Relations
$(a = b)(a \supset b) = \emptyset$	(T_{M038}) Conjunction of Binary Relations
$(a = b)(a \parallel b) = \emptyset$	(T_{M039}) Conjunction of Binary Relations
$(a \subset b)(a \supset b) = \emptyset$	(T_{M040}) Conjunction of Binary Relations
$(a \subset b)(a \parallel b) = \emptyset$	(T_{M041}) Conjunction of Binary Relations
$(a \supset b)(a \parallel b) = \emptyset$	(T_{M042}) Conjunction of Binary Relations

Example Proof of (T_{M037})

$$((a = b)(a \subset b) = \emptyset).((a = b)(a \supset b) = \emptyset).((a = b)(a \parallel b) = \emptyset).((a \subset b)(a \supset b) = \emptyset).((a \subset b)(a \parallel b) = \emptyset).((a \supset b)(a \parallel b) = \emptyset)$$

(A_{MX32})(Conjunction of Binary Relations Axiom)
Extraction

$(a \neq b) = \sim(a = b)$	(T_{M260}) Definition of $\neq$
$(a \sim \subset b) = \sim(a \subset b)$	(T_{M261}) Definition of $\sim \subset$
$(a \sim \supset b) = \sim(a \supset b)$	(T_{M262}) Definition of $\sim \supset$
$(a \not\subseteq b) = \sim(a \subseteq b)$	(T_{M259}) Definition of $\not\subseteq$
$(a \not\supseteq b) = \sim(a \supseteq b)$	(T_{M263}) Definition of $\not\supseteq$
$(a \not\parallel b) = \sim(a \parallel b)$	(T_{M264}) Definition of $\not\parallel$

Example Proof of (T_{M260})

$$(a \langle \sim o \rangle b) = \sim(a \langle o \rangle b)$$

(D_{M14}) (Definition of $\langle \sim o \rangle$)
Extraction

$$(a = \emptyset^1) \subseteq (a^1 = \emptyset^1) \quad (T_{M499})$$

Proof of (T_{M499})

$$\begin{aligned} \text{Let } a &= \emptyset^1 \\ a^1 &= \emptyset^{11} & (A_{MX19})(\text{Substitution}) \\ a^1 &= \emptyset^1 & (D_{M11})(\text{Definition of } a^{11}) \\ \text{Thus } (a = \emptyset^1) &\subseteq (a^1 = \emptyset^1) & (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(a \subset b) = (b \supset a) \quad (T_{M195}) \subset \text{Reverse of } \supset$$

Proof of (T_{M195})

$$\begin{aligned} (a \triangleleft b) &= (b \triangleright a) & (D_{M23})(\text{Definition of } \triangleleft)(\text{Eqn1}) \\ \{\triangleleft\} &= \{\subset, \supset, \triangleleft, \triangleright\} & (D_{S08})(\text{Definition of } \triangleleft) \\ \{\triangleleft\} &= \{\subset\} & \text{Extraction (Eqn2)} \\ (a \subset b) &= (b \supset a) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(c_1^1 \text{+++} c_E^1 = E) = (c_1 \text{+++} c_E = E) \quad (T_{M749})$$

Proof of (T_{M749})

$$\begin{aligned} c_1 \text{+++} c_E &= E & (D_{M10})(\text{Definition of Universal Set using Sets}) \\ (c_1 \text{+++} c_E = E) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn1}) \\ c_1^1 \text{+++} c_E^1 &= E & (D_{M11})(\text{Definition of Universal Set using Entities}) \\ (c_1^1 \text{+++} c_E^1 = E) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\ (c_1^1 \text{+++} c_E^1 = E) &= (c_1 \text{+++} c_E = E) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \end{aligned}$$

$$(x = a, b) + (x = b, c) = (x = a, b, c) \quad (T_{M754})$$

Proof of (T_{M754})

$$\text{Let } (x = a, b) + (x = b, c)$$

$(x = a) + (x = b) + (x = b) + (x = c)$ (D_{M15})(Definition of a Variable Set)
 $(x = a) + (x = b) + (x = c)$ ($AMX01$)(Disjunction Tautology)
 $(x = a, b, c)$ (D_{M15})(Definition of a Variable Set)
 Thus $(x = a, b) + (x = b, c) = (x = a, b, c)$ ($AMX02$)(Conditional Introduction) since this proof is reversible

$a \subseteq E$ (T_{M744}) Universal Set Theorem

Proof of (T_{M744})

$aE = a$ ($AMX17$)(Universal Set Conjunction Axiom)
 $a \subseteq E$ ($AMX44$)($\subseteq$ Equality Form)

$(a = b) = (b = a)$ (T_{M197}) Commutativity of $=$

Proof of (T_{M197})

$(a = b) = (a = b)$ (D_{M00})(Definition of $=$)
 $(a = b) = (b = a)$ ($AMX19$)(Substitution)

$\sim \emptyset^1 \neq \emptyset^1$ (T_{M650})

Proof of (T_{M650})

$a \neq a$ ($AMX08$)(Existence Axiom)(Eqn1)
 Let $a = \emptyset^1$ (Eqn2)
 $\emptyset^1 \neq \sim \emptyset^1$ ($AMX19$)(Substitution of Eqn2 into Eqn1)

$(a = a) = E$ (T_{M534})

Proof of (T_{M534})

$a = a$ (D_{M00})(Definition of $=$)
 $(a = a) = E$ ($AMX16$)(Order Axiom)

$(E = E) = E$ (T_{M342})

Proof of (T_{M342})

$E = E$ (D_{M00})(Definition of $=$)
 $(E = E) = E$ ($AMX16$)(Order Axiom)

$(\emptyset = \emptyset) = E$ (T_{M343})

Proof of (T_{M343})

$\emptyset = \emptyset$ (D_{M00})(Definition of $=$)
 $(\emptyset = \emptyset) = E$ ($AMX16$)(Order Axiom)

$a = (a = E) = ((a = E) = E) = (((a = E) = E) = E)$ and so on, ad infinitum. (T_{M003}) Higher Order Universality

Proof of (T_{M003})

Let a
 $a = E$ ($AMX16$)(Order Axiom)
 $(a = E) = E$ ($AMX16$)(Order Axiom)
 $((a = E) = E) = E$ ($AMX16$)(Order Axiom)

$(a = b)(c = d) \subseteq (a + c = b + d)$ (T_{M230}) Combination

Proof of (T_{M230})

Let $(a = b)$ (Eqn1)
 Let $(c = d)$ (Eqn2)
 $(a + c = a + c)$ (D_{M00})(Definition of $=$)(Eqn3)
 $(a + c = b + d)$ ($AMX19$)(Substitution of Eqns1&2 into Eqn3)
 Thus $(a = b)(c = d) \subseteq (a + c = b + d)$ ($AMX02$)(Conditional Introduction)

$$(a = b)(c = d) \subseteq (ac = bd) \quad (T_{M231}) \text{ Combination}$$

Proof of (T_{M231})

Let $(a = b)$	(Eqn1)
Let $(c = d)$	(Eqn2)
$(ac = ac)$	(D _{M00})(Definition of =)(Eqn3)
$(ac = bd)$	(A _{MX19})(Substitution of Eqns1&2 into Eqn3)
Thus $(a = b)(c = d) \subseteq (ac = bd)$	(A _{MX02})(Conditional Introduction)

$$(a \subseteq c)(c \subseteq b) \subseteq (a \subseteq b) \quad (T_{M139}) \text{ Hypothetical Syllogism}$$

Proof of (T_{M139})

Let $a \subseteq c$	
$a = ac$	(A _{MX44})($\subseteq$ Equality Form)(Eqn1)
Let $c \subseteq b$	
$c = bc$	(A _{MX44})($\subseteq$ Equality Form)(Eqn2)
$a = abc$	(A _{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)
$a = ab$	(A _{MX19})(Substitution of Eqn1 into Eqn3)
$a \subseteq b$	(A _{MX44})($\subseteq$ Equality Form)
Thus $(a \subseteq c)(c \subseteq b) \subseteq (a \subseteq b)$	(A _{MX02})(Conditional Introduction)

$$E^1 \subset E \quad (T_{M481}) \text{ Domain of the Universal Set Entity}$$

Proof of (T_{M481})

$a^1 \subset E$	(A _{MX37})(Plurality Axiom)(Eqn1)
Let $a^1 = E^1$	(Eqn2)
$E^1 \subset E$	(A _{MX19})(Substitution of Eqn2 into Eqn1)

$$\sim a^1 \neq (\sim a^1)^1 \quad (T_{M554})$$

Proof of (T_{M554})

$\sim a^1 \neq b^1$	(A _{MX18})(2nd Plurality Axiom)(Eqn1)
Let $b = \sim a^1$	(Eqn2)
$\sim a^1 \neq (\sim a^1)^1$	(A _{MX19})(Substitution of Eqn2 into Eqn1)

$$\emptyset^1 \subset E \quad (T_{M482}) \text{ Domain of the Null Set Entity}$$

Proof of (T_{M482})

$a^1 \subset E$	(A _{MX37})(Plurality Axiom)(Eqn1)
Let $a^1 = \emptyset^1$	(Eqn2)
$\emptyset^1 \subset E$	(A _{MX19})(Substitution of Eqn2 into Eqn1)

$$(a = E)(b \neq \emptyset) \subseteq (ab \neq \emptyset) \quad (T_{M110}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M110})

Let $a = E$	(Eqn1)
Let $b \neq \emptyset$	
$Eb \neq \emptyset$	(A _{MX11})(Universal Set Conjunction Axiom)(Eqn2)
$ab \neq \emptyset$	(A _{MX19})(Substitution of Eqn1 into Eqn2)
Thus $(a = E)(b \neq \emptyset) \subseteq (ab \neq \emptyset)$	(A _{MX02})(Conditional Introduction)

$$(a = b) \subseteq (a \neq \underline{b}) \quad (T_{M144})$$

Proof of (T_{M144})

Let $a = b$	(Eqn1)
$b \neq \underline{b}$	(A _{MX08})(Existence Axiom)(Eqn2)

$a \neq \underline{b}$ ($\mathbf{A}_{MX19}$)(Substitution of Eqn1 into Eqn2)
 Thus $(a = b) \subseteq (a \neq \underline{b})$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)

$a = a.(b^1 = b^1)$ ($\mathbf{T}_{M413}$) Identity Introduction

Proof of ($\mathbf{T}_{M413}$)

$a = a$ ($\mathbf{D}_{M00}$)(Definition of =)
 $a = aE$ ($\mathbf{A}_{MX11}$)(Universal Set Conjunction Axiom) (Eqn1)
 $(a = a) = E$ ($\mathbf{A}_{MX16}$)(Order Axiom)
 $(b^1 = b^1) = E$ Resymbolising (Eqn2)
 $a = a(b^1 = b^1)$ ($\mathbf{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$(a^1 \subseteq c).(a^1 = b^1) \subseteq (b^1 \subseteq c)$ ($\mathbf{T}_{M414}$) Identity Elimination

Proof of ($\mathbf{T}_{M414}$)

Let $(a^1 \subseteq c).(a^1 = b^1)$
 $(b^1 \subseteq c)$ ($\mathbf{A}_{MX19}$) (Substitution)
 Thus $(a^1 \subseteq c).(a^1 = b^1) \subseteq (b^1 \subseteq c)$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)

$((a = b) = E) \subseteq ((c = a) = (c = b))$ ($\mathbf{T}_{M467}$)

Proof of ($\mathbf{T}_{M467}$)

Let $a = b$ (Eqn1)
 $(c = a) = (c = a)$ ($\mathbf{D}_{M00}$)(Definition of =)(Eqn2)
 $(c = a) = (c = b)$ ($\mathbf{A}_{MX19}$) (Substitution of Eqn1 into Eqn2)
 Thus $((a = b) = E) \subseteq ((c = a) = (c = b))$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)

$((ab = c) = E) \subseteq ((a = E) \subseteq (b = c))$ ($\mathbf{T}_{M444c}$) Biconditional Exportation

Proof of ($\mathbf{T}_{M444c}$)

Let $a = E$ (Eqn1)
 Let $ab = c$ (Eqn2)
 $Eb = c$ ($\mathbf{A}_{MX19}$)(Substitution of Eqn1 into Eqn2)
 $b = c$ ($\mathbf{A}_{MX11}$)(Universal Set Conjunction Axiom)
 Thus $(a = E)(ab = c) \subseteq (b = c)$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)
 $((ab = c) = E) \subseteq ((a = E) \subseteq (b = c))$ ($\mathbf{A}_{MX35}$)(Exportation/Importation) for truth sets

$(a = E) \subseteq (a^1 = E^1)$ ($\mathbf{T}_{M488}$)

Proof of ($\mathbf{T}_{M488}$)

$(a = b) \subseteq (F(a) = F(b))$ ($\mathbf{A}_{MX19}$)(Substitution Axiom)(Eqn1)
 Let $F(a) = a^1$ (Eqn2)
 Thus $F(b) = b^1$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)(Eqn3)
 $(a = b) \subseteq (a^1 = b^1)$ ($\mathbf{A}_{MX19}$)(Substitution of Eqns2&3 into Eqn1)
 $(a = E) \subseteq (a^1 = E^1)$ Resymbolising

$\emptyset^1 \neq \emptyset$ ($\mathbf{T}_{M780}$) Existence of $\emptyset^1$

$E^1 \neq \emptyset$ ($\mathbf{T}_{M773}$) Existence of E^1

Example Proof of ($\mathbf{T}_{M780}$)

$((X = \emptyset^1, E^1) = E) \subseteq ((a = X) \subseteq (a \neq \emptyset))$ ($\mathbf{A}_{MX49}$)(Existence of E^1 and $\emptyset^1$)
 $(a = \emptyset^1) \subseteq (a \neq \emptyset)$ Extraction
 $\emptyset^1 \neq \emptyset$ ($\mathbf{A}_{MX19}$)(Substitution)

1.3 Proofs Using the Axioms and Definitions of Membership Theory and Relationship Theory Only

The following theorems use the axioms and definitions of relationship theory in their proof. However, because the latter are proven using axioms and definitions only, these proofs are not circular.

$$a + b = b + a \quad (T_{M403}) \text{ Commutative } +$$

$$ab = ba \quad (T_{M404}) \text{ Commutative } .$$

Example Proof of (T_{M403})

$$\begin{aligned} (\{+\}^1 \subseteq R_c) &= (a + b = b + a) & (D_{R40}) (\text{Definition of a Commutative Operator}) (o = \{+\}) (\text{Eqn1}) \\ (o = \{., \{+\}) \subseteq (o^1 \subseteq R_c R_v) & & (A_{R12}) (\text{Commutative \& Associative } . \& +) \\ \{+\}^1 \subseteq R_c & & \text{Extraction} \\ (\{+\}^1 \subseteq R_c) &= E & (A_{MX16}) (\text{Order Axiom}) (\text{Eqn2}) \\ E &= (a + b = b + a) & (A_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\ a + b &= b + a & (A_{MX16}) (\text{Order Axiom}) \\ \text{The proof of } (T_{M404}) (\text{Commutative } .) & \text{ is the same except that } \{.\}^1 \subseteq R_c \text{ is extracted} \end{aligned}$$

$$(a + b) + c = a + (b + c) \quad (T_{M405}) \text{ Associative } +$$

$$(a.b).c = a.(b.c) \quad (T_{M406}) \text{ Associative } .$$

Example Proof of (T_{M405})

$$\begin{aligned} (\{+\}^1 \subseteq R_v) &= ((a + b) + c = a + (b + c)) & (D_{R39}) (\text{Definition of an Associative Operator}) (o = \{+\}) (\text{Eqn1}) \\ (o = \{., \{+\}) \subseteq (o^1 \subseteq R_c R_v) & & (A_{R12}) (\text{Commutative \& Associative } . \& +) \\ \{+\}^1 \subseteq R_v & & \text{Extraction} \\ (\{+\}^1 \subseteq R_v) &= E & (A_{MX16}) (\text{Order Axiom}) (\text{Eqn2}) \\ E &= ((a + b) + c = a + (b + c)) & (A_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\ (a + b) + c &= a + (b + c) & (A_{MX16}) (\text{Order Axiom}) \\ \text{The proof of } (a.b).c = a.(b.c) \text{ (T}_{M406}) (\text{Associative } .) & \text{ is the same except that } \{.\}^1 \subseteq R_v \text{ is extracted.} \end{aligned}$$

$$\sim(a.b) = \underline{a} + \underline{b} \quad (T_{M025}) \text{ DeMorgan's Law}$$

Proof of (T_{M025})

$$\begin{aligned} (o = \{., \{+\}) : (\sim(a(o)b) &= \sim a^{\vee} o) \sim b) & (A_{S80}) (\text{DeMorgan's Law}) \\ \sim(a(\{.\})b) &= \sim a^{\vee} \{.\} \sim b & \text{Extraction} \\ \sim(a(\{.\})b) &= \sim a^{\vee} \{+\} \sim b & (D_{R51}) (\text{Definition of } \vee \{.\}) \\ \sim(a.b) &= \sim a + \sim b & (D_{A229}) (\text{Definition of } \langle \rangle) \\ \sim(a.b) &= \underline{a} + \underline{b} & (D_{M03}) (\text{Definition of } \sim) \end{aligned}$$

$$(a \subseteq b) = (a \subset b) + (a = b) \quad (T_{M266}) \text{ Definition of } \subseteq$$

$$(a \supseteq b) = (a \supset b) + (a = b) \quad (T_{M267}) \text{ Definition of } \supseteq$$

Example Proof of (T_{M266})

$$\begin{aligned} (a \trianglelefteq b) &= (a \triangleleft b) + (a = b) & (D_{M18}) (\text{Membership Definition of } \trianglelefteq) (\text{Eqn1}) \\ \{\triangleleft\} &= \{\subset, \supset, \triangleleft, \triangleleft\} & (D_{S08}) (\text{Definition of } \triangleleft) \\ \{\triangleleft\} &= \{\subset\} & \text{Extraction (Eqn2)} \\ (a \subseteq b) &= (a \subset b) + (a = b) & (A_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

1.4 Proofs using Theorems of Relationship Theory

$$\underline{ab} = \sim(a + b) \quad (T_{M026}) \text{ DeMorgan's Law}$$

Proof of (T_{M026})

$$\begin{aligned} (o = \{., \{+\}) : (\sim(a(o)b) &= \sim a^{\vee} o) \sim b) & (A_{S80}) (\text{DeMorgan's Law}) \\ \sim(a(\{+\})b) &= \sim a^{\vee} \{+\} \sim b & \text{Extraction} \\ \sim(a(\{+\})b) &= \sim a^{\vee} \{.\} \sim b & (T_{S583}) (\text{Definition of } \vee \{+\}) \end{aligned}$$

$$\begin{aligned}\sim(a + b) &= \sim a . \sim b & (D_{A229})(\text{Definition of } \langle \rangle) \\ \sim a . \sim b &= \sim(a + b) & (T_{M197})(\text{Commutativity of } =) \\ \underline{ab} &= \sim(a + b) & (D_{M03})(\text{Definition of } \sim)\end{aligned}$$

Circularity Check

(T_{S583}) above is proven later and uses (T_{R522}) and (T_{R012k}) in its proof. However, (T_{R522}) and (T_{R012k}) only use axioms and definitions in their proofs. The proof of (T_{M026}) above is not circular therefore.

1.5 Higher Tier Proofs

$$(a \subset b) = (b \subset a) \quad (T_{M010}) \text{ Inverse of a Proper Subset}$$

Proof of (T_{M010})

$$\begin{aligned}(o = \{=, \{ \subset, \{ \supset, \{ \parallel, \{ \subseteq, \{ \supseteq \}) : (a \langle o \rangle b) &= (\sim a \langle^V o \rangle \sim b) & (D_{R94})(\text{Definition of a Membership Relationship Set}) \\ (a \langle \{ \subset \rangle b) &= (\sim a \langle^V \{ \subset \rangle \sim b) & \text{Extraction} \\ (a \langle \{ \subset \rangle b) &= (\sim a \langle \{ \supset \rangle \sim b) & (D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set}) \\ (a \subset b) &= (\sim a \supset \sim b) & (D_{A229})(\text{Definition of } \langle \rangle) \\ (a \subset b) &= (\underline{b} \subset \underline{a}) & (T_{M195})(\subset \text{ Reverse of } \supset)\end{aligned}$$

$$a(b + c) = (ab) + (ac) \quad (T_{M407}) \text{ Distribution of } . \text{ over } +$$

Proof of (T_{M407})

$$\begin{aligned}(\{.\}^1 \subseteq R_D * \{+\}^1) &= ((b + c).a = (b.a) + (c.a)) & (D_{R41})(\text{Definition of a Distributive Operator})(o = \{.\})(p = \{+\})(\text{Eqn1}) \\ (\{.\}^1 \subseteq R_D * \{+\}^1) &= (a.(b + c) = (a.b) + (a.c)) & \\ & & (T_{M404})(\text{Commutative } .) \\ \{.\}^1 \subseteq R_D * \{+\}^1 & & (A_{R51})(\text{Distribution of } . \text{ over } + \text{ Axiom}) \\ (\{.\}^1 \subseteq R_D * \{+\}^1) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\ E &= (a.(b + c) = (a.b) + (a.c)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ a.(b + c) &= (a.b) + (a.c) & (A_{MX16})(\text{Order Axiom})\end{aligned}$$

$$a + \emptyset = a \quad (T_{M668}) \text{ Null Set Disjunction}$$

Proof of (T_{M668})

$$\begin{aligned}aE &= a & (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn1}) \\ \sim a &= \sim a & (D_{M00})(\text{Definition of } =)(\text{Eqn2}) \\ \sim(aE) &= \sim a & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\ \sim a + \sim E &= \sim a & (T_{M025})(\text{DeMorgan's Law}) \\ \sim a + \emptyset &= \sim a & (D_{M02})(\text{Definition of } \emptyset) \\ a + \emptyset &= a & \text{Resymbolising}\end{aligned}$$

$$aa = a \quad (T_{M667}) \text{ Conjunction Tautology}$$

Proof of (T_{M667})

$$\begin{aligned}a + a &= a & (A_{MX01})(\text{Disjunction Tautology})(\text{Eqn1}) \\ \sim a &= \sim a & (D_{M00})(\text{Definition of } =)(\text{Eqn2}) \\ \sim(a + a) &= \sim a & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\ \sim a . \sim a &= \sim a & (T_{M026})(\text{DeMorgan's Law}) \\ aa &= a & \text{Resymbolising}\end{aligned}$$

$$(a \subseteq b) = (b \supseteq a) \quad (T_{M196}) \subseteq \text{ Reverse of } \supseteq$$

Proof of (T_{M196})

$$\begin{aligned}(a \subset b) &= (b \supset a) & (T_{M195})(\subset \text{ Reverse of } \supset)(\text{Eqn1}) \\ (a = b) &= (b = a) & (T_{M197})(\text{Commutativity of } =)(\text{Eqn2}) \\ (a \subset b) + (a = b) &= (b \supset a) + (b = a) & (T_{M230})(\text{Combination of Eqns1\&2}) \\ (a \subseteq b) &= (b \supseteq a) & (T_{M266})(\text{Definition of } \subseteq) \text{ \& } (T_{M267})(\text{Definition of } \supseteq)\end{aligned}$$

$$(a \subseteq b)(b \subseteq a) = (a = b) \quad (\mathcal{T}_{M430}) \text{ Biconditional Introduction}$$

Proof of ($\mathcal{T}_{M430}$)

$$\begin{aligned} (a \subseteq b)(b \subseteq a) &= (a \subseteq b)(b \subseteq a) & (\mathcal{D}_{M00})(\text{Definition of } =) \\ (a \subseteq b)(b \subseteq a) &= ((a \subset b) + (a = b))((b \subset a) + (b = a)) & (\mathcal{T}_{M266})(\text{Definition of } \subseteq) \\ (a \subseteq b)(b \subseteq a) &= (a \subset b)(b \subset a) + (a = b)(b \subset a) + (a \subset b)(b = a) + (a = b)(b = a) & (\mathcal{T}_{M407})(\text{Distribution of } . \text{ over } +) \\ (a \subseteq b)(b \subseteq a) &= (a \subset b)(a \supset b) + (a = b)(a \supset b) + (a \subset b)(b = a) + (a = b)(b = a) & (\mathcal{T}_{M195})(\subset \text{ Reverse of } \supset) \\ (a \subseteq b)(b \subseteq a) &= (a \subset b)(a \supset b) + (a = b)(a \supset b) + (a \subset b)(a = b) + (a = b)(a = b) & (\mathcal{T}_{M197})(\text{Commutativity of } =) \\ (a \subseteq b)(b \subseteq a) &= \emptyset + \emptyset + \emptyset + (a = b)(a = b) & (\mathcal{T}_{M037}), (\mathcal{T}_{M038}) \& (\mathcal{T}_{M040})(\text{Conjunction of Binary Relations}) \\ (a \subseteq b)(b \subseteq a) &= (a = b)(a = b) & (\mathcal{T}_{M668})(\text{Null Set Disjunction}) \\ (a \subseteq b)(b \subseteq a) &= (a = b) & (\mathcal{T}_{M667})(\text{Conjunction Tautology}) \end{aligned}$$

$$((a = b) = E) \subseteq ((a \subseteq c) = (b \subseteq c)) \quad (\mathcal{T}_{M791})$$

Proof of ($\mathcal{T}_{M791}$)

$$\begin{aligned} \text{Let } a &= b & (\text{Eqn1}) \\ \text{Let } a &\subseteq c & (\text{Eqn2}) \\ b &\subseteq c & (\mathcal{A}_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\ \text{Thus } (a = b)(a \subseteq c) &\subseteq (b \subseteq c) & (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\ ((a = b) = E) &\subseteq ((a \subseteq c) \subseteq (b \subseteq c)) & (\mathcal{A}_{MX35})(\text{Exportation/Importation}) \text{ For Truth Sets (Eqn3)} \\ \text{Let } b &\subseteq c & (\text{Eqn4}) \\ a &\subseteq c & (\mathcal{A}_{MX19})(\text{Substitution of Eqn1 into Eqn4}) \\ \text{Thus } (a = b)(b \subseteq c) &\subseteq (a \subseteq c) & (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\ ((a = b) = E) &\subseteq ((b \subseteq c) \subseteq (a \subseteq c)) & (\mathcal{A}_{MX35})(\text{Exportation/Importation}) \text{ For Truth Sets (Eqn5)} \\ ((a = b) = E) &\subseteq ((a \subseteq c) \subseteq (b \subseteq c)) & (\mathcal{T}_{M430})(\text{Biconditional Introduction Eqns3\&5}) \end{aligned}$$

$$(a = b) = (\underline{a} = \underline{b}) \quad (\mathcal{T}_{M008}) \text{ Inverse of an Equality}$$

Proof of ($\mathcal{T}_{M008}$)

$$\begin{aligned} (a = b) &\subseteq (\underline{a} = \underline{b}) & (\mathcal{A}_{MX19})(\text{Substitution Axiom})(\text{Eqn1}) \\ (\underline{a} = \underline{b}) &\subseteq (\sim \underline{a} = \sim \underline{b}) & (\mathcal{A}_{MX19})(\text{Substitution Axiom}) \\ (\underline{a} = \underline{b}) &\subseteq (a = b) & (\mathcal{A}_{MX13})(\text{Double Complement Axiom})(\text{Eqn2}) \\ (a = b) &= (\underline{a} = \underline{b}) & (\mathcal{T}_{M430})(\text{Biconditional Introduction Eqns1\&2}) \end{aligned}$$

$$(a \parallel b) = (a \parallel \underline{b}) = (\underline{a} \parallel b) = (\underline{a} \parallel \underline{b}) \quad (\mathcal{T}_{M194}) \text{ Co-intersection}$$

Proof of ($\mathcal{T}_{M194}$)

$$\begin{aligned} (a \parallel b) &= (a \not\subseteq \underline{b})(a \not\subseteq b)(\underline{a} \not\subseteq \underline{b})(\underline{a} \not\subseteq b) & (\mathcal{D}_{M06})(\text{Definition of } \parallel)(\text{Eqn1}) \\ (a \parallel \underline{b}) &= (a \not\subseteq \underline{b})(a \not\subseteq \underline{b})(\underline{a} \not\subseteq \underline{b})(\underline{a} \not\subseteq \underline{b}) & (\text{Resymbolising Eqn1}) \\ (a \parallel b) &= (a \not\subseteq \underline{b})(a \not\subseteq b)(\underline{a} \not\subseteq \underline{b})(\underline{a} \not\subseteq \underline{b}) & (\mathcal{T}_{M404})(\text{Commutative } .)(\text{Eqn2}) \\ (\underline{a} \parallel b) &= (\underline{a} \not\subseteq \underline{b})(\underline{a} \not\subseteq b)(a \not\subseteq \underline{b})(a \not\subseteq b) & (\text{Resymbolising Eqn1}) \\ (\underline{a} \parallel \underline{b}) &= (\underline{a} \not\subseteq \underline{b})(\underline{a} \not\subseteq \underline{b})(\underline{a} \not\subseteq \underline{b})(\underline{a} \not\subseteq \underline{b}) & (\mathcal{T}_{M404})(\text{Commutative } .)(\text{Eqn3}) \\ (\underline{a} \parallel b) &= (\underline{a} \not\subseteq \underline{b})(\underline{a} \not\subseteq \underline{b})(a \not\subseteq \underline{b})(a \not\subseteq b) & (\text{Resymbolising Eqn1}) \\ (\underline{a} \parallel \underline{b}) &= (\underline{a} \not\subseteq \underline{b})(\underline{a} \not\subseteq b)(\underline{a} \not\subseteq \underline{b})(\underline{a} \not\subseteq \underline{b}) & (\mathcal{T}_{M404})(\text{Commutative } .)(\text{Eqn4}) \\ (a \parallel b) &= (a \parallel \underline{b}) = (\underline{a} \parallel b) = (\underline{a} \parallel \underline{b}) & (\mathcal{A}_{MX19})(\text{Substitution of Eqns1,2,3\&4}) \end{aligned}$$

$$(a = \emptyset)(b = \emptyset) = (a + b = \emptyset) \quad (\mathcal{T}_{M071}) \text{ Probabilistic Conjunction}$$

Proof of ($\mathcal{T}_{M071}$)

$$\begin{aligned} \text{Let } a &= \emptyset & (\text{Eqn1}) \\ \text{Let } b &= \emptyset & (\text{Eqn2}) \\ a + b &= a + b & (\mathcal{D}_{M00})(\text{Definition of } =)(\text{Eqn3}) \\ a + b &= \emptyset + \emptyset & (\mathcal{A}_{MX19})(\text{Substitution of Eqns1\&2 into Eqn3}) \end{aligned}$$

$a + b = \emptyset$	(A_{MX01})(Disjunction Tautology)
Thus $(a = \emptyset)(b = \emptyset) \subseteq (a + b = \emptyset)$	(A_{MX02})(Conditional Introduction)(Eqn4)
Let $a + b = \emptyset$	(Eqn5)
$a + b + b = \emptyset$	(A_{MX01})(Disjunction Tautology)(Eqn6)
$\emptyset + b = \emptyset$	(A_{MX19})(Substitution of Eqn5 into Eqn6)
$b = \emptyset$	(T_{M668})(Null Set Disjunction)
$a + a + b = \emptyset$	(A_{MX01})(Disjunction Tautology)(Eqn7)
$a + \emptyset = \emptyset$	(A_{MX19})(Substitution of Eqn5 into Eqn7)
$a = \emptyset$	(T_{M668})(Null Set Disjunction)
Thus $(a + b = \emptyset) \subseteq (a = \emptyset)(b = \emptyset)$	(A_{MX02})(Conditional Introduction)(Eqn8)
$(a = \emptyset)(b = \emptyset) = (a + b = \emptyset)$	(T_{M430})(Biconditional Introduction Eqns4&8)

$$(a = b) \subseteq (a = ab) \quad (T_{M772})$$

Proof of (T_{M772})

Let $a = b$	(Eqn1)
$a = bb$	(T_{M667})(Conjunction Tautology)(Eqn2)
$a = ab$	(A_{MX19})(Substitution of Eqn1 into Eqn2)
Thus $(a = b) \subseteq (a = ab)$	(A_{MX02})(Conditional Introduction)

$$(a \subseteq b) = (b \subseteq a) \quad (T_{M012}) \text{ Transposition}$$

Proof of (T_{M012})

$(a \subset b) = (\sim a \supset \sim b)$	(T_{M010})(Inverse of a Proper Subset)(Eqn1)
$(a = b) = (\sim a = \sim b)$	(T_{M008})(Inverse of an Equality)(Eqn2)
$(a \subset b) + (a = b) = (\sim a \supset \sim b) + (\sim a = \sim b)$	(A_{MX21})(Combination of Eqns1&2)
$(a \subseteq b) = (\sim a \supset \sim b) + (\sim a = \sim b)$	(T_{M266})(Definition of $\subseteq$)
$(a \subseteq b) = (\sim a \supseteq \sim b)$	(T_{M267})(Definition of $\supseteq$)
$(a \subseteq b) = (b \subseteq a)$	(T_{M196})($\subseteq$ Reverse of $\supseteq$)

$$(a \parallel b) = (b \parallel a) \quad (T_{M193}) \text{ Commutativity of } \parallel$$

Proof of (T_{M193})

$(a \parallel b) = (a \not\subseteq b)(a \not\subseteq b)(a \not\subseteq b)(a \not\subseteq b)$	(D_{M06})(Definition of $\parallel$)
$(a \parallel b) = (b \not\subseteq a)(b \not\subseteq a)(b \not\subseteq a)(b \not\subseteq a)$	(T_{M008})(Inverse of an Equality) & (T_{M012})(Transposition)
$(a \parallel b) = (b \parallel a)$	(D_{M06})(Definition of $\parallel$)

$$(a \subseteq b) = (b = b + a) \quad (T_{M764}) \subseteq \text{Equality Form}$$

Proof of (T_{M764})

$(a \subseteq b) = (a = ab)$	(A_{MX44})($\subseteq$ Equality Form)
$(\sim b \subseteq \sim a) = (a = ab)$	(T_{M012})(Transposition)
$(\sim b \subseteq \sim a) = (\sim a = \sim(ab))$	(T_{M008})(Inverse of an Equality)
$(\sim b \subseteq \sim a) = (\sim a = \sim a + \sim b)$	(T_{M025})(DeMorgan's Law)
$(a \subseteq b) = (b = b + a)$	Resymbolising

$$\emptyset \subseteq a \quad (T_{M670}) \text{ Null Set Theorem}$$

Proof of (T_{M670})

$a \subseteq E$	(T_{M744})(Universal Set Theorem)
$\sim E \subseteq \sim a$	(T_{M012})(Transposition)
$\emptyset \subseteq a$	(D_{M02})(Definition of $\emptyset$)
$\emptyset \subseteq a$	Resymbolising

$$(a^1 = b^1) = (a = b) \quad (T_{M479})$$

Proof of (T_{M479})

$(a = b) \subseteq (F(a) = F(b))$	$(\mathbf{A}_{MX19})$ (Substitution Axiom)(Eqn1)
Let $F(a) = a^1$	(Eqn2)
Thus $F(b) = b^1$	$(\mathbf{A}_{MX02})$ (Conditional Introduction)(Eqn3)
$(a = b) \subseteq (a^1 = b^1)$	$(\mathbf{A}_{MX19})$ (Substitution of Eqns2&3 into Eqn1)(Eqn4)
$(a^1 = b^1) \subseteq (F(a^1) = F(b^1))$	$(\mathbf{A}_{MX19})$ (Substitution Axiom)(Eqn5)
Let $F(a^1) = a$	(Eqn6)
Thus $F(b^1) = b$	$(\mathbf{A}_{MX02})$ (Conditional Introduction)(Eqn7)
$(a^1 = b^1) \subseteq (a = b)$	$(\mathbf{A}_{MX19})$ (Substitution of Eqns6&7 into Eqn5)(Eqn8)
$(a^1 = b^1) = (a = b)$	$(\mathbf{T}_{M430})$ (Biconditional Introduction Eqns4&8)

$$(a^1 = b) \subseteq (b = b^1) \quad (\mathbf{T}_{M782})$$

Proof of $(\mathbf{T}_{M782})$

Let $a^1 = b$	(Eqn1)
$a^{11} = b^1$	$(\mathbf{T}_{M479})$
$(a^1 \neq \emptyset) \subseteq (a^1 = b^1)$	$(\mathbf{D}_{M17})$ (Definition of a^{11})(Eqn2)
$(a^1 \neq \emptyset) \subseteq (b = b^1)$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn1 into Eqn2)
Thus $((a^1 = b) = E) \subseteq ((a^1 \neq \emptyset) \subseteq (b = b^1))$	$(\mathbf{A}_{MX02})$ (Conditional Introduction)
$(a^1 = b)(a^1 \neq \emptyset) \subseteq (b = b^1)$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets
$((a^1 \neq \emptyset) = E) \subseteq ((a^1 = b) \subseteq (b = b^1))$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets

$(a \trianglelefteq b) = (b \triangleright a)$ where $\triangleleft$ is any asymmetrical membership or ordering relation. $(\mathbf{T}_{M737})$ Definition of $\trianglelefteq$

Proof of $(\mathbf{T}_{M737})$

$(a = b) = (b = a)$	$(\mathbf{T}_{M197})$ (Commutativity of $=$)(Eqn1)
$(a \triangleleft b) = (b \triangleright a)$	$(\mathbf{D}_{M23})$ (Definition of $\triangleleft$)(Eqn2)
$(a \triangleleft b) + (a = b) = (b \triangleright a) + (b = a)$	$(\mathbf{A}_{MX21})$ (Combination of Eqns 1&2)
$(a \trianglelefteq b) = (b \triangleright a)$	$(\mathbf{D}_{M18})$ (Definition of $\trianglelefteq$)

$$\underline{\emptyset} = E \quad (\mathbf{T}_{M054}) \text{ Negative of Null Set}$$

Proof of $(\mathbf{T}_{M054})$

$(a = b) = (\underline{a} = \underline{b})$	$(\mathbf{T}_{M008})$ (Inverse of an Equality)(Eqn1)
Let $a = \emptyset$	(Eqn2)
Let $b = \underline{E}$	(Eqn3)
$(\emptyset = \underline{E}) = (\underline{\emptyset} = \sim \underline{E})$	$(\mathbf{A}_{MX19})$ (Substitution of Eqns2&3 into Eqn1)
$(\emptyset = \underline{E}) = (\underline{\emptyset} = E)$	$(\mathbf{A}_{MX13})$ (Double Complement Axiom)
$E = (\underline{\emptyset} = E)$	$(\mathbf{T}_{M001})$ (Universality of $(\mathbf{D}_{M02})$ Definition of $\emptyset$)
$\underline{\emptyset} = E$	$(\mathbf{A}_{MX16})$ (Order Axiom)

$$\emptyset^1 \subseteq E \quad (\mathbf{T}_{M542})$$

Proof of $(\mathbf{T}_{M542})$

$a \subseteq E$	$(\mathbf{T}_{M744})$ (Universal Set Theorem)(Eqn1)
Let $a = \emptyset^1$	(Eqn2)
$\emptyset^1 \subseteq E$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn2 into Eqn1)

$$(a = E) \subseteq (b \subseteq a) \quad (\mathbf{T}_{M440}) \text{ Implication Introduction}$$

Proof of $(\mathbf{T}_{M440})$

Let $(a = E)$	(Eqn1)
$(b \subseteq E)$	$(\mathbf{T}_{M744})$ (Universal Set Theorem)(Eqn2)
$(b \subseteq a)$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn1 into Eqn2)
Thus $(a = E) \subseteq (b \subseteq a)$	$(\mathbf{A}_{MX02})$ (Conditional Introduction)

$$(a \neq \underline{a}) = E \quad (\mathbf{T}_{M537})$$

$$(a = \underline{a}) = \emptyset \quad (T_{M535})$$

Proof of (T_{M537}) & (T_{M535})

$$\begin{aligned} a &\neq \underline{a} && (A_{MX08})(\text{Existence Axiom}) \\ (a \neq \underline{a}) &= E && (A_{MX16})(\text{Order Axiom}) \\ (a = \underline{a}) &= \sim E && (T_{M008})(\text{Inverse of an Equality}) \\ (a = \underline{a}) &= \emptyset && (D_{M02})(\text{Definition of } \emptyset) \end{aligned}$$

$$\sim a^1 \parallel b \quad (T_{M731})$$

Proof of (T_{M731})

$$\begin{aligned} a^1 \parallel b &&& (A_{MX15})(\text{Intersection with the Unit Set Axiom}) \\ \sim a^1 \parallel \sim b &&& (T_{M194})(\text{Co-intersection}) \& (T_{M008})(\text{Inverse of an Equality}) \\ \sim a^1 \parallel b &&& \text{Resymbolising} \end{aligned}$$

$$(a = E) = (\underline{a} = \emptyset) \quad (T_{M014})$$

Proof of (T_{M014})

$$\begin{aligned} \text{Let } a &= E \\ \underline{a} &= \sim E && (T_{M008})(\text{Inverse of an Equality}) \\ \underline{a} &= \emptyset && (D_{M02})(\text{Definition of } \emptyset) \\ \text{Thus } (a = E) &= (\underline{a} = \emptyset) && (A_{MX02})(\text{Conditional Introduction}) \text{ since the proof is reversible.} \end{aligned}$$

Alternative Proof of (T_{M014})

$$\begin{aligned} (a = b) &= (\underline{a} = \underline{b}) && (T_{M008})(\text{Inverse of an Equality})(\text{Eqn1}) \\ \text{Let } b &= E && (\text{Eqn2}) \\ \underline{b} &= \underline{E} && (T_{M008})(\text{Inverse of an Equality})(\text{Eqn3}) \\ \emptyset &= \underline{E} && (D_{M02})(\text{Definition of } \emptyset)(\text{Eqn4}) \\ \underline{b} &= \emptyset && (A_{MX19})(\text{Substitution of Eqn3 into Eqn4})(\text{Eqn5}) \\ (a = E) &= (\underline{a} = \emptyset) && (A_{MX19})(\text{Substitution of Eqns2\&5 into Eqn1}) \end{aligned}$$

$$(a \neq E) = (\underline{a} \neq \emptyset) \quad (T_{M416})$$

Proof of (T_{M416})

$$\begin{aligned} (a = E) &= (\underline{a} = \emptyset) && (T_{M014}) \\ (a \neq E) &= (\underline{a} \neq \emptyset) && (T_{M008})(\text{Inverse of an Equality}) \end{aligned}$$

$$(\underline{a} = E) = (a = \emptyset) \quad (T_{M417})$$

Proof of (T_{M417})

$$\begin{aligned} (a = E) &= (\underline{a} = \emptyset) && (T_{M014}) \\ (\underline{a} = E) &= (a = \emptyset) && \text{Resymbolising} \end{aligned}$$

$$(\underline{a} \neq E) = (a \neq \emptyset) \quad (T_{M415})$$

Proof of (T_{M415})

$$\begin{aligned} (a \neq E) &= (\underline{a} \neq \emptyset) && (T_{M416}) \\ (\underline{a} \neq E) &= (a \neq \emptyset) && \text{Resymbolising} \end{aligned}$$

$$(\underline{a} = E) \subseteq (a \neq E) \quad (T_{M462})$$

Proof of (T_{M462})

$$\begin{aligned} (a = b) &\subseteq (a \neq \underline{b}) && (T_{M144}) \\ (b = a) &\subseteq (a \neq \underline{b}) && (T_{M197})(\text{Commutativity of } =)(\text{Eqn 1}) \\ (a = b) &= (b = a) && (T_{M197})(\text{Commutativity of } =) \\ (a \neq b) &= (b \neq a) && (T_{M008})(\text{Inverse of an Equality}) \\ (a \neq \underline{b}) &= (\underline{b} \neq a) && \text{Resymbolising (Eqn 2)} \end{aligned}$$

$$\begin{array}{ll} (b = a) \subseteq (b \neq a) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (\underline{a} = E) \subseteq (a \neq E) & \text{Resymbolising} \end{array}$$

$$(a + b = E)(\underline{a} = E) \subseteq (b = E) \quad (\mathbf{T}_{M442}) \text{ Disjunctive Syllogism}$$

Proof of ($\mathbf{T}_{M442}$)

$$\begin{array}{ll} \text{Let } (a + b = E)(\underline{a} = E) & \\ (a + b = E)(a = \emptyset) & (\mathbf{T}_{M417}) \\ \emptyset + b = E & (\mathbf{A}_{MX19})(\text{Substitution}) \\ b = E & (\mathbf{T}_{M668})(\text{Null Set Disjunction}) \\ \text{Thus } (a + b = E)(\underline{a} = E) \subseteq (b = E) & (\mathbf{A}_{MX02})(\text{Conditional Introduction}) \end{array}$$

$$((a \not\subseteq b) = E) \subseteq ((a \subseteq b) \neq E) \quad (\mathbf{T}_{M463})$$

Proof of ($\mathbf{T}_{M463}$)

$$\begin{array}{ll} (\underline{a} = E) \subseteq (a \neq E) & (\mathbf{T}_{M462}) \\ ((a \not\subseteq b) = E) \subseteq ((a \subseteq b) \neq E) & \text{Resymbolising} \end{array}$$

$$(((a \subseteq b) \neq E) = E) \subseteq (((a \subseteq b) = E) \neq E) \quad (\mathbf{T}_{M464})$$

Proof of ($\mathbf{T}_{M464}$)

$$\begin{array}{ll} (\underline{a} = E) \subseteq (a \neq E) & (\mathbf{T}_{M462}) \\ (((a \subseteq b) \neq E) = E) \subseteq (((a \subseteq b) = E) \neq E) & \text{Resymbolising} \end{array}$$

$$(a = b) \subseteq (a + c = b + c) \quad (\mathbf{T}_{M232}) \text{ Deduction}$$

Proof of ($\mathbf{T}_{M232}$)

$$\begin{array}{ll} (a = b)(c = d) \subseteq (a + c = b + d) & (\mathbf{T}_{M230})(\text{Combination})(\text{Eqn1}) \\ \text{Let } d = c & (\text{Eqn2}) \\ (a = b)(c = c) \subseteq (a + c = b + c) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (a = b)E \subseteq (a + c = b + c) & (\mathbf{T}_{M001})(\text{Universality of } (\mathbf{D}_{M00}) \text{ Definition of } =) \\ (a = b) \subseteq (a + c = b + c) & (\mathbf{A}_{MX11})(\text{Universal Set Conjunction Axiom}) \end{array}$$

$$(a = b) \subseteq (ac = bc) \quad (\mathbf{T}_{M233}) \text{ Deduction}$$

Proof of ($\mathbf{T}_{M233}$)

$$\begin{array}{ll} (a = b)(c = d) \subseteq (ac = bd) & (\mathbf{T}_{M231})(\text{Combination})(\text{Eqn1}) \\ \text{Let } d = c & (\text{Eqn2}) \\ (a = b)(c = c) \subseteq (ac = bc) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (a = b)E \subseteq (ac = bc) & (\mathbf{T}_{M001})(\text{Universality of } (\mathbf{D}_{M00}) \text{ Definition of } =) \\ (a = b) \subseteq (ac = bc) & (\mathbf{A}_{MX11})(\text{Universal Set Conjunction Axiom}) \end{array}$$

$$\sim \emptyset^1 \neq (\sim \emptyset^1)^1 \quad (\mathbf{T}_{M548})$$

Proof of ($\mathbf{T}_{M548}$)

$$\begin{array}{ll} \sim a^1 \neq (\sim a^1)^1 & (\mathbf{T}_{M554})(\text{Eqn1}) \\ \text{Let } a^1 = \emptyset^1 & (\text{Eqn2}) \\ \sim \emptyset^1 \neq (\sim \emptyset^1)^1 & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{array}$$

$$\sim E^1 \neq (\sim E^1)^1 \quad (\mathbf{T}_{M547})$$

Proof of ($\mathbf{T}_{M547}$)

$$\begin{array}{ll} \sim a^1 \neq (\sim a^1)^1 & (\mathbf{T}_{M554})(\text{Eqn1}) \\ \text{Let } a^1 = E^1 & (\text{Eqn2}) \\ \sim E^1 \neq (\sim E^1)^1 & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{array}$$

$\sim(A_n) = \emptyset, \sim(D_n) = \emptyset$ & $\sim(T_n) = \emptyset$ (T_{M024}) Non-existence of Anti-Axioms & Definitions

Proof of (T_{M024})

$(A_n) = E$

(T_{M001}) (Universality of Axioms Definitions & Theorems)

$\sim(A_n) = \underline{E}$

(T_{M008}) (Inverse of an Equality)

$\sim(A_n) = \emptyset$

(D_{M02}) (Definition of $\emptyset$)

And similarly for definitions and theorems.

$\emptyset \neq E$

(T_{M352})

Proof of (T_{M352})

$a \neq a$

(A_{MX08}) (Existence Axiom) (Eqn1)

Let $a = \emptyset$

(Eqn2)

$\emptyset \neq \underline{\emptyset}$

(A_{MX19}) (Substitution of Eqn3 into Eqn4)

$\emptyset \neq E$

(T_{M054}) (Negative of Null Set)

$(a \neq a) = \emptyset$

(T_{M536})

Proof of (T_{M536})

$(a = a) = E$

(T_{M534})

$(a \neq a) = \sim E$

(T_{M008}) (Inverse of an Equality)

$(a \neq a) = \emptyset$

(D_{M02}) (Definition of $\emptyset$)

$(E \neq E) = \emptyset$

(T_{M345})

Proof of (T_{M345})

$(E = E) = E$

(T_{M342})

$(E \neq E) = \underline{E}$

(T_{M008}) (Inverse of an Equality)

$(E \neq E) = \emptyset$

(D_{M02}) (Definition of $\emptyset$)

$(\emptyset \neq \emptyset) = \emptyset$

(T_{M346})

Proof of (T_{M346})

$(\emptyset = \emptyset) = E$

(T_{M343})

$(\emptyset \neq \emptyset) = \underline{E}$

(T_{M008}) (Inverse of an Equality)

$(\emptyset \neq \emptyset) = \emptyset$

(D_{M02}) (Definition of $\emptyset$)

$(E \neq \emptyset) = E$

(T_{M344})

Proof of (T_{M344})

$\emptyset \neq E$

(T_{M352})

$(\emptyset \neq E) = E$

(A_{MX16}) (Order Axiom)

$(E = \emptyset) = \emptyset$

(T_{M347})

Proof of (T_{M347})

$(E \neq \emptyset) = E$

(T_{M344})

$(E = \emptyset) = \underline{E}$

(T_{M008}) (Inverse of an Equality)

$(E = \emptyset) = \emptyset$

(D_{M02}) (Definition of $\emptyset$)

$(a = \emptyset) \subseteq (a \neq E)$

(T_{M059})

Proof of (T_{M059})

Let $a = \emptyset$

(Eqn1)

$\emptyset \neq E$

(T_{M352}) (Eqn2)

$a \neq E$

(A_{MX19}) (Substitution of Eqn1 into Eqn2)

This proof cannot be reversed, i.e., $a \neq E$ and $\emptyset \neq E$ do not necessarily imply that $a = \emptyset$

Thus $(a = \emptyset) \subseteq (a \neq E)$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)

$(E \subseteq a) = (a = E)$ ($\mathbf{T}_{M275}$) Supersets of the Universal Set

Proof of ($\mathbf{T}_{M275}$)

Let $E \subseteq a$ (Eqn1)
 $a \subseteq E$ ($\mathbf{T}_{M744}$)(Universal Set Theorem)(Eqn2)
 $a = E$ ($\mathbf{T}_{M667}$)(Conjunction Tautology Eqns1&2)
 Thus $(E \subseteq a) \subseteq (a = E)$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)(Eqn3)
 Let $a = E$ (Eqn4)
 $a \subseteq E$ ($\mathbf{T}_{M744}$)(Universal Set Theorem)(Eqn5)
 $E \subseteq a$ ($\mathbf{A}_{MX19}$)(Substitution of Eqn4 into Eqn5) x2
 Thus $(a = E) \subseteq (E \subseteq a)$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)(Eqn6)
 $(E \subseteq a) = (a = E)$ ($\mathbf{T}_{M430}$)(Biconditional Introduction Eqns3&6)

$(E \subseteq a) = (a = \emptyset)$ ($\mathbf{T}_{M204}$) Supersets of the Universal Set

Proof of ($\mathbf{T}_{M204}$)

$(E \subseteq a) = (a = E)$ ($\mathbf{T}_{M275}$)(Supersets of the Universal Set)
 $(E \subseteq a) = (a = E)$ ($\mathbf{T}_{M008}$)(Inverse of an Equality)
 $(E \subseteq a) = (a = \emptyset)$ ($\mathbf{D}_{M02}$)(Definition of $\emptyset$)

$(a \subseteq \emptyset) = (a = \emptyset)$ ($\mathbf{T}_{M052}$) Subsets of the Null Set

Proof of ($\mathbf{T}_{M052}$)

$(E \subseteq a) = (a = E)$ ($\mathbf{T}_{M275}$)(Supersets of the Universal Set)
 $(\sim a \subseteq \sim E) = (a = E)$ ($\mathbf{T}_{M012}$)(Transposition)
 $(\sim a \subseteq \sim E) = (\sim a = \sim E)$ ($\mathbf{T}_{M008}$)(Inverse of an Equality)
 $(a \subseteq \emptyset) = (a = \emptyset)$ ($\mathbf{D}_{M02}$)(Definition of $\emptyset$)
 $(a \subseteq \emptyset) = (a = \emptyset)$ Resymbolising

$(a = E)(c = \emptyset)(ab \subseteq c) \subseteq (b = \emptyset)$ ($\mathbf{T}_{M790}$)

Proof of ($\mathbf{T}_{M790}$)

Let $a = E$ (Eqn1)
 Let $c = \emptyset$ (Eqn2)
 Let $ab \subseteq c$ (Eqn3)
 $Eb \subseteq \emptyset$ ($\mathbf{A}_{MX19}$)(Substitution of Eqns1&2 into Eqn3)
 $b \subseteq \emptyset$ ($\mathbf{A}_{MX11}$)(Universal Set Conjunction Axiom)
 $b = \emptyset$ ($\mathbf{T}_{M052}$)(Subsets of the Null Set)
 Thus $(a = E)(c = \emptyset)(ab \subseteq c) \subseteq (b = \emptyset)$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)

$(a = E)(a \subseteq b) \subseteq (b = E)$ ($\mathbf{T}_{M427}$) Modus Ponens

Proof of ($\mathbf{T}_{M427}$)

Let $(a = E)(a \subseteq b)$
 $(E \subseteq b)$ ($\mathbf{A}_{MX19}$)(Substitution)
 $(b = E)$ ($\mathbf{T}_{M275}$)(Supersets of the Universal Set)
 Thus $(a = E)(a \subseteq b) \subseteq (b = E)$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)

$(a \subseteq b)(c \subseteq d)(a + c = E) \subseteq (b + d = E)$ ($\mathbf{T}_{M446}$) Constructive Dilemma

Proof of ($\mathbf{T}_{M446}$)

Let $(a \subseteq b)(c \subseteq d)(a + c = E)$
 $(a + c \subseteq b + d)(a + c = E)$ ($\mathbf{A}_{MX21}$)(Combination Axiom)
 $E \subseteq b + d$ ($\mathbf{A}_{MX19}$)(Substitution)
 $E = b + d$ ($\mathbf{T}_{M275}$)(Supersets of the Universal Set)
 Thus $(a \subseteq b)(c \subseteq d)(a + c = E) \subseteq (b + d = E)$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)

$$((a \subseteq b) = E) = ((a = E) \subseteq (b = E))$$

(T_{M094}) Probabilistic Rule

Proof of (T_{M094})

$$(a = E)(a \subseteq b) \subseteq (b = E)$$

(T_{M427}) (Modus Ponens)

$$((a \subseteq b) = E) \subseteq ((a = E) \subseteq (b = E))$$

(A_{MX35})(Exportation/Importation) for truth sets (Eqn1)

$$\text{Let } (a = E) \subseteq (b = E)$$

$$a \subseteq b$$

(A_{MX16})(Order Axiom)

$$\text{Thus } ((a = E) \subseteq (b = E)) \subseteq ((a \subseteq b) = E)$$

(A_{MX02})(Conditional Introduction)(Eqn2)

$$((a \subseteq b) = E) = ((a = E) \subseteq (b = E))$$

(T_{M430})(Biconditional Introduction Eqns1&2)

$$((a \not\subseteq b) = E) \subseteq ((a = E) \subseteq (b \neq E))$$

(T_{M100}) Probabilistic Rule

Proof of (T_{M100})

$$\text{Let } (a \not\subseteq b)$$

(Eqn1)

$$\text{Let } (a = E)$$

(Eqn2)

$$(E \not\subseteq b)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(b \neq E)$$

(T_{M275}) & (T_{M008})

$$\text{Thus } (a \not\subseteq b)(a = E) \subseteq (b \neq E)$$

(A_{MX02})(Conditional Introduction)(Eqn3)

$$\text{Let } (a \not\subseteq b) = E$$

(Eqn4)

$$E(a = E) \subseteq (b \neq E)$$

(A_{MX19})(Substitution of Eqn4 into Eqn3)

$$(a = E) \subseteq (b \neq E)$$

(A_{MX17})(Universal Set Conjunction Axiom)

$$\text{Thus } ((a \not\subseteq b) = E) \subseteq ((a = E) \subseteq (b \neq E))$$

(A_{MX02})(Conditional Introduction)

$$((a \not\subseteq b) = E) \subseteq ((a = E) \subseteq (b \neq \emptyset))$$

(T_{M319}) Probabilistic Rule

Proof of (T_{M319})

$$((a \not\subseteq b) = E) \subseteq ((a = E) \subseteq (b \neq E))$$

(T_{M100})(Probabilistic Rule)

$$((a \not\subseteq b) = E) \subseteq ((a = E) \subseteq (b \neq E))$$

Resymbolising

$$((a \not\subseteq b) = E) \subseteq ((a = E) \subseteq (b \neq E))$$

(T_{M008}) & (T_{M008})

$$((a \not\subseteq b) = E) \subseteq ((a = E) \subseteq (b \neq \emptyset))$$

(D_{M02})(Definition of $\emptyset$)

$$(x \neq (a, b, c)) = (x \neq a).(x \neq b).(x \neq c)$$

(T_{M716})

Proof of (T_{M716})

$$(x = (a, b, c)) = (x = a) + (x = b) + (x = c)$$

(D_{M15})(Definition of a Variable Set)

$$(x \neq (a, b, c)) = \sim((x = a) + (x = b) + (x = c))$$

(T_{M008})(Inverse of an Equality)

$$(x \neq (a, b, c)) = (x \neq a).(x \neq b).(x \neq c)$$

(T_{M026})(DeMorgan's Law)

$$(x \neq \Omega) = (x \neq \sim\emptyset^1).(x \neq \sim\emptyset^1).(x \neq E^1).(x \neq \sim E^1)$$

(T_{M723})

Proof of (T_{M723})

$$(x \neq (\emptyset^1, \sim\emptyset^1, E^1, \sim E^1)) = (x \neq \sim\emptyset^1).(x \neq \sim\emptyset^1).(x \neq E^1).(x \neq \sim E^1)$$

(T_{M716})

$$(x \neq \Omega) = (x \neq \sim\emptyset^1).(x \neq \sim\emptyset^1).(x \neq E^1).(x \neq \sim E^1)$$

(D_{M16})(Definition of Ω)

$$(a \neq \Omega) = (a \neq \Omega)$$

(T_{M734})

Proof of (T_{M734})

$$\text{Let } a \neq \Omega$$

$$(a \neq \emptyset^1).(a \neq \sim\emptyset^1).(a \neq E^1).(a \neq \sim E^1)$$

(T_{M723})

$$(a \neq \sim\emptyset^1).(a \neq \emptyset^1).(a \neq \sim E^1).(a \neq E^1)$$

(T_{M008})(T_{M008})(Inverse of an Equality)

$$(a \neq \emptyset^1).(a \neq \sim\emptyset^1).(a \neq E^1).(a \neq \sim E^1)$$

(T_{M404})(Commutative .)

$$a \neq \Omega$$

(T_{M723})

$$\text{Thus } (a \neq \Omega) = (a \neq \Omega)$$

(A_{MX02})(Conditional Introduction) since this proof is reversible.

$$(a \not\subseteq b) = \sim(a \subseteq b) \sim (a = b) \quad (T_{M198})$$

Proof of (T_{M198})

$$\begin{aligned} (a \subseteq b) &= (a \subseteq b) + (a = b) & (T_{M266}) (\text{Definition of } \subseteq) \\ (a \not\subseteq b) &= \sim((a \subseteq b) + (a = b)) & (T_{M008}) (\text{Inverse of an Equality}) \\ (a \not\subseteq b) &= \sim(a \subseteq b) \sim (a = b) & (T_{M026}) (\text{DeMorgan's Law}) \end{aligned}$$

$$(a \not\supset b) = \sim(a \supset b) \sim (a = b) \quad (T_{M199})$$

Proof of (T_{M199})

$$\begin{aligned} (a \supset b) &= (a \supset b) + (a = b) & (T_{M267}) (\text{Definition of } \supset) \\ (a \not\supset b) &= \sim((a \supset b) + (a = b)) & (T_{M008}) (\text{Inverse of an Equality}) \\ (a \not\supset b) &= \sim(a \supset b) \sim (a = b) & (T_{M026}) (\text{DeMorgan's Law}) \end{aligned}$$

$$((c^1 \subseteq a) \subseteq (c^1 \subseteq b)) = ((c^1 \not\subseteq b) \subseteq (c^1 \not\subseteq a)) \quad (T_{M698})$$

Proof of (T_{M698})

$$\begin{aligned} (a \subseteq b) &= (\underline{b} \subseteq \underline{a}) & (T_{M012}) (\text{Transposition}) \\ ((c^1 \subseteq a) \subseteq (c^1 \subseteq b)) &= ((c^1 \not\subseteq b) \subseteq (c^1 \not\subseteq a)) & \text{Resymbolising} \end{aligned}$$

$$(a \subseteq b)(c \subseteq d)(\underline{b} + \underline{d} = E) \subseteq (\underline{a} + \underline{c} = E) \quad (T_{M447}) \text{ Destructive Dilemma}$$

Proof of (T_{M447})

$$\begin{aligned} \text{Let } (a \subseteq b)(c \subseteq d)(\underline{b} + \underline{d} = E) \\ (\underline{b} \subseteq \underline{a})(\underline{d} \subseteq \underline{c})(\underline{b} + \underline{d} = E) & (T_{M012}) (\text{Transposition}) \\ (\underline{b} + \underline{d} \subseteq \underline{a} + \underline{c})(\underline{b} + \underline{d} = E) & (A_{MX21}) (\text{Combination Axiom}) \\ E \subseteq \underline{a} + \underline{c} & (A_{MX19}) (\text{Substitution}) \\ E = \underline{a} + \underline{c} & (T_{M275}) (\text{Supersets of the Universal Set}) \\ \text{Thus } (a \subseteq b)(c \subseteq d)(\underline{b} + \underline{d} = E) \subseteq (\underline{a} + \underline{c} = E) & (A_{MX02}) (\text{Conditional Introduction}) \end{aligned}$$

$$(a \subseteq b)(c \subseteq d)(a + \underline{d} = E) \subseteq (b + \underline{c} = E) \quad (T_{M448}) \text{ Bidirectional Dilemma}$$

Proof of (T_{M448})

$$\begin{aligned} \text{Let } (a \subseteq b)(c \subseteq d)(a + \underline{d} = E) \\ (a \subseteq b)(\underline{d} \subseteq \underline{c})(a + \underline{d} = E) & (T_{M012}) (\text{Transposition}) \\ (a + \underline{d} \subseteq b + \underline{c})(a + \underline{d} = E) & (A_{MX21}) (\text{Combination Axiom}) \\ E \subseteq b + \underline{c} & (A_{MX19}) (\text{Substitution}) \\ E = b + \underline{c} & (T_{M275}) (\text{Supersets of the Universal Set}) \\ \text{Thus } (a \subseteq b)(c \subseteq d)(a + \underline{d} = E) \subseteq (b + \underline{c} = E) & (A_{MX02}) (\text{Conditional Introduction}) \end{aligned}$$

$$((a \not\subseteq b) = E) \subseteq ((b = E) \subseteq (a \neq E)) \quad (T_{M101}) \text{ Probabilistic Rule}$$

Proof of (T_{M101})

$$\begin{aligned} ((a \not\subseteq b) = E) &\subseteq ((a = E) \subseteq (b \neq E)) & (T_{M100}) (\text{Probabilistic Rule}) \\ ((a \not\subseteq b) = E) &\subseteq (\sim(b \neq E) \subseteq \sim(a = E)) & (T_{M012}) (\text{Transposition}) \\ ((a \not\subseteq b) = E) &\subseteq (\sim\sim(b = E) \subseteq \sim(a \neq E)) & (T_{M260}) (\text{Definition of } \neq) \\ ((a \not\subseteq b) = E) &\subseteq ((b = E) \subseteq (a \neq E)) & (A_{MX13}) (\text{Double Complement Axiom}) \end{aligned}$$

$$((a \not\subseteq b) = E) \subseteq ((b = \emptyset) \subseteq (a \neq \emptyset)) \quad (T_{M102}) \text{ Probabilistic Rule}$$

Proof of (T_{M102})

$$\begin{aligned} ((a \not\subseteq b) = E) &\subseteq ((a = E) \subseteq (b \neq E)) & (T_{M100}) (\text{Probabilistic Rule}) \\ ((\underline{b} \not\subseteq \underline{a}) = E) &\subseteq ((\underline{b} = E) \subseteq (\underline{a} \neq E)) & \text{Resymbolising} \\ ((a \not\subseteq b) = E) &\subseteq ((\underline{b} = E) \subseteq (\underline{a} \neq E)) & (T_{M012}) (T_{M008}) \\ ((a \not\subseteq b) = E) &\subseteq ((b = E) \subseteq (a \neq E)) & (T_{M008}) (\text{Inverse of an Equality}) \\ ((a \not\subseteq b) = E) &\subseteq ((b = \emptyset) \subseteq (a \neq \emptyset)) & (D_{M02}) (\text{Definition of } \emptyset) \end{aligned}$$

$$((a \not\subseteq b) = E) \subseteq ((a = \emptyset) \subseteq (b \neq \emptyset)) \quad (T_{M103}) \text{ Probabilistic Rule}$$

Proof of (T_{M103})

$$\begin{aligned} ((a \not\subseteq b) = E) &\subseteq ((b = \emptyset) \subseteq (a \neq \emptyset)) && (T_{M102}) \text{ (Probabilistic Rule)} \\ ((a \not\subseteq b) = E) &\subseteq (\sim(a \neq \emptyset) \subseteq \sim(b = \emptyset)) && (T_{M012}) \text{ (Transposition)} \\ ((a \not\subseteq b) = E) &\subseteq (\sim\sim(a = \emptyset) \subseteq (b \neq \emptyset)) && (T_{M260}) \text{ (Definition of } \neq) \\ ((a \not\subseteq b) = E) &\subseteq ((a = \emptyset) \subseteq (b \neq \emptyset)) && (A_{MX13}) \text{ (Double Complement Axiom)} \end{aligned}$$

$$((a = b) = E) = ((a = E) = (b = E)) \quad (T_{M094b}) \text{ Probabilistic Rule}$$

Proof of (T_{M094b})

$$\begin{aligned} \text{Let } (a = E) &= (b = E) \\ ((a = E) \subseteq (b = E)) &\cdot ((b = E) \subseteq (a = E)) && (T_{M430}) \text{ (Biconditional Introduction)} \\ ((a \subseteq b) = E) &\cdot ((b \subseteq a) = E) && (T_{M094}) \text{ (Probabilistic Rule)} \\ (a \subseteq b) &\cdot (b \subseteq a) && (A_{MX16}) \text{ (Order Axiom)} \\ (a = b) &&& (T_{M430}) \text{ (Biconditional Introduction)} \\ (a = b) &= E && (A_{MX16}) \text{ (Order Axiom)} \\ \text{Thus } ((a = b) = E) &= ((a = E) = (b = E)) && (A_{MX02}) \text{ (Conditional Introduction) since this proof is reversible} \end{aligned}$$

$$((a = b) = E) = ((a \neq \emptyset) = (b \neq \emptyset)) \quad (T_{M523}) \text{ Probabilistic Rule}$$

Proof of (T_{M523})

$$\begin{aligned} ((a = b) = E) &= ((a = E) = (b = E)) && (T_{M094b}) \text{ (Probabilistic Rule)} \\ ((a = b) = E) &= ((a = \sim E) = (b = \sim E)) && (T_{M008}) \text{ (Inverse of an Equality)} \\ ((a = b) = E) &= ((a = \emptyset) = (b = \emptyset)) && (D_{M02}) \text{ (Definition of } \emptyset) \\ ((a = b) = E) &= ((a \neq \emptyset) = (b \neq \emptyset)) && (T_{M008}) \text{ (Inverse of an Equality)} \\ ((a = b) = E) &= ((a \neq \emptyset) = (b \neq \emptyset)) && (T_{M008}) \text{ (Inverse of an Equality)} \\ ((a = b) = E) &= ((a \neq \emptyset) = (b \neq \emptyset)) && \text{Resymbolising} \end{aligned}$$

$$((a = E) \subseteq (b = E)) = ((b = \emptyset) \subseteq (a = \emptyset)) \quad (T_{M697})$$

Proof of (T_{M697})

$$\begin{aligned} (a \subseteq b) &= (b \subseteq a) && (T_{M012}) \text{ (Transposition)} \\ ((a \subseteq b) = (b \subseteq a)) &= E && (A_{MX16}) \text{ (Order Axiom)} \\ (((a \subseteq b) = E) = ((b \subseteq a) = E)) &&& (T_{M094b}) \text{ (Probabilistic Rule)} \\ ((a = E) \subseteq (b = E)) &= ((b = E) \subseteq (a = E)) && (T_{M094}) \text{ (Probabilistic Rule)} \\ ((a = E) \subseteq (b = E)) &= ((b = \sim E) \subseteq (a = \sim E)) && (T_{M008}) \text{ (Inverse of an Equality)} \\ ((a = E) \subseteq (b = E)) &= ((b = \emptyset) \subseteq (a = \emptyset)) && (D_{M02}) \text{ (Definition of } \emptyset) \end{aligned}$$

$$(a \subseteq b)(c \subseteq d) \subseteq (ac \subseteq bd) \quad (T_{M669}) \text{ Combination}$$

Proof of (T_{M669})

$$\begin{aligned} (a \subseteq b)(c \subseteq d) &\subseteq (a + c \subseteq b + d) && (A_{MX21}) \text{ (Combination Axiom)} \\ (b \subseteq a)(d \subseteq c) &\subseteq (b + d \subseteq a + c) && \text{Resymbolising} \\ (b \subseteq a)(d \subseteq c) &\subseteq (\sim(bd) \subseteq \sim(ac)) && (T_{M025}) \text{ (DeMorgan's Law)} \\ (a \subseteq b)(c \subseteq d) &\subseteq (ac \subseteq bd) && (T_{M012}) \text{ (Transposition)} \end{aligned}$$

$$(a \parallel b) \subseteq (a \neq \Omega)(b \neq \Omega) \quad (T_{M732})$$

Proof of (T_{M732})

$$\begin{aligned} \text{Let } a &= \emptyset^1 && \text{(Eqn1)} \\ \emptyset^1 \parallel b &&& (A_{MX15}) \text{ (Intersection with the Unit Set Axiom) (Eqn2)} \\ a \parallel b &&& (A_{MX19}) \text{ (Substitution of Eqn1 into Eqn2)} \\ \text{Thus } (a = \emptyset^1) &\subseteq (a \parallel b) && (A_{MX02}) \text{ (Conditional Introduction) (Eqn3)} \\ \text{Let } a &= \sim \emptyset^1 && \text{(Eqn4)} \\ \sim \emptyset^1 \parallel b &&& (T_{M731}) \text{ (Eqn5)} \\ a \parallel b &&& (A_{MX19}) \text{ (Substitution of Eqn4 into Eqn5)} \end{aligned}$$

Thus $(a = \sim\emptyset^1) \subseteq (a \parallel b)$	(A_{MX02}) (Conditional Introduction)(Eqn6)
Let $a = E^1$	(Eqn7)
$E^1 \parallel b$	(A_{MX15}) (Intersection with the Unit Set Axiom)(Eqn8)
$a \parallel b$	(A_{MX19}) (Substitution of Eqn7 into Eqn8)
Thus $(a = E^1) \subseteq (a \parallel b)$	(A_{MX02}) (Conditional Introduction)(Eqn9)
Let $a = \sim E^1$	(Eqn10)
$\sim E^1 \parallel b$	(T_{M731}) (Eqn11)
$a \parallel b$	(A_{MX19}) (Substitution of Eqn10 into Eqn11)
Thus $(a = \sim E^1) \subseteq (a \parallel b)$	(A_{MX02}) (Conditional Introduction)(Eqn12)
$(a = \emptyset^1) + (a = \sim\emptyset^1) + (a = E^1) + (a = \sim E^1) \subseteq (a \parallel b) + (a \parallel b) + (a \parallel b) + (a \parallel b)$	(A_{MX21}) (Combination of Eqns3,6,9&12)
$(a = \emptyset^1) + (a = \sim\emptyset^1) + (a = E^1) + (a = \sim E^1) \subseteq (a \parallel b)$	(A_{MX01}) (Disjunction Tautology)
$(a = \emptyset^1, \sim\emptyset^1, E^1, \sim E^1) \subseteq (a \parallel b)$	(D_{M15}) (Definition of a Variable Set)
$(a = \Omega) \subseteq (a \parallel b)$	(D_{M16}) (Definition of Ω)
$(a \parallel b) \subseteq (a \neq \Omega)$	(T_{M012}) (Transposition)(Eqn13)
$(b \parallel a) \subseteq (b \neq \Omega)$	Resymbolising
$(a \parallel b) \subseteq (b \neq \Omega)$	(T_{M193}) (Commutativity of $\parallel$)(Eqn14)
$(a \parallel b)(a \parallel b) \subseteq (a \neq \Omega)(b \neq \Omega)$	(T_{M669}) (Combination of Eqns13&14)
$(a \parallel b) \subseteq (a \neq \Omega)(b \neq \Omega)$	(T_{M667}) (Conjunction Tautology)

$$(a \subset E) = (\emptyset \subset a) \quad (T_{M016})$$

Proof of (T_{M016})

$(a \subset E)$	
$(E \subset a)$	(T_{M010}) (Inverse of a Proper Subset)
$(\emptyset \subset a)$	(D_{M02}) (Definition of $\emptyset$)
Thus $(a \subset E) = (\emptyset \subset a)$	(A_{MX02}) (Conditional Introduction) since this proof is reversible

$$(a \subseteq E) = (\emptyset \subseteq a) \quad (T_{M018})$$

Proof of (T_{M018})

$(a \subseteq b) = (b \subseteq a)$	(T_{M012}) (Transposition)(Eqn1)
Let $b = E$	(Eqn2)
$b = E$	(T_{M008}) (Inverse of an Equality)
$b = \emptyset$	(D_{M02}) (Definition of $\emptyset$)(Eqn3)
$(a \subseteq E) = (\emptyset \subseteq a)$	(A_{MX19}) (Substitution of Eqns2&3 into Eqn1)

$$(a = E) \subseteq (a \neq \emptyset) \quad (T_{M058})$$

Proof of (T_{M058})

$(a = \emptyset) \subseteq (a \neq E)$	(T_{M059})
$(a = E) \subseteq (a \neq \emptyset)$	(T_{M012}) (Transposition)

$$(a = E) \subseteq (a \neq E) \quad (T_{M060})$$

Proof of (T_{M060})

$(a = E) \subseteq (a \neq \emptyset)$	(T_{M058})
$(a = E) \subseteq (a \neq \emptyset)$	(T_{M008}) (Inverse of an Equality)
$(a = E) \subseteq (a \neq E)$	(T_{M054}) (Negative of Null Set)

$$(a = \emptyset) \subseteq (a \neq \emptyset) \quad (T_{M061})$$

Proof of (T_{M061})

$(a = \emptyset) \subseteq (a \neq E)$	(T_{M059})
$(a = \emptyset) \subseteq (a \neq E)$	(T_{M008}) (Inverse of an Equality)
$(a = \emptyset) \subseteq (a \neq \emptyset)$	(D_{M02}) (Definition of $\emptyset$)

$$(a \subseteq b) = (a \underline{b} = \emptyset) \quad (T_{M735}) \subseteq \text{Existential Form}$$

Proof of (T_{M735})

Let $a \underline{b} = \emptyset$

$$(a \underline{b} = \emptyset) = E$$

(A_{MX16})(Order Axiom) (LHS of (A_{MX35}))

$$(a = E) \subseteq (b \subseteq \emptyset)$$

(A_{MX35})(Exportation/Importation) since $a \underline{b} = \emptyset$ (RHS)

$$(a = E) \subseteq (b = \emptyset)$$

(T_{M052})(Subsets of the Null Set)

$$(a = E) \subseteq (b = \sim \emptyset)$$

(T_{M008})(Inverse of an Equality)

$$(a = E) \subseteq (b = E)$$

(T_{M054})(Negative of Null Set)

$$(a \subseteq b) = E$$

(T_{M094})(Probabilistic Rule)

$$(a \subseteq b)$$

(A_{MX16})(Order Axiom)

$$\text{Thus } (a \underline{b} = \emptyset) = (a \subseteq b)$$

(A_{MX02})(Conditional Introduction) since this proof is reversible

$$(a \subseteq b) = (a \underline{b} = \emptyset)$$

(T_{M197})(Commutativity of =)

$$(a \subseteq b) = (E = a + b) \quad (T_{M064}) \subseteq \text{Universal Form}$$

Proof of (T_{M064})

$$(a \subseteq b) = (a \underline{b} = \emptyset)$$

(T_{M735})($\subseteq$ Existential Form)

$$(a \subseteq b) = (\sim(a \underline{b}) = \sim \emptyset)$$

(T_{M008})(Inverse of an Equality)

$$(a \subseteq b) = (\sim(a \underline{b}) = E)$$

(T_{M054})(Negative of Null Set)

$$(a \subseteq b) = (a + b = E)$$

(T_{M025})(DeMorgan's Law)

$$E^1 \neq E \quad (T_{M492})$$

Proof of (T_{M492})

$$(E^1 = E)(E^1 \subset E) = \emptyset$$

(T_{M037})(Conjunction of Binary Relations)(Eqn1)

$$(E^1 \subset E) \subseteq (E^1 \neq E)$$

(T_{M735})($\subseteq$ Existential Form)

$$E^1 \subset E$$

(T_{M481})(Domain of the Universal Set Entity)

$$(E^1 \subset E) = E$$

(A_{MX16})(Order Axiom)(Eqn2)

$$E \subseteq (E^1 \neq E)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$E = (E^1 \neq E)$$

(T_{M275})(Supersets of the Universal Set)

$$E^1 \neq E$$

(A_{MX16})(Order Axiom)

$$\sim E^1 \neq \emptyset \quad (T_{M761})$$

Proof of (T_{M761})

$$E^1 \neq E$$

(T_{M492})

$$\sim E^1 \neq \sim E$$

(T_{M008})(Inverse of an Equality)

$$\sim E^1 \neq \emptyset$$

(D_{M02})(Definition of $\emptyset$)

$$(a \subset b) \subseteq (a \neq b) \quad (T_{M588})$$

$$(a \supset b) \subseteq (a \neq b) \quad (T_{M672})$$

$$(a \parallel b) \subseteq (a \neq b) \quad (T_{M673})$$

$$(a \subset b) \subseteq (a \sim \supset b) \quad (T_{M674})$$

Example Proof of (T_{M588})

$$(a = b)(a \subset b) = \emptyset$$

(T_{M037})(Conjunction of Binary Relations)

$$(a \subset b) \subseteq (a \neq b)$$

(T_{M735})($\subseteq$ Existential Form)

$$\emptyset^1 \neq E \quad (T_{M613})$$

Proof of (T_{M613})

$$\emptyset^1 \subset E$$

(T_{M482})(Domain of the Null Set Entity)

$$\emptyset^1 \neq E$$

(T_{M588})

$$\sim \emptyset^1 \neq \emptyset \quad (T_{M651})$$

Proof of (T_{M651})

$$\begin{aligned} \emptyset^1 &\neq E & (T_{M613}) \\ \sim \emptyset^1 &\neq \sim E & (T_{M008})(\text{Inverse of an Equality}) \\ \sim \emptyset^1 &\neq \emptyset & (D_{M02})(\text{Definition of } \emptyset) \end{aligned}$$

$$(a \subseteq \emptyset) = (\underline{a} = E) \quad (T_{M363}) \text{ Subsets of the Null Set}$$

Proof of (T_{M363})

$$\begin{aligned} (a \subseteq \emptyset) &= (a = \emptyset) & (T_{M052})(\text{Subsets of the Null Set}) \\ (a \subseteq \emptyset) &= (\underline{a} = \emptyset) & (T_{M008})(\text{Inverse of an Equality}) \\ (a \subseteq \emptyset) &= (\underline{a} = E) & (T_{M054})(\text{Negative of Null Set}) \end{aligned}$$

$$(a \subseteq b)(\underline{b} = E) \subseteq (\underline{a} = E) \quad (T_{M437}) \text{ Modus Tollens}$$

Proof of (T_{M437})

$$\begin{aligned} \text{Let } (a \subseteq b)(\underline{b} = E) \\ (a \subseteq b)(\underline{b} = \emptyset) & & (T_{M417}) \\ (a \subseteq \emptyset) & & (A_{MX19})(\text{Substitution}) \\ (a = \emptyset) & & (T_{M052})(\text{Subsets of the Null Set}) \\ (\underline{a} = E) & & (T_{M417}) \\ \text{Thus } (a \subseteq b)(\underline{b} = E) &\subseteq (\underline{a} = E) & (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(a \parallel b) = (ab \neq \emptyset)(\underline{ab} \neq \emptyset)(\underline{ab} \neq \emptyset)(\underline{ab} \neq \emptyset) \quad (T_{M733}) \text{ Definition of } \parallel$$

Proof of (T_{M733})

$$\begin{aligned} (a \parallel b) &= (a \not\subseteq \underline{b})(a \not\subseteq \underline{b})(\underline{a} \not\subseteq \underline{b})(\underline{a} \not\subseteq \underline{b}) & (D_{M06})(\text{Definition of } \parallel) \\ (a \parallel b) &= (ab \neq \emptyset)(\underline{ab} \neq \emptyset)(\underline{ab} \neq \emptyset)(\underline{ab} \neq \emptyset) & (T_{M735})(\subseteq \text{Existential Form}) \& (T_{M008})(\text{Inverse of an Equality}) \end{aligned}$$

$$(a \neq \emptyset) \subseteq (a^1 \neq \emptyset) \quad (T_{M503})$$

Proof of (T_{M503})

$$\begin{aligned} (a^1 = \emptyset) &\subseteq (a = \emptyset) & (A_{MX50})(\text{Components of a Null Entity}) \\ (a \neq \emptyset) &\subseteq (a^1 \neq \emptyset) & (T_{M012})(\text{Transposition}) \end{aligned}$$

$$(a^1 \neq \emptyset^1) \subseteq (a \neq \emptyset^1) \quad (T_{M505})$$

Proof of (T_{M505})

$$\begin{aligned} (a = \emptyset^1) &\subseteq (a^1 = \emptyset^1) & (T_{M499}) \\ (a^1 \neq \emptyset^1) &\subseteq (a \neq \emptyset^1) & (T_{M012})(\text{Transposition}) \end{aligned}$$

$$(a = b) = (\underline{ab} = \emptyset).(\underline{ab} = \emptyset) \quad (T_{M717}) \text{ Biconditional Theorem}$$

Proof of (T_{M717})

$$\begin{aligned} (a \subseteq b) &= (\underline{ab} = \emptyset) & (T_{M735})(\subseteq \text{Existential Form})(\text{Eqn1}) \\ (b \subseteq a) &= (\underline{ab} = \emptyset) & \text{Resymbolising (Eqn2)} \\ (a \subseteq b).(b \subseteq a) &= (\underline{ab} = \emptyset).(\underline{ab} = \emptyset) & (T_{M669})(\text{Combination of Eqns1\&2}) \\ (a = b) &= (\underline{ab} = \emptyset).(\underline{ab} = \emptyset) & (T_{M430})(\text{Biconditional Introduction}) \end{aligned}$$

$$(a = E) \subseteq (\underline{a} \subseteq b) \quad (T_{M424}) \text{ (Negation Elimination)}$$

Proof of (T_{M424})

$$\begin{aligned} \text{Let } a &= E \\ \underline{a} &= \sim E & (T_{M008})(\text{Inverse of an Equality}) \end{aligned}$$

$\underline{a} = \emptyset$	($\mathcal{D}_{M02}$)(Definition of $\emptyset$)(Eqn1)
$\emptyset \subseteq \mathbf{b}$	($\mathcal{T}_{M670}$)(Null Set Theorem)(Eqn2)
$\underline{a} \subseteq \mathbf{b}$	($\mathcal{A}_{MX19}$)(Substitution of Eqn1 into Eqn2)
Thus $(\mathbf{a} = \mathbf{E}) \subseteq (\underline{a} \subseteq \mathbf{b})$	($\mathcal{A}_{MX02}$)(Conditional Introduction)

$$(\mathbf{a} \subset \mathbf{a}) = \emptyset \quad (\mathcal{T}_{M665})$$

$$(\mathbf{a} \parallel \mathbf{a}) = \emptyset \quad (\mathcal{T}_{M666})$$

Example Proof of ($\mathcal{T}_{M665}$)

$(\mathbf{a} = \mathbf{b})(\mathbf{a} \subset \mathbf{b}) = \emptyset$	($\mathcal{T}_{M037}$)(Conjunction of Binary Relations) (Eqn1)
Let $\mathbf{b} = \mathbf{a}$	(Eqn2)
$(\mathbf{a} = \mathbf{a})(\mathbf{a} \subset \mathbf{a}) = \emptyset$	($\mathcal{A}_{MX19}$) (Substitution of Eqn2 Eqn1)
$\mathbf{E}(\mathbf{a} \subset \mathbf{a}) = \emptyset$	($\mathcal{T}_{M001}$)(Universality of ($\mathcal{D}_{M00}$)(Definition of $=$))
$(\mathbf{a} \subset \mathbf{a}) = \emptyset$	($\mathcal{A}_{MX11}$)(Universal Set Conjunction Axiom)

$$\mathbf{a} \sim \subset \mathbf{a} \quad (\mathcal{T}_{M392})$$

Proof of ($\mathcal{T}_{M392}$)

$(\mathbf{a} \subset \mathbf{a}) = \emptyset$	($\mathcal{T}_{M665}$)
$(\mathbf{a} \sim \subset \mathbf{a}) = \emptyset$	($\mathcal{T}_{M008}$)(Inverse of an Equality)
$(\mathbf{a} \sim \subset \mathbf{a}) = \mathbf{E}$	($\mathcal{T}_{M054}$)(Negative of the Null Set)
$\mathbf{a} \sim \subset \mathbf{a}$	($\mathcal{A}_{MX16}$)(Order Axiom)

$$\mathbf{a} \parallel \mathbf{a} \quad (\mathcal{T}_{M360})$$

Proof of ($\mathcal{T}_{M360}$)

$(\mathbf{a} \parallel \mathbf{a}) = \emptyset$	($\mathcal{T}_{M666}$)
$(\mathbf{a} \parallel \mathbf{a}) = \emptyset$	($\mathcal{T}_{M008}$)(Inverse of an Equality)
$(\mathbf{a} \parallel \mathbf{a}) = \mathbf{E}$	($\mathcal{T}_{M054}$)(Negative of the Null Set)
$\mathbf{a} \parallel \mathbf{a}$	($\mathcal{A}_{MX16}$)(Order Axiom)

$$\underline{a}\underline{a} = \emptyset \quad (\mathcal{T}_{M621}) \text{ Complement Conjunction}$$

Proof of ($\mathcal{T}_{M621}$)

$(\mathbf{a} = \mathbf{b}) = (\underline{a}\underline{b} = \emptyset)(\underline{a}\underline{b} = \emptyset)$	($\mathcal{T}_{M717}$)(Biconditional Theorem)(Eqn1)
Let $\mathbf{b} = \mathbf{a}$	(Eqn2)
$(\mathbf{a} \neq \Omega)$:	
$(\mathbf{a} = \mathbf{a}) = (\underline{a}\underline{a} = \emptyset)(\underline{a}\underline{a} = \emptyset)$	($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)
$\mathbf{E} = (\underline{a}\underline{a} = \emptyset)(\underline{a}\underline{a} = \emptyset)$	($\mathcal{T}_{M001}$)(Universality of ($\mathcal{D}_{M00}$))
$\mathbf{E} = (\underline{a}\underline{a} = \emptyset)$	($\mathcal{T}_{M667}$)(Conjunction Tautology)
$\underline{a}\underline{a} = \emptyset$	($\mathcal{A}_{MX16}$)(Order Axiom)

$$\sim(\underline{a}\underline{a}) = \mathbf{E} \quad (\mathcal{T}_{M434}) \text{ Law of Non-Contradiction}$$

Proof of ($\mathcal{T}_{M434}$)

$\underline{a}\underline{a} = \emptyset$	($\mathcal{T}_{M621}$)(Complement Conjunction)
$\sim(\underline{a}\underline{a}) = \sim\emptyset$	($\mathcal{T}_{M008}$)(Inverse of an Equality)
$\sim(\underline{a}\underline{a}) = \mathbf{E}$	($\mathcal{T}_{M054}$)(Negative of Null Set)

$$\mathbf{E} = \mathbf{a} + \underline{a} \quad (\mathcal{T}_{M671}) \text{ Tertium non Datur}$$

Proof of ($\mathcal{T}_{M671}$)

$\sim(\underline{a}\underline{b}) = \underline{a} + \underline{b}$	($\mathcal{T}_{M025}$)(DeMorgan's Law)(Eqn1)
Let $\mathbf{b} = \underline{a}$	(Eqn2)
$\sim(\underline{a}\underline{a}) = \underline{a} + \sim\sim\mathbf{a}$	($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)
$\sim(\underline{a}\underline{a}) = \underline{a} + \mathbf{a}$	($\mathcal{A}_{MX13}$)(Double Complement Axiom)

$$\begin{aligned}\sim\emptyset &= \underline{a} + a \\ E &= \underline{a} + a \\ E &= a + \underline{a}\end{aligned}$$

(T_{M621})(Complement Conjunction)
(T_{M054})(Negative of Null Set)
(T_{M403})(Commutative +)

$$E = (a \subset b) + (a = b) + (a \supset b) + (a \subset \underline{b}) + (a = \underline{b}) + (a \supset \underline{b}) + (a \parallel b) \quad (T_{M174}) \text{ Binary Relation Disjunction Rule}$$

Proof of (T_{M174})

$$\begin{aligned}E &= (a \subset b) + (a = b) + (a \supset b) + (a \subset \underline{b}) + (a = \underline{b}) + (a \supset \underline{b}) + \\ &\sim((a \subset b) + (a = b) + (a \supset b) + (a \subset \underline{b}) + (a = \underline{b}) + (a \supset \underline{b})) \quad (T_{M671})(\text{Tertium non Datur}) \\ E &= (a \subset b) + (a = b) + (a \supset b) + (a \subset \underline{b}) + (a = \underline{b}) + (a \supset \underline{b}) + \\ &(a \sim \underline{b}).(a \neq b).(a \sim \supset b).(a \sim \subset \underline{b}).(a \neq \underline{b}).(a \sim \supset \underline{b}) \quad (T_{M026})(\text{DeMorgan's Law}) \\ E &= (a \subset b) + (a = b) + (a \supset b) + (a \subset \underline{b}) + (a = \underline{b}) + (a \supset \underline{b}) + \\ &(a \sim \underline{b}).(a \neq b).(b \sim \subset a).(a \sim \subset \underline{b}).(a \neq \underline{b}).(\underline{b} \sim \subset a) \quad (T_{M008}) \& (T_{M195})(\subset \text{ Reverse of } \supset) \\ E &= (a \subset b) + (a = b) + (a \supset b) + (a \subset \underline{b}) + (a = \underline{b}) + (a \supset \underline{b}) + \\ &(a \sim \underline{b}).(a \neq b).(b \sim \subset a).(a \neq \underline{b}).(a \sim \subset \underline{b}).(a \neq \underline{b}).(\underline{b} \sim \subset a).(a \neq \underline{b}) \\ &\quad (T_{M667})(\text{Conjunction Tautology}) \\ E &= (a \subset b) + (a = b) + (a \supset b) + (a \subset \underline{b}) + (a = \underline{b}) + (a \supset \underline{b}) + (a \not\subseteq \underline{b}).(b \not\subseteq a).(a \not\subseteq \underline{b}).(\underline{b} \not\subseteq a) \\ &\quad (T_{M198}) \\ E &= (a \subset b) + (a = b) + (a \supset b) + (a \subset \underline{b}) + (a = \underline{b}) + (a \supset \underline{b}) + (a \not\subseteq \underline{b}).(\underline{a} \not\subseteq \underline{b}).(a \not\subseteq \underline{b}).(\underline{a} \not\subseteq \underline{b}) \\ &\quad (T_{M012})(\text{Transposition}) \\ E &= (a \subset b) + (a = b) + (a \supset b) + (a \subset \underline{b}) + (a = \underline{b}) + (a \supset \underline{b}) + (a \not\subseteq \underline{b}).(a \not\subseteq \underline{b}).(\underline{a} \not\subseteq \underline{b}).(\underline{a} \not\subseteq \underline{b}) \\ &\quad (T_{M404})(\text{Commutative .}) \\ E &= (a \subset b) + (a = b) + (a \supset b) + (a \subset \underline{b}) + (a = \underline{b}) + (a \supset \underline{b}) + (a \parallel b) \\ &\quad (D_{M06})(\text{Definition of } \parallel)\end{aligned}$$

$$a.\underline{a} \subseteq b \quad (T_{M441}) \text{ Deduction from a Contradiction}$$

Proof of (T_{M441})

$$\begin{aligned}\underline{a}\underline{a} &= \emptyset \quad (T_{M621})(\text{Complement Conjunction})(\text{Eqn1}) \\ \emptyset \subseteq b &\quad (T_{M670})(\text{Null Set Theorem})(\text{Eqn2}) \\ \underline{a}\underline{a} \subseteq b &\quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})\end{aligned}$$

$$E = (a_1 + \underline{a}_1).(a_2 + \underline{a}_2)...(a_n + \underline{a}_n) \quad (T_{M004}) \text{ General Universal Equation}$$

Proof of (T_{M004})

$$\begin{aligned}E &= E...E \quad (T_{M667})(\text{Conjunction Tautology})(\text{Eqn m}) \\ E &= a_1 + \underline{a}_1 \quad (T_{M671})(\text{Tertium non Datur})(\text{Eqn1}) \\ \dots & \\ E &= a_n + \underline{a}_n \quad (T_{M671})(\text{Tertium non Datur})(\text{Eqn n}) \\ E &= (a_1 + \underline{a}_1)...(a_n + \underline{a}_n) \quad (A_{MX19})(\text{Substitution of Eqns1 to n into Eqn m})\end{aligned}$$

$$b = b(a + \underline{a}) \quad (T_{M006}) \text{ Universal Pair Introduction/Elimination}$$

Proof of (T_{M006})

$$\begin{aligned}b &= bE \quad (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn1}) \\ E &= (a + \underline{a}) \quad (T_{M671})(\text{Tertium non Datur})(\text{Eqn2}) \\ b &= b(a + \underline{a}) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})\end{aligned}$$

$$\emptyset = a_1.\underline{a}_1 + a_2.\underline{a}_2 + \dots + a_n.\underline{a}_n \quad (T_{M005}) \text{ General Null Equation}$$

Proof of (T_{M005})

$$\begin{aligned}\emptyset &= \emptyset + \dots + \emptyset \quad (A_{MX01})(\text{Disjunction Tautology})(\text{Eqn m}) \\ \emptyset &= a_1.\underline{a}_1 \quad (T_{M621})(\text{Complement Conjunction})(\text{Eqn 1}) \\ \dots & \\ \emptyset &= a_n.\underline{a}_n \quad (T_{M621})(\text{Complement Conjunction})(\text{Eqn n}) \\ \emptyset &= a_1.\underline{a}_1 + \dots + a_n.\underline{a}_n \quad (A_{MX19})(\text{Substitution of Eqns1 to n into Eqn m})\end{aligned}$$

$(a \neq \emptyset)(b \neq \emptyset) \subseteq (ab = \text{indeterminate})$ (T_{M137}) Probabilistic Conjunction

Proof of (T_{M137})

Let $a \neq \emptyset$

$((a \neq \emptyset) = E)$

($AMX16$)(Order Axiom)(Eqn1)

Let $b \neq \emptyset$

$((b \neq \emptyset) = E)$

($AMX16$)(Order Axiom)(Eqn2)

$EE = E$

($TM667$)(Conjunction Tautology)(Eqn3)

$(a \neq \emptyset)(b \neq \emptyset) = E$

($AMX19$)(Substitution of Eqns1 & 2 into Eqn3)(Eqn5)

$E = (ab = c) + (ab \neq c)$

($TM671$)(Tertium non Datur)(Eqn6)

$(a \neq \emptyset)(b \neq \emptyset) = (ab = c) + (ab \neq c)$

($AMX19$)(Substitution of Eqn6 into Eqn5)

This equation cannot be simplified further. Thus if $(a \neq \emptyset)$ and $(b \neq \emptyset)$ then ab may have any value and is unconstrained by the conditions of a proposition.

$(a \neq E)(b \neq E) \subseteq (a + b = \text{indeterminate})$ (T_{M317}) Probabilistic Conjunction

Proof of (T_{M317})

$(a \neq \emptyset)(b \neq \emptyset) \subseteq (ab = \text{indeterminate})$

(T_{M137})(Probabilistic Conjunction)

$(a \neq \emptyset)(b \neq \emptyset) \subseteq (ab = \text{indeterminate})$

Resymbolising

$(a \neq \emptyset)(b \neq \emptyset) \subseteq (ab = \text{indeterminate})$

($TM008$)($TM008$)

$(a \neq \emptyset)(b \neq \emptyset) \subseteq (\sim(a + b) = \text{indeterminate})$

($TM026$)(DeMorgan's Law)

$(a \neq E)(b \neq E) \subseteq (\sim(a + b) = \text{indeterminate})$

($TM054$)(Negative of Null Set)

$(a \neq E)(b \neq E) \subseteq ((a + b) = \text{indeterminate})$

If a set is indeterminate then so is its complement

$\emptyset \subseteq E$ ($TM056$)

Proof of ($TM056$)

$a \subseteq E$

($TM744$)(Universal Set Theorem)(Eqn1)

Let $a = \emptyset$

(Eqn2)

$\emptyset \subseteq E$

($AMX19$)(Substitution of Eqn2 into Eqn1)

$(\emptyset = E) + (\emptyset \subseteq E)$

($TM266$)(Definition of $\subseteq$)

$(\emptyset = E)E + (\emptyset \subseteq E)$

($AMX11$)(Universal Set Conjunction Axiom) (Eqn3)

$\emptyset \neq E$

($TM352$)

$(\emptyset \neq E) = E$

($AMX16$)(Order Axiom) (Eqn4)

$(\emptyset = E)(\emptyset \neq E) + (\emptyset \subseteq E)$

($AMX19$)(Substitution of Eqn4 into Eqn3)

$\emptyset + (\emptyset \subseteq E)$

($TM621$)(Complement Conjunction)

$\emptyset \subseteq E$

($TM668$)(Null Set Disjunction)

$(a \subseteq b)(a = b) = (a = b)$ ($TM043$) Conjunction of Binary Relations

$(a \subseteq b)(a \subset b) = (a \subset b)$ ($TM044$) Conjunction of Binary Relations

$(a \supseteq b)(a = b) = (a = b)$ ($TM045$) Conjunction of Binary Relations

$(a \supseteq b)(a \supset b) = (a \supset b)$ ($TM046$) Conjunction of Binary Relations

Example Proof of ($TM043$)

Let $(a \subseteq b)(a = b)$

(Eqn1)

$(a \subseteq b) = (a \subset b) + (a = b)$

($TM266$)(Definition of $\subseteq$)(Eqn2)

$((a \subset b) + (a = b))(a = b)$

($AMX19$)(Substitution of Eqn2 into Eqn1)

$(a \subset b)(a = b) + (a = b)(a = b)$

($TM407$)(Distribution of $\cdot$ over $+$)

$\emptyset + (a = b)(a = b)$

($TM037$)(Conjunction of Binary Relations)

$(a = b)(a = b)$

($TM668$)(Null Set Disjunction)

$(a = b)$

($TM667$)(Conjunction Tautology)

Thus $(a \subseteq b)(a = b) = (a = b)$

($AMX02$)(Conditional Introduction) since this proof is reversible

$((a \not\subseteq b) \subseteq (a \neq \emptyset)(b \neq E))$ ($TM251$) Conditions for $\not\subseteq$

Proof of ($TM251$)

$(b = E) = (b = E)$	(D_{M00}) (Definition of $=$)
$(b = E) = (b = E)E$	(A_{MX11}) (Universal Set Conjunction Axiom)
$(b = E) = (b = E)(a \subseteq E)$	(T_{M007}) (Universality of (T_{M744}) Universal Set Theorem)
$(b = E) \subseteq (a \subseteq b)$	(A_{MX19}) (Substitution)
$(a \not\subseteq b) \subseteq (b \neq E)$	(T_{M012}) (Transposition)(Eqn1)
$(a = \emptyset) = (a = \emptyset)$	(D_{M00}) (Definition of $=$)
$(a = \emptyset) = (a = \emptyset)E$	(A_{MX11}) (Universal Set Conjunction Axiom)
$(a = \emptyset) = (a = \emptyset)(\emptyset \subseteq b)$	(T_{M007}) (Universality of (T_{M670}) Null Set Theorem)
$(a = \emptyset) \subseteq (a \subseteq b)$	(A_{MX19}) (Substitution)
$(a \not\subseteq b) \subseteq (a \neq \emptyset)$	(T_{M012}) (Transposition)(Eqn2)
$(a \not\subseteq b)(a \not\subseteq b) \subseteq (a \neq \emptyset)(b \neq E)$	(T_{M669}) (Combination of Eqns1&2)
$(a \not\subseteq b) \subseteq (a \neq \emptyset)(b \neq E)$	(T_{M667}) (Conjunction Tautology)

$((a \not\subseteq b) \subseteq (a \neq \emptyset)(b \neq \emptyset))$ (T_{M251b}) Conditions for $\not\subseteq$

Proof of (T_{M251b})

$(a \not\subseteq b) \subseteq (a \neq \emptyset)(b \neq E)$	(T_{M251}) (Conditions for $\not\subseteq$)
$(a \not\subseteq b) \subseteq (a \neq \emptyset)(b \neq E)$	Resymbolising
$(a \not\subseteq b) \subseteq (a \neq \emptyset)(b \neq \underline{E})$	(T_{M008}) (Inverse of an Equality) & (A_{MX13}) (Double Complement Axiom)
$(a \not\subseteq \underline{b}) \subseteq (a \neq \emptyset)(b \neq \emptyset)$	(D_{M02}) (Definition of $\emptyset$)

$(a \subseteq c) \subseteq (ab \subseteq c)$ (T_{M089}) Single Premise Syllogism

Proof of (T_{M089})

$(a \subseteq c)(b \subseteq d) \subseteq (ab \subseteq cd)$	(T_{M669}) (Combination)(Eqn1)
Let $d = E$	(Eqn2)
$(a \subseteq c)(b \subseteq E) \subseteq (ab \subseteq cE)$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)
$(a \subseteq c)E \subseteq (ab \subseteq cE)$	(T_{M001}) (Universality of (T_{M744}) Universal Set Theorem)
$(a \subseteq c) \subseteq (ab \subseteq c)$	(A_{MX11}) (Universal Set Conjunction Axiom)

$(c \subseteq a) \subseteq (c \subseteq a + b)$ (T_{M150})

Proof of (T_{M150})

$(a \subseteq c) \subseteq (ab \subseteq c)$	(T_{M089}) (Single Premise Syllogism)
$(a \subseteq c) \subseteq (ab \subseteq c)$	Resymbolising
$(a \subseteq c) \subseteq (\sim(a + b) \subseteq c)$	(T_{M026}) (DeMorgan's Law)
$(c \subseteq a) \subseteq (c \subseteq a + b)$	(T_{M012}) (Transposition)

$(a = \emptyset) \subseteq (ab = \emptyset)$ (T_{M350})

Proof of (T_{M350})

$(a \subseteq c) \subseteq (ab \subseteq c)$	(T_{M089}) (Single Premise Syllogism)(Eqn1)
Let $c = \emptyset$	(Eqn2)
$(a \subseteq \emptyset) \subseteq (ab \subseteq \emptyset)$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)
$(a = \emptyset) \subseteq (ab = \emptyset)$	(T_{M052}) (Subsets of the Null Set)

$a\emptyset = \emptyset$ (T_{M050}) Null Set Conjunction

Proof of (T_{M050})

$(a \subseteq c) \subseteq (ab \subseteq c)$	(T_{M089}) (Single Premise Syllogism)(Eqn1)
Let $a = \emptyset$	(Eqn2)
Let $c = \emptyset$	(Eqn3)
$(\emptyset \subseteq \emptyset) \subseteq (\emptyset b \subseteq \emptyset)$	(A_{MX19}) (Substitution of Eqns2&3 into Eqn1)
$(\emptyset = \emptyset) \subseteq (\emptyset b = \emptyset)$	(T_{M052}) (Subsets of the Null Set)
$E \subseteq (\emptyset b = \emptyset)$	(T_{M343})
$E = (\emptyset b = \emptyset)$	(T_{M275}) (Supersets of the Universal Set)
$\emptyset b = \emptyset$	(A_{MX16}) (Order Axiom)

$$\begin{aligned} b\emptyset &= \emptyset \\ a\emptyset &= \emptyset \end{aligned}$$

(T_{M404})(Commutative .)
Resymbolising

$$a + E = E \quad (T_{M055}) \text{ Universal Set Disjunction}$$

Proof of (T_{M055})

$$\begin{aligned} \emptyset &= a\emptyset & (T_{M050}) \text{ (Null Set Conjunction)} \\ \emptyset &= \sim(a\emptyset) & (T_{M008}) \text{ (Inverse of an Equality)} \\ E &= \sim(a\emptyset) & (T_{M054}) \text{ (Negative of Null Set)} \\ E &= a + \emptyset & (T_{M025}) \text{ (DeMorgan's Law)} \\ E &= a + E & (T_{M054}) \text{ (Negative of Null Set)} \\ E &= a + E & \text{Resymbolising} \end{aligned}$$

$$a = a + ab \quad (T_{M063}) \text{ Partial Redundancy}$$

Proof of (T_{M063})

$$\begin{aligned} (a = a + ab) &= (a = a + ab) & (D_{M00}) \text{ (Definition of =)} \\ (a = a + ab) &= (a = aE + ab) & (A_{MX11}) \text{ (Universal Set Conjunction Axiom)} \\ (a = a + ab) &= (a = a(E + b)) & (T_{M407}) \text{ (Distribution of . over +)(Eqn1)} \\ (a = a + ab) &= (a = aE) & (T_{M055}) \text{ (Universal Set Disjunction)} \\ (a = a + ab) &= (a = a) & (A_{MX11}) \text{ (Universal Set Conjunction Axiom)} \\ (a = a + ab) &= E & (T_{M001}) \text{ (Universality of (} D_{M00} \text{)(Definition of =))} \\ a &= a + ab & (A_{MX16}) \text{ (Order Axiom)} \end{aligned}$$

$$(a = E) \subseteq (a + b = E) \quad (T_{M159})$$

Proof of (T_{M159})

$$\begin{aligned} b + E &= E & (T_{M055}) \text{ (Eqn1)} \\ \text{Let } a &= E & \text{(Eqn2)} \\ b + a &= E & (A_{MX19}) \text{ (Substitution of Eqn2 into Eqn1)} \\ \text{Thus } (a = E) &\subseteq (a + b = E) & (A_{MX02}) \text{ (Conditional Introduction)} \end{aligned}$$

$$(a = \emptyset)(b = \emptyset) \subseteq (ab = \emptyset) \quad (T_{M122}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M122})

$$\begin{aligned} \text{Let } a &= \emptyset & \text{(Eqn1)} \\ \text{Let } b &= \emptyset & \text{(Eqn2)} \\ a\emptyset &= \emptyset & (T_{M050}) \text{ (Conjunction with the Null Set)(Eqn3)} \\ ab &= \emptyset & (A_{MX19}) \text{ (Substitution of Eqn2 into Eqn3)} \\ \text{Thus } (a = \emptyset)(b = \emptyset) &\subseteq (ab = \emptyset) & (A_{MX02}) \text{ (Conditional Introduction)} \end{aligned}$$

$$(a = E)(b = \emptyset) \subseteq (ab = \emptyset) \quad (T_{M111}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M111})

$$\begin{aligned} \text{Let } a &= E & \text{(Eqn1)} \\ \text{Let } b &= \emptyset & \text{(Eqn2)} \\ a\emptyset &= \emptyset & (T_{M050}) \text{ (Conjunction with the Null Set)(Eqn3)} \\ ab &= \emptyset & (A_{MX19}) \text{ (Substitution of Eqn2 into Eqn3)} \\ \text{Thus } (a = E)(b = \emptyset) &\subseteq (ab = \emptyset) & (A_{MX02}) \text{ (Conditional Introduction)} \end{aligned}$$

$$(a \neq E)(b = \emptyset) \subseteq (ab = \emptyset) \quad (T_{M115}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M115})

$$\begin{aligned} \text{Let } a &\neq E & \text{(Eqn1)} \\ \text{Let } b &= \emptyset & \text{(Eqn2)} \\ a\emptyset &= \emptyset & (T_{M050}) \text{ (Conjunction with the Null Set)(Eqn3)} \\ ab &= \emptyset & (A_{MX19}) \text{ (Substitution of Eqn2 into Eqn3)} \end{aligned}$$

Thus $(a \neq E)(b = \emptyset) \subseteq (ab = \emptyset)$ (A_{MX02})(Conditional Introduction)

$(a \neq \emptyset)(b = \emptyset) \subseteq (ab = \emptyset)$ (T_{M118}) Probabilistic Conjunction

Proof of (T_{M118})

Let $a \neq \emptyset$ (Eqn1)
 Let $b = \emptyset$ (Eqn2)
 $a\emptyset = \emptyset$ (T_{M050})(Conjunction with the Null Set)(Eqn3)
 $ab = \emptyset$ (A_{MX19})(Substitution of Eqn2 into Eqn3)
 Thus $(a \neq \emptyset)(b = \emptyset) \subseteq (ab = \emptyset)$ (A_{MX02})(Conditional Introduction)

$(a = E)(b = \emptyset) \subseteq (a + b = E)$ (T_{M310}) Probabilistic Conjunction

Proof of (T_{M310})

$(a = \emptyset)(b = E) \subseteq (ab = \emptyset)$ (T_{M111})(Probabilistic Conjunction)
 $(\underline{a} = \emptyset)(\underline{b} = E) \subseteq (\underline{ab} = \emptyset)$ Resymbolising
 $(a = \emptyset)(b = \underline{E}) \subseteq (\underline{ab} = \emptyset)$ (T_{M008})(Inverse of an Equality)
 $(a = \emptyset)(b = \underline{E}) \subseteq (\sim(a + b) = \emptyset)$ (T_{M026})(DeMorgan's Law)
 $(a = \emptyset)(b = \underline{E}) \subseteq (a + b = \emptyset)$ (T_{M008})(Inverse of an Equality)
 $(a = E)(b = \underline{E}) \subseteq (a + b = E)$ (T_{M054})(Negative of Null Set)
 $(a = E)(b = \emptyset) \subseteq (a + b = E)$ (D_{M02})(Definition of $\emptyset$)

$(a \neq E)(b = \emptyset) \subseteq (a + b \neq E)$ (T_{M311}) Probabilistic Conjunction

Proof of (T_{M311})

$(a = E)(b \neq \emptyset) \subseteq (ab \neq \emptyset)$ (T_{M110})(Probabilistic Conjunction)
 $(\underline{b} = E)(\underline{a} \neq \emptyset) \subseteq (\underline{ba} \neq \emptyset)$ Resymbolising
 $(\underline{a} \neq \emptyset)(\underline{b} = E) \subseteq (\underline{ab} \neq \emptyset)$ (T_{M404})(Commutative .)
 $(a \neq \emptyset)(b = E) \subseteq (\underline{ab} \neq \emptyset)$ (T_{M008})(T_{M008})
 $(a \neq \emptyset)(b = \underline{E}) \subseteq (\underline{ab} \neq \emptyset)$ (T_{M008})(Inverse of an Equality)
 $(a \neq \emptyset)(b = \underline{E}) \subseteq (\sim(a + b) \neq \emptyset)$ (T_{M026})(DeMorgan's Law)
 $(a \neq \emptyset)(b = \underline{E}) \subseteq (a + b \neq \emptyset)$ (T_{M008})(T_{M008})
 $(a \neq \emptyset)(b = \emptyset) \subseteq (a + b \neq \emptyset)$ (D_{M02})(Definition of $\emptyset$)
 $(a \neq E)(b = \emptyset) \subseteq (a + b \neq E)$ (T_{M054})(Negative of Null Set)

$(a = E)(b \neq \emptyset) \subseteq (a + b = E)$ (T_{M313}) Probabilistic Conjunction

Proof of (T_{M313})

$(a = \emptyset)(b \neq E) \subseteq (ab = \emptyset)$ (T_{M115})(Probabilistic Conjunction)
 $(\underline{a} = \emptyset)(\underline{b} \neq E) \subseteq (\underline{ab} = \emptyset)$ Resymbolising
 $(\underline{a} = \emptyset)(b \neq \underline{E}) \subseteq (\underline{ab} = \emptyset)$ (T_{M008})(T_{M008})
 $(a = \emptyset)(b \neq \underline{E}) \subseteq (\underline{ab} = \emptyset)$ (T_{M008})(Inverse of an Equality)
 $(a = \emptyset)(b \neq \underline{E}) \subseteq (\sim(a + b) = \emptyset)$ (T_{M026})(DeMorgan's Law)
 $(a = \emptyset)(b \neq \underline{E}) \subseteq (a + b = \emptyset)$ (T_{M008})(Inverse of an Equality)
 $(a = E)(b \neq \underline{E}) \subseteq (a + b = E)$ (T_{M054})(Negative of Null Set)
 $(a = E)(b \neq \emptyset) \subseteq (a + b = E)$ (D_{M02})(Definition of $\emptyset$)

$(a = E)(b = E) \subseteq (a + b = E)$ (T_{M315}) Probabilistic Conjunction

Proof of (T_{M315})

$(a = \emptyset)(b = \emptyset) \subseteq (ab = \emptyset)$ (T_{M122})(Probabilistic Conjunction)
 $(\underline{a} = \emptyset)(\underline{b} = \emptyset) \subseteq (\underline{ab} = \emptyset)$ Resymbolising
 $(a = \emptyset)(b = \emptyset) \subseteq (\underline{ab} = \emptyset)$ (T_{M008})(Inverse of an Equality)
 $(a = \emptyset)(b = \emptyset) \subseteq (\sim(a + b) = \emptyset)$ (T_{M026})(DeMorgan's Law)
 $(a = \emptyset)(b = \emptyset) \subseteq (a + b = \emptyset)$ (T_{M008})(Inverse of an Equality)
 $(a = E)(b = E) \subseteq (a + b = E)$ (T_{M054})(Negative of Null Set)

$$(a = E)(b \neq E) \subseteq (a + b = E) \quad (T_{M316}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M316})

$$\begin{aligned} (a \neq \emptyset)(b = \emptyset) &\subseteq (ab = \emptyset) && (T_{M118}) \text{ (Probabilistic Conjunction)} \\ (\underline{a} \neq \emptyset)(b = \emptyset) &\subseteq (\underline{ab} = \emptyset) && \text{Resymbolising} \\ (a \neq \emptyset)(b = \emptyset) &\subseteq (\underline{ab} = \emptyset) && (T_{M008})(T_{M008}) \\ (a \neq \emptyset)(b = \emptyset) &\subseteq (\underline{ab} = \emptyset) && (T_{M008}) \text{ (Inverse of an Equality)} \\ (a \neq \emptyset)(b = \emptyset) &\subseteq (\sim(a + b) = \emptyset) && (T_{M026}) \text{ (DeMorgan's Law)} \\ (a \neq \emptyset)(b = \emptyset) &\subseteq (a + b = \emptyset) && (T_{M008}) \text{ (Inverse of an Equality)} \\ (a \neq E)(b = E) &\subseteq (a + b = E) && (T_{M054}) \text{ (Negative of Null Set)} \\ (b \neq E)(a = E) &\subseteq (b + a = E) && \text{Resymbolising} \\ (a = E)(b \neq E) &\subseteq (b + a = E) && (T_{M404}) \text{ (Commutative .)} \\ (a = E)(b \neq E) &\subseteq (a + b = E) && (T_{M403}) \text{ (Commutative +)} \end{aligned}$$

$$(a \supseteq b) = (E = a + \underline{b}) \quad (T_{M362}) \supseteq \text{ Universal Form}$$

Proof of (T_{M362})

$$\begin{aligned} (a \subseteq b) &= (E = \underline{a} + b) && (T_{M064}) (\subseteq \text{ Universal Form}) \\ (b \subseteq a) &= (E = \underline{b} + a) && \text{Resymbolising} \\ (b \subseteq a) &= (E = a + \underline{b}) && (T_{M403}) \text{ (Commutative +)} \\ (a \supseteq b) &= (E = a + \underline{b}) && (T_{M196}) (\subseteq \text{ Reverse of } \supseteq) \end{aligned}$$

$$(a = b) = (a + \underline{b} = E)(\underline{a} + b = E) \quad (T_{M433}) \text{ Material Equivalence}$$

Proof of (T_{M433})

$$\begin{aligned} (a = b) &= (a \subseteq b)(b \subseteq a) && (T_{M430}) \text{ (Biconditional Introduction)} \\ (a = b) &= (a + \underline{b} = E)(\underline{a} + b = E) && (T_{M064}) (\subseteq \text{ Universal Form}) \end{aligned}$$

$$(a \not\subseteq c)(b \not\subseteq c) = \text{indeterminate} \quad (T_{M138}) \text{ Conjunction of Insufficiencies}$$

Proof of (T_{M138})

$$\begin{aligned} (a \not\subseteq c)(b \not\subseteq c) &\subseteq (abc = \text{indeterminate}) && (T_{M137}) \\ (a \not\subseteq c)(b \not\subseteq c) &\subseteq (abc = \text{indeterminate}) && (T_{M735}) (\subseteq \text{ Existential Form}) \end{aligned}$$

Since abc is indeterminate then so too is any equation comprising ab and c .

$(a \not\subseteq c)(b \not\subseteq c)$ yields some equation comprising ab and c .

Thus $(a \not\subseteq c)(b \not\subseteq c)$ is indeterminate.

$$(c \not\subseteq a)(c \not\subseteq b) = \text{indeterminate} \quad (T_{M368}) \text{ Conjunction of Unnecessities}$$

Proof of (T_{M368})

$$\begin{aligned} (a \not\subseteq c)(b \not\subseteq c) &= \text{indeterminate} && (T_{M138}) \text{ (Conjunction of Insufficiencies)} \\ (\underline{c} \not\subseteq \underline{a})(\underline{c} \not\subseteq \underline{b}) &= \text{indeterminate} && (T_{M012}) \text{ (Transposition)} \\ (c \not\subseteq a)(c \not\subseteq b) &= \text{indeterminate} && \text{Resymbolising} \end{aligned}$$

$$(a \subset b) = (a \subset b)(a \neq b) \quad (T_{M679})$$

Proof of (T_{M679})

$$\begin{aligned} (a \subset b) &\subseteq (a \neq b) && (T_{M588}) \\ (a \subset b) &= (a \subset b)(a \neq b) && (A_{MX44}) (\subseteq \text{ Equality Form}) \end{aligned}$$

$$(a \subset b) = (\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) \quad (T_{M718}) \text{ Definition of } \subset$$

Proof of (T_{M718})

$$\begin{aligned} (a \subseteq b) &= (\underline{ab} = \emptyset) && (T_{M735}) (\subseteq \text{ Existential Form}) \text{ (Eqn1)} \\ (a = b) &= (\underline{ab} = \emptyset).(\underline{ab} = \emptyset) && (T_{M717}) \text{ Biconditional Theorem} \end{aligned}$$

$$\begin{aligned}
(a \neq b) &= \sim((ab = \emptyset).(\underline{ab} = \emptyset)) & (T_{M008})(\text{Inverse of an Equality}) \\
(a \neq b) &= (a\underline{b} \neq \emptyset) + (\underline{ab} \neq \emptyset) & (T_{M025})(\text{DeMorgan's Law})(\text{Eqn2}) \\
(a \subseteq b).(a \neq b) &= (a\underline{b} = \emptyset)((\underline{ab} \neq \emptyset) + (\underline{ab} \neq \emptyset)) & (T_{M231})(\text{Combination of Eqns1\&2}) \\
((a \subset b) + (a = b)).(a \neq b) &= (a\underline{b} = \emptyset)((\underline{ab} \neq \emptyset) + (\underline{ab} \neq \emptyset)) & (T_{M266})(\text{Definition of } \subseteq) \\
(a \subset b).(a \neq b) + (a = b).(a \neq b) &= (a\underline{b} = \emptyset).(\underline{ab} \neq \emptyset) + (\underline{ab} = \emptyset).(\underline{ab} \neq \emptyset) & \\
(a \subset b).(a \neq b) + \emptyset &= (a\underline{b} = \emptyset).(\underline{ab} \neq \emptyset) + \emptyset & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
(a \subset b).(a \neq b) &= (a\underline{b} = \emptyset).(\underline{ab} \neq \emptyset) & (T_{M621})(\text{Complement Conjunction}) \\
(a \subset b) &= (a\underline{b} = \emptyset).(\underline{ab} \neq \emptyset) & (T_{M668})(\text{Null Set Disjunction}) \\
(a \subset b) &= (a\underline{b} = \emptyset).(\underline{ab} \neq \emptyset) & (T_{M679})
\end{aligned}$$

$$(a \supset b) = (a\underline{b} \neq \emptyset)(\underline{ab} = \emptyset) \quad (T_{M268}) \text{ Definition of } \supset$$

Proof of (T_{M268})

$$\begin{aligned}
(a \subset b) &= (a\underline{b} = \emptyset)(\underline{ab} \neq \emptyset) & (T_{M718})(\text{Definition of } \subset) \\
(b \subset a) &= (\underline{ba} = \emptyset)(\underline{ab} \neq \emptyset) & \text{Resymbolising} \\
(a \supset b) &= (\underline{ba} = \emptyset)(\underline{ab} \neq \emptyset) & (T_{M195})(\subset \text{ Reverse of } \supset) \\
(a \supset b) &= (a\underline{b} = \emptyset)(\underline{ab} \neq \emptyset) & (T_{M404})(\text{Commutative } .) \\
(a \supset b) &= (a\underline{b} \neq \emptyset)(\underline{ab} = \emptyset) & (T_{M404})(\text{Commutative } .)
\end{aligned}$$

$$(a \supseteq b) = (a\underline{b} = \emptyset) \quad (T_{M051b}) \supseteq \text{ Existential Form}$$

Proof of (T_{M051b})

$$\begin{aligned}
(a \supseteq b) &= (a \supset b) + (a = b) & (T_{M267})(\text{Definition of } \supseteq) (\text{Eqn1}) \\
(a = b) &= (a\underline{b} = \emptyset)(\underline{ab} = \emptyset) & (T_{M717})(\text{Biconditional Theorem})(\text{Eqn2}) \\
(a \supset b) &= (a\underline{b} \neq \emptyset)(\underline{ab} = \emptyset) & (T_{M268})(\text{Definition of } \supset) (\text{Eqn3}) \\
(a \supseteq b) &= (a\underline{b} \neq \emptyset)(\underline{ab} = \emptyset) + (a\underline{b} = \emptyset)(\underline{ab} = \emptyset) & (A_{MX19})(\text{Substitution of Eqns2 and 3 into Eqn1}) \\
(a \supseteq b) &= (a\underline{b} = \emptyset)((\underline{ab} \neq \emptyset) + (\underline{ab} = \emptyset)) & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
(a \supseteq b) &= (a\underline{b} = \emptyset)E & (T_{M671})(\text{Tertium non Datur}) \\
(a \supseteq b) &= (a\underline{b} = \emptyset) & (A_{MX11})(\text{Universal Set Conjunction Axiom})
\end{aligned}$$

$$a^1 \neq E \quad (T_{M575})$$

Proof of (T_{M575})

$$\begin{aligned}
a^1 \subset E & \quad (A_{MX37})(\text{Plurality Axiom}) \\
a^1 \neq E & \quad (T_{M588})
\end{aligned}$$

$$\sim a^1 \neq \emptyset \quad (T_{M691})$$

Proof of (T_{M691})

$$\begin{aligned}
a^1 \neq E & \quad (T_{M575}) \\
\sim a^1 \neq \emptyset & \quad (T_{M008})(\text{Inverse of an Equality})
\end{aligned}$$

$$(a \subseteq b) \subseteq (a \sim \supset b) \quad (T_{M562})$$

Proof of (T_{M562})

$$\begin{aligned}
(a \supset b) &\subseteq (a \neq b) & (T_{M672}) \\
(a = b) &\subseteq (a \sim \supset b) & (T_{M012})(\text{Transposition})(\text{Eqn1}) \\
(a \subset b) &\subseteq (a \sim \supset b) & (T_{M674})(\text{Eqn2}) \\
(a \subset b) + (a = b) &\subseteq (a \sim \supset b) + (a \sim \supset b) & (A_{MX21})(\text{Combination of Eqns1\&2}) \\
(a \subset b) + (a = b) &\subseteq (a \sim \supset b) & (A_{MX01})(\text{Disjunction Tautology}) \\
(a \subseteq b) &\subseteq (a \sim \supset b) & (T_{M266})(\text{Definition of } \subseteq)
\end{aligned}$$

$$(a \parallel \emptyset) = \emptyset \quad (T_{M253})$$

Proof of (T_{M253})

$$(a \supset b)(a \parallel b) = \emptyset \quad (T_{M042})(\text{Conjunction of Binary Relations})(\text{Eqn1})$$

$(a = b)(a \parallel b) = \emptyset$	(T_{M039}) (Conjunction of Binary Relations)(Eqn2)
$(a \supset b)(a \parallel b) + (a = b)(a \parallel b) = \emptyset + \emptyset$	(A_{MX21}) (Combination of Eqns1&2)
$(a \supset b)(a \parallel b) + (a = b)(a \parallel b) = \emptyset$	(A_{MX01}) (Disjunction Tautology)
$((a \supset b) + (a = b))(a \parallel b) = \emptyset$	(T_{M407}) (Distribution of $\cdot$ over $+$)
$(a \supseteq b)(a \parallel b) = \emptyset$	(T_{M267}) (Definition of $\supseteq$)(Eqn3)
Let $b = \emptyset$	(Eqn4)
$(a \supseteq \emptyset)(a \parallel \emptyset) = \emptyset$	(A_{MX19}) (Substitution of Eqn4 into Eqn3)
$E(a \parallel \emptyset) = \emptyset$	(T_{M001}) (Universality of (T_{M670}) (Null Set Theorem))
$(a \parallel \emptyset) = \emptyset$	(A_{MX11}) (Universal Set Conjunction Axiom)

$$(a \parallel E) = \emptyset \quad (T_{M098c})$$

Proof of (T_{M098c})

$(a \parallel \emptyset) = \emptyset$	(T_{M253})
$(a \parallel \emptyset) = \emptyset$	(T_{M194}) (Co-intersection)
$(a \parallel E) = \emptyset$	(T_{M054}) (Negative of Null Set)

$$(a \not\parallel \emptyset) = E \quad (T_{M053c})$$

Proof of (T_{M053c})

$(a \parallel \emptyset) = \emptyset$	(T_{M253})
$(a \not\parallel \emptyset) = \underline{\emptyset}$	(T_{M008}) (Inverse of an Equality)
$(a \not\parallel \emptyset) = E$	(T_{M054}) (Negative of Null Set)

$$(a \not\parallel E) = E \quad (T_{M098e})$$

Proof of (T_{M098e})

$(a \parallel E) = \emptyset$	(T_{M098c})
$\sim(a \parallel E) = \underline{\emptyset}$	(T_{M008}) (Inverse of an Equality)
$\sim(a \parallel E) = E$	(T_{M054}) (Negative of Null Set)
$(a \not\parallel E) = E$	(T_{M264}) (Definition of $\not\parallel$)

$$(a \subseteq \underline{a}) = (a = \emptyset) \quad (T_{M057})$$

Proof of (T_{M057})

$(a \neq \Omega)$:	
$(a \subseteq b) = (a = ab)$	(A_{MX44}) ($\subseteq$ Equality Form)(Eqn1)
Let $b = \underline{a}$	(Eqn2)
$(a \subseteq \underline{a}) = (a = a\underline{a})$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)
$(a \subseteq \underline{a}) = (a = \emptyset)$	(T_{M621}) (Complement Conjunction)

$$(\underline{a} \subseteq a) = (a = E) \quad (T_{M057b})$$

Proof of (T_{M057b})

$(a \subseteq \underline{a}) = (a = \emptyset)$	(T_{M057})
$(\underline{a} \subseteq a) = (\underline{a} = \emptyset)$	Resymbolising
$(\underline{a} \subseteq a) = (a = \underline{\emptyset})$	(T_{M008}) (Inverse of an Equality)
$(\underline{a} \subseteq a) = (a = E)$	(T_{M054}) (Negative of Null Set)

$$(ac = b) \subseteq (ac = bc) \quad (T_{M662}) \text{ Deduction}$$

Proof of (T_{M662})

Let $ac = b$	
$acc = bc$	(T_{M233}) (Deduction)
$ac = bc$	(T_{M667}) (Conjunction Tautology)
Thus $(ac = b) \subseteq (ac = bc)$	(A_{MX02}) (Conditional Introduction)

$$(a \subseteq b) \subseteq (ac \subseteq bc)$$

(T_{M029}) Deduction

Proof of (T_{M029})

$(a \subseteq b)(c \subseteq d) \subseteq (ac \subseteq bd)$	(T_{M669})(Combination)(Eqn1)
Let $d = c$	(Eqn2)
$(a \subseteq b)(c \subseteq c) \subseteq (ac \subseteq bc)$	($AMX19$)(Substitution of Eqn2 into Eqn1)(Eqn3)
$(c \subseteq c) = (c = c) + (c \subset c)$	(T_{M266})(Definition of $\subseteq$)
$(c \subseteq c) = E + (c \subset c)$	(T_{M001})(Universality of (D_{M00}) (Definition of $=$))
$(c \subseteq c) = E + \emptyset$	(T_{M665})
$(c \subseteq c) = E + \underline{E}$	(D_{M02})(Definition of $\emptyset$)
$(c \subseteq c) = E$	(T_{M671})(Tertium non Datur)(Eqn4)
$(a \subseteq b)E \subseteq (ac \subseteq bc)$	($AMX19$)(Substitution of Eqn4 into Eqn3)
$(a \subseteq b) \subseteq (ac \subseteq bc)$	($AMX11$)(Universal Set Conjunction Axiom)

$$(a \subseteq b) \subseteq (a + c \subseteq b + c)$$

(T_{M030}) Deduction

Proof of (T_{M030})

$(a \subseteq b)(c \subseteq d) \subseteq (a + c \subseteq b + d)$	($AMX21$)(Combination Axiom)(Eqn1)
Let $d = c$	(Eqn2)
$(a \subseteq b)(c \subseteq c) \subseteq (a + c \subseteq b + c)$	($AMX19$)(Substitution of Eqn2 into Eqn1)(Eqn3)
$(c \subseteq c) = (c = c) + (c \subset c)$	(T_{M266})(Definition of $\subseteq$)
$(c \subseteq c) = E + (c \subset c)$	(T_{M001})(Universality of (D_{M00}) (Definition of $=$))
$(c \subseteq c) = E + \emptyset$	(T_{M665})
$(c \subseteq c) = E + \underline{E}$	(D_{M02})(Definition of $\emptyset$)
$(c \subseteq c) = E$	(T_{M671})(Tertium non Datur)(Eqn4)
$(a \subseteq b)E \subseteq (a + c \subseteq b + c)$	($AMX19$)(Substitution of Eqn4 into Eqn3)
$(a \subseteq b) \subseteq (a + c \subseteq b + c)$	($AMX11$)(Universal Set Conjunction Axiom)

$$(a \subset b) = (a = ab)(a \neq b)$$

(T_{M766}) $\subset$ Equality Form

Proof of (T_{M766})

$(a \subseteq b) = (a = ab)$	($AMX44$)($\subseteq$ Equality Form)
$(a \subseteq b)(a \neq b) = (a = ab)(a \neq b)$	(T_{M029})(Deduction)
$((a \subset b) + (a = b))(a \neq b) = (a = ab)(a \neq b)$	(T_{M266})(Definition of $\subseteq$)
$(a \subset b)(a \neq b) + (a = b)(a \neq b) = (a = ab)(a \neq b)$	(T_{M407})(Distribution of $.$ over $+$)
$(a \subset b)(a \neq b) + \emptyset = (a = ab)(a \neq b)$	(T_{M621})(Complement Conjunction)
$(a \subset b)(a \neq b) = (a = ab)(a \neq b)$	(T_{M668})(Null Set Disjunction)(Eqn1)
$(a = b)(a \subset b) = \emptyset$	(T_{M037})(Conjunction of Binary Relations)
$(a \subset b) \subseteq (a \neq b)$	(T_{M735})($\subseteq$ Existential Form)
$(a \subset b) = (a \subset b)(a \neq b)$	($AMX44$)($\subseteq$ Equality Form)(Eqn2)
$(a \subset b) = (a = ab)(a \neq b)$	($AMX19$)(Substitution of Eqn2 into Eqn1)

$$(a \subset b) = (b = a + b)(a \neq b)$$

(T_{M765}) $\subset$ Equality Form

Proof of (T_{M765})

$(a \subseteq b) = (b = a + b)$	(T_{M764})($\subseteq$ Equality Form)
$(a \subseteq b)(a \neq b) = (b = a + b)(a \neq b)$	(T_{M029})(Deduction)
$((a \subset b) + (a = b))(a \neq b) = (b = a + b)(a \neq b)$	(T_{M266})(Definition of $\subseteq$)
$(a \subset b)(a \neq b) + (a = b)(a \neq b) = (b = a + b)(a \neq b)$	(T_{M407})(Distribution of $.$ over $+$)
$(a \subset b)(a \neq b) + \emptyset = (b = a + b)(a \neq b)$	(T_{M621})(Complement Conjunction)
$(a \subset b)(a \neq b) = (b = a + b)(a \neq b)$	(T_{M668})(Null Set Disjunction)(Eqn1)
$(a = b)(a \subset b) = \emptyset$	(T_{M037})(Conjunction of Binary Relations)
$(a \subset b) \subseteq (a \neq b)$	(T_{M735})($\subseteq$ Existential Form)
$(a \subset b) = (a \subset b)(a \neq b)$	($AMX44$)($\subseteq$ Equality Form)(Eqn2)
$(a \subset b) = (b = a + b)(a \neq b)$	($AMX19$)(Substitution of Eqn2 into Eqn1)

$ab \subseteq a$ (T_{M083}) Conjunction Elimination

Proof of (T_{M083})

$b \subseteq E$	(T_{M744})(Universal Set Theorem)
$ab \subseteq aE$	(T_{M029})(Deduction)
$ab \subseteq a$	($AMX11$)(Universal Set Conjunction Axiom)

$(a \subseteq b) \subseteq (a \subseteq ab)$ (T_{M438}) Absorbtion

Proof of (T_{M438})

Let $(a \subseteq b)$	
$(aa \subseteq ab)$	(T_{M029})(Deduction)
$(a \subseteq ab)$	(T_{M667})(Conjunction Tautology)
Thus $(a \subseteq b) \subseteq (a \subseteq ab)$	($AMX02$)(Conditional Introduction)

$(a \subset E) = (a \neq E)$ (T_{M020}) Definition of $\neq E$

Proof of (T_{M020})

$(a \neq E) + (a = E) = E$	(T_{M671})(Tertium non Datur)
$(a \subset E)((a \neq E) + (a = E)) = (a \subset E)E$	(T_{M029})(Deduction)
$(a \subset E)(a \neq E) + (a \subset E)(a = E) = (a \subset E)E$	(T_{M407})(Distribution of $\cdot$ over $+$)
$(a \subset E)(a \neq E) + (a \subset E)(a = E) = (a \subset E)$	($AMX11$)(Universal Set Conjunction Axiom)
$(a \subset E)(a \neq E) + \emptyset = (a \subset E)$	(T_{M037})(Conjunction of Binary Relations)
$(a \subset E)(a \neq E) = (a \subset E)$	(T_{M668})(Null Set Disjunction)
$(a \subset E) \subseteq (a \neq E)$	($AMX44$)($\subseteq$ Equality Form)(Eqn1)
$a \subseteq E$	(T_{M744})(Universal Set Theorem)
$(a \subseteq E) = E$	($AMX16$)(Order Axiom)
$(a \subset E) + (a = E) = E$	(T_{M266})(Definition of $\subseteq$)
$(a \neq E)((a \subset E) + (a = E)) = (a \neq E)E$	(T_{M029})(Deduction)
$(a \neq E)(a \subset E) + (a \neq E)(a = E) = (a \neq E)E$	(T_{M407})(Distribution of $\cdot$ over $+$)
$(a \neq E)(a \subset E) + (a \neq E)(a = E) = (a \neq E)$	($AMX11$)(Universal Set Conjunction Axiom)
$(a \neq E)(a \subset E) + \emptyset = (a \neq E)$	(T_{M621})(Complement Conjunction)
$(a \neq E)(a \subset E) = (a \neq E)$	(T_{M668})(Null Set Disjunction)
$(a \neq E) \subseteq (a \subset E)$	($AMX44$)($\subseteq$ Equality Form)(Eqn2)
$(a \subset E) = (a \neq E)$	(T_{M430})(Biconditional Introduction Eqns1&2)

$(\emptyset \subset a) = (a \neq \emptyset)$ (T_{M021}) Definition of $\neq \emptyset$

Proof of (T_{M021})

$(a \subset E) = (a \neq E)$	(T_{M020})(Definition of $\neq E$)
$(\sim E \subset \sim a) = (a \neq E)$	(T_{M010})(Inverse of a Proper Subset)
$(\sim E \subset \sim a) = (\sim a \neq \sim E)$	(T_{M008}) & (T_{M008})(Inverse of an Equality)
$(\emptyset \subset a) = (a \neq \emptyset)$	($DM02$)(Definition of $\emptyset$)
$(\emptyset \subset a) = (a \neq \emptyset)$	Resymbolising

$(a \sim \subset E) = (a = E)$ (T_{M022}) Definition of $\sim \subset E$

Proof of (T_{M022})

$(a \subset E) = (a \neq E)$	(T_{M020})(Definition of $\neq E$)
$\sim(a \subset E) = \sim(a \neq E)$	(T_{M008})(Inverse of an Equality)
$(a \sim \subset E) = \sim(a \neq E)$	(T_{M261})(Definition of $\sim \subset$)
$(a \sim \subset E) = \sim \sim(a = E)$	(T_{M260})(Definition of $\neq$)
$(a \sim \subset E) = (a = E)$	($AMX13$)(Double Complement Axiom)

$(\emptyset \sim \subset a) = (a = \emptyset)$ (T_{M023}) Definition of $\sim \subset \emptyset$

Proof of (T_{M023})

$(\emptyset \subset a) = (a \neq \emptyset)$	(T_{M021}) (Definition of $\neq \emptyset$)
$\sim(\emptyset \subset a) = \sim(a \neq \emptyset)$	(T_{M008}) (Inverse of an Equality)
$(\emptyset \sim \subset a) = \sim(a \neq \emptyset)$	(T_{M261}) (Definition of $\sim \subset$)
$(\emptyset \sim \subset a) = \sim\sim(a = \emptyset)$	(T_{M260}) (Definition of $\neq$)
$(\emptyset \sim \subset a) = (a = \emptyset)$	(A_{MX13}) (Double Complement Axiom)

$(a = b)(a \neq b) = \emptyset$	(T_{M031}) Complement Conjunction
$(a \subset b)(a \sim \subset b) = \emptyset$	(T_{M032}) Complement Conjunction
$(a \supset b)(a \sim \supset b) = \emptyset$	(T_{M033}) Complement Conjunction
$(a \parallel b)(a \not\parallel b) = \emptyset$	(T_{M034}) Complement Conjunction
$(a \subseteq b)(a \not\subseteq b) = \emptyset$	(T_{M035}) Complement Conjunction
$(a \supseteq b)(a \not\supseteq b) = \emptyset$	(T_{M036}) Complement Conjunction

Example Proof of (T_{M031})

$(a = b)(a \neq b)$	
$(a = b)\sim(a = b)$	(T_{M260}) (Definition of $\neq$)
$\emptyset$	(T_{M621}) (Complement Conjunction)
Thus $(a = b)(a \neq b) = \emptyset$	(A_{MX02}) (Conditional Introduction)

$$(a \sim \subset b)(a \subseteq b) = (a = b) \quad (T_{M578})$$

Proof of (T_{M578})

$(a \sim \subset b)(a \subseteq b) = (a \sim \subset b)(a \subseteq b)$	(D_{M00}) (Definition of $=$)
$(a \sim \subset b)(a \subseteq b) = (a \sim \subset b)((a \subset b) + (a = b))$	(T_{M266}) (Definition of $\subseteq$)
$(a \sim \subset b)(a \subseteq b) = (a \sim \subset b)(a \subset b) + (a \sim \subset b)(a = b)$	(T_{M407}) (Distribution of $\cdot$ over $+$)
$(a \sim \subset b)(a \subseteq b) = \emptyset + (a \sim \subset b)(a = b)$	(T_{M621}) (Complement Conjunction)
$(a \sim \subset b)(a \subseteq b) = (a \sim \subset b)(a = b)$	(T_{M668}) (Null Set Disjunction)(Eqn1)
$(a = b)(a \subset b) = \emptyset$	(T_{M037}) (Conjunction of Binary Relations)
$(a = b) \subseteq (a \sim \subset b)$	(T_{M735}) ($\subseteq$ Existential Form)
$(a = b) = (a = b)(a \sim \subset b)$	(A_{MX44}) ($\subseteq$ Equality Form)(Eqn2)
$(a \sim \subset b)(a \subseteq b) = (a = b)$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)

$$(a \subset \emptyset) = \emptyset \quad (T_{M053}) \text{ Proper Subset of the Null Set}$$

Proof of (T_{M053})

$(a \subset b) \subseteq (a \neq b)$	(T_{M588}) (Eqn1)
Let $b = \emptyset$	(Eqn2)
$(a \subset \emptyset) \subseteq (a \neq \emptyset)$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)(Eqn3)
$(a \subseteq \emptyset) = (a = \emptyset)$	(T_{M052}) (Subsets of the Null Set)
$(a \subset \emptyset) + (a = \emptyset) = (a = \emptyset)$	(T_{M266}) (Definition of $\subseteq$)(Eqn4)
$(a \subset \emptyset)((a \subset \emptyset) + (a = \emptyset)) = (a \neq \emptyset)(a = \emptyset)$	(T_{M669}) (Combination of Eqns3&4)
$(a \subset \emptyset)((a \subset \emptyset) + (a = \emptyset)) = \emptyset$	(T_{M031}) (Complement Conjunction)
$(a \subset \emptyset)(a \subset \emptyset) + (a \subset \emptyset)(a = \emptyset) = \emptyset$	(T_{M407}) (Distribution of $\cdot$ over $+$)
$(a \subset \emptyset)(a \subset \emptyset) + \emptyset = \emptyset$	(T_{M037}) (Conjunction of Binary Relations)
$(a \subset \emptyset)(a \subset \emptyset) = \emptyset$	(T_{M668}) (Null Set Disjunction)
$(a \subset \emptyset) = \emptyset$	(T_{M667}) (Conjunction Tautology)

$$(a \sim \subset \emptyset) = E \quad (T_{M053b})$$

Proof of (T_{M053b})

$(a \subset \emptyset) = \emptyset$	(T_{M053}) (Proper Subset of the Null Set)
$(a \sim \subset \emptyset) = \emptyset$	(T_{M008}) (Inverse of an Equality)
$(a \sim \subset \emptyset) = E$	(T_{M054}) (Negative of Null Set)

$$E^1 \sim \subset \emptyset \quad (T_{M757})$$

Proof of (T_{M757})

$$\begin{aligned} (a \sim \subset \emptyset) &= E & (T_{M053b}) \\ a \sim \subset \emptyset & & (A_{MX16})(\text{Order Axiom})(\text{Eqn1}) \\ \text{Let } a &= E^1 & (\text{Eqn2}) \\ E^1 \sim \subset \emptyset & & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(a \subset b) \subseteq (b \neq \emptyset) \quad (T_{M540})$$

Proof of (T_{M540})

$$\begin{aligned} \text{Let } b &= \emptyset & (\text{Eqn1}) \\ \text{Let } a &\subset b & (\text{Eqn2}) \\ a &\subset \emptyset & (A_{MX19})(\text{Substitution Eqn1 into Eqn2}) \\ \emptyset & & (T_{M053})(\text{Proper Subset of the Null Set}) \\ \text{Thus } (b = \emptyset)(a \subset b) &= \emptyset & (A_{MX02})(\text{Conditional Introduction}) \\ (a \subset b) &\subseteq (b \neq \emptyset) & (T_{M735})(\subseteq \text{ Existential Form}) \end{aligned}$$

$$(E \subset a) = \emptyset \quad (T_{M098b}) \text{ Proper Superset of the Universal Set}$$

Proof of (T_{M098b})

$$\begin{aligned} (a \subset \emptyset) &= \emptyset & (T_{M053})(\text{Proper Subset of the Null Set}) \\ (\emptyset \subset a) &= \emptyset & (T_{M010})(\text{Inverse of a Proper Subset}) \\ (E \subset a) &= \emptyset & (T_{M054})(\text{Negative of Null Set}) \\ (E \subset a) &= \emptyset & \text{Resymbolising} \end{aligned}$$

$$(E \sim \subset a) = E \quad (T_{M098d})$$

Proof of (T_{M098d})

$$\begin{aligned} (E \subset a) &= \emptyset & (T_{M098b})(\text{Proper Superset of the Universal Set}) \\ \sim(E \subset a) &= \emptyset & (T_{M008})(\text{Inverse of an Equality}) \\ \sim(E \subset a) &= E & (T_{M054})(\text{Negative of Null Set}) \\ (E \sim \subset a) &= E & (T_{M261})(\text{Definition of } \sim \subset) \end{aligned}$$

$$\emptyset^1 \sim \subset \emptyset \quad (T_{M605})$$

Proof of (T_{M605})

$$\begin{aligned} (a \subset b)(a \supset b) &= \emptyset & (T_{M040})(\text{Conjunction of Binary Relations}) \\ (a \supset b) &\subseteq (a \sim \subset b) & (T_{M735})(\subseteq \text{ Existential Form}) \\ (b \subset a) &\subseteq (a \sim \subset b) & (T_{M195})(\subset \text{ Reverse of } \supset)(\text{Eqn1}) \\ (a = b)(a \supset b) &= \emptyset & (T_{M038})(\text{Conjunction of Binary Relations}) \\ (a = b) &\subseteq (a \sim \subset b) & (T_{M735})(\subseteq \text{ Existential Form}) \\ (b = a) &\subseteq (a \sim \subset b) & (T_{M197})(\text{Commutativity of } =)(\text{Eqn2}) \\ (b \subset a) + (b = a) &\subseteq (a \sim \subset b) + (a \sim \subset b) & (A_{MX21})(\text{Combination of Eqns1\&2}) \\ (b \subset a) + (b = a) &\subseteq (a \sim \subset b) & (A_{MX01})(\text{Disjunction Tautology}) \\ (b \subseteq a) &\subseteq (a \sim \subset b) & (T_{M266})(\text{Definition of } \subseteq)(\text{Eqn3}) \\ \text{Let } a &= \emptyset^1 & (\text{Eqn4}) \\ \text{Let } b &= \emptyset & (\text{Eqn5}) \\ (\emptyset \subseteq \emptyset^1) &\subseteq (\emptyset^1 \sim \subset \emptyset) & (A_{MX19})(\text{Substitution of Eqns4\&5 into Eqn3}) \\ E &\subseteq (\emptyset^1 \sim \subset \emptyset) & (T_{M001})(\text{Universality of } (T_{M670}) \text{ Null Set Theorem}) \\ E &= (\emptyset^1 \sim \subset \emptyset) & (T_{M275})(\text{Supersets of the Universal Set}) \\ \emptyset^1 &\sim \subset \emptyset & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$(\emptyset^1 \subset a) \subseteq (a \neq \emptyset) \quad (T_{M652})$$

Proof of (T_{M652})

$$\text{Let } a = \emptyset \quad (\text{Eqn1})$$

$\emptyset^1 \sim \subset \emptyset$	(T_{M605})(Eqn2)
$\emptyset^1 \sim \subset a$	(A_{MX19})(Substitution of Eqn1 into Eqn2)
Thus $(a = \emptyset) \subseteq (\emptyset^1 \sim \subset a)$	(A_{MX02})(Conditional Introduction)
$(\emptyset^1 \subset a) \subseteq (a \neq \emptyset)$	(T_{M012})(Transposition)

$$(a^1 b = \emptyset) = (a^1 \subseteq b) \quad (T_{M688})$$

<u>Proof of</u> (T_{M688})	
$(a \subseteq b) = (ab = \emptyset)$	(T_{M735})($\subseteq$ Existential Form)(Eqn1)
Let $a = a^1$	(Eqn2)
$(a^1 \subseteq b) = (a^1 b = \emptyset)$	(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)
$(a^1 \subseteq b) = (a^1 b = \emptyset)$	Resymbolising

$$(\underline{a}^1 b = \emptyset) = (\underline{a}^1 \subseteq b) \quad (T_{M689})$$

<u>Proof of</u> (T_{M689})	
$(a \subseteq b) = (ab = \emptyset)$	(T_{M735})($\subseteq$ Existential Form)(Eqn1)
Let $a = \underline{a}^1$	(Eqn2)
$(\underline{a}^1 \subseteq b) = (\underline{a}^1 b = \emptyset)$	(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)
$(\underline{a}^1 \subseteq b) = (\underline{a}^1 b = \emptyset)$	Resymbolising

$$E \sim \subset \sim \emptyset^1 \quad (T_{M675})$$

<u>Proof of</u> (T_{M675})	
$\emptyset^1 \sim \subset \emptyset$	(T_{M605})
$\sim \emptyset \sim \subset \sim \emptyset^1$	(T_{M010})(Inverse of a Proper Subset)
$E \sim \subset \sim \emptyset^1$	(T_{M054})(Negative of Null Set)

$$a + ab = a + b \quad (T_{M612})$$

<u>Proof of</u> (T_{M612})	
$a + b = a + b$	(D_{M00})(Definition of =)
$a + b = a + b.E$	(A_{MX11})(Universal Set Conjunction Axiom)
$a + b = a + b(a + \underline{a})$	(T_{M671})(Tertium non Datur)
$a + b = a + ab + \underline{ab}$	(T_{M407})(Distribution of . over +)
$a + b = a + \underline{ab}$	(T_{M063})(Partial Redundancy)

$$a + \underline{ab} = \underline{ab} + b \quad (T_{M611})$$

<u>Proof of</u> (T_{M611})	
$a + \underline{ab} = a + b$	(T_{M612})(Eqn1)
$b + \underline{ba} = b + a$	Resymbolising (Eqn2)
$a + \underline{ab} = b + \underline{ba}$	(A_{MX19})(Substitution of Eqn2 into Eqn1)
$a + \underline{ab} = b + \underline{ab}$	(T_{M404})(Commutative .)
$a + \underline{ab} = \underline{ab} + b$	(T_{M403})(Commutative +)

$$a + (b.c) = (a + b)(a + c) \quad (T_{M273}) \text{ Distribution of } + \text{ over } .$$

<u>Proof of</u> (T_{M273})	
$(a + b)(a + c) = (a + b)(a + c)$	(D_{M00})(Definition of =)
$a(a + c) + b(a + c) = (a + b)(a + c)$	(T_{M407})(Distribution of . over +)
$aa + ac + ba + bc = (a + b)(a + c)$	(T_{M407})(Distribution of . over +)
$a + ac + ba + bc = (a + b)(a + c)$	(T_{M667})(Conjunction Tautology)
$a + bc = (a + b)(a + c)$	(T_{M063})(Partial Redundancy)

$$((a = b) = E) \subseteq ((c \subseteq a) = (c \subseteq b)) \quad (T_{M269})$$

Proof of (T_{M269})

Let $a \subseteq c$	(Eqn1)
Let $c = b$	(Eqn2)
$a \subseteq b$	($AMX19$)(Substitution of Eqn2 into Eqn1)
Thus $(a \subseteq c)(c = b) \subseteq (a \subseteq b)$	($AMX02$)(Conditional Introduction)
$((c = b) = E) \subseteq ((a \subseteq c) \subseteq (a \subseteq b))$	($AMX35$)(Exportation/Importation)(Eqn3) for truth sets
Let $a \subseteq b$	(Eqn4)
Let $c = b$	(Eqn5)
$a \subseteq c$	($AMX19$)(Substitution of Eqn5 into Eqn4)
Thus $(a \subseteq b)(c = b) \subseteq (a \subseteq c)$	($AMX02$)(Conditional Introduction)
$((c = b) = E) \subseteq ((a \subseteq b) \subseteq (a \subseteq c))$	($AMX35$)(Exportation/Importation)(Eqn6) for truth sets
$((c = b) = E)((c = b) = E) \subseteq ((a \subseteq c) \subseteq (a \subseteq b))((a \subseteq b) \subseteq (a \subseteq c))$	(T_{M669})(Combination of Eqns3&6)
$((c = b) = E) \subseteq ((a \subseteq c) \subseteq (a \subseteq b))((a \subseteq b) \subseteq (a \subseteq c))$	(T_{M667})(Conjunction Tautology)
$((c = b) = E) \subseteq ((a \subseteq c) = (a \subseteq b))$	(T_{M430})(Biconditional Introduction)
$((a = b) = E) \subseteq ((c \subseteq a) = (c \subseteq b))$	Resymbolising

$$((a \subseteq b) = E) \subseteq ((b \subseteq a) = (a = b)) \quad (T_{M426}) \text{ Biconditional Introduction}$$

Proof of (T_{M426})

$(a \subseteq b)(b \subseteq a) = (a = b)$	(T_{M430})(Biconditional Introduction)
$((a \subseteq b) = E) \subseteq ((b \subseteq a) = (a = b))$	(T_{M444c})(Biconditional Exportation)

$$((a = E) = E) \subseteq ((a \subseteq b) \subseteq (b = E)) \quad (T_{M435}) \text{ Modus Ponens}$$

Proof of (T_{M435})

$(a = E)(a \subseteq b) \subseteq (b = E)$	(T_{M427})(Modus Ponens)
$((a = E) = E) \subseteq ((a \subseteq b) \subseteq (b = E))$	($AMX35$)(Exportation/ Importation) for truth sets

$$((c \neq E) = E) \subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b)) \quad (T_{M141}) \text{ Conditional Syllogism}$$

Proof of (T_{M141})

Let $(a \subseteq c)$	
$E = (a + c)$	(T_{M064})($\subseteq$ Universal form)(Eqn1)
Let $(b \subseteq c)$	
$E = (b + c)$	(T_{M064})($\subseteq$ Universal form)(Eqn2)
Let $c \neq \emptyset$	
$EEc \neq \emptyset$	($AMX11$)(Universal Set Conjunction Axiom)(Eqn3)
$(a + c)(b + c)c \neq \emptyset$	($AMX19$)(Substitution of Eqns1 and 2 into Eqn3)
$abc + bcc + acc + ccc \neq \emptyset$	(T_{M407})(Distribution of . over +)
$abc + b\emptyset + a\emptyset + c\emptyset \neq \emptyset$	(T_{M621})(Complement Conjunction)
$abc + \emptyset + \emptyset + \emptyset \neq \emptyset$	(T_{M050})(Null Set Conjunction)
$abc \neq \emptyset$	(T_{M668})(Null Set Disjunction)
$ab \neq \emptyset$	(T_{M350})(T_{M012})
$a \not\subseteq b$	(T_{M735})($\subseteq$ Existential Form)&(T_{M008})
Thus $(c \neq \emptyset)(a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b)$	($AMX02$)(Conditional Introduction)
$(c \neq \sim\emptyset)(a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b)$	(T_{M008})(T_{M008})
$(c \neq E)(a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b)$	(T_{M054})(Negative of Null Set)
$((c \neq E) = E) \subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b))$	($AMX35$)(Exportation/Importation) for truth sets

$$((ac \subseteq bc) = E) \subseteq ((c = E) \subseteq (a \subseteq b)) \quad (T_{M091}) \text{ Conditional Subsets Rule}$$

Proof of (T_{M091})

Let $(c = E)$	(Eqn1)
Let $(ac \subseteq bc)$	(Eqn2)
$(aE \subseteq bE)$	($AMX19$)(Substitution of Eqn1 into Eqn2)
$(a \subseteq b)$	($AMX11$)(Universal Set Conjunction Axiom)

Thus $(c = E)(ac \subseteq bc) \subseteq (a \subseteq b)$
 $((ac \subseteq bc) = E) \subseteq ((c = E) \subseteq (a \subseteq b))$

(A_{MX02}) (Conditional Introduction)
 (A_{MX35}) (Exportation/Importation) for truth sets

$((ac = bc) = E) \subseteq ((c = E) \subseteq (a = b))$ (T_{M091b}) Conditional Subsets Rule

Proof of (T_{M091b})

$((ac \subseteq bc) = E) \subseteq ((c = E) \subseteq (a \subseteq b))$
 $(ac \subseteq bc)(c = E) \subseteq (a \subseteq b)$
 $(bc \subseteq ac)(c = E) \subseteq (b \subseteq a)$
 $(ac \subseteq bc)(c = E)(bc \subseteq ac)(c = E) \subseteq (a \subseteq b)(b \subseteq a)$
 $(ac \subseteq bc)(bc \subseteq ac)(c = E) \subseteq (a \subseteq b)(b \subseteq a)$
 $(ac = bc)(c = E) \subseteq (a = b)$
 $((ac = bc) = E) \subseteq ((c = E) \subseteq (a = b))$

(T_{M091}) (Conditional Subsets Rule)
 (A_{MX35}) (Exportation/Importation)(Eqn1) for truth sets
 Resymbolising (Eqn2)
 (T_{M669}) (Combination of Eqns1&2)
 (T_{M667}) (Conjunction Tautology)
 (T_{M430}) (Biconditional Introduction)
 (A_{MX35}) (Exportation/Importation) for truth sets

$((a \subseteq b) = E) \subseteq ((c \subseteq a) \subseteq (c \subseteq b))$ (T_{M272}) Deductive Rule

Proof of (T_{M272})

$(c \subseteq a)(a \subseteq b) \subseteq (c \subseteq b)$
 $((a \subseteq b) = E) \subseteq ((c \subseteq a) \subseteq (c \subseteq b))$

(T_{M139}) (Hypothetical Syllogism)
 (A_{MX35}) (Exportation/Importation) for truth sets

$((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) \subseteq (c^1 \subseteq b))$ (T_{M099}) Entity Mode / Set Mode Rule

Proof of (T_{M099})

$((a \subseteq b) = E) \subseteq ((c \subseteq a) \subseteq (c \subseteq b))$
 Let $c = c^1$
 $((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) \subseteq (c^1 \subseteq b))$

(T_{M272}) (Deductive Rule)(Eqn1)
 (Eqn2)
 (A_{MX19}) (Substitution of Eqn2 into Eqn1)

$(c \subseteq ab) \subseteq (c \subseteq a)$ (T_{M073}) Single Premise Syllogism

Proof of (T_{M073})

$ab \subseteq a$
 $(c \subseteq ab) \subseteq (c \subseteq a)$

(T_{M083}) (Conjunction Elimination)
 (T_{M272}) (Deductive Rule)

$(a \subset c)(c \subset b) \subseteq (a \subset b)$ (T_{M400})

Proof of (T_{M400})

$(a \subset c) = (a = ac)(a \neq c)$
 $(c \subset b) = (c = cb)(c \neq b)$
 $(a \subset c)(c \subset b) = (a = ac)(c = cb)(a \neq c)(c \neq b)$
 $(a \subset c)(c \subset b) \subseteq (a = ac)(a = acb)(a \neq c)(c \neq b)$
 $(a \subset c)(c \subset b) \subseteq (a = ab)(a \neq c)(c \neq b)$
 Let $b = a$
 Let $(a \subset c)(c \subset b)$
 $(a \subset c)(c \subset a)$
 $\emptyset$
 Thus $(b = a)(a \subset c)(c \subset b) = \emptyset$
 $(a \subset c)(c \subset b) \subseteq (b \neq a)$
 $(a \subset c)(c \subset b)(a \subset c)(c \subset b) \subseteq (b = a + b)(b \neq a)(a \neq c)(c \neq b)$
 $(a \subset c)(c \subset b) \subseteq (b = a + b)(b \neq a)(a \neq c)(c \neq b)$
 $(a \subset c)(c \subset b) \subseteq (a \subset b)(a \neq c)(c \neq b)$
 $(a \subset c)(c \subset b) \subseteq (a \subset b)$

(T_{M765}) ($\subset$ Equality Form)(Eqn1)
 Resymbolising (Eqn2)
 (T_{M231}) (Combination of Eqns1&2)
 (A_{MX19}) (Substitution)
 (A_{MX19}) (Substitution)(Eqn3)
 (Eqn4)
 (Eqn5)
 (A_{MX19}) (Substitution of Eqn4 into Eqn5)
 (T_{M040}) (Conjunction of Binary Relations)
 (A_{MX02}) (Conditional Introduction)
 (T_{M735}) ($\subseteq$ Existential Form)(Eqn6)
 (T_{M669}) (Combination of Eqns3&6)
 (T_{M667}) (Conjunction Tautology)
 (T_{M765}) ($\subset$ Equality Form)
 (T_{M073}) (Single Premise Syllogism)

$(a \subset c)(c \subset b) + (a \subset c)(c = b) + (a = c)(c \subset b) \subseteq (a \subset b)$ (T_{M680})

Proof of (T_{M680})

$(a \subset c)(c \subset b) \subseteq (a \subset b)$
 Let $a \subset c$

(T_{M400}) (Proper Syllogism)(Eqn1)
 (Eqn2)

Let $c = b$	(Eqn3)
$a \subset b$	($\mathcal{A}_{MX19}$)(Substitution of Eqn3 into Eqn2)
Thus $(a \subset c)(c = b) \subseteq (a \subset b)$	($\mathcal{A}_{MX02}$)(Conditional Introduction)(Eqn4)
Let $a = c$	(Eqn5)
Let $c \subset b$	(Eqn6)
$a \subset b$	($\mathcal{A}_{MX19}$)(Substitution of Eqn5 into Eqn6)
Thus $(a = c)(c \subset b) \subseteq (a \subset b)$	($\mathcal{A}_{MX02}$)(Conditional Introduction)(Eqn7)
$(a \subset c)(c \subset b) + (a \subset c)(c = b) + (a = c)(c \subset b) \subseteq (a \subset b) + (a \subset b) + (a \subset b)$	($\mathcal{A}_{MX21}$)(Combination of Eqns1,4&7)
$(a \subset c)(c \subset b) + (a \subset c)(c = b) + (a = c)(c \subset b) \subseteq (a \subset b)$	($\mathcal{A}_{MX01}$)(Disjunction Tautology)

$(a + b \subseteq c) \subseteq (a \subseteq c)$ (T_{M072}) Single Premise Syllogism

Proof of (T_{M072})

$(c \subseteq ab) \subseteq (c \subseteq a)$	(T_{M073})(Single Premise Syllogism)
$(\sim(ab) \subseteq c) \subseteq (a \subseteq c)$	(T_{M012})(Transposition)
$(a + b \subseteq c) \subseteq (a \subseteq c)$	(T_{M025})(DeMorgan's Law)
$(a + b \subseteq c) \subseteq (a \subseteq c)$	Resymbolising

$(a \subseteq c)(b \subseteq c) = (a + b \subseteq c)$ (T_{M131b}) Conjunction of Sufficiencies

Proof of (T_{M131b})

Let $a \subseteq c$	(Eqn1)
Let $b \subseteq c$	(Eqn2)
$a + b \subseteq c + c$	($\mathcal{A}_{MX21}$)(Combination of Eqns1&2)
$a + b \subseteq c$	($\mathcal{A}_{MX01}$)(Disjunction Tautology)
Thus $(a \subseteq c)(b \subseteq c) \subseteq (a + b \subseteq c)$	($\mathcal{A}_{MX02}$)(Conditional Introduction)(Eqn3)
Let $a + b \subseteq c$	
$a \subseteq c$	(T_{M072})(Single Premise Syllogism)
$b \subseteq c$	(T_{M072})(Single Premise Syllogism)
Thus $(a + b \subseteq c) \subseteq (a \subseteq c)(b \subseteq c)$	($\mathcal{A}_{MX02}$)(Conditional Introduction)(Eqn4)
$(a \subseteq c)(b \subseteq c) = (a + b \subseteq c)$	(T_{M430})(Biconditional Introduction Eqns3&4)

$(c \neq \emptyset):((c \subseteq a) \subseteq (c \subseteq b)) \subseteq ((a \subseteq b) = E)$ (T_{M747}) Deductive Rule

Proof of (T_{M747})

Let $c_1^{+++} c_E^1 = E$	i.e., c_1 to c_E are all possible sets
$c_1^{+++} c_E = E$	(T_{M749})(Eqn1)
Let $c = c_1, \dots, c_E$	(Eqn2) i.e. c is any possible set
Let $(c \subseteq a) \subseteq (c \subseteq b)$	(Eqn3)
For $r = 1$ to E $(c_r \subseteq a) \subseteq (c_r \subseteq b)$	($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn3)(Eqns4)
$(c_1 \subseteq a) \dots (c_E \subseteq a) \subseteq (c_1 \subseteq b) \dots (c_E \subseteq b)$	(T_{M669})(Combination of Eqns4) For $r = 1$ to E
$(c_1^{+++} c_E \subseteq a) \subseteq (c_1^{+++} c_E \subseteq b)$	(T_{M131b})(Conjunction of Sufficiencies)(Eqn5)
$(E \subseteq a) \subseteq (E \subseteq b)$	($\mathcal{A}_{MX19}$)(Substitution of Eqn1 into Eqn5)
$(E = a) \subseteq (E = b)$	(T_{M275})(Supersets of the Universal Set)
$(a \subseteq b) = E$	(T_{M094})(Probabilistic Rule)
Thus $((c \subseteq a) \subseteq (c \subseteq b)) \subseteq ((a \subseteq b) = E)$	($\mathcal{A}_{MX02}$)(Conditional Introduction)

NB this is subject to the condition $(c \neq \emptyset)$ as implied by Eqn2. If $\emptyset$ is substituted for c in this theorem then it simplifies to $a \subseteq b$ irrespective of the values of a and b and this is clearly false.

$(c^1 \neq \emptyset):((c^1 \subseteq a) \subseteq (c^1 \subseteq b)) \subseteq ((a \subseteq b) = E)$ (T_{M748}) Entity Mode / Set Mode Rule

Proof of (T_{M748})

$(c \neq \emptyset):$	
$((c \subseteq a) \subseteq (c \subseteq b)) \subseteq ((a \subseteq b) = E)$	(T_{M747})(Deductive Rule)(Eqn1)
Let $c = c^1$	(Eqn2)
$(c^1 \neq \emptyset):$	

$$((c^1 \subseteq a) \subseteq (c^1 \subseteq b)) \subseteq ((a \subseteq b) = E)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(c \neq \emptyset):((c \subseteq a) \subseteq (c \subseteq b)) = ((a \subseteq b) = E)$$

(T_{M750}) Deductive Rule

Proof of (T_{M750})

($c \neq \emptyset$):

$$((c \subseteq a) \subseteq (c \subseteq b)) \subseteq ((a \subseteq b) = E)$$

(T_{M747})(Deductive Rule)(Eqn1)

$$((a \subseteq b) = E) \subseteq ((c \subseteq a) \subseteq (c \subseteq b))$$

(T_{M272})(Deductive Rule)(Eqn2)

$$((c \subseteq a) \subseteq (c \subseteq b)) = ((a \subseteq b) = E)$$

(T_{M430})(Biconditional Introduction Eqns1&2)

$$(c^1 \neq \emptyset):((c^1 \subseteq a) \subseteq (c^1 \subseteq b)) = ((a \subseteq b) = E)$$

(T_{M751}) Entity Mode / Set Mode Rule

Proof of (T_{M751})

($c^1 \neq \emptyset$):

$$((c^1 \subseteq a) \subseteq (c^1 \subseteq b)) \subseteq ((a \subseteq b) = E)$$

(T_{M748})(Entity Mode / Set Mode Rule)(Eqn1)

$$((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) \subseteq (c^1 \subseteq b))$$

(T_{M099})(Entity Mode / Set Mode Rule)(Eqn2)

$$((c^1 \subseteq a) \subseteq (c^1 \subseteq b)) = ((a \subseteq b) = E)$$

(T_{M430})(Biconditional Introduction Eqns1&2)

$$(a \neq \Omega) + (ab = \emptyset):((a = E) \subseteq (b = c)) \subseteq ((ab \subseteq c) = E)$$

(T_{M444d}) Biconditional Importation

Proof of (T_{M444d})

($a \neq \Omega$) + ($ab = \emptyset$):

$$((ab \subseteq c) = E) = ((a = E) \subseteq (b \subseteq c))$$

(A_{MX35})(Exportation/Importation)(Eqn1)

$$(b \subseteq c)(b = c) = (b = c)$$

(T_{M043})(Conjunction of Binary Relations)

$$(b = c) \subseteq (b \subseteq c)$$

(A_{MX44})($\subseteq$ Equality Form)

$$((a = E) \subseteq (b = c)) \subseteq ((a = E) \subseteq (b \subseteq c))$$

(T_{M272})(Deductive Rule)(Eqn2)

$$((a = E) \subseteq (b = c)) \subseteq ((ab \subseteq c) = E)$$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

$$(a \neq \Omega) + (ab = \emptyset):(((a = E) \subseteq (b = c)) = ((ab = ac) = E))$$

(T_{M444b}) Biconditional Exportation/Importation

Proof of (T_{M444b})

($a \neq \Omega$) + ($ab = \emptyset$):

$$((a = E) \subseteq (b = c)) \subseteq ((ab \subseteq c) = E)$$

(T_{M444d})(Biconditional Importation)(Eqn1)

$$((a = E) \subseteq (b = c)) \subseteq ((ac \subseteq b) = E)$$

(T_{M444d})(Biconditional Importation)(Eqn2)

$$((a = E) \subseteq (b = c)).((a = E) \subseteq (b = c)) \subseteq ((ab \subseteq c) = E).((ac \subseteq b) = E)$$

(T_{M669})(Combination of Eqns1&2)

$$((a = E) \subseteq (b = c)) \subseteq ((ab \subseteq c) = E).((ac \subseteq b) = E)$$

(T_{M667})(Conjunction Tautology)(Eqn3)

$$\text{Let } (ab \subseteq c) = E$$

(Eqn4)

$$\text{Let } (ac \subseteq b) = E$$

(Eqn5)

$$EE = E$$

(T_{M667})(Conjunction Tautology)(Eqn6)

$$(ab \subseteq c)(ac \subseteq b) = E$$

(A_{MX19})(Substitution of Eqns4&5 into Eqn6)

$$\text{Thus } ((ab \subseteq c) = E)((ac \subseteq b) = E) \subseteq ((ab \subseteq c)(ac \subseteq b) = E)$$

(A_{MX02})(Conditional Introduction)(Eqn7)

$$((a = E) \subseteq (b = c)) \subseteq ((ab \subseteq c)(ac \subseteq b) = E)$$

(T_{M139})(Hypothetical Syllogism Eqns3&7 for truth sets (Eqn8)

$$\text{Let } (ab \subseteq c)$$

(T_{M029})(Deduction)

$$aab \subseteq ac$$

(T_{M667})(Conjunction Tautology)

$$ab \subseteq ac$$

(A_{MX02})(Conditional Introduction)(Eqn9)

$$\text{Thus } (ab \subseteq c) \subseteq (ab \subseteq ac)$$

$$\text{Let } (ac \subseteq b)$$

(T_{M029})(Deduction)

$$aac \subseteq ab$$

(T_{M667})(Conjunction Tautology)

$$ac \subseteq ab$$

(A_{MX02})(Conditional Introduction)(Eqn10)

$$\text{Thus } (ac \subseteq b) \subseteq (ac \subseteq ab)$$

(T_{M669})(Combination of Eqns9&10)

$$(ab \subseteq c)(ac \subseteq b) \subseteq (ab \subseteq ac)(ac \subseteq ab)$$

(T_{M430})(Biconditional Introduction)(Eqn11)

$$(ab \subseteq c)(ac \subseteq b) \subseteq (ab = ac)$$

(Eqn12)

$$\text{Let } (ab \subseteq c)(ac \subseteq b) = E$$

(A_{MX19})(Substitution of Eqn12 into Eqn11)

$$E \subseteq (ab = ac)$$

(T_{M275}) (Supersets of the Universal Set)

$$(ab = ac) = E$$

Thus $((ab \subseteq c)(ac \subseteq b) = E) \subseteq ((ab = ac) = E)$
 $((a = E) \subseteq (b = c)) \subseteq ((ab = ac) = E)$

Let $a = E$
 Let $ab = ac$
 $Eb = Ec$
 $b = c$

Thus $(a = E)(ab = ac) \subseteq (b = c)$
 $((ab = ac) = E) \subseteq ((a = E) \subseteq (b = c))$
 $((a = E) \subseteq (b = c)) = ((ab = ac) = E)$

($AMX02$)(Conditional Introduction)(Eqn13)
 ($TM139$)(Hypothetical Syllogism Eqns8&13)
 for truth sets (Eqn14)
 (Eqn15)
 (Eqn16)
 ($AMX19$)(Substitution of Eqn15 into Eqn16)
 ($AMX11$)(Universal Set Conjunction Axiom)
 ($AMX02$)(Conditional Introduction)
 ($AMX35$)(Exportation/Importation)(Eqn17) for truth sets
 ($TM430$)(Biconditional Introduction Eqns14&17)

$(a \supseteq b) = (b = ab)$

($TM361$) $\supseteq$ Equality Form

Proof of ($TM361$)

$(a \subseteq b) = (a = ab)$ ($AMX44$)($\subseteq$ Equality Form)
 $(b \subseteq a) = (b = ba)$ Resymbolising
 $(b \subseteq a) = (b = ab)$ ($TM404$)(Commutative .)
 $(a \supseteq b) = (b = ab)$ ($TM196$)($\subseteq$ Reverse of $\supseteq$)

$(ab \subseteq c) = (ab \subseteq ac)$

($TM256$) Conditional Subsets Rule

Proof of ($TM256$)

Let $ab \subseteq ac$
 $ab = abac$ ($AMX44$)($\subseteq$ Equality Form)
 $ab = abc$ ($TM667$)(Conjunction Tautology)
 $ab \subseteq c$ ($AMX44$)($\subseteq$ Equality Form)
 Thus $(ab \subseteq c) = (ab \subseteq ac)$ ($AMX02$)(Conditional Introduction) as this proof is reversible

$((ac \subseteq b) = E) \subseteq ((c = E) \subseteq (a \subseteq b))$

($TM257$) Conditional Subsets Rule

Proof of ($TM257$)

$((ac \subseteq bc) = E) \subseteq ((c = E) \subseteq (a \subseteq b))$ ($TM091$)(Conditional Subsets Rule)
 $((ac \subseteq b) = E) \subseteq ((c = E) \subseteq (a \subseteq b))$ ($TM256$)(Conditional Subsets Rule)

$(ab \subseteq c) = (a \subseteq \underline{b} + c)$

($TM468$)

Proof of ($TM468$)

Let $a \subseteq \underline{b} + c$
 $a = a(\underline{b} + c)$ ($AMX44$)($\subseteq$ Equality Form)
 $a = a\underline{b} + ac$ ($TM407$)(Distribution of . over +)
 $ab = b(a\underline{b} + ac)$ ($TM233$)(Deduction)
 $ab = ab\underline{b} + abc$ ($TM407$)(Distribution of . over +)
 $ab = a\emptyset + abc$ ($TM621$)(Complement Conjunction)
 $ab = \emptyset + abc$ ($TM050$)(Null Set Conjunction)
 $ab = abc$ ($TM668$)(Null Set Disjunction)
 $ab \subseteq c$ ($AMX44$)($\subseteq$ Equality Form)
 Thus $(a \subseteq \underline{b} + c) \subseteq (ab \subseteq c)$ ($AMX02$)(Conditional Introduction)(Eqn1)
 Let $ab \subseteq c$
 $ab = abc$ ($AMX44$)($\subseteq$ Equality Form)
 $ab + ab = ab + abc$ ($TM232$)(Deduction)
 $a(b + \underline{b}) = ab + abc$ ($TM407$)(Distribution of . over +)
 $aE = ab + abc$ ($TM671$)(Tertium non Datur)
 $a = ab + abc$ ($AMX11$)(Universal Set Conjunction Axiom)
 $a = a(\underline{b} + bc)$ ($TM407$)(Distribution of . over +)
 $a \subseteq \underline{b} + bc$ ($AMX44$)($\subseteq$ Equality Form)
 $a \subseteq \underline{b} + \underline{b}c + bc$ ($TM063$)(Partial Redundancy)
 $a \subseteq \underline{b} + c(\underline{b} + b)$ ($TM407$)(Distribution of . over +)
 $a \subseteq \underline{b} + cE$ ($TM671$)(Tertium non Datur)
 $a \subseteq \underline{b} + c$ ($AMX11$)(Universal Set Conjunction Axiom)

Thus $(\mathbf{ab} \subseteq \mathbf{c}) \subseteq (\mathbf{a} \subseteq \mathbf{b} + \mathbf{c})$
 $(\mathbf{ab} \subseteq \mathbf{c}) = (\mathbf{a} \subseteq \mathbf{b} + \mathbf{c})$

$(\mathbf{A}_{MX02})$ (Conditional Introduction)(Eqn2)
 $(\mathbf{T}_{M430})$ (Biconditional Introduction Eqns1&2)

$(\mathbf{a}^1 = \emptyset^1) \subseteq (\mathbf{a} = \emptyset) + (\mathbf{a} = \emptyset^1)$ $(\mathbf{T}_{M497})$

$((\mathbf{a} \neq \emptyset^1) = \mathbf{E}) \subseteq ((\mathbf{a}^1 = \emptyset^1) \subseteq (\mathbf{a} = \emptyset))$ $(\mathbf{T}_{M532})$

Proof of $(\mathbf{T}_{M497})$ & $(\mathbf{T}_{M532})$

$(\mathbf{a}^1 = \emptyset^1) \subseteq (\mathbf{a} = \emptyset)$	$(\mathbf{T}_{M479})$ (Eqn1)
$(\mathbf{a}^1 = \emptyset^{11}) \subseteq (\mathbf{a} = \emptyset^1)$	$(\mathbf{T}_{M479})$
$(\mathbf{a}^1 = \emptyset^1) \subseteq (\mathbf{a} = \emptyset^1)$	$(\mathbf{D}_{M17})$ (Definition of $\mathbf{a}^{11}$)(Eqn2)
$(\mathbf{a}^1 = \emptyset^1) + (\mathbf{a}^1 = \emptyset^1) \subseteq (\mathbf{a} = \emptyset) + (\mathbf{a} = \emptyset^1)$	$(\mathbf{A}_{MX21})$ (Combination of Eqns1&2)
$(\mathbf{a}^1 = \emptyset^1) \subseteq (\mathbf{a} = \emptyset) + (\mathbf{a} = \emptyset^1)$	$(\mathbf{A}_{MX01})$ (Disjunction Tautology)
$(\mathbf{a} \neq \emptyset^1)(\mathbf{a}^1 = \emptyset^1) \subseteq (\mathbf{a} = \emptyset)$	$(\mathbf{T}_{M468})$
$((\mathbf{a} \neq \emptyset^1) = \mathbf{E}) \subseteq ((\mathbf{a}^1 = \emptyset^1) \subseteq (\mathbf{a} = \emptyset))$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets

$(\mathbf{a} = \emptyset) \subseteq (\mathbf{a}^1 = \emptyset) + (\mathbf{a}^1 = \emptyset^1)$ $(\mathbf{T}_{M496})$

$((\mathbf{a}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} = \emptyset) \subseteq (\mathbf{a}^1 = \emptyset^1))$ $(\mathbf{T}_{M532b})$

Proof of $(\mathbf{T}_{M496})$ & $(\mathbf{T}_{M532b})$

$(\mathbf{a} = \emptyset) \subseteq (\mathbf{a}^1 = \emptyset^1)$	$(\mathbf{T}_{M479})$
$(\mathbf{a} = \emptyset) \subseteq (\mathbf{a}^1 = \emptyset^1) + (\mathbf{a}^1 = \emptyset^1)$	$(\mathbf{A}_{MX01})$ (Disjunction Tautology)
$(\mathbf{a} = \emptyset) \subseteq (\mathbf{a}^1 = \emptyset^1) + (\mathbf{a}^{11} = \emptyset^1)$	$(\mathbf{D}_{M17})$ (Definition of $\mathbf{a}^{11}$)(Eqn1)
$(\mathbf{a}^{11} = \emptyset^1) \subseteq (\mathbf{a}^1 = \emptyset)$	$(\mathbf{T}_{M479})$
$(\mathbf{a}^1 = \emptyset^1) + (\mathbf{a}^{11} = \emptyset^1) \subseteq (\mathbf{a}^1 = \emptyset^1) + (\mathbf{a}^1 = \emptyset)$	$(\mathbf{T}_{M030})$ (Deduction)(Eqn2)
$(\mathbf{a} = \emptyset) \subseteq (\mathbf{a}^1 = \emptyset^1) + (\mathbf{a}^1 = \emptyset)$	$(\mathbf{T}_{M139})$ (Hypothetical Syllogism Eqns1&2)
$(\mathbf{a}^1 \neq \emptyset)(\mathbf{a} = \emptyset) \subseteq (\mathbf{a}^1 = \emptyset^1)$	$(\mathbf{T}_{M468})$
$((\mathbf{a}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} = \emptyset) \subseteq (\mathbf{a}^1 = \emptyset^1))$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets

$(\mathbf{a}^1 = \mathbf{E}^1) \subseteq (\mathbf{a} = \mathbf{E}) + (\mathbf{a} = \mathbf{E}^1)$ $(\mathbf{T}_{M488b})$

$((\mathbf{a} \neq \mathbf{E}^1) = \mathbf{E}) \subseteq ((\mathbf{a}^1 = \mathbf{E}^1) \subseteq (\mathbf{a} = \mathbf{E}))$ $(\mathbf{T}_{M532c})$

Proof of $(\mathbf{T}_{M488b})$ & $(\mathbf{T}_{M532c})$

$(\mathbf{a}^1 = \mathbf{E}^1) \subseteq (\mathbf{a} = \mathbf{E})$	$(\mathbf{T}_{M479})$ (Eqn1)
$(\mathbf{a}^1 = \mathbf{E}^{11}) \subseteq (\mathbf{a} = \mathbf{E}^1)$	$(\mathbf{T}_{M479})$
$(\mathbf{a}^1 = \mathbf{E}^1) \subseteq (\mathbf{a} = \mathbf{E}^1)$	$(\mathbf{D}_{M17})$ (Definition of $\mathbf{a}^{11}$)(Eqn2)
$(\mathbf{a}^1 = \mathbf{E}^1) + (\mathbf{a}^1 = \mathbf{E}^1) \subseteq (\mathbf{a} = \mathbf{E}) + (\mathbf{a} = \mathbf{E}^1)$	$(\mathbf{A}_{MX21})$ (Combination of Eqns1&2)
$(\mathbf{a}^1 = \mathbf{E}^1) \subseteq (\mathbf{a} = \mathbf{E}) + (\mathbf{a} = \mathbf{E}^1)$	$(\mathbf{A}_{MX01})$ (Disjunction Tautology)
$(\mathbf{a} \neq \mathbf{E}^1)(\mathbf{a}^1 = \mathbf{E}^1) \subseteq (\mathbf{a} = \mathbf{E})$	$(\mathbf{T}_{M468})$
$((\mathbf{a} \neq \mathbf{E}^1) = \mathbf{E}) \subseteq ((\mathbf{a}^1 = \mathbf{E}^1) \subseteq (\mathbf{a} = \mathbf{E}))$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets

$((\mathbf{a} \subseteq \mathbf{c}) = \mathbf{E}) \subseteq ((\mathbf{a} \subseteq \mathbf{b}) = (\mathbf{a} + \mathbf{c} \subseteq \mathbf{b} + \mathbf{c}))$ $(\mathbf{T}_{M048})$ Reversible Deduction

Proof of $(\mathbf{T}_{M048})$

Let $\mathbf{a} \subseteq \mathbf{c}$	
$\mathbf{c} \subseteq \mathbf{a}$	$(\mathbf{T}_{M012})$ (Transposition)
$\mathbf{c} = \mathbf{ca}$	$(\mathbf{A}_{MX44})$ ($\subseteq$ Equality Form)(Eqn1)
Let $\mathbf{a} + \mathbf{c} \subseteq \mathbf{b} + \mathbf{c}$	(Eqn2)
$\mathbf{a} + \mathbf{ca} \subseteq \mathbf{b} + \mathbf{ca}$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn1 into Eqn2)
$\mathbf{a}(\mathbf{a} + \mathbf{ca}) \subseteq \mathbf{a}(\mathbf{b} + \mathbf{ca})$	$(\mathbf{T}_{M029})$ (Deduction)
$\mathbf{aa} + \mathbf{aca} \subseteq \mathbf{ab} + \mathbf{aca}$	$(\mathbf{T}_{M407})$ (Distribution of $\cdot$ over $+$)
$\mathbf{aa} + \mathbf{c}\emptyset \subseteq \mathbf{ab} + \mathbf{c}\emptyset$	$(\mathbf{T}_{M621})$ (Complement Conjunction)
$\mathbf{aa} + \emptyset \subseteq \mathbf{ab} + \emptyset$	$(\mathbf{T}_{M050})$ (Null Set Conjunction)
$\mathbf{aa} \subseteq \mathbf{ab}$	$(\mathbf{T}_{M668})$ (Null Set Disjunction)
$\mathbf{a} \subseteq \mathbf{ab}$	$(\mathbf{T}_{M667})$ (Conjunction Tautology)
$\mathbf{ab} \subseteq \mathbf{abb}$	$(\mathbf{T}_{M029})$ (Deduction)
$\mathbf{ab} \subseteq \mathbf{a}\emptyset$	$(\mathbf{T}_{M621})$ (Complement Conjunction)

$ab \subseteq \emptyset$	(T_{M050})(Null Set Conjunction)
$ab = \emptyset$	(T_{M052})(Subsets of the Null Set)
$a \subseteq b$	(T_{M735})($\subseteq$ Existential Form)
Thus $(a \subseteq c)(a + c \subseteq b + c) \subseteq (a \subseteq b)$	($AMX02$)(Conditional Introduction)
$((a \subseteq c) = E) \subseteq ((a + c \subseteq b + c) \subseteq (a \subseteq b))$	($AMX35$)(Exportation/Importation)(Eqn3) for truth sets
$(a \subseteq b) \subseteq (a + c \subseteq b + c)$	(T_{M030})(Deduction)
$E = ((a \subseteq b) \subseteq (a + c \subseteq b + c))$	($AMX16$)(Order Axiom)(Eqn4)
$((a \subseteq c) = E) \subseteq E$	(T_{M744})(Universal Set Theorem)(Eqn5)
$((a \subseteq c) = E) \subseteq ((a \subseteq b) \subseteq (a + c \subseteq b + c))$	($AMX19$)(Substitution of Eqn4 into Eqn5)(Eqn6)
$((a \subseteq c) = E).((a \subseteq c) = E) \subseteq ((a + c \subseteq b + c) \subseteq (a \subseteq b)).((a \subseteq b) \subseteq (a + c \subseteq b + c))$	(T_{M669})(Combination of Eqns3&6)
$((a \subseteq c) = E) \subseteq ((a + c \subseteq b + c) \subseteq (a \subseteq b)).((a \subseteq b) \subseteq (a + c \subseteq b + c))$	(T_{M667})(Conjunction Tautology)
$((a \subseteq c) = E) \subseteq ((a \subseteq b) = (a + c \subseteq b + c))$	(T_{M430})(Biconditional Introduction)

$$((a \not\subseteq b) = (a \not\subseteq b)(a \neq \emptyset)(b \neq E)) \quad (T_{M278}) \text{ Conditions for } \not\subseteq$$

Proof of (T_{M278})

$(a \not\subseteq b) \subseteq (a \neq \emptyset)(b \neq E)$	(T_{M251})(Conditions for $\not\subseteq$)
$(a \not\subseteq b) = (a \not\subseteq b)(a \neq \emptyset)(b \neq E)$	($AMX44$)($\subseteq$ Equality Form)

$$(a \subseteq b)(c \subseteq d) = (E = (a + b)(c + d)) \quad (T_{M067}) \text{ Universal Form of Two Subsets}$$

Proof of (T_{M067})

Let $a \subseteq b$	
$a = ab$	($AMX44$)(Equality Form)(Eqn1)
Let $c \subseteq d$	
$c = cd$	($AMX44$)(Equality Form)(Eqn2)
$E = (a + a)(b + b)(c + c)(d + d)$	(T_{M004})(General Universal Equation)(Eqn3)
$E = (ab + a)(b + b)(cd + c)(d + d)$	($AMX19$)(Substitution of Eqn1&2 into Eqn3)
$E = (abb + abb + ab + ab)(cdd + cdd + cd + cd)$	(T_{M407})(Distribution of . over +)
$E = (abb + a\emptyset + ab + ab)(cdd + c\emptyset + cd + cd)$	(T_{M621})(Complement Conjunction)
$E = (abb + \emptyset + ab + ab)(cdd + \emptyset + cd + cd)$	(T_{M050})(Null Set Conjunction)
$E = (abb + ab + ab)(cdd + cd + cd)$	(T_{M668})(Null Set Disjunction)
$E = (ab + ab + ab)(cd + cd + cd)$	(T_{M667})(Conjunction Tautology)
$E = (ab + ab + ab + ab)(cd + cd + cd + cd)$	($AMX01$)(Disjunction Tautology)
$E = (b(a + a) + a(b + b))(d(c + c) + c(d + d))$	(T_{M407})(Distribution of . over +)
$E = (b + a)(d + c)$	(T_{M006})(Universal Pair Elimination)
$E = (a + b)(c + d)$	(T_{M403})(Commutative +)
Thus $(a \subseteq b)(c \subseteq d) = (E = (a + b)(c + d))$	($AMX02$)(Conditional Introduction)

$$((a \subseteq d)(b \subseteq c) = E) \subseteq ((a \subseteq b)(c \subseteq d) = (ac \subseteq bd)) \quad (T_{M074b}) \text{ Reversible Combination}$$

Proof of (T_{M074b})

Let $a \subseteq b$	(Eqn1)
Let $c \subseteq d$	(Eqn2)
$E = (a + b)(c + d)$	(T_{M067})(Universal Form of two Subsets Eqns1&2)
$E = ac + bc + ad + bd$	(T_{M407})(Distribution of . over +)
$E = ac + bc + \emptyset + ad + \emptyset + bd$	(T_{M668})(Null Set Disjunction)(Eqn3)
Let $a \subseteq d$	
$ad = \emptyset$	(T_{M735})($\subseteq$ Existential Form)(Eqn4)
Let $c \subseteq b$	
$bc = \emptyset$	(T_{M735})($\subseteq$ Existential Form)(Eqn5)
$E = ac + bc + bc + ad + ad + bd$	($AMX19$)(Substitution of Eqns4 & 5 into Eqn3)
$E = ac + c(b + b) + a(d + d) + bd$	(T_{M407})(Distribution of . over +)
$E = ac + c + a + bd$	(T_{M006})(Universal Pair Elimination)
$E = a + c + bd$	(T_{M063})(Partial Redundancy)

$\sim(\underline{a} + \underline{c}) \subseteq \underline{b}\underline{d}$	(T_{M064})($\subseteq$ Universal Form)
$\sim\sim(\underline{a}\underline{c}) \subseteq \underline{b}\underline{d}$	(T_{M025})(DeMorgan's Law)
$\underline{a}\underline{c} \subseteq \underline{b}\underline{d}$	(AMX_{13})(Double Complement Axiom)
Thus $(\underline{a} \subseteq \underline{b})(\underline{c} \subseteq \underline{d})(\underline{a} \subseteq \underline{d})(\underline{c} \subseteq \underline{b}) = (\underline{a}\underline{c} \subseteq \underline{b}\underline{d})$	(AMX_{02})(Conditional Introduction)
$(\underline{a} \subseteq \underline{b})(\underline{c} \subseteq \underline{d})(\underline{a} \subseteq \underline{d})(\underline{b} \subseteq \underline{c}) = (\underline{a}\underline{c} \subseteq \underline{b}\underline{d})$	(T_{M012})(Transposition)
$(\underline{a} \subseteq \underline{b})(\underline{c} \subseteq \underline{d})(\underline{a} \subseteq \underline{d})(\underline{b} \subseteq \underline{c})(\underline{a} \subseteq \underline{d})(\underline{b} \subseteq \underline{c}) = (\underline{a} \subseteq \underline{d})(\underline{b} \subseteq \underline{c})(\underline{a}\underline{c} \subseteq \underline{b}\underline{d})$	(T_{M029})(Deduction)
$(\underline{a} \subseteq \underline{b})(\underline{c} \subseteq \underline{d})(\underline{a} \subseteq \underline{d})(\underline{b} \subseteq \underline{c}) = (\underline{a} \subseteq \underline{d})(\underline{b} \subseteq \underline{c})(\underline{a}\underline{c} \subseteq \underline{b}\underline{d})$	(T_{M667})(Conjunction Tautology)
$((\underline{a} \subseteq \underline{b})(\underline{c} \subseteq \underline{d})(\underline{a} \subseteq \underline{d})(\underline{b} \subseteq \underline{c}) = (\underline{a} \subseteq \underline{d})(\underline{b} \subseteq \underline{c})(\underline{a}\underline{c} \subseteq \underline{b}\underline{d})) = \underline{E}$	(AMX_{16})(Order Axiom)
$((\underline{a} \subseteq \underline{d})(\underline{b} \subseteq \underline{c}) = \underline{E}) \subseteq ((\underline{a} \subseteq \underline{b})(\underline{c} \subseteq \underline{d}) = (\underline{a}\underline{c} \subseteq \underline{b}\underline{d}))$	(T_{M091b})(Conditional Subsets Rule)

$$(\underline{a} \subseteq \underline{b}) = (\underline{b} = \underline{a} + \underline{b}) \quad (T_{M521})$$

Proof of (T_{M521})

$(\underline{a} \subseteq \underline{b}) = (\underline{a} = \underline{a}\underline{b})$	(AMX_{44})($\subseteq$ Equality Form)
$(\underline{b} \subseteq \underline{a}) = (\underline{b} = \underline{b}\underline{a})$	Resymbolising
$(\underline{a} \subseteq \underline{b}) = (\underline{b} = \underline{b}\underline{a})$	(T_{M012})(Transposition)
$(\underline{a} \subseteq \underline{b}) = (\underline{b} = \sim(\underline{b}\underline{a}))$	(T_{M008})(Inverse of an Equality)
$(\underline{a} \subseteq \underline{b}) = (\underline{b} = \underline{a} + \underline{b})$	(T_{M025})(DeMorgan's Law)

$$(\underline{a} \not\subseteq \underline{c})(\underline{b} \subseteq \underline{c}) \subseteq (\underline{a} \not\subseteq \underline{b}) \quad (T_{M692}) \text{ 2nd Transposed Syllogism}$$

Proof of (T_{M692})

$(\underline{a} \subseteq \underline{b})(\underline{b} \subseteq \underline{c}) \subseteq (\underline{a} \subseteq \underline{c})$	(T_{M139})(Hypothetical Syllogism)
$((\underline{a} \subseteq \underline{b}) = \underline{E}) \subseteq ((\underline{b} \subseteq \underline{c}) \subseteq (\underline{a} \subseteq \underline{c}))$	(AMX_{35})(Exportation/Importation) for truth sets
$((\underline{a} \subseteq \underline{b}) = \underline{E}) \subseteq ((\underline{a} \not\subseteq \underline{c}) \subseteq (\underline{b} \not\subseteq \underline{c}))$	(T_{M012})(Transposition)
$(\underline{a} \subseteq \underline{b})(\underline{a} \not\subseteq \underline{c}) \subseteq (\underline{b} \not\subseteq \underline{c})$	(AMX_{35})(Exportation/Importation) for truth sets
$(\underline{c} \subseteq \underline{b})(\underline{c} \not\subseteq \underline{a}) \subseteq (\underline{b} \not\subseteq \underline{a})$	Resymbolising
$(\underline{b} \subseteq \underline{c})(\underline{a} \not\subseteq \underline{c}) \subseteq (\underline{a} \not\subseteq \underline{b})$	(T_{M012})(Transposition)
$(\underline{a} \not\subseteq \underline{c})(\underline{b} \subseteq \underline{c}) \subseteq (\underline{a} \not\subseteq \underline{b})$	(T_{M404})(Commutative .)

$$(\underline{a} \subseteq \underline{c})(\underline{b} \not\subseteq \underline{a}) \subseteq (\underline{b} \not\subseteq \underline{c}) \quad (T_{M140}) \text{ 1st Transposed Syllogism}$$

Proof of (T_{M140})

$(\underline{a} \not\subseteq \underline{c})(\underline{b} \subseteq \underline{c}) \subseteq (\underline{a} \not\subseteq \underline{b})$	(T_{M692})(2nd Transposed Syllogism)
$(\underline{b} \not\subseteq \underline{c})(\underline{a} \subseteq \underline{c}) \subseteq (\underline{b} \not\subseteq \underline{a})$	Resymbolising
$(\underline{a} \subseteq \underline{c})(\underline{b} \not\subseteq \underline{c}) \subseteq (\underline{b} \not\subseteq \underline{a})$	(T_{M404})(Commutative .)

$$(\underline{a} = \underline{c})(\underline{c} = \underline{b}) \subseteq (\underline{a} = \underline{b}) \quad (T_{M142}) \text{ Equality Syllogism}$$

$$(\underline{a} = \underline{c})(\underline{c} \neq \underline{b}) \subseteq (\underline{a} \neq \underline{b}) \quad (T_{M143}) \text{ Inequality Syllogism}$$

Proof of (T_{M142}) & (T_{M143})

$(\underline{a} \subseteq \underline{c})(\underline{c} \subseteq \underline{b}) \subseteq (\underline{a} \subseteq \underline{b})$	(T_{M139})(Hypothetical Syllogism)(Eqn1)
$(\underline{a} \subseteq \underline{c})(\underline{c} \subseteq \underline{b}) \subseteq (\underline{a} \subseteq \underline{b})$	Resymbolising
$(\underline{c} \subseteq \underline{a})(\underline{b} \subseteq \underline{c}) \subseteq (\underline{b} \subseteq \underline{a})$	(T_{M008})(Transposition)(Eqn2)
$(\underline{a} \subseteq \underline{c})(\underline{c} \subseteq \underline{b})(\underline{c} \subseteq \underline{a})(\underline{b} \subseteq \underline{c}) \subseteq (\underline{a} \subseteq \underline{b})(\underline{b} \subseteq \underline{a})$	(T_{M669})(Combination of Eqns1&2)
$(\underline{a} = \underline{c})(\underline{c} = \underline{b}) \subseteq (\underline{a} = \underline{b})$	(T_{M430})(Biconditional Introduction)
$((\underline{a} = \underline{c}) = \underline{E}) \subseteq ((\underline{c} = \underline{b}) \subseteq (\underline{a} = \underline{b}))$	(AMX_{35})(Exportation/Importation)(Eqn3) for truth sets
$((\underline{c} = \underline{a}) = \underline{E}) \subseteq ((\underline{a} = \underline{b}) \subseteq (\underline{c} = \underline{b}))$	Resymbolising(Eqn4)
$((\underline{a} = \underline{c}) = \underline{E})((\underline{c} = \underline{a}) = \underline{E}) \subseteq ((\underline{c} = \underline{b}) \subseteq (\underline{a} = \underline{b}))((\underline{a} = \underline{b}) \subseteq (\underline{c} = \underline{b}))$	(T_{M669})(Combination of Eqns3&4)
$((\underline{a} = \underline{c}) = \underline{E}) \subseteq ((\underline{c} = \underline{b}) \subseteq (\underline{a} = \underline{b}))((\underline{a} = \underline{b}) \subseteq (\underline{c} = \underline{b}))$	(T_{M667})(Conjunction Tautology)
$((\underline{a} = \underline{c}) = \underline{E}) \subseteq ((\underline{a} = \underline{b}) = (\underline{c} = \underline{b}))$	(T_{M430})(Biconditional Introduction)
$(\underline{a} = \underline{c})(\underline{c} = \underline{b}) \subseteq (\underline{a} = \underline{b})$	(T_{M444d})(Biconditional Importation) for truth sets
$((\underline{a} = \underline{c}) = \underline{E}) \subseteq ((\underline{c} \neq \underline{b}) = (\underline{a} \neq \underline{b}))$	(T_{M008})(Inverse of an Equality)
$(\underline{a} = \underline{c})(\underline{c} \neq \underline{b}) \subseteq (\underline{a} \neq \underline{b})$	(T_{M444d})(Biconditional Importation) for truth sets

$(a \subseteq c)(cd \subseteq b) \subseteq (ad \subseteq b)$ (T_{M357}) Partial Syllogism

Proof of (T_{M357})

Let $(a \subseteq c)$
 $(ad \subseteq cd)$ (T_{M029})(Deduction)(Eqn1)
 Let $(cd \subseteq b)$ (Eqn2)
 $(ad \subseteq b)$ (T_{M139})(Hypothetical Syllogism Eqns1&2)
 Thus $(a \subseteq c)(cd \subseteq b) \subseteq (ad \subseteq b)$ ($AMX02$)(Conditional Introduction)

$(a = E) \subseteq (b \subseteq ab)$ (T_{M427}) Conjunction Introduction

Proof of (T_{M427})

$(a = E) \subseteq (b \subseteq a)$ (T_{M440})(Implication Introduction)(Eqn1)
 $(b \subseteq a) \subseteq (bb \subseteq ab)$ (T_{M029})(Deduction)(Eqn2)
 $(a = E) \subseteq (bb \subseteq ab)$ (T_{M139})(Hypothetical Syllogism Eqns1&2)
 $(a = E) \subseteq (b \subseteq ab)$ (T_{M667})(Conjunction Tautology)

$((a \not\subseteq b) = E) = E \subseteq (((a \subseteq b) = E) \neq E)$ (T_{M465})

Proof of (T_{M465})

$((a \not\subseteq b) = E) \subseteq ((a \subseteq b) \neq E)$ (T_{M463})
 $((a \not\subseteq b) = E) = E \subseteq (((a \subseteq b) \neq E) = E)$ ($AMX16$)(Order Axiom)(Eqn 1)
 $((a \subseteq b) \neq E) = E \subseteq (((a \subseteq b) = E) \neq E)$ (T_{M464})(Eqn2)
 $((a \not\subseteq b) = E) = E \subseteq (((a \subseteq b) = E) \neq E)$ (T_{M139})(Hypothetical Syllogism)

$(a \subseteq c)(c = b) \subseteq (a \subseteq b)$ (T_{M700}) Substitution Syllogism

Proof of (T_{M700})

$((c = b) = E) \subseteq ((a \subseteq c) = (a \subseteq b))$ (T_{M269})
 $(a \subseteq c)(c = b) \subseteq (a \subseteq b)$ (T_{M444d})(Biconditional Importation) for truth sets

$((ab \subseteq c) = E) \subseteq ((b \subseteq a) \subseteq (b \subseteq c))$ (T_{M709})

Proof of (T_{M709})

$(ab \subseteq c)(b = ab) \subseteq (b \subseteq c)$ (T_{M700})(Substitution Syllogism)
 $(ab \subseteq c)(b \subseteq a) \subseteq (b \subseteq c)$ ($AMX44$)($\subseteq$ Equality Form)(Eqn1)
 $((ab \subseteq c) = E) \subseteq ((b \subseteq a) \subseteq (b \subseteq c))$ ($AMX35$)(Exportation/Importation) for truth sets

$(a = b)(b = \emptyset) \subseteq (a = \emptyset)$ (T_{M539})

$(a = b)(b \neq \emptyset) \subseteq (a \neq \emptyset)$ (T_{M538})

Proof of (T_{M538}) & (T_{M539})

Let $a = b$ (Eqn1)
 Let $b \neq \emptyset$ (Eqn2)
 $a \neq \emptyset$ ($AMX19$)(Substitution Eqn1 into Eqn2)
 Thus $(a = b)(b \neq \emptyset) \subseteq (a \neq \emptyset)$ ($AMX02$)(Conditional Introduction)
 $((a = b) = E) \subseteq ((b \neq \emptyset) \subseteq (a \neq \emptyset))$ ($AMX35$)(Exportation/Importation)(Eqn3)
 for truth sets
 Let $a = b$ (Eqn4)
 Let $a \neq \emptyset$ (Eqn5)
 $b \neq \emptyset$ ($AMX19$)(Substitution Eqn4 into Eqn5)
 Thus $(a = b)(a \neq \emptyset) \subseteq (b \neq \emptyset)$ ($AMX02$)(Conditional Introduction)
 $((a = b) = E) \subseteq ((a \neq \emptyset) \subseteq (b \neq \emptyset))$ ($AMX35$)(Exportation/Importation)(Eqn6)
 for truth sets
 $((a = b) = E).((a = b) = E) \subseteq ((a \neq \emptyset) \subseteq (b \neq \emptyset)).((b \neq \emptyset) \subseteq (a \neq \emptyset))$
 (T_{M669})(Combination of Eqns3&6)

$((a = b) = E) \subseteq ((a \neq \emptyset) \subseteq (b \neq \emptyset)).((b \neq \emptyset) \subseteq (a \neq \emptyset))$	(T_{M667})(Conjunction Tautology)
$((a = b) = E) \subseteq ((b \neq \emptyset) = (a \neq \emptyset))$	(T_{M430})(Biconditional Introduction)
$((a = b) = E) \subseteq ((b = \emptyset) = (a = \emptyset))$	(T_{M008})(Inverse of an Equality)
$(a = b)(b = \emptyset) \subseteq (a = \emptyset)$	(T_{M444d})(Biconditional Importation) for truth sets

$$(b = E)((a.b)^1 \neq \emptyset) \subseteq ((a.E)^1 \neq \emptyset) \quad (T_{M571})$$

Proof of (T_{M571})

$((a.b)^1 \neq \emptyset) = ((a.b)^1 \neq \emptyset)$	(D_{M00})(Definition of =)(Eqn1)
Let $b = E$	(Eqn2)
$((a.b)^1 \neq \emptyset) = ((a.E)^1 \neq \emptyset)$	($AMX19$)(Substitution of Eqn2 into Eqn1)
Thus $((b = E) = E) \subseteq (((a.b)^1 \neq \emptyset) = ((a.E)^1 \neq \emptyset))$	($AMX02$)(Conditional Introduction)
$(b = E)((a.b)^1 \neq \emptyset) \subseteq ((a.E)^1 \neq \emptyset)$	(T_{M444d})(Biconditional Importation) for truth sets

$$((a \subseteq b) = E) \subseteq ((b \subseteq c) \subseteq (a \subseteq c)) \quad (T_{M364}) \text{ Deductive Rule}$$

Proof of (T_{M364})

$(a \subseteq b)(b \subseteq c) \subseteq (a \subseteq c)$	(T_{M139})(Hypothetical Syllogism)
$((a \subseteq b) = E) \subseteq ((b \subseteq c) \subseteq (a \subseteq c))$	($AMX35$)(Exportation/Importation) for truth sets

$$(c^1 \subseteq a)(a \subseteq b) \subseteq (c^1 a = c^1 b) \quad (T_{M579})$$

Proof of (T_{M579})

$((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) \subseteq (c^1 \subseteq b))$	(T_{M272})(Deductive Rule)
$((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) = (c^1 \subseteq a) (c^1 \subseteq b))$	($AMX44$)($\subseteq$ Equality Form)
$((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) = (c^1 = c^1 a)(c^1 = c^1 b))$	($AMX44$)($\subseteq$ Equality Form)(Eqn1)
$(c^1 = c^1 a)(c^1 = c^1 b) \subseteq (c^1 a = c^1 b)$	(T_{M142})(Equality Syllogism)(Eqn2)
$((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) \subseteq (c^1 a = c^1 b))$	($AMX19$)(Substitution of Eqn1 into Eqn2)
$(c^1 \subseteq a)(a \subseteq b) \subseteq (c^1 a = c^1 b)$	($AMX35$)(Exportation/Importation) for truth sets

Chapter 2 Proofs of Membership Theorems – Part 2

2.1 Theorem Proofs

$((a \subseteq b) = E) = ((b \neq E) \subseteq (a \neq E))$ (T_{M095}) Probabilistic Rule

Proof of (T_{M095})

$((a \subseteq b) = E) = ((a = E) \subseteq (b = E))$ (T_{M094}) (Probabilistic Rule)
 $((a \subseteq b) = E) = (\sim(b = E) \subseteq \sim(a = E))$ (T_{M012}) (Transposition)
 $((a \subseteq b) = E) = ((b \neq E) \subseteq (a \neq E))$ (T_{M260}) (Definition of $\neq$)

$((a \subseteq b) = E) = ((b = \emptyset) \subseteq (a = \emptyset))$ (T_{M097}) Probabilistic Rule

Proof of (T_{M097})

$((a \subseteq b) = E) = ((a = E) \subseteq (b = E))$ (T_{M094}) (Probabilistic Rule)
 $((a \subseteq b) = E) = ((a = \sim E) \subseteq (b = \sim E))$ (T_{M008}) (Inverse of an Equality)
 $((a \subseteq b) = E) = ((a = \emptyset) \subseteq (b = \emptyset))$ (D_{M02}) (Definition of $\emptyset$)
 $((b \subseteq a) = E) = ((a = \emptyset) \subseteq (b = \emptyset))$ (T_{M012}) (Transposition)
 $((a \subseteq b) = E) = ((b = \emptyset) \subseteq (a = \emptyset))$ Resymbolising

$((a \subseteq b) = E) = ((a \neq \emptyset) \subseteq (b \neq \emptyset))$ (T_{M096}) Probabilistic Rule

Proof of (T_{M096})

$((a \subseteq b) = E) = ((b = \emptyset) \subseteq (a = \emptyset))$ (T_{M097})
 $((a \subseteq b) = E) = (\sim(a = \emptyset) \subseteq \sim(b = \emptyset))$ (T_{M012}) (Transposition)
 $((a \subseteq b) = E) = ((a \neq \emptyset) \subseteq (b \neq \emptyset))$ (T_{M260}) (Definition of $\neq$)

$(a \neq \emptyset)(a \subseteq b) \subseteq (b \neq \emptyset)$ (T_{M777})

Proof of (T_{M777})

$((a \subseteq b) = E) \subseteq ((a \neq \emptyset) \subseteq (b \neq \emptyset))$ (T_{M096}) (Probabilistic Rule)
 $((a \neq \emptyset)(a \subseteq b) \subseteq (b \neq \emptyset)) = E$ (A_{MX35}) (Exportation/Importation) for truth sets
 $(a \neq \emptyset)(a \subseteq b) \subseteq (b \neq \emptyset)$ (A_{MX16}) (Order Axiom)

$((a \subseteq \underline{b}) = E) = ((a = E) \subseteq (b = \emptyset))$ (T_{M324}) Probabilistic Rule

Proof of (T_{M324})

$((a \subseteq b) = E) = ((a = E) \subseteq (b = E))$ (T_{M094}) (Probabilistic Rule)
 $((a \subseteq \underline{b}) = E) = ((a = E) \subseteq (\underline{b} = E))$ Resymbolising
 $((a \subseteq \underline{b}) = E) = ((a = E) \subseteq (b = \underline{E}))$ (T_{M008}) (Inverse of an Equality)
 $((a \subseteq \underline{b}) = E) = ((a = E) \subseteq (b = \emptyset))$ (D_{M02}) (Definition of $\emptyset$)

$((a \subseteq \underline{b}) = E) = ((a \neq \emptyset) \subseteq (b \neq E))$ (T_{M325}) Probabilistic Rule

Proof of (T_{M325})

$((a \subseteq b) = E) = ((a \neq \emptyset) \subseteq (b \neq \emptyset))$ (T_{M096}) (Probabilistic Rule)
 $((a \subseteq \underline{b}) = E) = ((a \neq \emptyset) \subseteq (\underline{b} \neq \emptyset))$ Resymbolising
 $((a \subseteq \underline{b}) = E) = ((a \neq \emptyset) \subseteq (b \neq \underline{\emptyset}))$ (T_{M008}) (Inverse of an Equality)
 $((a \subseteq \underline{b}) = E) = ((a \neq \emptyset) \subseteq (b \neq E))$ (T_{M054}) (Negative of the Null Set)

$((\underline{a} \subseteq b) = E) = ((a = \emptyset) \subseteq (b = E))$ (T_{M326}) Probabilistic Rule

Proof of (T_{M326})

$((a \subseteq b) = E) = ((a = E) \subseteq (b = E))$ (T_{M094}) (Probabilistic Rule)

$$\begin{aligned}
((a \subseteq b) = E) &= ((a = E) \subseteq (b = E)) && \text{Resymbolising} \\
((a \subseteq b) = E) &= ((a = \underline{E}) \subseteq (b = E)) && (T_{M008})(\text{Inverse of an Equality}) \\
((a \subseteq b) = E) &= ((a = \emptyset) \subseteq (b = E)) && (D_{M02})(\text{Definition of } \emptyset)
\end{aligned}$$

$$((a \subseteq b) = E) = ((a \neq E) \subseteq (b \neq \emptyset)) \quad (T_{M327}) \text{ Probabilistic Rule}$$

Proof of (T_{M327})

$$\begin{aligned}
((a \subseteq b) = E) &= ((a \neq \emptyset) \subseteq (b \neq \emptyset)) && (T_{M096})(\text{Probabilistic Rule}) \\
((a \subseteq b) = E) &= ((a \neq \emptyset) \subseteq (b \neq \emptyset)) && \text{Resymbolising} \\
((a \subseteq b) = E) &= ((a \neq \emptyset) \subseteq (b \neq \emptyset)) && (T_{M008})(\text{Inverse of an Equality}) \\
((a \subseteq b) = E) &= ((a \neq E) \subseteq (b \neq \emptyset)) && (T_{M054})(\text{Negative of the Null Set})
\end{aligned}$$

$$((a \subseteq b) = E) = ((a = \emptyset) \subseteq (b = \emptyset)) \quad (T_{M332}) \text{ Probabilistic Rule}$$

Proof of (T_{M332})

$$\begin{aligned}
((a \subseteq b) = E) &= ((a = E) \subseteq (b = E)) && (T_{M094})(\text{Probabilistic Rule}) \\
((a \subseteq b) = E) &= ((a = E) \subseteq (b = E)) && \text{Resymbolising} \\
((a \subseteq b) = E) &= ((a = \underline{E}) \subseteq (b = \underline{E})) && (T_{M008})(\text{Inverse of an Equality}) \\
((a \subseteq b) = E) &= ((a = \emptyset) \subseteq (b = \emptyset)) && (D_{M02})(\text{Definition of } \emptyset)
\end{aligned}$$

$$((a \subseteq b) = E) = ((a \neq E) \subseteq (b \neq E)) \quad (T_{M333}) \text{ Probabilistic Rule}$$

Proof of (T_{M333})

$$\begin{aligned}
((a \subseteq b) = E) &= ((a \neq \emptyset) \subseteq (b \neq \emptyset)) && (T_{M096})(\text{Probabilistic Rule}) \\
((a \subseteq b) = E) &= ((a \neq \emptyset) \subseteq (b \neq \emptyset)) && \text{Resymbolising} \\
((a \subseteq b) = E) &= ((a \neq \emptyset) \subseteq (b \neq \emptyset)) && (T_{M008})(\text{Inverse of an Equality}) \\
((a \subseteq b) = E) &= ((a \neq E) \subseteq (b \neq E)) && (T_{M054})(\text{Negative of the Null Set})
\end{aligned}$$

$$((a \subseteq b) = E) \subseteq ((a = E) \subseteq (b \neq \emptyset)) \quad (T_{M247}) \text{ Probabilistic Rule}$$

Proof of (T_{M247})

$$\begin{aligned}
\text{Let } (a = E) &\subseteq (b = E) && (\text{Eqn1}) \\
(b = E) &\subseteq (b \neq \emptyset) && (T_{M058})(\text{Eqn2}) \\
(a = E) &\subseteq (b \neq \emptyset) && (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\
\text{Thus } ((a = E) &\subseteq (b = E)) \subseteq ((a = E) \subseteq (b \neq \emptyset)) && (A_{MX02})(\text{Conditional Introduction})(\text{Eqn3}) \\
((a \subseteq b) = E) &= ((a = E) \subseteq (b = E)) && (T_{M094})(\text{Probabilistic Rule})(\text{Eqn4}) \\
((a \subseteq b) = E) &\subseteq ((a = E) \subseteq (b \neq \emptyset)) && (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})
\end{aligned}$$

$$((a \subseteq b) = E) \subseteq ((b = \emptyset) \subseteq (a \neq E)) \quad (T_{M248}) \text{ Probabilistic Rule}$$

Proof of (T_{M248})

$$\begin{aligned}
((a \subseteq b) = E) &\subseteq ((a = E) \subseteq (b \neq \emptyset)) && (T_{M247})(\text{Probabilistic Rule}) \\
((a \subseteq b) = E) &\subseteq (\sim(b \neq \emptyset) \subseteq \sim(a = E)) && (T_{M008})(\text{Inverse of an Equality}) \\
((a \subseteq b) = E) &\subseteq (\sim\sim(b = \emptyset) \subseteq (a \neq E)) && (T_{M260})(\text{Definition of } \neq) \\
((a \subseteq b) = E) &\subseteq ((b = \emptyset) \subseteq (a \neq E)) && (A_{MX13})(\text{Double Complement Axiom})
\end{aligned}$$

$$((a \not\subseteq b) = E) \subseteq ((a = E) \not\subseteq (b = E)) \quad (T_{M320}) \text{ Probabilistic Rule}$$

Proof of (T_{M320})

$$\begin{aligned}
((a \subseteq b) = E) &= ((a = E) \subseteq (b = E)) && (T_{M094})(\text{Probabilistic Rule}) \\
((a \subseteq b) \neq E) &= ((a = E) \not\subseteq (b = E)) && (T_{M008})(\text{Inverse of an Equality})(\text{Eqn1}) \\
((a \subseteq b) = \emptyset) &\subseteq ((a \subseteq b) \neq E) && (T_{M059}) \\
((a \not\subseteq b) = \emptyset) &\subseteq ((a \subseteq b) \neq E) && (T_{M008})(\text{Inverse of an Equality}) \\
((a \not\subseteq b) = E) &\subseteq ((a \subseteq b) \neq E) && (T_{M054})(\text{Negative of the Null Set})(\text{Eqn2}) \\
((a \not\subseteq b) = E) &\subseteq ((a = E) \not\subseteq (b = E)) && (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})
\end{aligned}$$

$$((a \not\subseteq b) = E) \subseteq ((a \neq \emptyset) \not\subseteq (b \neq \emptyset)) \quad (T_{M321}) \text{ Probabilistic Rule}$$

Proof of (T_{M321})

$$\begin{aligned} ((a \subseteq b) = E) &= ((a \neq \emptyset) \subseteq (b \neq \emptyset)) & (T_{M096}) & \text{(Probabilistic Rule)} \\ ((a \subseteq b) \neq E) &= ((a \neq \emptyset) \not\subseteq (b \neq \emptyset)) & (T_{M008}) & \text{(Inverse of an Equality) (Eqn1)} \\ ((a \subseteq b) = \emptyset) &\subseteq ((a \subseteq b) \neq E) & (T_{M059}) & \\ ((a \not\subseteq b) = \emptyset) &\subseteq ((a \subseteq b) \neq E) & (T_{M008}) & \text{(Inverse of an Equality)} \\ ((a \not\subseteq b) = E) &\subseteq ((a \subseteq b) \neq E) & (T_{M054}) & \text{(Negative of the Null Set) (Eqn2)} \\ ((a \not\subseteq b) = E) &\subseteq ((a \neq \emptyset) \not\subseteq (b \neq \emptyset)) & (A_{MX19}) & \text{(Substitution of Eqn1 into Eqn2)} \end{aligned}$$

$$((a \not\subseteq b) = E) \subseteq ((a = E) \not\subseteq (b = \emptyset)) \quad (T_{M322}) \text{ Probabilistic Rule}$$

Proof of (T_{M322})

$$\begin{aligned} ((a \not\subseteq b) = E) &\subseteq ((a = E) \not\subseteq (b = E)) & (T_{M320}) & \text{(Probabilistic Rule)} \\ ((a \not\subseteq b) = E) &\subseteq ((a = E) \not\subseteq (b = E)) & & \text{Resymbolising} \\ ((a \not\subseteq b) = E) &\subseteq ((a = E) \not\subseteq (b = \underline{E})) & (T_{M008}) & \text{(Inverse of an Equality)} \\ ((a \not\subseteq b) = E) &\subseteq ((a = E) \not\subseteq (b = \emptyset)) & (D_{M02}) & \text{(Definition of } \emptyset) \end{aligned}$$

$$((a \not\subseteq b) = E) \subseteq ((a \neq \emptyset) \not\subseteq (b \neq E)) \quad (T_{M323}) \text{ Probabilistic Rule}$$

Proof of (T_{M323})

$$\begin{aligned} ((a \not\subseteq b) = E) &\subseteq ((a \neq \emptyset) \not\subseteq (b \neq \emptyset)) & (T_{M321}) & \text{(Probabilistic Rule)} \\ ((a \not\subseteq b) = E) &\subseteq ((a \neq \emptyset) \not\subseteq (b \neq \emptyset)) & & \text{Resymbolising} \\ ((a \not\subseteq b) = E) &\subseteq ((a \neq \emptyset) \not\subseteq (b \neq \underline{\emptyset})) & (T_{M008}) & \text{(Inverse of an Equality)} \\ ((a \not\subseteq b) = E) &\subseteq ((a \neq \emptyset) \not\subseteq (b \neq E)) & (T_{M054}) & \text{(Negative of the Null Set)} \end{aligned}$$

$$((a \not\subseteq b) = E) \subseteq ((a = \emptyset) \not\subseteq (b = E)) \quad (T_{M328}) \text{ Probabilistic Rule}$$

Proof of (T_{M328})

$$\begin{aligned} ((a \not\subseteq b) = E) &\subseteq ((a = E) \not\subseteq (b = E)) & (T_{M320}) & \text{(Probabilistic Rule)} \\ ((a \not\subseteq b) = E) &\subseteq ((a = E) \not\subseteq (b = E)) & & \text{Resymbolising} \\ ((a \not\subseteq b) = E) &\subseteq ((a = \underline{E}) \not\subseteq (b = E)) & (T_{M008}) & \text{(Inverse of an Equality)} \\ ((a \not\subseteq b) = E) &\subseteq ((a = \emptyset) \not\subseteq (b = E)) & (D_{M02}) & \text{(Definition of } \emptyset) \end{aligned}$$

$$((a \not\subseteq b) = E) \subseteq ((a \neq E) \not\subseteq (b \neq \emptyset)) \quad (T_{M329}) \text{ (Probabilistic Rule)}$$

Proof of (T_{M329})

$$\begin{aligned} ((a \not\subseteq b) = E) &\subseteq ((a \neq \emptyset) \not\subseteq (b \neq \emptyset)) & (T_{M321}) & \text{(Probabilistic Rule)} \\ ((a \not\subseteq b) = E) &\subseteq ((a \neq \emptyset) \not\subseteq (b \neq \emptyset)) & & \text{Resymbolising} \\ ((a \not\subseteq b) = E) &\subseteq ((a \neq \underline{\emptyset}) \not\subseteq (b \neq \emptyset)) & (T_{M008}) & \text{(Inverse of an Equality)} \\ ((a \not\subseteq b) = E) &\subseteq ((a \neq E) \not\subseteq (b \neq \emptyset)) & (T_{M054}) & \text{(Negative of the Null Set)} \end{aligned}$$

$$((a \not\subseteq b) = E) \subseteq ((a = \emptyset) \not\subseteq (b = \emptyset)) \quad (T_{M330}) \text{ Probabilistic Rule}$$

Proof of (T_{M330})

$$\begin{aligned} ((a \not\subseteq b) = E) &\subseteq ((a = E) \not\subseteq (b = E)) & (T_{M320}) & \text{(Probabilistic Rule)} \\ ((a \not\subseteq b) = E) &\subseteq ((a = E) \not\subseteq (b = E)) & & \text{Resymbolising} \\ ((a \not\subseteq b) = E) &\subseteq ((a = \underline{E}) \not\subseteq (b = \underline{E})) & (T_{M008}) & \text{(Inverse of an Equality)} \\ ((a \not\subseteq b) = E) &\subseteq ((a = \emptyset) \not\subseteq (b = \emptyset)) & (D_{M02}) & \text{(Definition of } \emptyset) \end{aligned}$$

$$((a \not\subseteq b) = E) \subseteq ((a \neq E) \not\subseteq (b \neq E)) \quad (T_{M331}) \text{ Probabilistic Rule}$$

Proof of (T_{M331})

$$\begin{aligned} ((a \not\subseteq b) = E) &\subseteq ((a \neq \emptyset) \not\subseteq (b \neq \emptyset)) & (T_{M321}) & \text{(Probabilistic Rule)} \\ ((a \not\subseteq b) = E) &\subseteq ((a \neq \emptyset) \not\subseteq (b \neq \emptyset)) & & \text{Resymbolising} \\ ((a \not\subseteq b) = E) &\subseteq ((a \neq \underline{\emptyset}) \not\subseteq (b \neq \underline{\emptyset})) & (T_{M008}) & (T_{M008}) \end{aligned}$$

$$((\underline{a} \not\subseteq \underline{b}) = E) \subseteq ((a \neq E) \not\subseteq (b \neq E)) \quad (T_{M054})(\text{Negative of the Null Set})$$

$$(c^1 \neq \emptyset): (((a \subseteq b) = E) = E) \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = E)) \quad (T_{M303})$$

Proof of (T_{M303})

$(c^1 \neq \emptyset):$

$$((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) \subseteq (c^1 \subseteq b))$$

(T_{M099})(Entity Mode / Set Mode Rule)

$$(((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) \subseteq (c^1 \subseteq b))) = E$$

(A_{MX16})(Order Axiom)

$$(((a \subseteq b) = E) = E) \subseteq ((c^1 \subseteq a) \subseteq (c^1 \subseteq b)) = E$$

(T_{M094})(Probabilistic Rule)

$$(((a \subseteq b) = E) = E) \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = E))$$

(T_{M094})(Probabilistic Rule)

$$(c^1 \neq \emptyset): (((a = E) = E) = E) \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq a) = E)) \quad (T_{M454})$$

Proof of (T_{M454})

$(c^1 \neq \emptyset):$

$$(((a \subseteq b) = E) = E) \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = E))$$

(T_{M303})(Eqn1)

Let $a = E$

(Eqn2)

$$(((E \subseteq b) = E) = E) \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq b) = E))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(((b = E) = E) = E) \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq b) = E))$$

(T_{M275})(Supersets of the Universal Set)

$$(((a = E) = E) = E) \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq a) = E))$$

Resymbolising

$$(c^1 \neq \emptyset): (((a = E) \subseteq (b = E)) = E) \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = E)) \quad (T_{M458})$$

Proof of (T_{M458})

$(c^1 \neq \emptyset):$

$$(((a \subseteq b) = E) = E) \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = E))$$

(T_{M303})(Eqn1)

$$((a \subseteq b) = E) = ((a = E) \subseteq (b = E))$$

(T_{M094})(Probabilistic Rule)(Eqn2)

$$(((a = E) \subseteq (b = E)) = E) \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = E))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + (a = \emptyset)(b \neq \emptyset) \quad (T_{M657})$$

Proof of (T_{M657})

$$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset)$$

(T_{M718})(Definition of $\subset$)

$$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset)E$$

(A_{MX11})(Universal Set Conjunction Axiom)

$$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset)((a \neq \emptyset) + (a = \emptyset))$$

(T_{M671})(Tertium non Datur)

$$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + (a = \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset)$$

(T_{M407})(Distribution of $\cdot$ over $+$)

$$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + (a = \emptyset)(\emptyset \neq \emptyset)(\sim \emptyset \cdot b \neq \emptyset)$$

(A_{MX19})(Substitution)

$$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + (a = \emptyset)(\emptyset = \emptyset)(\sim \emptyset \cdot b \neq \emptyset)$$

(T_{M050})(Null Set Conjunction)

$$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + (a = \emptyset)E(\sim \emptyset \cdot b \neq \emptyset)$$

(T_{M343})

$$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + (a = \emptyset)(\sim \emptyset \cdot b \neq \emptyset)$$

(A_{MX11})(Universal Set Conjunction Axiom)

$$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + (a = \emptyset)(E \cdot b \neq \emptyset)$$

(T_{M054})(Negative of the Null Set)

$$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + (a = \emptyset)(b \neq \emptyset)$$

(A_{MX11})(Universal Set Conjunction Axiom)

$$(a \supset b) = (b \neq \emptyset)(\underline{ab} \neq \emptyset)(\underline{ab} = \emptyset) + (a \neq \emptyset)(b = \emptyset) \quad (T_{M660})$$

Proof of (T_{M660})

$$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + (a = \emptyset)(b \neq \emptyset)$$

(T_{M657})

$$(b \subset a) = (b \neq \emptyset)(\underline{ba} = \emptyset)(\underline{ba} \neq \emptyset) + (b = \emptyset)(a \neq \emptyset)$$

Resymbolising

$$(a \supset b) = (b \neq \emptyset)(\underline{ba} = \emptyset)(\underline{ba} \neq \emptyset) + (b = \emptyset)(a \neq \emptyset)$$

(T_{M195})($\subset$ Reverse of $\supset$)

$$(a \supset b) = (b \neq \emptyset)(\underline{ab} \neq \emptyset)(\underline{ab} = \emptyset) + (a \neq \emptyset)(b = \emptyset)$$

(T_{M404})(Commutative $\cdot$)

$$((a \neq \emptyset) = E) \subseteq ((a \subset b) = (\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset)) \quad (T_{M655})$$

Proof of (T_{M655})

$$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + (a = \emptyset)(b \neq \emptyset)$$

(T_{M657})(Eqn1)

Let $a \neq \emptyset$

$$(a \neq \emptyset) = E$$

(A_{MX16})(Order Axiom)(Eqn2)

$(a = \emptyset) = \sim E$	(T_{M008}) (Inverse of an Equality)
$(a = \emptyset) = \emptyset$	(D_{M02}) (Definition of $\emptyset$)(Eqn3)
$(a \subset b) = E(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + \emptyset(b \neq \emptyset)$	(A_{MX19}) (Substitution of Eqns2&3 into Eqn1)
$(a \subset b) = (\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + \emptyset(b \neq \emptyset)$	(A_{MX11}) (Universal Set Conjunction Axiom)
$(a \subset b) = (\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + \emptyset$	(T_{M050}) (Null Set Conjunction)
$(a \subset b) = (\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset)$	(T_{M668}) (Null Set Disjunction)
Thus $((a \neq \emptyset) = E) \subseteq ((a \subset b) = (\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset))$	(A_{MX02}) (Conditional Introduction)

$$((b \neq \emptyset) = E) \subseteq ((a \supset b) = (\underline{ab} \neq \emptyset)(\underline{ab} = \emptyset)) \quad (T_{M658})$$

Proof of (T_{M658})

$((a \neq \emptyset) = E) \subseteq ((a \subset b) = (\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset))$	(T_{M655})
$((b \neq \emptyset) = E) \subseteq ((b \subset a) = (\underline{ba} = \emptyset)(\underline{ba} \neq \emptyset))$	Resymbolising
$((b \neq \emptyset) = E) \subseteq ((a \supset b) = (\underline{ba} = \emptyset)(\underline{ba} \neq \emptyset))$	(T_{M195}) ($\subset$ Reverse of $\supset$)
$((b \neq \emptyset) = E) \subseteq ((a \supset b) = (\underline{ab} \neq \emptyset)(\underline{ab} = \emptyset))$	(T_{M404}) (Commutative .)

$$((a = \emptyset) = E) \subseteq ((a \subset b) = (b \neq \emptyset)) \quad (T_{M656})$$

Proof of (T_{M656})

$(a \subset b) = (a \neq \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + (a = \emptyset)(b \neq \emptyset)$	(T_{M657}) (Eqn1)
Let $a = \emptyset$	
$(a = \emptyset) = E$	(A_{MX16}) (Order Axiom)(Eqn2)
$(a \neq \emptyset) = \sim E$	(T_{M008}) (Inverse of an Equality)
$(a \neq \emptyset) = \emptyset$	(D_{M02}) (Definition of $\emptyset$)(Eqn3)
$(a \subset b) = \emptyset(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + E(b \neq \emptyset)$	(A_{MX19}) (Substitution of Eqns2&3 into Eqn1)
$(a \subset b) = \emptyset(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) + (b \neq \emptyset)$	(A_{MX11}) (Universal Set Conjunction Axiom)
$(a \subset b) = \emptyset + (b \neq \emptyset)$	(T_{M050}) (Null Set Conjunction)
$(a \subset b) = (b \neq \emptyset)$	(T_{M668}) (Null Set Disjunction)
Thus $((a = \emptyset) = E) \subseteq ((a \subset b) = (b \neq \emptyset))$	(A_{MX02}) (Conditional Introduction)

$$((b = \emptyset) = E) \subseteq ((a \supset b) = (a \neq \emptyset)) \quad (T_{M659})$$

Proof of (T_{M659})

$((a = \emptyset) = E) \subseteq ((a \subset b) = (b \neq \emptyset))$	(T_{M656})
$((b = \emptyset) = E) \subseteq ((b \subset a) = (a \neq \emptyset))$	Resymbolising
$((b = \emptyset) = E) \subseteq ((a \supset b) = (a \neq \emptyset))$	(T_{M195}) ($\subset$ Reverse of $\supset$)

$$(a + b \subseteq c) \subseteq (ab \subseteq c) \quad (T_{M151})$$

Proof of (T_{M151})

$(a + b \subseteq c) \subseteq (a \subseteq c)$	(T_{M072}) (Eqn1)
$(a \subseteq c) \subseteq (ab \subseteq c)$	(T_{M089}) (Single Premise Syllogism)(Eqn2)
$(a + b \subseteq c) \subseteq (ab \subseteq c)$	(T_{M139}) (Hypothetical Syllogism Eqns1&2)

$$(a + b = c) \subseteq (a \subseteq c) \quad (T_{M072b}) \text{ Single Premise Syllogism}$$

Proof of (T_{M072b})

$(a + b \subseteq c) \subseteq (a \subseteq c)$	(T_{M072}) (Single Premise Syllogism)
$(a + b \subset c) + (a + b = c) \subseteq (a \subseteq c)$	(T_{M266}) (Definition of $\subseteq$)
$(a + b = c) \subseteq (a \subseteq c)$	(T_{M072}) (Single Premise Syllogism)

$$(ab = c) \subseteq (a \subseteq \underline{b} + c) \quad (T_{M786})$$

Proof of (T_{M786})

$(ab \subseteq c) = (a \subseteq \underline{b} + c)$	(T_{M468})
$(ab \subset c) + (ab = c) = (a \subseteq \underline{b} + c)$	(T_{M266}) (Definition of $\subseteq$)
$(ab = c) \subseteq (a \subseteq \underline{b} + c)$	(T_{M072b}) (Single Premise Syllogism)

$$(a \subset c)(c \subseteq b) \subseteq (a \subset b) \quad (T_{M401})$$

Proof of (T_{M401})

Let $a \subset c$	(Eqn1)
Let $c = b$	(Eqn2)
$a \subset b$	(A_{MX19})(Substitution of Eqn2 into Eqn1)
Thus $(a \subset c)(c = b) \subseteq (a \subset b)$	(A_{MX02})(Conditional Introduction)(Eqn3)
$(a \subset c)(c \subseteq b) \subseteq (a \subset b)$	(T_{M400})(Proper Syllogism)(Eqn4)
$(a \subset c)(c \subseteq b) + (a \subset c)(c = b) \subseteq (a \subset b) + (a \subset b)$	(A_{MX21})(Combination Axiom Eqns 3 & 4)
$(a \subset c)(c \subseteq b) + (a \subset c)(c = b) \subseteq (a \subset b)$	(A_{MX01})(Disjunction Tautology)
$(a \subset c)((c \subseteq b) + (c = b)) \subseteq (a \subset b)$	(T_{M407})(Distribution of . over +)
$(a \subset c)(c \subseteq b) \subseteq (a \subset b)$	(T_{M266})(Definition of $\subseteq$)

$$(a \subseteq c)(c \subset b) \subseteq (a \subset b) \quad (T_{M402})$$

Proof of (T_{M402})

Let $a = c$	(Eqn1)
Let $c \subset b$	(Eqn2)
$a \subset b$	(A_{MX19})(Substitution of Eqn2 into Eqn1)
Thus $(a = c)(c \subset b) \subseteq (a \subset b)$	(A_{MX02})(Conditional Introduction)(Eqn3)
$(a \subset c)(c \subset b) \subseteq (a \subset b)$	(T_{M400})(Proper Syllogism)(Eqn4)
$(a \subset c)(c \subset b) + (a = c)(c \subset b) \subseteq (a \subset b) + (a \subset b)$	(A_{MX21})(Combination Axiom Eqns 3 & 4)
$(a \subset c)(c \subset b) + (a = c)(c \subset b) \subseteq (a \subset b)$	(A_{MX01})(Disjunction Tautology)
$((a \subset c) + (a = c))(c \subset b) \subseteq (a \subset b)$	(T_{M407})(Distribution of . over +)
$(a \subseteq c)(c \subset b) \subseteq (a \subset b)$	(T_{M266})(Definition of $\subseteq$)

$$(c = ab) \subseteq (c \subseteq a) \quad (T_{M073b}) \text{ Single Premise Syllogism}$$

Proof of (T_{M073b})

$(c \subseteq ab) \subseteq (c \subseteq a)$	(T_{M073})(Single Premise Syllogism)
$(c \subset ab) + (c = ab) \subseteq (c \subseteq a)$	(T_{M266})(Definition of $\subseteq$)
$(c = ab) \subseteq (c \subseteq a)$	(T_{M072})(Single Premise Syllogism)

$$(c \subseteq ab) \subseteq (c \subseteq a + b) \quad (T_{M152}) \text{ Single Premise Syllogism}$$

Proof of (T_{M152})

$(c \subseteq ab) \subseteq (c \subseteq a)$	(T_{M073})(Eqn1)
$(c \subseteq a) \subseteq (c \subseteq a + b)$	(T_{M150})(Eqn2)
$(c \subseteq ab) \subseteq (c \subseteq a + b)$	(T_{M139})(Hypothetical Syllogism Eqns1 & 2)

$$(a = b) \subseteq (a \subseteq b) \quad (T_{M425}) \text{ Biconditional Elimination}$$

Proof of (T_{M425})

$(a \subseteq b)(a = b) = (a = b)$	(T_{M043})(Conjunction of Binary Relations)
$(a = b) \subseteq (a \subseteq b)$	(T_{M073b})(Single Premise Syllogism)

Alternative Proof of (T_{M425})

$(a = b) = (a \subseteq b)(b \subseteq a)$	(T_{M430})(Biconditional Introduction)
$(a = b) \subseteq (a \subseteq b)$	(T_{M073b})(Single Premise Syllogism)

$$(a \subset \emptyset^1) = (a = \emptyset) \quad (T_{M711})$$

Proof of (T_{M711})

$(a \subset b^1) = (a = \emptyset)(b^1 \neq \emptyset)$	(A_{MX40})(Proper Subset of Unit Set Axiom)(Eqn1)
Let $b^1 = \emptyset^1$	(Eqn2)
$(a \subset \emptyset^1) = (a = \emptyset)(\emptyset^1 \neq \emptyset)$	(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)
$(a \subset \emptyset^1) = (a = \emptyset)E$	(T_{M780})(Existence of $\emptyset^1$)

$$(a \subset \emptyset^1) = (a = \emptyset)$$

($\mathbf{A_{MX11}}$)(Universal Set Conjunction Axiom)

$$(a \subset \mathbf{E}^1) = (a = \emptyset)$$

($\mathbf{T_{M727}}$)

Proof of ($\mathbf{T_{M727}}$)

$$(a \subset \mathbf{b}^1) = (a = \emptyset)(\mathbf{b}^1 \neq \emptyset)$$

($\mathbf{A_{MX40}}$)(Proper Subset of Unit Set Axiom)(Eqn1)

$$\text{Let } \mathbf{b}^1 = \mathbf{E}^1$$

(Eqn2)

$$(a \subset \mathbf{E}^1) = (a = \emptyset)(\mathbf{E}^1 \neq \emptyset)$$

($\mathbf{A_{MX19}}$)(Substitution of Eqn2 into Eqn1)

$$(a \subset \mathbf{E}^1) = (a = \emptyset)\mathbf{E}$$

($\mathbf{T_{M773}}$)(Existence of $\mathbf{E}^1$)

$$(a \subset \mathbf{E}^1) = (a = \emptyset)$$

($\mathbf{A_{MX11}}$)(Universal Set Conjunction Axiom)

$$(\mathbf{a}^1 \subseteq \emptyset^1)(\mathbf{a}^1 \neq \emptyset^1) \subseteq (a = \emptyset)$$

($\mathbf{T_{M785}}$)

Proof of ($\mathbf{T_{M785}}$)

$$(\mathbf{a}^1 = \emptyset) \subseteq (a = \emptyset)$$

($\mathbf{A_{MX50}}$)(Components of a Null Entity)(Eqn1)

$$(\mathbf{a}^1 \subset \emptyset^1) \subseteq (\mathbf{a}^1 = \emptyset)$$

($\mathbf{T_{M711}}$)(Eqn2)

$$(\mathbf{a}^1 \subset \emptyset^1) \subseteq (a = \emptyset)$$

($\mathbf{T_{M139}}$)(Hypothetical Syllogism Eqns2&1)(Eqn3)

$$(a \subset b) = (a = ab)(a \neq b)$$

($\mathbf{T_{M766}}$)

$$(a \subset b) = (a \subseteq b)(a \neq b)$$

($\mathbf{A_{MX44}}$)($\subseteq$ Equality Form)(Eqn4)

$$\text{Let } a = \mathbf{a}^1$$

(Eqn5)

$$\text{Let } b = \emptyset^1$$

(Eqn6)

$$(\mathbf{a}^1 \subset \emptyset^1) = (\mathbf{a}^1 \subseteq \emptyset^1)(\mathbf{a}^1 \neq \emptyset^1)$$

($\mathbf{A_{MX19}}$)(Substitution of Eqns5&6 into Eqn4)(Eqn7)

$$(\mathbf{a}^1 \subseteq \emptyset^1)(\mathbf{a}^1 \neq \emptyset^1) \subseteq (a = \emptyset)$$

($\mathbf{A_{MX19}}$)(Substitution of Eqn7 into Eqn3)

$$(c^1 \subseteq a).(c^1 \subseteq b).(c^1 \neq \emptyset) \subseteq (ac^1 \neq \emptyset)$$

($\mathbf{T_{M411}}$) Existential Quantifier Introduction

Proof of ($\mathbf{T_{M411}}$)

$$\text{Let } (c^1 \subseteq a).(c^1 \subseteq b).(c^1 \neq \emptyset)$$

$$(c^1 = c^1 a).(c^1 \subseteq b).(c^1 \neq \emptyset)$$

($\mathbf{A_{MX44}}$)($\subseteq$ Equality Form)

$$(c^1 \subseteq b).(c^1 a \neq \emptyset)$$

($\mathbf{A_{MX19}}$)(Substitution)

$$\text{Thus } (c^1 \subseteq a).(c^1 \subseteq b).(c^1 \neq \emptyset) \subseteq (c^1 \subseteq b).(c^1 a \neq \emptyset)$$

($\mathbf{A_{MX02}}$)(Conditional Introduction)

$$(c^1 \subseteq a).(c^1 \subseteq b).(c^1 \neq \emptyset) \subseteq (c^1 a \neq \emptyset)$$

($\mathbf{T_{M073}}$)(Single Premise Syllogism)

$$(a \subseteq \underline{b} + c) = (ab \subseteq ac)$$

($\mathbf{T_{M088}}$) Conditional Subsets Rule

Proof of ($\mathbf{T_{M088}}$)

$$\text{Let } ab \subseteq ac$$

$$ab \subseteq c$$

($\mathbf{T_{M073}}$)(Single Premise Syllogism)

$$a \subseteq \underline{b} + c$$

($\mathbf{T_{M468}}$)

$$\text{Thus } (ab \subseteq ac) \subseteq (a \subseteq \underline{b} + c)$$

($\mathbf{A_{MX02}}$)(Conditional Introduction)(Eqn1)

$$\text{Let } a \subseteq \underline{b} + c$$

$$ab \subseteq c$$

($\mathbf{T_{M468}}$)

$$aab \subseteq ac$$

($\mathbf{T_{M029}}$)(Deduction)

$$ab \subseteq ac$$

($\mathbf{T_{M667}}$)(Conjunction Tautology)

$$\text{Thus } (a \subseteq \underline{b} + c) \subseteq (ab \subseteq ac)$$

($\mathbf{A_{MX02}}$)(Conditional Introduction)(Eqn2)

$$(a \subseteq \underline{b} + c) = (ab \subseteq ac)$$

($\mathbf{T_{M430}}$) (Biconditional Introduction Eqns1&2)

$$((a \subseteq \underline{b} + c) = \mathbf{E}) \subseteq ((a = \mathbf{E}) \subseteq (b \subseteq c))$$

($\mathbf{T_{M367}}$) Conditional Subsets Rule

Proof of ($\mathbf{T_{M367}}$)

$$((ab \subseteq ac) = \mathbf{E}) \subseteq ((a = \mathbf{E}) \subseteq (b \subseteq c))$$

($\mathbf{T_{M091}}$)(Conditional Subsets Rule)

$$((a \subseteq \underline{b} + c) = \mathbf{E}) \subseteq ((a = \mathbf{E}) \subseteq (b \subseteq c))$$

($\mathbf{T_{M088}}$)(Conditional Subsets Rule)

$$(b \subseteq ac) \subseteq (ab \subseteq ac)$$

($\mathbf{T_{M249}}$) Conditional Subsets Rule

Proof of ($\mathbf{T_{M249}}$)

$$(a \subseteq bc) \subseteq (a \subseteq c)$$

($\mathbf{T_{M073}}$)(Single Premise Syllogism)

$(a \subseteq bc) = (a \subseteq bc)(a \subseteq c)$	$(\mathbf{A}_{MX44})(\subseteq \text{Equality Form})$
$(a \subseteq bc) = (a \subseteq bc)(a = ac)$	$(\mathbf{A}_{MX44})(\subseteq \text{Equality Form})$
$(a \subseteq bc) \subseteq (ac \subseteq bc)$	$(\mathbf{A}_{MX19})(\text{Substitution})$
$(b \subseteq ac) \subseteq (ab \subseteq ac)$	Resymbolising

$(b \subseteq ac) \subseteq (ab \subseteq c)$ $(\mathbf{T}_{M249b})$ Conditional Subsets Rule

Proof of $(\mathbf{T}_{M249b})$

$(b \subseteq ac) \subseteq (ab \subseteq ac)$	$(\mathbf{T}_{M249})(\text{Eqn1})$
$(ab \subseteq ac) \subseteq (ab \subseteq c)$	$(\mathbf{T}_{M073})(\text{Single Premise Syllogism})(\text{Eqn2})$
$(b \subseteq ac) \subseteq (ab \subseteq c)$	$(\mathbf{T}_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$

$((a \subseteq bc) = E) \subseteq ((c = E) \subseteq (a \subseteq b))$ $(\mathbf{T}_{M250})$ Conditional Subsets Rule

Proof of $(\mathbf{T}_{M250})$

$((ac \subseteq bc) = E) \subseteq ((c = E) \subseteq (a \subseteq b))$	$(\mathbf{T}_{M091})(\text{Conditional Subsets Rule})(\text{Eqn1})$
$(a \subseteq bc) \subseteq (ac \subseteq bc)$	$(\mathbf{T}_{M249})(\text{Conditional Subsets Rule})$
$((a \subseteq bc) \subseteq (ac \subseteq bc)) = E$	$(\mathbf{A}_{MX16})(\text{Order Axiom})$
$((a \subseteq bc) = E) \subseteq ((ac \subseteq bc) = E)$	$(\mathbf{T}_{M094})(\text{Probabilistic Rule})(\text{Eqn2})$
$((a \subseteq bc) = E) \subseteq ((c = E) \subseteq (a \subseteq b))$	$(\mathbf{T}_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$

$(c \subseteq a)(c \subseteq b) = (c \subseteq ab)$ $(\mathbf{T}_{M132})$ Composition

Proof of $(\mathbf{T}_{M132})$

Let $c \subseteq a$	(Eqn1)
Let $c \subseteq b$	(Eqn2)
$cc \subseteq ab$	$(\mathbf{T}_{M669})(\text{Combination of Eqns1\&2})$
$c \subseteq ab$	$(\mathbf{T}_{M667})(\text{Conjunction Tautology})$
Thus $(c \subseteq a)(c \subseteq b) \subseteq (c \subseteq ab)$	$(\mathbf{A}_{MX02})(\text{Conditional Introduction})(\text{Eqn3})$
Let $c \subseteq ab$	
$c \subseteq a$	$(\mathbf{T}_{M073})(\text{Single Premise Syllogism})$
$c \subseteq b$	$(\mathbf{T}_{M073})(\text{Single Premise Syllogism})$
Thus $(c \subseteq ab) \subseteq (c \subseteq a)(c \subseteq b)$	$(\mathbf{A}_{MX02})(\text{Conditional Introduction})(\text{Eqn4})$
$(c \subseteq a)(c \subseteq b) = (c \subseteq ab)$	$(\mathbf{T}_{M430})(\text{Biconditional Introduction Eqns3\&4})$

$(a \subseteq c)(c \subseteq b) \subseteq (a \subseteq bc)$ $(\mathbf{T}_{M787})$

Proof of $(\mathbf{T}_{M787})$

Let $(a \subseteq c)$	(Eqn1)
Let $(c \subseteq b)$	(Eqn2)
$(a \subseteq b)$	$(\mathbf{T}_{M139})(\text{Hypothetical Syllogism})(\text{Eqn3})$
$(a \subseteq bc)$	$(\mathbf{T}_{M132})(\text{Composition of Eqns1\&3})$
Thus $(a \subseteq c)(c \subseteq b) \subseteq (a \subseteq bc)$	$(\mathbf{A}_{MX02})(\text{Conditional Introduction})$

$(a \neq \emptyset)(a \subseteq bc) \subseteq (a \subseteq c)(c \not\subseteq b)$ $(\mathbf{T}_{M788})$

Proof of $(\mathbf{T}_{M788})$

$(a \subseteq bc) \subseteq (a \subseteq c)$	$(\mathbf{T}_{M073})(\text{Single Premise Syllogism})$
$(a \neq \emptyset)(a \subseteq bc) \subseteq (a \subseteq c)$	$(\mathbf{T}_{M089})(\text{Single Premise Syllogism})(\text{Eqn1})$
$((a \subseteq bc) = E) = ((a \neq \emptyset) \subseteq (bc \neq \emptyset))$	$(\mathbf{T}_{M096})(\text{Probabilistic Rule})$
$(a \neq \emptyset)(a \subseteq bc) \subseteq (bc \neq \emptyset)$	$(\mathbf{A}_{MX35})(\text{Exportation/Importation}) \text{ for truth sets}$
$(a \neq \emptyset)(a \subseteq bc) \subseteq (c \not\subseteq b)$	$(\mathbf{T}_{M735})(\subseteq \text{Existential Form}) \text{ \& } (\mathbf{T}_{M012})(\text{Transposition})(\text{Eqn2})$
$(a \neq \emptyset)(a \subseteq bc) \subseteq (a \subseteq c)(c \not\subseteq b)$	$(\mathbf{T}_{M132})(\text{Composition of Eqns1\&2})$

$(c \subseteq a)(c \subseteq b) = (a + b \subseteq c)$ $(\mathbf{T}_{M132b})$ Composition

Proof of $(\mathbf{T}_{M132b})$

$(c \subseteq a)(c \subseteq b) = (c \subseteq ab)$	$(\mathbf{T}_{M132})(\text{Composition})$

$$(c \subseteq a)(c \subseteq b) = (\sim(ab) \subseteq c) \quad (T_{M012})(\text{Transposition})$$

$$(c \subseteq a)(c \subseteq b) = (\underline{a} + \underline{b} \subseteq c) \quad (T_{M025})(\text{DeMorgan's Law})$$

$$(a \neq \emptyset)(a \subseteq c)(a \subseteq b) \subseteq (b \not\subseteq \sim c) \quad (T_{M776})$$

Proof of (T_{M776})

$$(a \neq \emptyset)(a \subseteq bc) \subseteq (bc \neq \emptyset) \quad (T_{M777})$$

$$(a \neq \emptyset)(a \subseteq bc) \subseteq (b \not\subseteq \sim c) \quad (T_{M735})(\subseteq \text{Existential Form})$$

$$(a \neq \emptyset)(a \subseteq c)(a \subseteq b) \subseteq (b \not\subseteq \sim c) \quad (T_{M132})(\text{Composition})$$

$$((a = E) \subseteq (b \subseteq c)) \subseteq ((a \subseteq b) \subseteq (a \subseteq c)) \quad (T_{M420}) \text{ Distribution of Implication over Implication}$$

Proof of (T_{M420})

$$\text{Let } ((a = E) \subseteq (b \subseteq c))$$

$$((a = E)(a \subseteq b) \subseteq (a \subseteq b)(b \subseteq c)) \quad (T_{M029})(\text{Deduction})$$

$$((a = E)(a \subseteq b) \subseteq (a \subseteq c)) \quad (T_{M139})(\text{Hypothetical Syllogism})$$

$$((a \subseteq bE) \subseteq (a \subseteq c)) \quad (T_{M132})(\text{Composition})$$

$$((a \subseteq b) \subseteq (a \subseteq c)) \quad (A_{MX11})(\text{Universal Set Conjunction Axiom})$$

$$\text{Thus } ((a = E) \subseteq (b \subseteq c)) \subseteq ((a \subseteq b) \subseteq (a \subseteq c)) \quad (A_{MX02})(\text{Conditional Introduction})$$

$$(c \not\subseteq a) + (c \not\subseteq b) = (c \not\subseteq ab) \quad (T_{M173}) \text{ Disjunction of Unnecessities}$$

Proof of (T_{M173})

$$(c \subseteq a)(c \subseteq b) = (c \subseteq ab) \quad (T_{M132})(\text{Composition})$$

$$\sim((c \subseteq a)(c \subseteq b)) = \sim(c \subseteq ab) \quad (T_{M008})(\text{Inverse of an Equality})$$

$$\sim(c \subseteq a) + \sim(c \subseteq b) = \sim(c \subseteq ab) \quad (T_{M026})(\text{DeMorgan's Law})$$

$$(c \not\subseteq a) + (c \not\subseteq b) = (c \not\subseteq ab) \quad (T_{M259})(\text{Definition of } \not\subseteq)$$

$$(c \not\subseteq a) + (c \not\subseteq b) = (\underline{a} + \underline{b} \not\subseteq c) \quad (T_{M371}) \text{ Disjunction of Unnecessities}$$

Proof of (T_{M371})

$$(c \not\subseteq a) + (c \not\subseteq b) = (c \not\subseteq ab) \quad (T_{M173})(\text{Disjunction of Unnecessities})$$

$$(c \not\subseteq a) + (c \not\subseteq b) = (\sim(ab) \not\subseteq c) \quad (T_{M012})(\text{Transposition})$$

$$(c \not\subseteq a) + (c \not\subseteq b) = (\underline{a} + \underline{b} \not\subseteq c) \quad (T_{M026})(\text{DeMorgan's Law})$$

$$(a = E^1) \subseteq (a^1 = E^1) \quad (T_{M736})$$

Proof of (T_{M736})

$$(a = b) \subseteq (F(a) = F(b)) \quad (A_{MX19})(\text{Substitution Axiom})(\text{Eqn1})$$

$$\text{Let } b = E^1 \quad (\text{Eqn2})$$

$$(a = E^1) \subseteq (F(a) = F(E^1)) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3})$$

$$\text{Let } F(a) = a^1 \quad (\text{Eqn4})$$

$$\text{Thus } F(E^1) = E^{11} \quad (A_{MX02})(\text{Conditional Introduction})(\text{Eqn5})$$

$$\text{NB The conditions } (F(a) = a^1) \subseteq (a^1 \neq \emptyset)(a^1 \neq E) \text{ are met}$$

$$(a = E^1) \subseteq (a^1 = E^{11}) \quad (A_{MX19})(\text{Substitution of Eqn4\&5 into Eqn3})$$

$$(a = E^1) \subseteq (a^1 = E^1) \quad (D_{M17})(\text{Definition of } a^{11}) \text{ since } E^1 \neq \emptyset$$

$$(a \neq \emptyset)(a \subseteq b^1) \subseteq (a = b^1) \quad (T_{M577})$$

Proof of (T_{M577})

$$(a \subset b^1) = (b^1 \neq \emptyset)(a = \emptyset) \quad (A_{MX40})(\text{Proper Subset of Unit Set Axiom})$$

$$(a \subset b^1) \subseteq (a = \emptyset) \quad (T_{M073b})(\text{Single Premise Syllogism})$$

$$(a \neq \emptyset) \subseteq (a \sim \subset b^1) \quad (T_{M012})(\text{Transposition})$$

$$(a \neq \emptyset)(a \subseteq b^1) \subseteq (a \sim \subset b^1)(a \subseteq b^1) \quad (T_{M029})(\text{Deduction})$$

$$(a \neq \emptyset)(a \subseteq b^1) \subseteq (a = b^1) \quad (T_{M578})$$

$$(a^1 \neq \emptyset) \subseteq \sim(a^1 \subset b^1) \quad (T_{M348})$$

Proof of (T_{M348})

$$\begin{aligned} (a^1 \subset b^1) &= (b^1 \neq \emptyset)(a^1 = \emptyset) & (A_{MX40}) & \text{(Proper Subset of Unit Set Axiom)} \\ (a^1 \subset b^1) &\subseteq (a^1 = \emptyset) & (T_{M073b}) & \text{(Single Premise Syllogism)} \\ (a^1 \neq \emptyset) &\subseteq \sim(a^1 \subset b^1) & (T_{M012}) & \text{(Transposition)} \end{aligned}$$

$$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 + b^1 = c^1) = (a^1 = c^1)(b^1 = c^1)) \quad (T_{M589})$$

Proof of (T_{M589})

$$\begin{aligned} \text{Let } a^1 &= c^1 & (\text{Eqn1}) \\ \text{Let } b^1 &= c^1 & (\text{Eqn2}) \\ a^1 + b^1 &= c^1 + c^1 & (T_{M230}) & \text{(Combination of Eqns1\&2)} \\ a^1 + b^1 &= c^1 & (A_{MX01}) & \text{(Disjunction Tautology)} \\ \text{Thus } (a^1 = c^1)(b^1 = c^1) &\subseteq (a^1 + b^1 = c^1) & (A_{MX02}) & \text{(Conditional Introduction)(Eqn3)} \\ \text{Let } a^1 + b^1 &= c^1 \\ a^1 &\subseteq c^1 & (T_{M072b}) & \text{(Single Premise Syllogism)} \\ (a^1 \subset c^1) + (a^1 = c^1) & & (T_{M266}) & \text{(Definition of } \subseteq \text{)(Eqn4)} \\ \text{Let } a^1 &\neq \emptyset \\ \sim(a^1 \subset c^1) & & (T_{M348}) \\ \sim(a^1 \subset c^1) &= E & (A_{MX16}) & \text{(Order Axiom)} \\ (a^1 \subset c^1) &= \sim E & (T_{M008}) & \text{(Inverse of an Equality)} \\ (a^1 \subset c^1) &= \emptyset & (D_{M02}) & \text{(Definition of } \emptyset \text{)(Eqn5)} \\ \emptyset + (a^1 = c^1) & & (A_{MX19}) & \text{(Substitution of Eqn5 into Eqn4)} \\ a^1 &= c^1 & (T_{M668}) & \text{(Null Set Disjunction)} \\ b^1 &\subseteq c^1 & (T_{M072b}) & \text{(Single Premise Syllogism)} \\ (b^1 \subset c^1) + (b^1 = c^1) & & (T_{M266}) & \text{(Definition of } \subseteq \text{)(Eqn6)} \\ \text{Let } b^1 &\neq \emptyset \\ \sim(b^1 \subset c^1) & & (T_{M348}) \\ \sim(b^1 \subset c^1) &= E & (A_{MX16}) & \text{(Order Axiom)} \\ (b^1 \subset c^1) &= \sim E & (T_{M008}) & \text{(Inverse of an Equality)} \\ (b^1 \subset c^1) &= \emptyset & (D_{M02}) & \text{(Definition of } \emptyset \text{)(Eqn7)} \\ \emptyset + (b^1 = c^1) & & (A_{MX19}) & \text{(Substitution of Eqn7 into Eqn6)} \\ b^1 &= c^1 & (T_{M668}) & \text{(Null Set Disjunction)} \\ \text{Thus } (a^1 \neq \emptyset)(b^1 \neq \emptyset)(a^1 + b^1 = c^1) &\subseteq (a^1 = c^1)(b^1 = c^1) & (A_{MX02}) & \text{(Conditional Introduction)} \\ ((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) &\subseteq ((a^1 + b^1 = c^1) \subseteq (a^1 = c^1)(b^1 = c^1)) & (A_{MX35}) & \text{(Exportation/Importation)(Eqn8) for truth sets} \\ ((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) &\subseteq ((a^1 + b^1 = c^1) = (a^1 = c^1)(b^1 = c^1)) & (T_{M430}) & \text{(Biconditional Introduction Eqns3\&8)} \end{aligned}$$

$$(a \neq \emptyset) \subseteq (a \sim \subset a^1) \quad (T_{M560})$$

Proof of (T_{M560})

$$\begin{aligned} (a \subset a^1) &= (a^1 \neq \emptyset)(a = \emptyset) & (A_{MX40}) & \text{(Proper Subset of Unit Set Axiom)} \\ (a \subset a^1) &\subseteq (a = \emptyset) & (T_{M073b}) & \text{(Single Premise Syllogism)} \\ (a \neq \emptyset) &\subseteq (a \sim \subset a^1) & (T_{M012}) & \text{(Transposition)} \end{aligned}$$

$$((a \not\subseteq b)(a \not\subseteq b) = E) \subseteq ((a \subseteq b) = \emptyset) \quad (T_{M584})$$

Proof of (T_{M584})

$$\begin{aligned} \text{Let } a &\not\subseteq b \\ (a \not\subseteq b) &= E & (A_{MX16}) & \text{(Order Axiom)} \\ (a \subseteq b) &= E & (T_{M008}) & \text{(Inverse of an Equality)} \\ (a \subseteq b) &= \emptyset & (T_{M054}) & \text{(Negative of Null Set)} \\ \text{Thus } ((a \not\subseteq b) = E) &\subseteq ((a \subseteq b) = \emptyset) & (A_{MX02}) & \text{(Conditional Introduction)(Eqn1)} \\ (a \not\subseteq b)(a \not\subseteq b) &\subseteq (a \not\subseteq b) & (T_{M083}) & \text{(Conjunction Elimination)} \\ ((a \not\subseteq b)(a \not\subseteq b) \subseteq (a \not\subseteq b)) &= E & (A_{MX16}) & \text{(Order Axiom)} \\ ((a \not\subseteq b)(a \not\subseteq b) = E) &\subseteq ((a \not\subseteq b) = E) & (T_{M094}) & \text{(Probabilistic Rule)(Eqn2)} \end{aligned}$$

$$((a \not\subseteq b)(a \not\subseteq b) = E) \subseteq ((a \subseteq b) = \emptyset) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$$

$$(ab = E) \subseteq (a = E) \quad (T_{M084})$$

Proof of (T_{M084})

$$\begin{aligned} ab &\subseteq a && (T_{M083})(\text{Conjunction Elimination}) \\ (ab \subseteq a) &= E && (A_{MX16})(\text{Order Axiom}) \\ (ab = E) &\subseteq (a = E) && (T_{M094})(\text{Probabilistic Rule}) \end{aligned}$$

$$(ab = E) \subseteq (ab = \emptyset) \quad (T_{M372})$$

Proof of (T_{M372})

$$\begin{aligned} \text{Let } ab &= E \\ b &= E && (T_{M084}) \\ \underline{b} &= \underline{E} && (T_{M008})(\text{Inverse of an Equality}) \\ \underline{b} &= \emptyset && (D_{M02})(\text{Definition of } \emptyset) \\ ab &= \emptyset && (T_{M350}) \\ \text{Thus } (ab = E) &\subseteq (ab = \emptyset) && (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(a + b = \emptyset) \subseteq (a + \underline{b} = E) \quad (T_{M373})$$

Proof of (T_{M373})

$$\begin{aligned} (ab = E) &\subseteq (ab = \emptyset) && (T_{M372}) \\ (\sim(ab) = E) &\subseteq (\sim(ab) = \emptyset) && (T_{M008})(\text{Inverse of an Equality}) \\ (\sim(ab) = \emptyset) &\subseteq (\sim(ab) = \emptyset) && (D_{M02})(\text{Definition of } \emptyset) \\ (\sim(ab) = \emptyset) &\subseteq (\sim(ab) = E) && (T_{M054})(\text{Negative of Null Set}) \\ (\underline{a} + \underline{b} = \emptyset) &\subseteq (\underline{a} + \underline{b} = E) && (T_{M025})(\text{DeMorgan's Law}) \\ (a + b = \emptyset) &\subseteq (a + \underline{b} = E) && \text{Resymbolising} \end{aligned}$$

$$(a = E)(b = E) = (ab = E) \quad (T_{M108}) \text{ Conjunction Introduction}$$

Proof of (T_{M108})

$$\begin{aligned} \text{Let } (a = E)(b = E) \\ (aE = E)(b = E) &&& (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\ (ab = E) &&& (A_{MX19})(\text{Substitution Axiom}) \\ \text{Thus } (a = E)(b = E) &\subseteq (ab = E) && (A_{MX02})(\text{Conditional Introduction})(\text{Eqn1}) \\ (ab = E) &\subseteq (a = E) && (T_{M084})(\text{Eqn2}) \\ (ab = E) &\subseteq (b = E) && \text{Resymbolising (Eqn3)} \\ (ab = E)(ab = E) &\subseteq (a = E)(b = E) && (T_{M669})(\text{Combination of Eqns2\&3}) \\ (ab = E) &\subseteq (a = E)(b = E) && (T_{M667})(\text{Conjunction Tautology})(\text{Eqn4}) \\ (a = E)(b = E) &= (ab = E) && (T_{M430})(\text{Biconditional Introduction Eqns1\&4}) \end{aligned}$$

$$((a = b) = E) \subseteq ((c^1 \subseteq a) = (c^1 \subseteq b)) \quad (T_{M775}) \text{ Entity Mode / Set Mode Rule}$$

Proof of (T_{M775})

$$\begin{aligned} ((a \subseteq b) = E) &\subseteq ((c^1 \subseteq a) \subseteq (c^1 \subseteq b)) && (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn1}) \\ ((b \subseteq a) = E) &\subseteq ((c^1 \subseteq b) \subseteq (c^1 \subseteq a)) && \text{Resymbolising (Eqn2)} \\ ((a \subseteq b) = E)((b \subseteq a) = E) &\subseteq ((c^1 \subseteq a) \subseteq (c^1 \subseteq b))((c^1 \subseteq b) \subseteq (c^1 \subseteq a)) \\ &&& (T_{M669})(\text{Combination of Eqns1\&2}) \\ ((a \subseteq b)(b \subseteq a) = E) &\subseteq ((c^1 \subseteq a) \subseteq (c^1 \subseteq b))((c^1 \subseteq b) \subseteq (c^1 \subseteq a)) \\ &&& (T_{M108})(\text{Conjunction Introduction}) \\ ((a = b) = E) &\subseteq ((c^1 \subseteq a) = (c^1 \subseteq b)) && (T_{M430})(\text{Biconditional Introduction}) \end{aligned}$$

$$((a = E) = E) \subseteq ((b = E) = (ab = E)) \quad (T_{M443}) \text{ Theorem Introduction}$$

Proof of (T_{M443})

$$\begin{aligned} (a = E)(b = E) &= (ab = E) && (T_{M108})(\text{Conjunction Introduction}) \\ ((a = E) = E) &\subseteq ((b = E) = (ab = E)) && (T_{M444c})(\text{Biconditional Exportation}) \end{aligned}$$

$$(a \neq E)(b \neq E) \subseteq (ab \neq E) \quad (T_{M113}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M113})

$$\begin{aligned} \text{Let } a \neq E & \quad (\text{Eqn1}) \\ \text{Let } b \neq E & \quad (\text{Eqn2}) \\ ab \neq E & \quad (T_{M084})(T_{M012}) \\ \text{Thus } (a \neq E)(b \neq E) \subseteq (ab \neq E) & \quad (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(a = E)(b \neq E) \subseteq (ab \neq E) \quad (T_{M109}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M109})

$$\begin{aligned} \text{Let } a = E & \quad (\text{Eqn1}) \\ \text{Let } b \neq E & \quad (\text{Eqn2}) \\ ab \neq E & \quad (T_{M084})(T_{M012}) \\ \text{Thus } (a = E)(b \neq E) \subseteq (ab \neq E) & \quad (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(a \neq E)(b \neq \emptyset) \subseteq (ab \neq E) \quad (T_{M117}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M117})

$$\begin{aligned} \text{Let } a \neq E & \quad (\text{Eqn1}) \\ \text{Let } b \neq \emptyset & \quad (\text{Eqn2}) \\ ab \neq E & \quad (T_{M084})(T_{M012}) \\ \text{Thus } (a \neq E)(b \neq \emptyset) \subseteq (ab \neq E) & \quad (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(a \neq \emptyset)(b = \emptyset) \subseteq (a + b \neq \emptyset) \quad (T_{M309}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M309})

$$\begin{aligned} (a = E)(b \neq E) \subseteq (ab \neq E) & \quad (T_{M109})(\text{Probabilistic Conjunction}) \\ (b = E)(a \neq E) \subseteq (ba \neq E) & \quad \text{Resymbolising} \\ (a \neq E)(b = E) \subseteq (ab \neq E) & \quad (T_{M404})(\text{Commutative .}) \\ (a \neq E)(b = \underline{E}) \subseteq (ab \neq E) & \quad (T_{M008})(\text{Inverse of an Equality}) \\ (a \neq \underline{E})(b = \underline{E}) \subseteq (ab \neq E) & \quad (T_{M008})(T_{M008}) \\ (a \neq \underline{E})(b = \underline{E}) \subseteq (\sim(a + b) \neq E) & \quad (T_{M026})(\text{DeMorgan's Law}) \\ (a \neq \underline{E})(b = \underline{E}) \subseteq (a + b \neq \underline{E}) & \quad (T_{M008})(T_{M008}) \\ (a \neq \emptyset)(b = \emptyset) \subseteq (a + b \neq \emptyset) & \quad (D_{M02})(\text{Definition of } \emptyset) \end{aligned}$$

$$(a \neq \emptyset)(b \neq \emptyset) \subseteq (a + b \neq \emptyset) \quad (T_{M312}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M312})

$$\begin{aligned} (a \neq E)(b \neq E) \subseteq (ab \neq E) & \quad (T_{M113})(\text{Probabilistic Conjunction}) \\ (a \neq E)(b \neq E) \subseteq (ab \neq E) & \quad \text{Resymbolising} \\ (a \neq \underline{E})(b \neq \underline{E}) \subseteq (ab \neq E) & \quad (T_{M008})(T_{M008}) \\ (a \neq \underline{E})(b \neq \underline{E}) \subseteq (\sim(a + b) \neq E) & \quad (T_{M026})(\text{DeMorgan's Law}) \\ (a \neq \underline{E})(b \neq \underline{E}) \subseteq (a + b \neq \underline{E}) & \quad (T_{M008})(T_{M008}) \\ (a \neq \emptyset)(b \neq \emptyset) \subseteq (a + b \neq \emptyset) & \quad (D_{M02})(\text{Definition of } \emptyset) \end{aligned}$$

$$(a \neq E)(b \neq \emptyset) \subseteq (a + b \neq \emptyset) \quad (T_{M314}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M314})

$$\begin{aligned} (a \neq \emptyset)(b \neq E) \subseteq (ab \neq E) & \quad (T_{M117})(\text{Probabilistic Conjunction}) \\ (a \neq \emptyset)(b \neq E) \subseteq (ab \neq E) & \quad \text{Resymbolising} \\ (a \neq \emptyset)(b \neq \underline{E}) \subseteq (ab \neq E) & \quad (T_{M008})(T_{M008}) \\ (a \neq \emptyset)(b \neq \underline{E}) \subseteq (\sim(a + b) \neq E) & \quad (T_{M026})(\text{DeMorgan's Law}) \\ (a \neq \emptyset)(b \neq \underline{E}) \subseteq (a + b \neq \underline{E}) & \quad (T_{M008})(T_{M008}) \\ (a \neq \emptyset)(b \neq \emptyset) \subseteq (a + b \neq \emptyset) & \quad (D_{M02})(\text{Definition of } \emptyset) \end{aligned}$$

$$(a \neq E)(b \neq \emptyset) \subseteq (a + b \neq \emptyset) \quad (T_{M054})(\text{Negative of Null Set})$$

$$(a^1 \subseteq b) \subseteq ((a^1 b)^1 = a^1 b) \quad (T_{M555})$$

Proof of (T_{M555})

Let $a^1 \subseteq b$

$$a^1 = a^1 b \quad (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn1})$$

$$a^{11} = (a^1 b)^1 \quad (T_{M479})$$

$$a^1 = (a^1 b)^1 \quad (D_{M17})(\text{Definition of } a^{11})(\text{Eqn2})$$

$$(a^1 b)^1 = a^1 b \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$$

$$\text{Thus } (a^1 \subseteq b) \subseteq ((a^1 b)^1 = a^1 b) \quad (A_{MX02})(\text{Conditional Introduction})$$

$$(a^1 E)^1 = a^1 E \quad (T_{M570})$$

Proof of (T_{M570})

$$(a^1 \subseteq b) \subseteq ((a^1 b)^1 = a^1 b) \quad (T_{M555})(\text{Eqn1})$$

$$\text{Let } b = E \quad (\text{Eqn2})$$

$$(a^1 \subseteq E) \subseteq ((a^1 E)^1 = a^1 E) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3})$$

$$a^1 \subseteq E \quad (T_{M744})(\text{Universal Set Theorem})$$

$$(a^1 \subseteq E) = E \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn4})$$

$$E \subseteq ((a^1 E)^1 = a^1 E) \quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})$$

$$E = ((a^1 E)^1 = a^1 E) \quad (T_{M275})(\text{Supersets of the Universal Set})$$

$$(a^1 E)^1 = a^1 E \quad (A_{MX16})(\text{Order Axiom})$$

$$((a \subseteq c)(b \subseteq c) = E) \subseteq ((a = b) = (a + c = b + c)) \quad (T_{M007}) \text{ Reversible Deduction}$$

Proof of (T_{M007})

$$((a \subseteq c) = E) \subseteq ((a \subseteq b) = (a + c \subseteq b + c)) \quad (T_{M048})(\text{Reversible Deduction})$$

$$((a \subseteq c) = E)((b \subseteq c) = E) \subseteq ((a \subseteq b) = (a + c \subseteq b + c)) \quad (T_{M089})(\text{Single Premise Syllogism})$$

$$((a \subseteq c)(b \subseteq c) = E) \subseteq ((a \subseteq b) = (a + c \subseteq b + c)) \quad (T_{M108})(\text{Conjunction Introduction})(\text{Eqn1})$$

$$(a \neq \Omega)(b \neq \Omega):$$

$$((b \subseteq c)(a \subseteq c) = E) \subseteq ((b \subseteq a) = (b + c \subseteq a + c)) \quad \text{Resymbolising}$$

$$((a \subseteq c)(b \subseteq c) = E) \subseteq ((b \subseteq a) = (b + c \subseteq a + c)) \quad (T_{M404})(\text{Commutative } \cdot)(\text{Eqn2})$$

$$((a \subseteq c)(b \subseteq c) = E).((a \subseteq c)(b \subseteq c) = E) \subseteq ((a \subseteq b) = (a + c \subseteq b + c)).((b \subseteq a) = (b + c \subseteq a + c))$$

$$(T_{M669})(\text{Combination of Eqns1\&2})$$

$$((a \subseteq c)(b \subseteq c) = E) \subseteq ((a \subseteq b) = (a + c \subseteq b + c)).((b \subseteq a) = (b + c \subseteq a + c))$$

$$(T_{M667})(\text{Conjunction Tautology})(\text{Eqn3})$$

$$((a \subseteq b) = (a + c \subseteq b + c)).((b \subseteq a) = (b + c \subseteq a + c)) \subseteq ((a \subseteq b)(b \subseteq a) = (a + c \subseteq b + c)(b + c \subseteq a + c))$$

$$(T_{M669})(\text{Combination})$$

$$((a \subseteq b) = (a + c \subseteq b + c)).((b \subseteq a) = (b + c \subseteq a + c)) \subseteq ((a = b) = (a + c = b + c))$$

$$(T_{M043})(\text{Conjunction of Membership Relationship Categories})(\text{Eqn4})$$

$$((a \subseteq c)(b \subseteq c) = E) \subseteq ((a = b) = (a + c = b + c)) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns3\&4})$$

$$(a^1 \not\subseteq b) \subseteq (a^1 \subseteq b) \quad (T_{M604})$$

Proof of (T_{M604})

$$E = (a \subset b) + (a = b) + (a \supset b) + (a \subset \underline{b}) + (a = \underline{b}) + (a \supset \underline{b}) + (a \parallel b)$$

$$(T_{M174})(\text{Binary Relation Disjunction Rule})(\text{Eqn1})$$

$$\text{Let } a = a^1 \quad (\text{Eqn2})$$

$$E = (a^1 \subset b) + (a^1 = b) + (a^1 \supset b) + (a^1 \subset \underline{b}) + (a^1 = \underline{b}) + (a^1 \supset \underline{b}) + (a^1 \parallel b)$$

$$(A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3})$$

$$E = (a^1 \subset b) + (a^1 = b) + (a^1 \supset b) + (a^1 \subset \underline{b}) + (a^1 = \underline{b}) + (a^1 \supset \underline{b}) + (a^1 \parallel b)$$

$$a^1 \not\parallel b \quad (A_{MX15})(\text{Intersection with the Unit Set Axiom})$$

$$(a^1 \not\parallel b) = E \quad (A_{MX16})(\text{Order Axiom})$$

$$(a^1 \parallel b) = \emptyset \quad (T_{M008})(\text{Inverse of an Equality})(\text{Eqn4})$$

$$E = (a^1 \subset b) + (a^1 = b) + (a^1 \supset b) + (a^1 \subset \underline{b}) + (a^1 = \underline{b}) + (a^1 \supset \underline{b}) + \emptyset$$

$$(A_{MX19})(\text{Substitution of Eqn4 into Eqn3})$$

$$E = (a^1 \subset b) + (a^1 = b) + (a^1 \supset b) + (a^1 \subset \underline{b}) + (a^1 = \underline{b}) + (a^1 \supset \underline{b})$$

$E = (a^1 \subset b) + (a^1 = b) + (b = \emptyset)(a^1 \neq \emptyset) + (a^1 \subset \underline{b}) + (a^1 = \underline{b}) + (\underline{b} = \emptyset)(a^1 \neq \emptyset)$	(T_{M668})(Null Set Disjunction)
$E = (a^1 \subset b) + (a^1 = b) + (b = \emptyset)(a^1 \neq \emptyset) + (a^1 \subset \underline{b}) + (a^1 = \underline{b}) + (b = \sim \emptyset)(a^1 \neq \emptyset)$	(A_{MX40})(Proper Subset of Unit Set Axiom)
$E = (a^1 \subset b) + (a^1 = b) + (b = \emptyset)(a^1 \neq \emptyset) + (a^1 \subset \underline{b}) + (a^1 = \underline{b}) + (b = E)(a^1 \neq \emptyset)$	(T_{M008})(Inverse of an Equality)
	(T_{M054})(Negative of Null Set)(Eqn5)
Let $(b \neq \emptyset)$	
$(b \neq \emptyset) = E$	(A_{MX16})(Order Axiom)
$(b = \emptyset) = \sim E$	(T_{M008})(Inverse of an Equality)
$(b = \emptyset) = \emptyset$	(D_{M02})(Definition of $\emptyset$)(Eqn6)
Let $(b \neq E)$	
$(b \neq E) = E$	(A_{MX16})(Order Axiom)
$(b = E) = \sim E$	(T_{M008})(Inverse of an Equality)
$(b = E) = \emptyset$	(D_{M02})(Definition of $\emptyset$)(Eqn7)
$E = (a^1 \subset b) + (a^1 = b) + \emptyset(a^1 \neq \emptyset) + (a^1 \subset \underline{b}) + (a^1 = \underline{b}) + \emptyset(a^1 \neq \emptyset)$	(A_{MX19})(Substitution of Eqns6&7 into Eqn5)
$E = (a^1 \subset b) + (a^1 = b) + \emptyset + (a^1 \subset \underline{b}) + (a^1 = \underline{b}) + \emptyset$	(T_{M050})(Null Set Conjunction)
$E = (a^1 \subset b) + (a^1 = b) + (a^1 \subset \underline{b}) + (a^1 = \underline{b})$	(T_{M668})(Null Set Disjunction)
Thus $((b \neq \emptyset)(b \neq E) = E) \subseteq (E = (a^1 \subset b) + (a^1 = b) + (a^1 \subset \underline{b}) + (a^1 = \underline{b}))$	(A_{MX02})(Conditional Introduction)
$((b \neq \emptyset)(b \neq E) \subseteq (a^1 \subset b) + (a^1 = b) + (a^1 \subset \underline{b}) + (a^1 = \underline{b})) = E$	(T_{M094})(Probabilistic Rule)
$(b \neq \emptyset)(b \neq E) \subseteq (a^1 \subset b) + (a^1 = b) + (a^1 \subset \underline{b}) + (a^1 = \underline{b})$	(A_{MX16})(Order Axiom)
$(b \neq \emptyset)(b \neq E) \subseteq (a^1 \subseteq b) + (a^1 \subseteq \underline{b})$	(T_{M266})(Definition of $\subseteq$)(Eqn8)
Let $b = \emptyset$	(Eqn9)
Let $(a^1 \not\subseteq b) \not\subseteq (a^1 \subseteq \underline{b})$	(Eqn10)
$(a^1 \not\subseteq b) \not\subseteq (a^1 \subseteq \sim \emptyset)$	(A_{MX19})(Substitution of Eqn9 into Eqn10)
$(a^1 \not\subseteq b) \not\subseteq (a^1 \subseteq E)$	(T_{M054})(Negative of Null Set)
$(a^1 \not\subseteq b) \not\subseteq E$	(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)
$\sim E$	(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)
$\emptyset$	(D_{M02})(Definition of $\emptyset$)
Thus $((b = \emptyset) = E)((a^1 \not\subseteq b) \not\subseteq (a^1 \subseteq \underline{b})) = \emptyset$	(A_{MX02})(Conditional Introduction)
$((b = \emptyset) = E) \subseteq ((a^1 \not\subseteq b) \subseteq (a^1 \subseteq \underline{b}))$	(T_{M735})($\subseteq$ Existential Form)
$((b = \emptyset) = E) \subseteq (E = (a^1 \subseteq b) + (a^1 \subseteq \underline{b}))$	(T_{M064})($\subseteq$ Universal Form)
$((b = \emptyset) \subseteq (a^1 \subseteq b) + (a^1 \subseteq \underline{b})) = E$	(T_{M094})(Probabilistic Rule)
$(b = \emptyset) \subseteq (a^1 \subseteq b) + (a^1 \subseteq \underline{b})$	(A_{MX16})(Order Axiom)(Eqn11)
Let $b = E$	(Eqn12)
Let $(a^1 \not\subseteq b) \not\subseteq (a^1 \subseteq \underline{b})$	(Eqn13)
$(a^1 \not\subseteq E) \not\subseteq (a^1 \subseteq \underline{b})$	(A_{MX19})(Substitution of Eqn12 into Eqn13)
$\sim E \not\subseteq (a^1 \subseteq \underline{b})$	(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)
$\emptyset \not\subseteq (a^1 \subseteq \underline{b})$	(D_{M02})(Definition of $\emptyset$)
$\sim E$	(T_{M001})(Universality of (T_{M670}) Null Set Theorem)
$\emptyset$	(D_{M02})(Definition of $\emptyset$)
Thus $((b = E) = E)((a^1 \not\subseteq b) \not\subseteq (a^1 \subseteq \underline{b})) = \emptyset$	(A_{MX02})(Conditional Introduction)
$((b = E) = E) \subseteq ((a^1 \not\subseteq b) \subseteq (a^1 \subseteq \underline{b}))$	(T_{M735})($\subseteq$ Existential Form)
$((b = E) = E) \subseteq (E = (a^1 \subseteq b) + (a^1 \subseteq \underline{b}))$	(T_{M064})($\subseteq$ Universal Form)
$((b = E) \subseteq (a^1 \subseteq b) + (a^1 \subseteq \underline{b})) = E$	(T_{M094})(Probabilistic Rule)
$(b = E) \subseteq (a^1 \subseteq b) + (a^1 \subseteq \underline{b})$	(A_{MX16})(Order Axiom)(Eqn14)
$(b = \emptyset) + (b = E) \subseteq (a^1 \subseteq b) + (a^1 \subseteq \underline{b}) + (a^1 \subseteq b) + (a^1 \subseteq \underline{b})$	(A_{MX27})(Combination of Eqns11&14)
$(b = \emptyset) + (b = E) \subseteq (a^1 \subseteq b) + (a^1 \subseteq \underline{b})$	(A_{MX07})(Disjunction Tautology)
$\sim((b \neq \emptyset)(b \neq E)) \subseteq (a^1 \subseteq b) + (a^1 \subseteq \underline{b})$	(T_{M026})(DeMorgan's Law)(Eqn15)
$(b \neq \emptyset)(b \neq E) + \sim((b \neq \emptyset)(b \neq E)) \subseteq (a^1 \subseteq b) + (a^1 \subseteq \underline{b}) + (a^1 \subseteq b) + (a^1 \subseteq \underline{b})$	(A_{MX27})(Combination of Eqns8&15)
$(b \neq \emptyset)(b \neq E) + \sim((b \neq \emptyset)(b \neq E)) \subseteq (a^1 \subseteq b) + (a^1 \subseteq \underline{b})$	

$$E \subseteq (a^1 \subseteq b) + (a^1 \subseteq \underline{b})$$

$$(a^1 \not\subseteq b) \subseteq (a^1 \subseteq \underline{b})$$

($\mathbf{A}_{MX01}$)(Disjunction Tautology)
 ($\mathbf{T}_{M671}$)(Tertium non Datur)
 ($\mathbf{T}_{M064}$)($\subseteq$ Universal Form)

$$E = (a^1.b = \emptyset) + (a^1.b = a^1) \quad (\mathbf{T}_{M572})$$

Proof of ($\mathbf{T}_{M572}$)

$$(a^1 \not\subseteq b) \subseteq (a^1 \subseteq \underline{b}) \quad (\mathbf{T}_{M604})$$

$$(a^1 \not\subseteq b) = (a^1 b = \emptyset) \quad (\mathbf{T}_{M735})(\subseteq \text{Existential Form})(\text{Eqn1})$$

$$(a^1 \subseteq b) = (a^1 b = a^1) \quad (\mathbf{A}_{MX44})(\subseteq \text{Equality Form})(\text{Eqn2})$$

$$(a^1 \subseteq b) + (a^1 \not\subseteq b) = (a^1 b = a^1) + (a^1 b = \emptyset) \quad (\mathbf{A}_{MX21})(\text{Combination of Eqns1\&2})$$

$$E = (a^1 b = a^1) + (a^1 b = \emptyset) \quad (\mathbf{T}_{M671})(\text{Tertium non Datur})$$

$$E = (a.b^1 = \emptyset) + (a.b^1 = b^1) \quad (\mathbf{T}_{M573})$$

Proof of ($\mathbf{T}_{M573}$)

$$E = (a^1.b = \emptyset) + (a^1.b = a^1) \quad (\mathbf{T}_{M572})$$

$$E = (b^1.a = \emptyset) + (b^1.a = b^1) \quad \text{Resymbolising}$$

$$E = (a.b^1 = \emptyset) + (a.b^1 = b^1) \quad (\mathbf{T}_{M404})(\text{Commutative .})$$

$$E = (a^1.b^1 = \emptyset) + (a^1.b^1 = a^1) + (a^1.b^1 = b^1) \quad (\mathbf{T}_{M574})$$

Proof of ($\mathbf{T}_{M574}$)

$$E = (a^1.b^1 = \emptyset) + (a^1.b^1 = a^1) \quad (\mathbf{T}_{M572})(\text{Eqn1})$$

$$E = (a^1.b^1 = \emptyset) + (a^1.b^1 = b^1) \quad (\mathbf{T}_{M573})(\text{Eqn2})$$

$$E + E = (a^1.b^1 = \emptyset) + (a^1.b^1 = a^1) + (a^1.b^1 = \emptyset) + (a^1.b^1 = b^1) \quad (\mathbf{A}_{MX21})(\text{Combination of Eqns1\&2})$$

$$E = (a^1.b^1 = \emptyset) + (a^1.b^1 = a^1) + (a^1.b^1 = b^1) \quad (\mathbf{A}_{MX01})(\text{Disjunction Tautology})$$

$$E = (a^1 \subseteq b) + (a^1 = b) + (a^1 \subseteq \underline{b}) + (a^1 = \underline{b}) \quad (\mathbf{T}_{M519})$$

Proof of ($\mathbf{T}_{M519}$)

$$(a^1 \not\subseteq b) \subseteq (a^1 \subseteq \underline{b}) \quad (\mathbf{T}_{M604})$$

$$(a^1 \subseteq b) + (a^1 \not\subseteq b) \subseteq (a^1 \subseteq b) + (a^1 \subseteq \underline{b}) \quad (\mathbf{T}_{M030})(\text{Deduction})$$

$$E \subseteq (a^1 \subseteq b) + (a^1 \subseteq \underline{b}) \quad (\mathbf{T}_{M671})(\text{Tertium non Datur})$$

$$E = (a^1 \subseteq b) + (a^1 \subseteq \underline{b}) \quad (\mathbf{T}_{M275})(\text{Supersets of the Universal Set})$$

$$E = (a^1 \subseteq b) + (a^1 = b) + (a^1 \subseteq \underline{b}) + (a^1 = \underline{b}) \quad (\mathbf{T}_{M266})(\text{Definition of } \subseteq)$$

$$E = (c^1 \subseteq ab) + (c^1 \subseteq \underline{ab}) + (c^1 \subseteq a\bar{b}) + (c^1 \subseteq \bar{a}b) \quad (\mathbf{T}_{M525})$$

Proof of ($\mathbf{T}_{M525}$)

$$E = (a^1 \subseteq b) + (a^1 = b) + (a^1 \subseteq \underline{b}) + (a^1 = \underline{b}) \quad (\mathbf{T}_{M519})$$

$$E = (a^1 \subseteq b) + (a^1 \subseteq \underline{b}) \quad (\mathbf{T}_{M266})(\text{Definition of } \subseteq)$$

$$E = (c^1 \subseteq a) + (c^1 \subseteq \underline{a}) \quad \text{Resymbolising (Eqn1)}$$

$$E = (c^1 \subseteq b) + (c^1 \subseteq \underline{b}) \quad \text{Resymbolising (Eqn2)}$$

$$E.E = ((c^1 \subseteq a) + (c^1 \subseteq \underline{a})).((c^1 \subseteq b) + (c^1 \subseteq \underline{b})) \quad (\mathbf{T}_{M231})(\text{Combination of Eqns1\&2})$$

$$E = ((c^1 \subseteq a) + (c^1 \subseteq \underline{a})).((c^1 \subseteq b) + (c^1 \subseteq \underline{b})) \quad (\mathbf{A}_{MX11})(\text{Universal Set Conjunction Axiom})$$

$$E = (c^1 \subseteq a)(c^1 \subseteq b) + (c^1 \subseteq \underline{a})(c^1 \subseteq b) + (c^1 \subseteq a)(c^1 \subseteq \underline{b}) + (c^1 \subseteq \underline{a})(c^1 \subseteq \underline{b}) \quad (\mathbf{T}_{M407})(\text{Distribution of . over +})$$

$$E = (c^1 \subseteq ab) + (c^1 \subseteq \underline{ab}) + (c^1 \subseteq a\bar{b}) + (c^1 \subseteq \bar{a}b) \quad (\mathbf{T}_{M132})(\text{Composition})$$

$$(((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = E)) = (((c^1 \subseteq b) = \emptyset) \subseteq ((c^1 \subseteq a) = \emptyset)) \quad (\mathbf{T}_{M699})$$

Proof of ($\mathbf{T}_{M699}$)

$$((c^1 \subseteq a) \subseteq (c^1 \subseteq b)) = ((c^1 \not\subseteq b) \subseteq (c^1 \not\subseteq a)) \quad (\mathbf{T}_{M698})$$

$$(((c^1 \subseteq a) \subseteq (c^1 \subseteq b)) = ((c^1 \not\subseteq b) \subseteq (c^1 \not\subseteq a))) = E \quad (\mathbf{A}_{MX16})(\text{Order Axiom})$$

$$(((c^1 \subseteq a) \subseteq (c^1 \subseteq b)) = E) = (((c^1 \not\subseteq b) \subseteq (c^1 \not\subseteq a)) = E) \quad (\mathbf{T}_{M094b})(\text{Probabilistic Rule})$$

$$(((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = E)) = (((c^1 \not\subseteq b) \subseteq (c^1 \not\subseteq a)) = E) \quad (\mathbf{T}_{M094a})(\text{Probabilistic Rule})$$

$$(((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = E)) = (((c^1 \subseteq b) = \emptyset) \subseteq ((c^1 \subseteq a) = \emptyset)) \quad (T_{M097})(\text{Probabilistic Rule})$$

$$(a \neq E) + (b \neq E) = (ab \neq E) \quad (T_{M163}) \text{ Probabilistic Disjunction}$$

Proof of (T_{M163})

$$\begin{aligned} (a = E)(b = E) &= (ab = E) & (T_{M108})(\text{Conjunction Introduction}) \\ \sim((a = E)(b = E)) &= \sim(ab = E) & (T_{M008})(\text{Inverse of an Equality}) \\ \sim(a = E) + \sim(b = E) &= \sim(ab = E) & (T_{M025})(\text{DeMorgan's Law}) \text{ for truth sets} \\ (a \neq E) + (b \neq E) &= (ab \neq E) & (T_{M260})(\text{Definition of } \neq) \end{aligned}$$

$$(a \neq E) + (b \neq \emptyset) = (a\bar{b} \neq E) \quad (T_{M382}) \text{ Probabilistic Disjunction}$$

Proof of (T_{M382})

$$\begin{aligned} (a \neq E) + (b \neq E) &= (ab \neq E) & (T_{M163})(\text{Probabilistic Disjunction}) \\ (a \neq E) + (\bar{b} \neq E) &= (a\bar{b} \neq E) & \text{Resymbolising} \\ (a \neq E) + (b \neq \bar{E}) &= (a\bar{b} \neq E) & (T_{M008})(T_{M008}) \\ (a \neq E) + (b \neq \emptyset) &= (a\bar{b} \neq E) & (D_{M02})(\text{Definition of } \emptyset) \end{aligned}$$

$$(a \neq \emptyset) + (b \neq \emptyset) = (a\bar{b} \neq E) \quad (T_{M383}) \text{ Probabilistic Disjunction}$$

Proof of (T_{M383})

$$\begin{aligned} (a \neq E) + (b \neq E) &= (ab \neq E) & (T_{M163})(\text{Probabilistic Disjunction}) \\ (\bar{a} \neq E) + (\bar{b} \neq E) &= (a\bar{b} \neq E) & \text{Resymbolising} \\ (a \neq \bar{E}) + (b \neq \bar{E}) &= (a\bar{b} \neq E) & (T_{M008})(T_{M008}) \\ (a \neq \emptyset) + (b \neq \emptyset) &= (a\bar{b} \neq E) & (D_{M02})(\text{Definition of } \emptyset) \end{aligned}$$

$$(a \neq E) + (b \neq E) = (\bar{a} + \bar{b} \neq \emptyset) \quad (T_{M387}) \text{ Probabilistic Disjunction}$$

Proof of (T_{M387})

$$\begin{aligned} (a \neq E) + (b \neq E) &= (ab \neq E) & (T_{M163})(\text{Probabilistic Disjunction}) \\ (a \neq E) + (b \neq E) &= (\sim(ab) \neq \bar{E}) & (T_{M008})(T_{M008}) \\ (a \neq E) + (b \neq E) &= (\sim(ab) \neq \emptyset) & (D_{M02})(\text{Definition of } \emptyset) \\ (a \neq E) + (b \neq E) &= (\bar{a} + \bar{b} \neq \emptyset) & (T_{M025})(\text{DeMorgan's Law}) \end{aligned}$$

$$(a \neq E) + (b \neq \emptyset) = (\bar{a} + \bar{b} \neq \emptyset) \quad (T_{M388}) \text{ Probabilistic Disjunction}$$

Proof of (T_{M388})

$$\begin{aligned} (a \neq E) + (b \neq E) &= (\bar{a} + \bar{b} \neq \emptyset) & (T_{M387})(\text{Probabilistic Disjunction}) \\ (a \neq E) + (\bar{b} \neq E) &= (\bar{a} + \bar{b} \neq \emptyset) & \text{Resymbolising} \\ (a \neq E) + (b \neq \bar{E}) &= (\bar{a} + \bar{b} \neq \emptyset) & (T_{M008})(T_{M008}) \\ (a \neq E) + (b \neq \emptyset) &= (\bar{a} + \bar{b} \neq \emptyset) & (D_{M02})(\text{Definition of } \emptyset) \end{aligned}$$

$$(a = E)(b = \emptyset) = (a\bar{b} = E) \quad (T_{M374}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M374})

$$\begin{aligned} (a = E)(b = E) &= (ab = E) & (T_{M108})(\text{Conjunction Introduction}) \\ (a = E)(\bar{b} = \bar{E}) &= (ab = E) & (T_{M008})(\text{Inverse of an Equality}) \\ (a = E)(\bar{b} = \emptyset) &= (ab = E) & (D_{M02})(\text{Definition of } \emptyset) \\ (a = E)(b = \emptyset) &= (a\bar{b} = E) & \text{Resymbolising} \end{aligned}$$

$$(a = \emptyset)(b = \emptyset) = (a\bar{b} = E) \quad (T_{M375}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M375})

$$\begin{aligned} (a = E)(b = E) &= (ab = E) & (T_{M108})(\text{Conjunction Introduction}) \\ (\bar{a} = \bar{E})(\bar{b} = \bar{E}) &= (ab = E) & (T_{M008})(\text{Inverse of an Equality}) \\ (\bar{a} = \emptyset)(\bar{b} = \emptyset) &= (ab = E) & (D_{M02})(\text{Definition of } \emptyset) \end{aligned}$$

$$(a = \emptyset)(b = \emptyset) = (ab = E)$$

Resymbolising

$$(a \neq \emptyset).(a \neq \emptyset^1) \subseteq (a^1 \neq \emptyset^1)$$

(T_{M502})Proof of (T_{M502})

$$(a^1 = \emptyset^1) \subseteq (a = \emptyset) + (a = \emptyset^1)$$

(T_{M497})

$$\sim((a = \emptyset) + (a = \emptyset^1)) \subseteq (a^1 \neq \emptyset^1)$$

(T_{M012})(Transposition)

$$(a \neq \emptyset).(a \neq \emptyset^1) \subseteq (a^1 \neq \emptyset^1)$$

(T_{M026})(DeMorgan's Law)

$$(a^1 \neq \emptyset^1).(a^1 \neq \emptyset) \subseteq (a \neq \emptyset)$$

(T_{M501})Proof of (T_{M501})

$$(a = \emptyset) \subseteq (a^1 = \emptyset) + (a^1 = \emptyset^1)$$

(T_{M496})

$$\sim((a^1 = \emptyset) + (a^1 = \emptyset^1)) \subseteq (a \neq \emptyset)$$

(T_{M012})(Transposition)

$$(a^1 \neq \emptyset^1).(a^1 \neq \emptyset) \subseteq (a \neq \emptyset)$$

(T_{M026})(DeMorgan's Law)

$$(a \neq \emptyset).(a \neq \emptyset^1) = (a^1 \neq \emptyset).(a^1 \neq \emptyset^1)$$

(T_{M618})Proof of (T_{M618})

$$(a \neq \emptyset).(a \neq \emptyset^1) \subseteq (a^1 \neq \emptyset^1)$$

(T_{M502})(Eqn1)

$$(a \neq \emptyset) \subseteq (a^1 \neq \emptyset)$$

(T_{M503})(Eqn2)

$$(a \neq \emptyset).(a \neq \emptyset).(a \neq \emptyset^1) \subseteq (a^1 \neq \emptyset).(a^1 \neq \emptyset^1)$$

(T_{M669})(Combination of Eqns1&2)

$$(a \neq \emptyset).(a \neq \emptyset^1) \subseteq (a^1 \neq \emptyset).(a^1 \neq \emptyset^1)$$

(T_{M667})(Conjunction Tautology)(Eqn3)

$$(a^1 \neq \emptyset^1).(a^1 \neq \emptyset) \subseteq (a \neq \emptyset)$$

(T_{M501})

$$(a^1 \neq \emptyset^1).(a^1 \neq \emptyset)(a \neq \emptyset^1) \subseteq (a \neq \emptyset)(a \neq \emptyset^1)$$

(T_{M029})(Deduction)(Eqn4)

$$(a = \emptyset^1) \subseteq (a^1 = \emptyset^1)$$

(T_{M499})

$$(a^1 \neq \emptyset^1) \subseteq (a \neq \emptyset^1)$$

(T_{M012})(Transposition)

$$(a^1 \neq \emptyset^1) = (a^1 \neq \emptyset^1).(a \neq \emptyset^1)$$

(A_{MX44})($\subseteq$ Equality Form)(Eqn5)

$$(a^1 \neq \emptyset^1).(a^1 \neq \emptyset) \subseteq (a \neq \emptyset)(a \neq \emptyset^1)$$

(A_{MX19})(Substitution of Eqn5 into Eqn4)(Eqn6)

$$(a \neq \emptyset).(a \neq \emptyset^1) = (a^1 \neq \emptyset).(a^1 \neq \emptyset^1)$$

(T_{M430})(Biconditional Introduction Eqns3&6)

$$(a^1 \neq E^1) \subseteq (a \neq E)$$

(T_{M514})Proof of (T_{M514})

$$(a = E) \subseteq (a^1 = E^1)$$

(T_{M488})

$$(a^1 \neq E^1) \subseteq (a \neq E)$$

(T_{M012})(Transposition)

$$(a^1 + b^1 = E) \subseteq (b^1 \neq \emptyset)$$

(T_{M576})Proof of (T_{M576})

$$(a^1 \neq E)(b^1 = \emptyset) \subseteq (a^1 + b^1 \neq E)$$

(T_{M311})(Probabilistic Conjunction)

$$(a^1 + b^1 = E) \subseteq \sim((a^1 \neq E)(b^1 = \emptyset))$$

(T_{M012})(Transposition)

$$(a^1 + b^1 = E) \subseteq (a^1 = E) + (b^1 \neq \emptyset)$$

(T_{M025})(DeMorgan's Law)(Eqn1)

$$a^1 \neq E$$

(T_{M575})(Plurality of Universal Set)

$$(a^1 \neq E) = E$$

(A_{MX16})(Order Axiom)

$$(a^1 = E) = \sim E$$

(T_{M008})(Inverse of an Equality)

$$(a^1 = E) = \emptyset$$

(D_{M02})(Definition of $\emptyset$)(Eqn2)

$$(a^1 + b^1 = E) \subseteq \emptyset + (b^1 \neq \emptyset)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(a^1 + b^1 = E) \subseteq (b^1 \neq \emptyset)$$

(T_{M668})(Null Set Disjunction)

$$a \subseteq a + b \quad (T_{M082}) \text{ Disjunction Introduction}$$

Proof of (T_{M082})

$$(a + b \subseteq c) \subseteq (a \subseteq c)$$

(T_{M072})

$$(a + b \subseteq c) + (a + b = c) \subseteq (a \subseteq c)$$

(T_{M266})(Definition of $\subseteq$)

$$(a + b = c) \subseteq (a \subseteq c)$$

(T_{M072})

$$\begin{array}{ll}
 (a + b = c) = (a \subseteq c)(a + b = c) & (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn1}) \\
 \text{Let } a + b = c & (\text{Eqn2}) \\
 (a + b = a + b) = (a \subseteq a + b)(a + b = a + b) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
 E = (a \subseteq a + b)E & (T_{M007})(\text{Universality of } (D_{M00})) \\
 E = (a \subseteq a + b) & (A_{MX17})(\text{Universal Set Conjunction Axiom}) \\
 a \subseteq a + b & (A_{MX16})(\text{Order Axiom})
 \end{array}$$

$$(a \subseteq \emptyset^1) = (a = \emptyset) + (a = \emptyset^1) \quad (T_{M598})$$

Proof of (T_{M598})

$$\begin{array}{ll}
 (a \subseteq \emptyset^1) = (a \subseteq \emptyset^1) + (a = \emptyset^1) & (T_{M266})(\text{Definition of } \subseteq)(\text{Eqn1}) \\
 (a \subseteq \emptyset^1) = (a = \emptyset) & (T_{M717})(\text{Eqn2}) \\
 (a \subseteq \emptyset^1) = (a = \emptyset) + (a = \emptyset^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
 \end{array}$$

$$ab \subseteq a + b \quad (T_{M146}) \text{ Conjunction a Subset of Disjunction}$$

Proof of (T_{M146})

$$\begin{array}{ll}
 ab \subseteq a & (T_{M083})(\text{Conjunction Elimination})(\text{Eqn1}) \\
 a \subseteq a + b & (T_{M082})(\text{Disjunction Introduction})(\text{Eqn2}) \\
 ab \subseteq a + b & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})
 \end{array}$$

$$(x = a, b) \subseteq (x = a, b, c) \quad (T_{M753})$$

Proof of (T_{M753})

$$\begin{array}{ll}
 (x = a) + (x = b) \subseteq (x = a) + (x = b) + (x = c) & (T_{M082})(\text{Disjunction Introduction}) \\
 (x = a, b) \subseteq (x = a, b, c) & (D_{M15})(\text{Definition of a Variable Set})
 \end{array}$$

$$(a \neq b) = (a^1 \neq b^1) \quad (T_{M480})$$

Proof of (T_{M480})

$$\begin{array}{ll}
 (a^1 = b^1) = (a = b) & (T_{M479}) \\
 (a \neq b) = (a^1 \neq b^1) & (T_{M008})(\text{Inverse of an Equality})
 \end{array}$$

$$\emptyset^1 \neq E^1 \quad (T_{M784})$$

Proof of (T_{M784})

$$\begin{array}{ll}
 \emptyset \neq E & (T_{M352}) \\
 \emptyset^1 \neq E^1 & (T_{M480})
 \end{array}$$

$$(a^1 = b^1) = (a^1 = b) \quad (T_{M479b})$$

Proof of (T_{M479b})

$$\begin{array}{ll}
 (a^1 = b^1) = (a = b) & (T_{M479})(\text{Eqn1}) \\
 \text{Let } a = a^1 & (\text{Eqn2}) \\
 (a^{11} = b^1) = (a^1 = b) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
 (a^1 = b^1) = (a^1 = b) & (D_{M17})(\text{Definition of } a^{11})
 \end{array}$$

$$(a + b = \emptyset) \subseteq (a = \emptyset) \quad (T_{M086})$$

Proof of (T_{M086})

$$\begin{array}{ll}
 a \subseteq a + b & (T_{M082})(\text{Disjunction Introduction}) \\
 (a \subseteq a + b) = E & (A_{MX16})(\text{Order Axiom}) \\
 (a + b = \emptyset) \subseteq (a = \emptyset) & (T_{M097})(\text{Probabilistic Rule})
 \end{array}$$

$$((ac \subseteq bc) = E) \subseteq ((c \subseteq a) \subseteq (c \subseteq b)) \quad (T_{M234}) \text{ Conditional Subsets Rule}$$

Proof of (T_{M234})

$((a \subseteq b) = E) \subseteq ((c \subseteq a) \subseteq (c \subseteq b))$	(T_{M272}) (Deductive Rule)(Eqn1)
Let $a = ac$	(Eqn2)
Let $b = bc$	(Eqn3)
$((ac \subseteq bc) = E) \subseteq ((c \subseteq ac) \subseteq (c \subseteq bc))$	(A_{MX19}) (Substitution of Eqns2&3 into Eqn1)
$((ac \subseteq bc) = E) \subseteq ((c \subseteq a)(c \subseteq c) \subseteq (c \subseteq b)(c \subseteq c))$	(T_{M132}) (Composition)(Eqn4)
$c = c$	(D_{M00}) (Definition of $=$)
$c \subseteq c$	(T_{M425}) (Biconditional Elimination)
$(c \subseteq c) = E$	(A_{MX16}) (Order Axiom)(Eqn5)
$((ac \subseteq bc) = E) \subseteq ((c \subseteq a)E \subseteq (c \subseteq b)E)$	(A_{MX19}) (Substitution of Eqn5 into Eqn4)
$((ac \subseteq bc) = E) \subseteq ((c \subseteq a) \subseteq (c \subseteq b))$	(A_{MX11}) (Universal Set Conjunction Axiom)

$$((ac = bc) = E) \subseteq ((c \subseteq a) = (c \subseteq b)) \quad (T_{M234b}) \text{ Conditional Subsets Rule}$$

Proof of (T_{M234b})

$((ac \subseteq bc) = E) \subseteq ((c \subseteq a) \subseteq (c \subseteq b))$	(T_{M234}) (Conditional Subsets Rule)(Eqn1)
$((bc \subseteq ac) = E) \subseteq ((c \subseteq b) \subseteq (c \subseteq a))$	Resymbolising (Eqn2)
$((ac \subseteq bc) = E)((bc \subseteq ac) = E) \subseteq ((c \subseteq a) \subseteq (c \subseteq b))((c \subseteq b) \subseteq (c \subseteq a))$	(T_{M669}) (Combination of Eqns 1 & 2)
$((ac \subseteq bc)(bc \subseteq ac) = E) \subseteq ((c \subseteq a) \subseteq (c \subseteq b))((c \subseteq b) \subseteq (c \subseteq a))$	(T_{M108}) (Conjunction Introduction)
$((ac = bc) = E) \subseteq ((c \subseteq a) \subseteq (c \subseteq b))((c \subseteq b) \subseteq (c \subseteq a))$	(T_{M430}) (Biconditional Introduction)
$((ac = bc) = E) \subseteq ((c \subseteq a) = (c \subseteq b))$	(T_{M430}) (Biconditional Introduction)

$$(a \subset b) \subseteq (a \subseteq b) \quad (T_{M592})$$

Proof of (T_{M592})

$a \subseteq a + b$	(T_{M082}) (Disjunction Introduction)(Eqn1)
$(a \subset b) \subseteq (a \subset b) + (a = b)$	Resymbolising
$(a \subset b) \subseteq (a \subseteq b)$	(T_{M266}) (Definition of $\subseteq$)

$$ac + bd \subseteq (a + b)(c + d) \quad (T_{M522})$$

Proof of (T_{M522})

$(a + b)(c + d) = (a + b)(c + d)$	(D_{M00}) (Definition of $=$)
$(a + b).c + (a + b).d = (a + b)(c + d)$	(T_{M407}) (Distribution of $.$ over $+$)
$ac + bc + ad + bd = (a + b)(c + d)$	(T_{M407}) (Distribution of $.$ over $+$)(Eqn1)
$ac + bd \subseteq ac + bc + ad + bd$	(T_{M082}) (Disjunction Introduction)(Eqn2)
$ac + bd \subseteq (a + b)(c + d)$	(A_{MX19}) (Substitution of Eqn1 into Eqn2)

$$(a \subseteq c)(b \subseteq c) = (c \subseteq ab) \quad (T_{M131}) \text{ Conjunction of Sufficiencies}$$

Proof of (T_{M131})

$(a \subseteq c)(b \subseteq c) = (a + b \subseteq c)$	(T_{M131b}) (Conjunction of Sufficiencies)
$(a \subseteq c)(b \subseteq c) = (c \subseteq \sim(a + b))$	(T_{M012}) (Transposition)
$(a \subseteq c)(b \subseteq c) = (c \subseteq ab)$	(T_{M026}) (DeMorgan's Law)

$$((a \subseteq c) = E) \subseteq ((b \subseteq c) = (a + b \subseteq c)) \quad (T_{M422}) \text{ Disjunction Elimination}$$

Proof of (T_{M422})

$(a \subseteq c)(b \subseteq c) = (a + b \subseteq c)$	(T_{M131b}) (Conjunction of Sufficiencies)
$((a \subseteq c) = E) \subseteq ((b \subseteq c) = (a + b \subseteq c))$	(T_{M444c}) (Biconditional Exportation)

$$(a + b = E)(a \subseteq c)(b \subseteq c) \subseteq (c = E) \quad (T_{M429}) \text{ Disjunction Elimination}$$

Proof of (T_{M429})

Let $(a + b = E)(a \subseteq c)(b \subseteq c)$	
$(a + b = E)(a + b \subseteq c)$	(T_{M131b}) (Conjunction of Sufficiencies)

$(E \subseteq c)$	(A_{MX19}) (Substitution)
$(c = E)$	(T_{M275}) (Supersets of the Universal Set)
Thus $(a + b = E)(a \subseteq c)(b \subseteq c) \subseteq (c = E)$	(A_{MX02}) (Conditional Introduction)

$$(a = E) + (b = E) \subseteq (a + b = E) \quad (T_{M160}) \text{ Probabilistic Disjunction}$$

Proof of (T_{M160})

$(a = E) \subseteq (a + b = E)$	(T_{M159}) (Eqn1)
$(b = E) \subseteq (a + b = E)$	Resymbolising (Eqn2)
$(a = E) + (b = E) \subseteq (a + b = E)$	(T_{M131b}) (Conjunction of Sufficiencies Eqns1&2)

$$(a = \emptyset) + (b = \emptyset) \subseteq (ab = \emptyset) \quad (T_{M161}) \text{ Probabilistic Disjunction}$$

Proof of (T_{M161})

$(a = E) + (b = E) \subseteq (a + b = E)$	(T_{M160}) (Probabilistic Disjunction)
$(\underline{a} = E) + (\underline{b} = E) \subseteq (\underline{a} + \underline{b} = E)$	Resymbolising
$(a = \emptyset) + (b = \emptyset) \subseteq (\sim(\underline{a} + \underline{b}) = E)$	(T_{M008}) (Inverse of an Equality)
$(a = \emptyset) + (b = \emptyset) \subseteq (\sim(\underline{a} + \underline{b}) = \emptyset)$	(D_{M02}) (Definition of $\emptyset$)
$(a = \emptyset) + (b = \emptyset) \subseteq (\sim \underline{a} \sim \underline{b} = \emptyset)$	(T_{M026}) (DeMorgan's Law)
$(a = \emptyset) + (b = \emptyset) \subseteq (ab = \emptyset)$	(A_{MX13}) (Double Complement Axiom)

$$(a = E) + (b = E) \subseteq (\underline{ab} = \emptyset) \quad (T_{M378}) \text{ Probabilistic Disjunction}$$

Proof of (T_{M378})

$(a = \emptyset) + (b = \emptyset) \subseteq (ab = \emptyset)$	(T_{M161}) (Probabilistic Disjunction)
$(\underline{a} = \emptyset) + (\underline{b} = \emptyset) \subseteq (ab = \emptyset)$	(T_{M008}) (Inverse of an Equality)
$(\underline{a} = E) + (\underline{b} = E) \subseteq (ab = \emptyset)$	(T_{M054}) (Negative of Null Set)
$(a = E) + (b = E) \subseteq (\underline{ab} = \emptyset)$	Resymbolising

$$(a = E) + (b = \emptyset) \subseteq (\underline{ab} = \emptyset) \quad (T_{M381}) \text{ Probabilistic Disjunction}$$

Proof of (T_{M381})

$(a = \emptyset) + (b = \emptyset) \subseteq (ab = \emptyset)$	(T_{M161}) (Probabilistic Disjunction)
$(\underline{a} = \emptyset) + (\underline{b} = \emptyset) \subseteq (ab = \emptyset)$	(T_{M008}) (Inverse of an Equality)
$(\underline{a} = E) + (\underline{b} = \emptyset) \subseteq (ab = \emptyset)$	(T_{M054}) (Negative of Null Set)
$(a = E) + (b = \emptyset) \subseteq (\underline{ab} = \emptyset)$	Resymbolising

$$(ab \neq \emptyset) \subseteq (a \neq \emptyset)(b \neq \emptyset) \quad (T_{M242})$$

Proof of (T_{M242})

$(a = \emptyset) + (b = \emptyset) \subseteq (ab = \emptyset)$	(T_{M161}) (Probabilistic Disjunction)
$\sim(ab = \emptyset) \subseteq \sim((a = \emptyset) + (b = \emptyset))$	(T_{M008}) (Inverse of an Equality)
$(ab \neq \emptyset) \subseteq \sim((a = \emptyset) + (b = \emptyset))$	(T_{M260}) (Definition of $\neq$)
$(ab \neq \emptyset) \subseteq (a \neq \emptyset)(b \neq \emptyset)$	(T_{M026}) (DeMorgan's Law)

$$(c^1 \subseteq a).(c^1 \subseteq b).(c^1 \neq \emptyset) \subseteq (ab \neq \emptyset) \quad (T_{M410}) \text{ Existential Quantifier Introduction}$$

Proof of (T_{M410})

Let $(c^1 \subseteq a).(c^1 \subseteq b).(c^1 \neq \emptyset)$	
$(c^1 \subseteq ab).(c^1 \neq \emptyset)$	(T_{M132}) (Composition)
$(c^1 = c^1 ab).(c^1 \neq \emptyset)$	(A_{MX44}) ($\subseteq$ Equality Form)
$(c^1 ab \neq \emptyset)$	(A_{MX19}) (Substitution)
$(c^1 \neq \emptyset)(ab \neq \emptyset)$	(T_{M242})
Thus $(c^1 \subseteq a).(c^1 \subseteq b).(c^1 \neq \emptyset) \subseteq (c^1 \neq \emptyset)(ab \neq \emptyset)$	(A_{MX02}) (Conditional Introduction)
$(c^1 \subseteq a).(c^1 \subseteq b).(c^1 \neq \emptyset) \subseteq (ab \neq \emptyset)$	(T_{M073}) (Single Premise Syllogism)

$$(a = E) + (b \neq \emptyset) \subseteq (a + b \neq \emptyset)$$

(T_{M164}) Probabilistic Disjunction

Proof of (T_{M164})

$$(a \neq \emptyset) \subseteq (a + b \neq \emptyset)$$

(T_{M086})(T_{M012})(Eqn1)

$$(b \neq \emptyset) \subseteq (a + b \neq \emptyset)$$

Resymbolising (Eqn2)

$$(a = E) \subseteq (a \neq \emptyset)$$

(T_{M058})(Eqn3)

$$(a = E) \subseteq (a + b \neq \emptyset)$$

(T_{M139})(Hypothetical Syllogism Eqns1&3)(Eqn4)

$$(b \neq \emptyset) + (a = E) \subseteq (a + b \neq \emptyset)$$

(T_{M131b})(Conjunction of Sufficiencies Eqns2&4)

$$(a = E) + (b \neq E) \subseteq (a + \underline{b} \neq \emptyset)$$

(T_{M385}) Probabilistic Disjunction

Proof of (T_{M385})

$$(a = E) + (b \neq \emptyset) \subseteq (a + b \neq \emptyset)$$

(T_{M164})(Probabilistic Disjunction)

$$(a = E) + (\underline{b} \neq \emptyset) \subseteq (a + \underline{b} \neq \emptyset)$$

Resymbolising

$$(a = E) + (b \neq \emptyset) \subseteq (a + \underline{b} \neq \emptyset)$$

(T_{M008})(T_{M008})

$$(a = E) + (b \neq E) \subseteq (a + \underline{b} \neq \emptyset)$$

(T_{M054})(Negative of Null Set)

$$(a \neq E) + (b = \emptyset) \subseteq (\underline{a} + \underline{b} \neq \emptyset)$$

(T_{M389}) Probabilistic Disjunction

Proof of (T_{M389})

$$(a = E) + (b \neq \emptyset) \subseteq (a + b \neq \emptyset)$$

(T_{M164})(Probabilistic Disjunction)

$$(\underline{b} = E) + (\underline{a} \neq \emptyset) \subseteq (\underline{b} + \underline{a} \neq \emptyset)$$

Resymbolising

$$(\underline{a} \neq \emptyset) + (\underline{b} = E) \subseteq (\underline{a} + \underline{b} \neq \emptyset)$$

(T_{M403})(Commutative +)

$$(\underline{a} \neq \emptyset) + (b = E) \subseteq (\underline{a} + \underline{b} \neq \emptyset)$$

(T_{M008})(Inverse of an Equality)

$$(a \neq \emptyset) + (b = E) \subseteq (\underline{a} + \underline{b} \neq \emptyset)$$

(T_{M008})(T_{M008})

$$(a \neq E) + (b = E) \subseteq (\underline{a} + \underline{b} \neq \emptyset)$$

(T_{M054})(Negative of Null Set)

$$(a \neq E) + (b = \emptyset) \subseteq (\underline{a} + \underline{b} \neq \emptyset)$$

(D_{M02})(Definition of $\emptyset$)

$$(a \neq \emptyset) + (b = \emptyset) \subseteq (a + \underline{b} \neq \emptyset)$$

(T_{M390}) Probabilistic Disjunction

Proof of (T_{M390})

$$(a \neq E) + (b = \emptyset) \subseteq (\underline{a} + \underline{b} \neq \emptyset)$$

(T_{M389})(Probabilistic Disjunction)

$$(\underline{a} \neq E) + (b = \emptyset) \subseteq (\underline{a} + \underline{b} \neq \emptyset)$$

Resymbolising

$$(a \neq E) + (b = \emptyset) \subseteq (\underline{a} + \underline{b} \neq \emptyset)$$

(T_{M008})(T_{M008})

$$(a \neq \emptyset) + (b = \emptyset) \subseteq (\underline{a} + \underline{b} \neq \emptyset)$$

(D_{M02})(Definition of $\emptyset$)

$$(a \neq E) + (b = \emptyset) \subseteq (ab \neq E)$$

(T_{M165}) Probabilistic Disjunction

Proof of (T_{M165})

$$(a \neq E) \subseteq (ab \neq E)$$

(T_{M084})(T_{M012})(Eqn1)

$$(b \neq E) \subseteq (ab \neq E)$$

Resymbolising (Eqn2)

$$(b = \emptyset) \subseteq (b \neq E)$$

(T_{M059})(Eqn3)

$$(b = \emptyset) \subseteq (ab \neq E)$$

(T_{M139})(Hypothetical Syllogism Eqns2&3) (Eqn4)

$$(a \neq E) + (b = \emptyset) \subseteq (ab \neq E)$$

(T_{M131b})(Conjunction of Sufficiencies Eqns1&4)

$$(a = E) + (b \neq E) \subseteq (\underline{ab} \neq E)$$

(T_{M379}) Probabilistic Disjunction

Proof of (T_{M379})

$$(a \neq E) + (b = \emptyset) \subseteq (ab \neq E)$$

(T_{M165})(Probabilistic Disjunction)

$$(b \neq E) + (\underline{a} = \emptyset) \subseteq (\underline{ba} \neq E)$$

Resymbolising

$$(\underline{a} = \emptyset) + (b \neq E) \subseteq (\underline{ba} \neq E)$$

(T_{M403})(Commutative +)

$$(\underline{a} = \emptyset) + (b \neq E) \subseteq (\underline{ab} \neq E)$$

(T_{M404})(Commutative .)

$$(a = \emptyset) + (b \neq E) \subseteq (\underline{ab} \neq E)$$

(T_{M008})(Inverse of an Equality)

$$(a = E) + (b \neq E) \subseteq (\underline{ab} \neq E)$$

(T_{M054})(Negative of Null Set)

$$(a = E) + (b \neq \emptyset) \subseteq (\underline{ab} \neq E)$$

(T_{M380}) Probabilistic Disjunction

Proof of (T_{M380})

$(a \neq E) + (b = \emptyset) \subseteq (ab \neq E)$	(T_{M165})(Probabilistic Disjunction)
$(\underline{b} \neq E) + (\underline{a} = \emptyset) \subseteq (\underline{ba} \neq E)$	Resymbolising
$(\underline{a} = \emptyset) + (\underline{b} \neq E) \subseteq (\underline{ba} \neq E)$	(T_{M403})(Commutative +)
$(\underline{a} = \emptyset) + (\underline{b} \neq E) \subseteq (\underline{ab} \neq E)$	(T_{M404})(Commutative .)
$(a = \emptyset) + (\underline{b} \neq E) \subseteq (\underline{ab} \neq E)$	(T_{M008})(Inverse of an Equality)
$(a = \emptyset) + (\underline{b} \neq E) \subseteq (\underline{ab} \neq E)$	(T_{M008})(T_{M008})
$(a = E) + (\underline{b} \neq E) \subseteq (\underline{ab} \neq E)$	(T_{M054})(Negative of Null Set)
$(a = E) + (\underline{b} \neq \emptyset) \subseteq (\underline{ab} \neq E)$	(D_{M02})(Definition of $\emptyset$)

$$(a \neq \emptyset) + (b = \emptyset) \subseteq (\underline{ab} \neq E) \quad (T_{M384}) \text{ Probabilistic Disjunction}$$

Proof of (T_{M384})

$(a \neq E) + (b = \emptyset) \subseteq (ab \neq E)$	(T_{M165})(Probabilistic Disjunction)
$(\underline{a} \neq E) + (b = \emptyset) = (\underline{ab} \neq E)$	Resymbolising
$(a \neq E) + (b = \emptyset) = (\underline{ab} \neq E)$	(T_{M008})(T_{M008})
$(a \neq \emptyset) + (b = \emptyset) = (\underline{ab} \neq E)$	(D_{M02})(Definition of $\emptyset$)

$$(a = E) + (b = \emptyset) \subseteq (a + \underline{b} = E) \quad (T_{M386}) \text{ Probabilistic Disjunction}$$

Proof of (T_{M386})

$(a = E) + (b = E) \subseteq (a + b = E)$	(T_{M160})(Probabilistic Disjunction)
$(a = E) + (\underline{b} = E) \subseteq (a + \underline{b} = E)$	Resymbolising
$(a = E) + (b = \underline{E}) \subseteq (a + \underline{b} = E)$	(T_{M008})(Inverse of an Equality)
$(a = E) + (b = \emptyset) \subseteq (a + \underline{b} = E)$	(D_{M02})(Definition of $\emptyset$)

$$(a = \emptyset) + (b = \emptyset) \subseteq (\underline{a} + \underline{b} = E) \quad (T_{M391}) \text{ Probabilistic Disjunction}$$

Proof of (T_{M391})

$(a = E) + (b = E) \subseteq (a + b = E)$	(T_{M160})(Probabilistic Disjunction)
$(\underline{a} = E) + (\underline{b} = E) \subseteq (\underline{a} + \underline{b} = E)$	Resymbolising
$(a = \underline{E}) + (b = \underline{E}) \subseteq (\underline{a} + \underline{b} = E)$	(T_{M008})(Inverse of an Equality)
$(a = \emptyset) + (b = \emptyset) \subseteq (\underline{a} + \underline{b} = E)$	(D_{M02})(Definition of $\emptyset$)

$$(a + b \neq E) \subseteq (a \neq E)(b \neq E) \quad (T_{M240})$$

Proof of (T_{M240})

$(a = E) + (b = E) \subseteq (a + b = E)$	(T_{M160})(Probabilistic Disjunction)
$\sim(a + b = E) \subseteq \sim((a = E) + (b = E))$	(T_{M008})(Inverse of an Equality)
$(a + b \neq E) \subseteq \sim((a = E) + (b = E))$	(T_{M260})(Definition of $\neq$)
$(a + b \neq E) \subseteq (a \neq E)(b \neq E)$	(T_{M026})(DeMorgans Law)

$$(a \subseteq c) + (b \subseteq c) \subseteq (ab \subseteq c) \quad (T_{M169}) \text{ Disjunction of Sufficiencies}$$

Proof of (T_{M169})

Let $a \subseteq c$	(Eqn1)
$ab \subseteq a$	(T_{M083})(Conjunction Elimination)(Eqn2)
$ab \subseteq c$	(T_{M139})(Hypothetical Syllogism Eqns1&2)
Thus $(a \subseteq c) \subseteq (ab \subseteq c)$	(A_{MX02})(Conditional Introduction)(Eqn3)
Let $b \subseteq c$	(Eqn4)
$ab \subseteq b$	(T_{M083})(Conjunction Elimination)(Eqn5)
$ab \subseteq c$	(T_{M139})(Hypothetical Syllogism Eqns4&5)
Thus $(b \subseteq c) \subseteq (ab \subseteq c)$	(A_{MX02})(Conditional Introduction)(Eqn6)
$(a \subseteq c) + (b \subseteq c) \subseteq (ab \subseteq c) + (ab \subseteq c)$	(A_{MX21})(Combination of Eqns3&6)
$(a \subseteq c) + (b \subseteq c) \subseteq (ab \subseteq c)$	(A_{MX01})(Disjunction Tautology)

$$(a \subseteq c) + (b \subseteq c) \subseteq (c \subseteq a + b) \quad (T_{M369}) \text{ Disjunction of Sufficiencies}$$

Proof of (T_{M369})

$$\begin{aligned} (a \subseteq c) + (b \subseteq c) &\subseteq (ab \subseteq c) & (T_{M169}) & \text{(Disjunction of Sufficiencies)} \\ (a \subseteq c) + (b \subseteq c) &\subseteq (c \subseteq \sim(ab)) & (T_{M012}) & \text{(Transposition)} \\ (a \subseteq c) + (b \subseteq c) &\subseteq (c \subseteq a + b) & (T_{M025}) & \text{(DeMorgan's Law)} \end{aligned}$$

$$(c \subseteq a) + (c \subseteq b) \subseteq (c \subseteq a + b) \quad (T_{M170}) \text{ Disjunction of Necessities}$$

Proof of (T_{M170})

$$\begin{aligned} \text{Let } c &\subseteq a & (\text{Eqn1}) \\ a &\subseteq a + b & (T_{M082}) & \text{(Disjunction Introduction)(Eqn2)} \\ c &\subseteq a + b & (T_{M139}) & \text{(Hypothetical Syllogism Eqns1\&2)} \\ \text{Thus } (c &\subseteq a) &\subseteq (c \subseteq a + b) & (A_{MX02}) & \text{(Conditional Introduction)(Eqn3)} \\ \text{Let } c &\subseteq b & (\text{Eqn4}) \\ b &\subseteq a + b & (T_{M082}) & \text{(Disjunction Introduction)(Eqn5)} \\ c &\subseteq a + b & (T_{M139}) & \text{(Hypothetical Syllogism Eqns4\&5)} \\ \text{Thus } (c &\subseteq b) &\subseteq (c \subseteq a + b) & (A_{MX02}) & \text{(Conditional Introduction)(Eqn6)} \\ (c \subseteq a) + (c &\subseteq b) &\subseteq (c \subseteq a + b) + (c \subseteq a + b) & (A_{MX21}) & \text{(Combination of Eqns3\&6)} \\ (c \subseteq a) + (c &\subseteq b) &\subseteq (c \subseteq a + b) & (A_{MX01}) & \text{(Disjunction Tautology)} \end{aligned}$$

$$(c \subseteq a) + (c \subseteq b) \subseteq (\underline{ab} \subseteq c) \quad (T_{M370}) \text{ Disjunction of Necessities}$$

Proof of (T_{M370})

$$\begin{aligned} (c \subseteq a) + (c \subseteq b) &\subseteq (c \subseteq a + b) & (T_{M170}) & \text{(Disjunction of Necessities)} \\ (c \subseteq a) + (c \subseteq b) &\subseteq (\sim(a + b) \subseteq c) & (T_{M012}) & \text{(Transposition)} \\ (c \subseteq a) + (c \subseteq b) &\subseteq (\underline{ab} \subseteq c) & (T_{M026}) & \text{(DeMorgan's Law)} \end{aligned}$$

$$(a \not\subseteq c) + (b \not\subseteq c) = (c \not\subseteq \underline{ab}) \quad (T_{M171}) \text{ Disjunction of Insufficiencies}$$

Proof of (T_{M171})

$$\begin{aligned} (a \subseteq c)(b \subseteq c) &= (c \subseteq \underline{ab}) & (T_{M131}) & \text{(Conjunction of Sufficiencies)} \\ \sim((a \subseteq c)(b \subseteq c)) &= \sim(c \subseteq \underline{ab}) & (T_{M008}) & \text{(Inverse of an Equality)} \\ \sim(a \subseteq c) + \sim(b \subseteq c) &= \sim(c \subseteq \underline{ab}) & (T_{M026}) & \text{(DeMorgan's Law)} \\ (a \not\subseteq c) + (b \not\subseteq c) &= (c \not\subseteq \underline{ab}) & (T_{M259}) & \text{(Definition of } \not\subseteq) \end{aligned}$$

$$(a \not\subseteq c) + (b \not\subseteq c) = (a + b \not\subseteq c) \quad (T_{M172}) \text{ Disjunction of Insufficiencies}$$

Proof of (T_{M172})

$$\begin{aligned} (a \not\subseteq c) + (b \not\subseteq c) &= (c \not\subseteq \underline{ab}) & (T_{M171}) & \text{(Disjunction of Insufficiencies)} \\ (a \not\subseteq c) + (b \not\subseteq c) &= (\sim(\underline{ab}) \not\subseteq c) & (T_{M012}) & \text{(Transposition)} \\ (a \not\subseteq c) + (b \not\subseteq c) &= (a + b \not\subseteq c) & (T_{M025}) & \text{(DeMorgan's Law)} \end{aligned}$$

$$(a = b) = (E = ab + \underline{ab}) \quad (T_{M068}) \text{ Material Equivalence}$$

Proof of (T_{M068})

$$\begin{aligned} (a = b) &= (\underline{ab} = \emptyset)(\underline{ab} = \emptyset) & (T_{M717}) & \text{(Biconditional Theorem)} \\ (a = b) &= (\underline{ab} + \underline{ab} = \emptyset) & (T_{M071}) & \text{(Probabilistic Conjunction)} \\ (a = b) &= (\sim(\underline{ab} + \underline{ab}) = \sim\emptyset) & (T_{M008}) & \text{(Inverse of an Equality)} \\ (a = b) &= (\sim(\underline{ab} + \underline{ab}) = E) & (T_{M054}) & \text{(Negative of Null Set)} \\ (a = b) &= (\sim(\underline{ab}).\sim(\underline{ab}) = E) & (T_{M026}) & \text{(DeMorgan's Law)} \\ (a = b) &= ((\underline{a} + \underline{b}).(\underline{a} + \underline{b}) = E) & (T_{M025}) & \text{(DeMorgan's Law)} \\ (a = b) &= (\underline{aa} + \underline{ab} + \underline{ab} + \underline{bb} = E) & (T_{M407}) & \text{(Distribution of } \cdot \text{ over } +) \\ (a = b) &= (\emptyset + \underline{ab} + \underline{ab} + \emptyset = E) & (T_{M621}) & \text{(Complement Conjunction)} \\ (a = b) &= (\underline{ab} + \underline{ab} = E) & (T_{M668}) & \text{(Null Set Disjunction)} \\ (a = b) &= (E = \underline{ab} + \underline{ab}) & (T_{M197}) & \text{(Commutativity of } =) \end{aligned}$$

$$(a = b + c) = (E = \underline{ab} + \underline{ac} + \underline{abc}) \quad (T_{M070})$$

Proof of (T_{M070})

Let $a = b + c$

$$E = a(b + c) + \underline{a} \cdot \sim(b + c) \quad (T_{M068})(\text{Material Equivalence})$$

$$E = a(b + c) + \underline{abc} \quad (T_{M026})(\text{DeMorgan's Law})$$

$$E = ab + ac + \underline{abc} \quad (T_{M407})(\text{Distribution of } \cdot \text{ over } +)$$

$$(a = b)(a = \underline{b}) = \emptyset \quad (T_{M212}) \text{ Conjunction of Membership Relationship Categories}$$

Proof of (T_{M212})

$$(b = \underline{b}) = \emptyset \quad (T_{M535})(\text{Eqn1})$$

Let $a = b$ (Eqn2)

$$(a = \underline{b}) = \emptyset \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$\text{Thus } ((a = b) = E) \subseteq ((a = \underline{b}) = \emptyset) \quad (A_{MX02})(\text{Conditional Introduction})$$

$$(a = b)(a = \underline{b}) \subseteq \emptyset \quad (T_{M444d})(\text{Biconditional Importation}) \text{ for truth sets}$$

$$(a = b)(a = \underline{b}) = \emptyset \quad (T_{M052})(\text{Subsets of the Null Set})$$

$$((a \neq \emptyset) + (b \neq E) = E) \subseteq ((a \subset b)(a = \underline{b}) = \emptyset) \quad (T_{M208})$$

Proof of (T_{M208})

$$(a \subseteq \underline{a}) = (a = \emptyset) \quad (T_{M057})$$

$$(a \subset \underline{a}) + (a = \underline{a}) = (a = \emptyset) \quad (T_{M266})(\text{Definition of } \subseteq)$$

$$(a \subset \underline{a}) + \emptyset = (a = \emptyset) \quad (T_{M535})$$

$$(a \subset \underline{a}) = (a = \emptyset) \quad (T_{M668})(\text{Null Set Disjunction})(\text{Eqn1})$$

$$(\underline{b} \subseteq b) = (b = E) \quad (T_{M057b})$$

$$(\underline{b} \subset b) + (\underline{b} = b) = (b = E) \quad (T_{M266})(\text{Definition of } \subseteq)$$

$$(\underline{b} \subset b) + \emptyset = (b = E) \quad (T_{M535})$$

$$(\underline{b} \subset b) = (b = E) \quad (T_{M668})(\text{Null Set Disjunction})(\text{Eqn2})$$

$$(a \subset \underline{a})(\underline{b} \subset b) = (a = \emptyset)(b = E) \quad (T_{M231})(\text{Combination of Eqns1\&2})(\text{Eqn3})$$

Let $a = \underline{b}$ (Eqn4)

$$(a \subset b)(a \subset b) = (a = \emptyset)(b = E) \quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})$$

$$(a \subset b) = (a = \emptyset)(b = E) \quad (T_{M667})(\text{Conjunction Tautology})$$

$$\text{Thus } ((a = \underline{b}) = E) \subseteq ((a \subset b) = (a = \emptyset)(b = E)) \quad (A_{MX02})(\text{Conditional Introduction})$$

$$(a \subset b)(a = \underline{b}) \subseteq (a = \emptyset)(b = E) \quad (T_{M444d})(\text{Biconditional Importation}) \text{ for truth sets}$$

$$\sim((a = \emptyset)(b = E)) \subseteq \sim((a \subset b)(a = \underline{b})) \quad (T_{M012})(\text{Transposition})$$

$$(\sim((a = \emptyset)(b = E))) \subseteq (\sim((a \subset b)(a = \underline{b}))) = E \quad (A_{MX16})(\text{Order Axiom})$$

$$(\sim((a = \emptyset)(b = E))) = E \subseteq (\sim((a \subset b)(a = \underline{b}))) = E \quad (T_{M094})(\text{Probabilistic Rule})$$

$$(\sim((a = \emptyset)(b = E))) = E \subseteq ((a \subset b)(a = \underline{b}) = \sim E) \quad (T_{M008})(\text{Inverse of an Equality})$$

$$(\sim((a = \emptyset)(b = E))) = E \subseteq ((a \subset b)(a = \underline{b}) = \emptyset) \quad (D_{M02})(\text{Definition of } \emptyset)$$

$$((a \neq \emptyset) + (b \neq E) = E) \subseteq ((a \subset b)(a = \underline{b}) = \emptyset) \quad (T_{M025})(\text{DeMorgan's Law})$$

$$((a \neq \emptyset) + (b \neq \emptyset) = E) \subseteq ((a = b)(a \subset \underline{b}) = \emptyset) \quad (T_{M211a})$$

Proof of (T_{M211a})

$$((a \neq \emptyset) + (b \neq E) = E) \subseteq ((a \subset b)(a = \underline{b}) = \emptyset) \quad (T_{M208})$$

$$((a \neq \emptyset) + (\underline{b} \neq E) = E) \subseteq ((a \subset \underline{b})(a = b) = \emptyset) \quad \text{Resymbolising}$$

$$((a \neq \emptyset) + (b \neq \underline{E}) = E) \subseteq ((a \subset \underline{b})(a = b) = \emptyset) \quad (T_{M008})(T_{M008})$$

$$((a \neq \emptyset) + (b \neq \emptyset) = E) \subseteq ((a \subset \underline{b})(a = b) = \emptyset) \quad (D_{M02})(\text{Definition of } \emptyset)$$

$$((a \neq E) + (b \neq E) = E) \subseteq ((a \supset \underline{b})(a = b) = \emptyset) \quad (T_{M213})$$

Proof of (T_{M213})

$$((a \neq \emptyset) + (b \neq E) = E) \subseteq ((a \subset b)(a = \underline{b}) = \emptyset) \quad (T_{M208})$$

$$((\underline{a} \neq \emptyset) + (b \neq E) = E) \subseteq ((\underline{a} \subset b)(\underline{a} = \underline{b}) = \emptyset) \quad \text{Resymbolising}$$

$$((\underline{a} \neq \emptyset) + (b \neq E) = E) \subseteq ((\underline{a} \subset b)(a = b) = \emptyset) \quad (T_{M008})(\text{Inverse of an Equality})$$

$$((\underline{a} \neq \emptyset) + (b \neq E) = E) \subseteq ((\underline{b} \subset a)(a = b) = \emptyset) \quad (T_{M010})(\text{Inverse of a Proper Subset})$$

$$((\underline{a} \neq \emptyset) + (b \neq E) = E) \subseteq ((a \supset \underline{b})(a = b) = \emptyset) \quad (T_{M195})(\supset \text{ Reverse of } \supset)$$

$$\begin{aligned} ((a \neq \emptyset) + (b \neq E) = E) &\subseteq ((a \supset \underline{b})(a = \underline{b}) = \emptyset) & (T_{M008})(T_{M008}) \\ ((a \neq E) + (b \neq E) = E) &\subseteq ((a \supset \underline{b})(a = \underline{b}) = \emptyset) & (T_{M054})(\text{Negative of Null Set}) \end{aligned}$$

$$((b \neq \emptyset) + (a \neq E) = E) \subseteq ((a \supset b)(a = \underline{b}) = \emptyset) \quad (T_{M216})$$

Proof of (T_{M216})

$$\begin{aligned} ((a \neq \emptyset) + (b \neq E) = E) &\subseteq ((a \subset b)(a = \underline{b}) = \emptyset) & (T_{M208}) \\ ((b \neq \emptyset) + (a \neq E) = E) &\subseteq ((b \subset a)(b = \underline{a}) = \emptyset) & \text{Resymbolising} \\ ((b \neq \emptyset) + (a \neq E) = E) &\subseteq ((b \subset a)(a = \underline{b}) = \emptyset) & (T_{M008})(\text{Inverse of an Equality}) \\ ((b \neq \emptyset) + (a \neq E) = E) &\subseteq ((a \supset b)(a = \underline{b}) = \emptyset) & (T_{M195})(\subset \text{ Reverse of } \supset) \end{aligned}$$

$$((a \neq \emptyset) + (b = \emptyset) + (b = E) = E) = ((a \subset b)(a \subset \underline{b}) = \emptyset) \quad (T_{M207})$$

Proof of (T_{M207})

$$\begin{aligned} \text{Let } (a \subset b)(a \subset \underline{b}) & & (\text{Eqn1}) \\ (a \subset b) = (\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) & & (T_{M718})(\text{Definition of } \subset)(\text{Eqn2}) \\ (a \subset \underline{b}) = (\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset) & & \text{Resymbolising (Eqn3)} \\ (\underline{ab} = \emptyset)(\underline{ab} = \emptyset)(\underline{ab} \neq \emptyset)(\underline{ab} \neq \emptyset) & & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\ (\underline{ab} + \underline{ab} = \emptyset)(\underline{ab} \neq \emptyset)(\underline{ab} \neq \emptyset) & & (T_{M071})(\text{Probabilistic Conjunction}) \\ (a(\underline{b} + \underline{b}) = \emptyset)(\underline{ab} \neq \emptyset)(\underline{ab} \neq \emptyset) & & (T_{M407})(\text{Distribution of } . \text{ over } +) \\ (aE = \emptyset)(\underline{ab} \neq \emptyset)(\underline{ab} \neq \emptyset) & & (T_{M671})(\text{Tertium non Datur}) \\ (a = \emptyset)(\underline{ab} \neq \emptyset)(\underline{ab} \neq \emptyset) & & (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\ (a = \emptyset)(a = \emptyset)(\underline{ab} \neq \emptyset)(\underline{ab} \neq \emptyset) & & (T_{M667})(\text{Conjunction Tautology}) \\ (a = \emptyset)(a = \emptyset)(\underline{ab} \neq \emptyset)(\underline{ab} \neq \emptyset) & & (T_{M008})(\text{Inverse of an Equality}) \\ (a = \emptyset)(a = E)(\underline{ab} \neq \emptyset)(\underline{ab} \neq \emptyset) & & (T_{M054})(\text{Negative of Null Set}) \\ (a = \emptyset)(Eb \neq \emptyset)(Eb \neq \emptyset) & & (A_{MX19})(\text{Substitution}) \\ (a = \emptyset)(b \neq \emptyset)(b \neq \emptyset) & & (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\ (a = \emptyset)(b \neq \emptyset)(b \neq \emptyset) & & (T_{M008})(\text{Inverse of an Equality}) \\ (a = \emptyset)(b \neq \emptyset)(b \neq E) & & (T_{M054})(\text{Negative of Null Set}) \\ \text{Thus } (a \subset b)(a \subset \underline{b}) = (a = \emptyset)(b \neq \emptyset)(b \neq E) & & (A_{MX02})(\text{Conditional Introduction}) \\ & & \text{since this proof is reversible. (Eqn4)} \\ \text{Let } (a \subset b)(a \subset \underline{b}) = \emptyset & & (\text{Eqn5}) \\ (a = \emptyset)(b \neq \emptyset)(b \neq E) = \emptyset & & (A_{MX19})(\text{Substitution of Eqn4 into Eqn5}) \\ \sim((a = \emptyset)(b \neq \emptyset)(b \neq E)) = \underline{\emptyset} & & (T_{M008})(\text{Inverse of an Equality}) \\ \sim((a = \emptyset)(b \neq \emptyset)(b \neq E)) = E & & (T_{M054})(\text{Negative of Null Set}) \\ \sim(a = \emptyset) + \sim(b \neq \emptyset) + \sim(b \neq E) = E & & (T_{M025})(\text{DeMorgan's Law}) \\ (a \neq \emptyset) + (b = \emptyset) + (b = E) = E & & (D_{M14})(\text{Definition of } \sim \text{ and } \neq) \\ \text{Thus } ((a \neq \emptyset) + (b = \emptyset) + (b = E) = E) = ((a \subset b)(a \subset \underline{b}) = \emptyset) & & (A_{MX02})(\text{Conditional Introduction}) \\ & & \text{since the proof is reversible.} \end{aligned}$$

$$((b \neq E) + (a = E) + (a = \emptyset) = E) = ((a \subset b)(a \supset \underline{b}) = \emptyset) \quad (T_{M209})$$

Proof of (T_{M209})

$$\begin{aligned} ((a \neq \emptyset) + (b = \emptyset) + (b = E) = E) &= ((a \subset b)(a \supset \underline{b}) = \emptyset) & (T_{M207}) \\ ((b \neq \emptyset) + (a = \emptyset) + (a = E) = E) &= ((b \subset a)(b \supset \underline{a}) = \emptyset) & \text{Resymbolising} \\ ((b \neq \emptyset) + (a = \emptyset) + (a = E) = E) &= ((a \subset b)(b \supset \underline{a}) = \emptyset) & (T_{M010})(\text{Inverse of a Proper Subset}) \\ ((b \neq \emptyset) + (a = \emptyset) + (a = E) = E) &= ((a \subset b)(a \supset \underline{b}) = \emptyset) & (T_{M195})(\subset \text{ Reverse of } \supset) \\ ((b \neq \emptyset) + (a = \emptyset) + (a = E) = E) &= ((a \subset b)(a \supset \underline{b}) = \emptyset) & (T_{M008})(T_{M008}) \\ ((b \neq \emptyset) + (a = \emptyset) + (a = E) = E) &= ((a \subset b)(a \supset \underline{b}) = \emptyset) & (T_{M008})(\text{Inverse of an Equality}) \\ ((b \neq E) + (a = E) + (a = \underline{E}) = E) &= ((a \subset b)(a \supset \underline{b}) = \emptyset) & (T_{M054})(\text{Negative of Null Set}) \\ ((b \neq E) + (a = E) + (a = \emptyset) = E) &= ((a \subset b)(a \supset \underline{b}) = \emptyset) & (D_{M02})(\text{Definition of } \emptyset) \end{aligned}$$

$$((b \neq \emptyset) + (a = \emptyset) + (a = E) = E) = ((a \supset b)(a \subset \underline{b}) = \emptyset) \quad (T_{M215})$$

Proof of (T_{M215})

$$((a \neq \emptyset) + (b = \emptyset) + (b = E) = E) = ((a \subset b)(a \subset \underline{b}) = \emptyset) \quad (T_{M207})$$

$$\begin{aligned}
((b \neq \emptyset) + (a = \emptyset) + (a = E) = E) &= ((b \subset a)(b \subset \underline{a}) = \emptyset) && \text{Resymbolising} \\
((b \neq \emptyset) + (a = \emptyset) + (a = E) = E) &= ((b \subset a)(a \subset \underline{b}) = \emptyset) && (T_{M010})(\text{Inverse of a Proper Subset}) \\
((b \neq \emptyset) + (a = \emptyset) + (a = E) = E) &= ((a \supset b)(a \subset \underline{b}) = \emptyset) && (T_{M195})(\subset \text{ Reverse of } \supset)
\end{aligned}$$

$$((a \neq E) + (b = E) + (b = \emptyset) = E) = ((a \supset b)(a \supset \underline{b}) = \emptyset) \quad (T_{M217})$$

Proof of (T_{M217})

$$\begin{aligned}
((a \neq \emptyset) + (b = \emptyset) + (b = E) = E) &= ((a \subset b)(a \subset \underline{b}) = \emptyset) && (T_{M207}) \\
((a \neq \emptyset) + (\underline{b} = \emptyset) + (\underline{b} = E) = E) &= ((a \subset \underline{b})(a \subset b) = \emptyset) && \text{Resymbolising} \\
((a \neq \emptyset) + (\underline{b} = \emptyset) + (\underline{b} = E) = E) &= ((b \subset a)(\underline{b} \subset a) = \emptyset) && (T_{M010})(\text{Inverse of a Proper Subset}) \\
((a \neq \emptyset) + (\underline{b} = \emptyset) + (\underline{b} = E) = E) &= ((a \supset b)(a \supset \underline{b}) = \emptyset) && (T_{M195})(\subset \text{ Reverse of } \supset) \\
((a \neq \emptyset) + (b = \emptyset) + (\underline{b} = E) = E) &= ((a \supset b)(a \supset \underline{b}) = \emptyset) && (T_{M008})(T_{M008}) \\
((a \neq \emptyset) + (b = \emptyset) + (\underline{b} = E) = E) &= ((a \supset b)(a \supset \underline{b}) = \emptyset) && (T_{M008})(\text{Inverse of an Equality}) \\
((a \neq E) + (b = E) + (b = \underline{E}) = E) &= ((a \supset b)(a \supset \underline{b}) = \emptyset) && (T_{M054})(\text{Negative of Null Set}) \\
((a \neq E) + (b = E) + (b = \emptyset) = E) &= ((a \supset b)(a \supset \underline{b}) = \emptyset) && (D_{M02})(\text{Definition of } \emptyset)
\end{aligned}$$

$$(a = E)(b = \emptyset) = (\underline{a} + b = \emptyset) \quad (T_{M377}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M377})

$$\begin{aligned}
(a = \emptyset)(b = \emptyset) &= (a + b = \emptyset) && (T_{M071})(\text{Probabilistic Conjunction}) \\
(\underline{a} = \underline{\emptyset})(b = \emptyset) &= (a + b = \emptyset) && (T_{M008})(\text{Inverse of an Equality}) \\
(\underline{a} = E)(b = \emptyset) &= (a + b = \emptyset) && (T_{M054})(\text{Negative of Null Set}) \\
(a = E)(b = \emptyset) &= (\underline{a} + b = \emptyset) && \text{Resymbolising}
\end{aligned}$$

$$(a = E)(b = E) = (\underline{a} + \underline{b} = \emptyset) \quad (T_{M376}) \text{ Probabilistic Conjunction}$$

Proof of (T_{M376})

$$\begin{aligned}
(a = \emptyset)(b = \emptyset) &= (a + b = \emptyset) && (T_{M071})(\text{Probabilistic Conjunction}) \\
(\underline{a} = \underline{\emptyset})(\underline{b} = \underline{\emptyset}) &= (a + b = \emptyset) && (T_{M008})(\text{Inverse of an Equality}) \\
(\underline{a} = E)(\underline{b} = E) &= (a + b = \emptyset) && (T_{M054})(\text{Negative of Null Set}) \\
(a = E)(b = E) &= (\underline{a} + \underline{b} = \emptyset) && \text{Resymbolising}
\end{aligned}$$

$$(a \neq \emptyset) + (b \neq \emptyset) = (a + b \neq \emptyset) \quad (T_{M238}) \text{ Probabilistic Disjunction}$$

Proof of (T_{M238})

$$\begin{aligned}
(a + b = \emptyset) &= (a = \emptyset)(b = \emptyset) && (T_{M071})(\text{Probabilistic Conjunction}) \\
\sim(a + b = \emptyset) &= \sim((a = \emptyset)(b = \emptyset)) && (T_{M008})(\text{Inverse of an Equality}) \\
(a + b \neq \emptyset) &= \sim((a = \emptyset)(b = \emptyset)) && (T_{M260})(\text{Definition of } \neq) \\
(a + b \neq \emptyset) &= (a \neq \emptyset) + (b \neq \emptyset) && (T_{M025})(\text{DeMorgan's Law}) \text{ for truth sets}
\end{aligned}$$

$$((a = b + c) = E) = ((a \neq \emptyset) = (b \neq \emptyset) + (c \neq \emptyset)) \quad (T_{M524})$$

Proof of (T_{M524})

$$\begin{aligned}
((a = b) = E) &= ((a \neq \emptyset) = (b \neq \emptyset)) && (T_{M523})(\text{Probabilistic Rule})(\text{Eqn1}) \\
\text{Let } b &= b + c && (\text{Eqn2}) \\
((a = b + c) = E) &= ((a \neq \emptyset) = (b + c \neq \emptyset)) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\
(b \neq \emptyset) + (c \neq \emptyset) &= (b + c \neq \emptyset) && (T_{M238})(\text{Probabilistic Disjunction})(\text{Eqn4}) \\
((a = b + c) = E) &= ((a \neq \emptyset) = (b \neq \emptyset) + (c \neq \emptyset)) && (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})
\end{aligned}$$

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq b) \subseteq (a \not\subseteq \underline{b})) \quad (T_{M175})$$

Proof of (T_{M175})

$$\begin{aligned}
((a \neq \emptyset) + (b = \emptyset) + (b = E) = E) &= ((a \subset b)(a \subset \underline{b}) = \emptyset) && (T_{M207})(\text{Eqn1}) \\
((a \neq \emptyset) + (b \neq E) = E) &\subseteq ((a \subset b)(a = \underline{b}) = \emptyset) && (T_{M208})(\text{Eqn2}) \\
((a \neq \emptyset) + (b \neq \emptyset) = E) &\subseteq ((a = b)(a \subset \underline{b}) = \emptyset) && (T_{M211a})(\text{Eqn3})
\end{aligned}$$

$$\begin{aligned}
& (a = b)(a = \underline{b}) = \emptyset & (T_{M212}) \\
& E = ((a = b)(a = \underline{b}) = \emptyset) & (A_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\
& ((a \neq \emptyset) + (b = \emptyset) + (b = E) = E)((a \neq \emptyset) + (b \neq E) = E)((a \neq \emptyset) + (b \neq \emptyset) = E)E \subseteq \\
& ((a \subset b)(a \subset \underline{b}) = \emptyset)((a \subset b)(a = \underline{b}) = \emptyset)((a = b)(a \subset \underline{b}) = \emptyset)((a = b)(a = \underline{b}) = \emptyset) & (T_{M231})(\text{Combination of Eqns1, 2, 3 and 4})(\text{Eqn5}) \\
& \text{Let } c = ((a \neq \emptyset) + (b = \emptyset) + (b = E) = E)((a \neq \emptyset) + (b \neq E) = E)((a \neq \emptyset) + (b \neq \emptyset) = E)E & (\text{Eqn6}) \\
& \text{Let } d = ((a \subset b)(a \subset \underline{b}) = \emptyset)((a \subset b)(a = \underline{b}) = \emptyset)((a = b)(a \subset \underline{b}) = \emptyset)((a = b)(a = \underline{b}) = \emptyset) & (\text{Eqn7}) \\
& c \subseteq d & (A_{MX19})(\text{Substitution of Eqns6\&7 into Eqn5})(\text{Eqn8}) \\
& c = ((a \neq \emptyset) + (b = \emptyset) + (b = E) = E)((a \neq \emptyset) + (b \neq E) = E)((a \neq \emptyset) + (b \neq \emptyset) = E) & (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\
& c = ((a \neq \emptyset) + (b = \emptyset) + (b = E))((a \neq \emptyset) + (b \neq E))((a \neq \emptyset) + (b \neq \emptyset)) = E & (T_{M108})(\text{Conjunction Introduction}) \\
& c = (((a \neq \emptyset) + (b = \emptyset) + (b = E))(a \neq \emptyset) + ((a \neq \emptyset) + (b = \emptyset) + (b = E))(b \neq E))((a \neq \emptyset) + (b \neq \emptyset)) = E & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
& c = (((a \neq \emptyset)(a \neq \emptyset) + (b = \emptyset)(a \neq \emptyset) + (b = E)(a \neq \emptyset) + (a \neq \emptyset)(b \neq E) + (b = \emptyset)(b \neq E) + & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
& (b = E)(b \neq E))((a \neq \emptyset) + (b \neq \emptyset)) = E & (T_{M667})(\text{Conjunction Tautology}) \\
& c = (((a \neq \emptyset) + (b = \emptyset)(a \neq \emptyset) + (b = E)(a \neq \emptyset) + (a \neq \emptyset)(b \neq E) + (b = \emptyset)(b \neq E) + & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
& (b = E)(b \neq E))((a \neq \emptyset) + (b \neq \emptyset)) = E & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
& c = (((a \neq \emptyset) + (b = \emptyset)(a \neq \emptyset) + (a \neq \emptyset)E + (b = \emptyset)(b \neq E) + (b = E)(b \neq E))((a \neq \emptyset) + (b \neq \emptyset)) = E & (T_{M671})(\text{Tertium non Datur}) \\
& c = (((a \neq \emptyset) + (b = \emptyset)(a \neq \emptyset) + (a \neq \emptyset) + (b = \emptyset)(b \neq E) + (b = E)(b \neq E))((a \neq \emptyset) + (b \neq \emptyset)) = E & (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\
& c = (((a \neq \emptyset) + (b = \emptyset)(a \neq \emptyset) + (b = \emptyset)(b \neq E) + (b = E)(b \neq E))((a \neq \emptyset) + (b \neq \emptyset)) = E & (A_{MX01})(\text{Disjunction Tautology}) \\
& c = (((a \neq \emptyset) + (b = \emptyset)(a \neq \emptyset) + (b = \emptyset)(b \neq E) + \emptyset)((a \neq \emptyset) + (b \neq \emptyset)) = E & (T_{M621})(\text{Complement Conjunction}) \\
& c = (((a \neq \emptyset) + (b = \emptyset)(b \neq E) + \emptyset)((a \neq \emptyset) + (b \neq \emptyset)) = E & (T_{M063})(\text{Partial Redundancy}) \\
& c = (((a \neq \emptyset) + (b = \emptyset)(b \neq E))((a \neq \emptyset) + (b \neq \emptyset)) = E & (T_{M668})(\text{Null Set Disjunction}) \\
& c = ((a \neq \emptyset)(a \neq \emptyset) + (a \neq \emptyset)(b = \emptyset)(b \neq E) + (b \neq \emptyset)(a \neq \emptyset) + (b \neq \emptyset)(b = \emptyset)(b \neq E) = E & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
& c = ((a \neq \emptyset) + (a \neq \emptyset)(b = \emptyset)(b \neq E) + (b \neq \emptyset)(a \neq \emptyset) + (b \neq \emptyset)(b = \emptyset)(b \neq E) = E & (T_{M621})(\text{Complement Conjunction}) \\
& c = ((a \neq \emptyset) + (b \neq \emptyset)(b = \emptyset)(b \neq E) = E & (T_{M063})(\text{Partial Redundancy}) \\
& c = ((a \neq \emptyset) + \emptyset(b \neq E) = E & (T_{M621})(\text{Complement Conjunction}) \\
& c = ((a \neq \emptyset) + \emptyset = E & (T_{M050})(\text{Null Set Conjunction}) \\
& c = ((a \neq \emptyset) = E & (T_{M668})(\text{Null Set Disjunction})(\text{Eqn9}) \\
& d = ((a \subset b)(a \subset \underline{b}) + (a \subset b)(a = \underline{b}) + (a = b)(a \subset \underline{b}) + (a = b)(a = \underline{b}) = \emptyset) & (T_{M071})(\text{Probabilistic Conjunction}) \\
& d = ((a \subset b)((a \subset \underline{b}) + (a = \underline{b})) + (a = b)((a \subset \underline{b}) + (a = \underline{b})) = \emptyset) & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
& d = (((a \subset b) + (a = b))((a \subset \underline{b}) + (a = \underline{b})) = \emptyset) & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
& d = ((a \subseteq b)(a \subseteq \underline{b}) = \emptyset) & (T_{M266})(\text{Definition of } \subseteq) \\
& d = ((a \subseteq b) \subseteq (a \not\subseteq \underline{b})) & (T_{M735})(\subseteq \text{ Existential Form})(\text{Eqn10}) \\
& ((a \neq \emptyset) = E) \subseteq ((a \subseteq b) \subseteq (a \not\subseteq \underline{b})) & (A_{MX19})(\text{Substitution of Eqns9 \& 10 into Eqn8})
\end{aligned}$$

$$((a \subseteq b) = E) \subseteq ((a \subseteq \underline{b}) \subseteq (\underline{a} = E)) \quad (T_{M423}) \text{ Reductio ad Absurdam}$$

Proof of (T_{M423})

$$\begin{aligned}
& ((a \neq \emptyset) = E) \subseteq ((a \subseteq b) \subseteq (a \not\subseteq \underline{b})) & (T_{M175})(\text{Negation Introduction}) \\
& ((a \neq \emptyset) = E) \subseteq ((a \subseteq b)(a \subseteq \underline{b}) = \emptyset) & (T_{M735})(\subseteq \text{ Existential Form}) \\
& (a \neq \emptyset)(a \subseteq b)(a \subseteq \underline{b}) \subseteq \emptyset & (T_{M444d})(\text{Biconditional Importation}) \text{ for truth sets} \\
& (a \neq \emptyset)(a \subseteq b)(a \subseteq \underline{b}) = \emptyset & (T_{M052})(\text{Subsets of the Null Set}) \\
& ((a \subseteq b) = E) \subseteq ((a \neq \emptyset)(a \subseteq \underline{b}) = \emptyset) & (T_{M444c})(\text{Biconditional Exportation})
\end{aligned}$$

$$\begin{aligned}
((a \subseteq b) = E) &\subseteq ((a \subseteq \underline{b}) \subseteq (a = \emptyset)) & (T_{M735})(\subseteq \text{Existential Form}) \\
((a \subseteq b) = E) &\subseteq ((a \subseteq \underline{b}) \subseteq (a = \sim\emptyset)) & (T_{M008})(\text{Inverse of an Equality}) \\
((a \subseteq b) = E) &\subseteq ((a \subseteq \underline{b}) \subseteq (a = E)) & (T_{M054})(\text{Negative of Null Set})
\end{aligned}$$

NB The version of Reductio Ad Absurdum used in these proofs is $(\underline{s} = \emptyset) = (s \subseteq t)$ where s and t are truth Sets.

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq b) \subseteq (b \not\subseteq \underline{a})) \quad (T_{M693})$$

Proof of (T_{M693})

$$\begin{aligned}
((a \neq \emptyset) = E) &\subseteq ((a \subseteq b) \subseteq (a \not\subseteq \underline{b})) & (T_{M175})(\text{Negation Introduction}) \\
((a \neq \emptyset) = E) &\subseteq ((a \subseteq b) \subseteq (b \not\subseteq \underline{a})) & (T_{M012})(\text{Transposition})
\end{aligned}$$

$$((b \neq E) = E) \subseteq ((a \subseteq b) \subseteq (\underline{a} \not\subseteq b)) \quad (T_{M771})$$

Proof of (T_{M771})

$$\begin{aligned}
((a \neq \emptyset) = E) &\subseteq ((a \subseteq b) \subseteq (b \not\subseteq \underline{a})) & (T_{M693}) \\
((a \neq \emptyset) = E) &\subseteq ((\underline{b} \subseteq \underline{a}) \subseteq (b \not\subseteq \underline{a})) & (T_{M012})(\text{Transposition}) \\
((\underline{b} \neq \emptyset) = E) &\subseteq ((a \subseteq b) \subseteq (\underline{a} \not\subseteq b)) & \text{Resymbolising} \\
((b \neq \sim\emptyset) = E) &\subseteq ((a \subseteq b) \subseteq (\underline{a} \not\subseteq b)) & (T_{M012})(\text{Transposition}) \\
((b \neq E) = E) &\subseteq ((a \subseteq b) \subseteq (\underline{a} \not\subseteq b)) & (T_{M054})(\text{Negative of Null Set})
\end{aligned}$$

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq b) = (a \not\subseteq \underline{b})(a \subseteq b)) \quad (T_{M596})$$

Proof of (T_{M596})

$$\begin{aligned}
((a \neq \emptyset) = E) &\subseteq ((a \subseteq b) \subseteq (a \not\subseteq \underline{b})) & (T_{M175})(\text{Negation Introduction}) \\
((a \neq \emptyset) = E) &\subseteq ((a \subseteq b) = (a \not\subseteq \underline{b})(a \subseteq b)) & (A_{MX44})(\subseteq \text{Equality Form})
\end{aligned}$$

$$((b \neq \emptyset) = E) \subseteq ((a \supseteq b) = (a \not\subseteq \underline{b})(a \subseteq \underline{b})) \quad (T_{M595})$$

Proof of (T_{M595})

$$\begin{aligned}
((a \neq \emptyset) = E) &\subseteq ((a \subseteq b) = (a \not\subseteq \underline{b})(a \subseteq b)) & (T_{M596}) \\
((b \neq \emptyset) = E) &\subseteq ((b \subseteq a) = (b \not\subseteq \underline{a})(b \subseteq a)) & \text{Resymbolising} \\
((b \neq \emptyset) = E) &\subseteq ((b \subseteq a) = (a \not\subseteq \underline{b})(a \subseteq \underline{b})) & (T_{M012})(\text{Transposition}) \\
((b \neq \emptyset) = E) &\subseteq ((a \supseteq b) = (a \not\subseteq \underline{b})(a \subseteq \underline{b})) & (T_{M196})(\subseteq \text{Reverse of } \supseteq)
\end{aligned}$$

$$(((a \neq \emptyset) = E) = E) \subseteq (((a = E) \subseteq (b = E)) \subseteq ((b = E) \subseteq (a \neq \emptyset))) \quad (T_{M694})$$

Proof of (T_{M694})

$$\begin{aligned}
((a \neq \emptyset) = E) &\subseteq ((a \subseteq b) \subseteq (b \not\subseteq \underline{a})) & (T_{M693}) \\
(((a \neq \emptyset) = E) \subseteq ((a \subseteq b) \subseteq (b \not\subseteq \underline{a}))) &= E & (A_{MX16})(\text{Order Axiom}) \\
(((a \neq \emptyset) = E) = E) &\subseteq (((a \subseteq b) \subseteq (b \not\subseteq \underline{a})) = E) & (T_{M094})(\text{Probabilistic Rule}) \\
(((a \neq \emptyset) = E) = E) &\subseteq (((a \subseteq b) = E) \subseteq ((b \not\subseteq \underline{a}) = E)) & (T_{M094})(\text{Probabilistic Rule})(\text{Eqn1}) \\
((a \subseteq b) = E) &= ((a = E) \subseteq (b = E)) & (T_{M094})(\text{Probabilistic Rule})(\text{Eqn2}) \\
((b \not\subseteq \underline{a}) = E) &\subseteq ((b = E) \subseteq (a \neq \emptyset)) & (T_{M319})(\text{Probabilistic Rule})(\text{Eqn3}) \\
(((a \neq \emptyset) = E) = E) &\subseteq (((a = E) \subseteq (b = E)) \subseteq ((b \not\subseteq \underline{a}) = E)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn4}) \\
(((a \neq \emptyset) = E) = E) &\subseteq (((a = E) \subseteq (b = E)) \subseteq ((b = E) \subseteq (a \neq \emptyset))) & (T_{M139})(\text{Hypothetical Syllogism Eqns3\&4})
\end{aligned}$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq b) \subseteq (a^1 \not\subseteq \underline{b})) \quad (T_{M606})$$

Proof of (T_{M606})

$$\begin{aligned}
((a \neq \emptyset) = E) &\subseteq ((a \subseteq b) \subseteq (a \not\subseteq \underline{b})) & (T_{M175}) \\
((a^1 \neq \emptyset) = E) &\subseteq ((a^1 \subseteq b) \subseteq (a^1 \not\subseteq \underline{b})) & \text{Resymbolising}
\end{aligned}$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq \underline{b}) = (a^1 \subseteq b)) \quad (T_{M090}) \text{ Inverse Unit Subset Rule}$$

Proof of (T_{M090})

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq b) \subseteq (a^1 \not\subseteq \underline{b}))$$

(T_{M606})(Inverse Unit Subset Rule)(Eqn1)

$$(a^1 \not\subseteq \underline{b}) \subseteq (a^1 \subseteq \underline{b})$$

(T_{M604})

$$E = ((a^1 \not\subseteq \underline{b}) \subseteq (a^1 \subseteq \underline{b}))$$

(A_{MX16})(Order Axiom)

$$E = ((a^1 \not\subseteq \underline{b}) \subseteq (a^1 \subseteq \underline{b}))$$

Resymbolising (Eqn2)

$$((a^1 \neq \emptyset) = E).E \subseteq ((a^1 \subseteq b) \subseteq (a^1 \not\subseteq \underline{b})).((a^1 \not\subseteq \underline{b}) \subseteq (a^1 \subseteq \underline{b}))$$

(T_{M669})(Combination of Eqns1&2)

$$((a^1 \neq \emptyset) = E).E \subseteq ((a^1 \not\subseteq \underline{b}) = (a^1 \subseteq \underline{b}))$$

(T_{M430})(Biconditional Introduction)

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq \underline{b}) = (a^1 \subseteq \underline{b}))$$

(A_{MX11})(Universal Set Conjunction Axiom)

$$(a^1 \subseteq b + c) = (a^1 \subseteq b) + (a^1 \subseteq c)$$

(T_{M789})

Proof of (T_{M789})

$$(a^1 \subseteq bc) = (a^1 \subseteq b)(a^1 \subseteq c)$$

(T_{M132})(Composition)

$$(a^1 \not\subseteq bc) = \sim((a^1 \subseteq b)(a^1 \subseteq c))$$

(T_{M008})(Inverse of an Equality)

$$(a^1 \not\subseteq bc) = (a^1 \not\subseteq b) + (a^1 \not\subseteq c)$$

(T_{M025})(DeMorgan's Law)

Let $a^1 \neq \emptyset$

$$(a^1 \subseteq \sim(bc)) = (a^1 \subseteq \underline{b}) + (a^1 \subseteq \underline{c})$$

(T_{M090})(Inverse Unit Subset Rule)

$$(a^1 \subseteq \underline{b} + \underline{c}) = (a^1 \subseteq \underline{b}) + (a^1 \subseteq \underline{c})$$

(T_{M025})(DeMorgan's Law)

$$(a^1 \subseteq b + c) = (a^1 \subseteq b) + (a^1 \subseteq c)$$

Resymbolising $\underline{b}$ and $\underline{c}$ only

$$\text{Thus } ((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq b + c) = (a^1 \subseteq b) + (a^1 \subseteq c))$$

(A_{MX02})(Conditional Introduction)

$$(a^1 \neq \emptyset)(a^1 \subseteq b + c) = (a^1 \neq \emptyset)((a^1 \subseteq b) + (a^1 \subseteq c))$$

(T_{M444b})(Biconditional Exportation/Importation)
for truth sets (Eqn1)

Let $a^1 = \emptyset$

(Eqn2)

$$\text{Let } (a^1 \subseteq b + c) \neq (a^1 \subseteq b) + (a^1 \subseteq c)$$

(Eqn3)

$$(\emptyset \subseteq b + c) \neq (\emptyset \subseteq b) + (\emptyset \subseteq c)$$

(A_{MX19})(Substitution of Eqn2 into Eqn3)

$$E \neq E + E$$

(T_{M001})(Universality of (T_{M670}) Null Set Theorem)

$$E \neq E$$

(A_{MX01})(Disjunction Tautology)

$$\emptyset$$

(T_{M345})

$$\text{Thus } ((a^1 = \emptyset) = E)((a^1 \subseteq b + c) \neq (a^1 \subseteq b) + (a^1 \subseteq c)) = \emptyset$$

(A_{MX02})(Conditional Introduction)

$$((a^1 = \emptyset) = E) \subseteq ((a^1 \subseteq b + c) = (a^1 \subseteq b) + (a^1 \subseteq c))$$

(T_{M735})($\subseteq$ Existential Form)

$$(a^1 = \emptyset)(a^1 \subseteq b + c) = (a^1 = \emptyset)((a^1 \subseteq b) + (a^1 \subseteq c))$$

(T_{M444b})(Biconditional Exportation/Importation)
for truth sets (Eqn4)

$$(a^1 = \emptyset)(a^1 \subseteq b + c) + (a^1 \neq \emptyset)(a^1 \subseteq b + c) = (a^1 = \emptyset)((a^1 \subseteq b) + (a^1 \subseteq c)) + (a^1 \neq \emptyset)((a^1 \subseteq b) + (a^1 \subseteq c))$$

(T_{M230})(Combination of Eqns1&2)

$$((a^1 = \emptyset) + (a^1 \neq \emptyset))(a^1 \subseteq b + c) = ((a^1 = \emptyset) + (a^1 \neq \emptyset))((a^1 \subseteq b) + (a^1 \subseteq c))$$

(T_{M407})(Distribution of . over +)

$$E(a^1 \subseteq b + c) = E((a^1 \subseteq b) + (a^1 \subseteq c))$$

(T_{M671})(Tertium non Datur)

$$(a^1 \subseteq b + c) = (a^1 \subseteq b) + (a^1 \subseteq c)$$

(A_{MX11})(Universal Set Conjunction Axiom)

$$((a \subseteq b) = E) \subseteq ((c^1 \not\subseteq \underline{a}) \subseteq (c^1 \subseteq \underline{b}))$$

(T_{M353}) Entity Mode / Set Mode Rule

Proof of (T_{M353})

$$((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) \subseteq (c^1 \subseteq b))$$

(T_{M099})(Entity Mode / Set Mode Rule)

$$((a \subseteq b) = E) = ((c^1 \not\subseteq \underline{a}) \subseteq (c^1 \subseteq \underline{b}))$$

(T_{M090})(Inverse Unit Subset Rule)

$$((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) \subseteq (c^1 \not\subseteq \underline{b}))$$

(T_{M354}) Entity Mode / Set Mode Rule

Proof of (T_{M354})

$$((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) \subseteq (c^1 \subseteq b))$$

(T_{M099})(Entity Mode / Set Mode Rule)

$$((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) \subseteq (c^1 \not\subseteq \underline{b}))$$

(T_{M090})(Inverse Unit Subset Rule)

$$((a \subseteq b) = E) \subseteq ((c^1 \not\subseteq \underline{a}) \subseteq (c^1 \not\subseteq \underline{b}))$$

(T_{M355}) Entity Mode / Set Mode Rule

Proof of (T_{M355})

$$((a \subseteq b) = E) \subseteq ((c^1 \subseteq a) \subseteq (c^1 \subseteq b))$$

(T_{M099})(Entity Mode / Set Mode Rule)

$$((a \subseteq b) = E) \subseteq ((c^1 \not\subseteq \underline{a}) \subseteq (c^1 \not\subseteq \underline{b}))$$

(T_{M090})(Inverse Unit Subset Rule)

$$(((a \subseteq b) = E) = E) \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = \emptyset)) \quad (TM306)$$

Proof of (TM306)

$$\begin{aligned} ((a \subseteq b) = E) &= E \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = E)) & (TM303) \\ ((a \subseteq b) = E) &= E \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = E)) & \text{Resymbolising} \\ ((a \subseteq b) = E) &= E \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \not\subseteq b) = \sim E)) & (TM008)(\text{Inverse of an Equality}) \\ ((a \subseteq b) = E) &= E \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \not\subseteq b) = \emptyset)) & (DM02)(\text{Definition of } \emptyset) \\ ((a \subseteq b) = E) &= E \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = \emptyset)) & (TM090)(\text{Inverse Unit Subset Rule}) \end{aligned}$$

$$((a = \emptyset) = E) = E \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq a) = \emptyset)) \quad (TM457)$$

Proof of (TM457)

$$\begin{aligned} ((a \subseteq b) = E) &= E \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) = \emptyset)) & (TM306)(\text{Eqn1}) \\ \text{Let } a &= E & (\text{Eqn2}) \\ ((E \subseteq b) = E) &= E \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq b) = \emptyset)) & (AMX19)(\text{Substitution of Eqn2 into Eqn1}) \\ ((b = E) = E) &= E \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq b) = \emptyset)) & (TM275)(\text{Supersets of the Universal Set}) \\ ((b = \emptyset) = E) &= E \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq b) = \emptyset)) & (TM014) \\ ((a = \emptyset) = E) &= E \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq a) = \emptyset)) & \text{Resymbolising} \end{aligned}$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq S_x) = (a^1 \subseteq \sim S_x)) \quad (TM639)$$

Proof of (TM639)

$$\begin{aligned} ((a^1 \neq \emptyset) = E) &\subseteq ((a^1 \not\subseteq b) = (a^1 \subseteq b)) & (TM090)(\text{Inverse Unit Subset Rule})(\text{Eqn1}) \\ \text{Let } b &= \sim S_x & (\text{Eqn2}) \\ ((a^1 \neq \emptyset) = E) &\subseteq ((a^1 \not\subseteq S_x) = (a^1 \subseteq \sim S_x)) & (AMX19)(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \neq b^1) = (a^1 b^1 = \emptyset)) \quad (TM205) \text{ Unit Set Conjunction Rule}$$

Proof of (TM205)

$$\begin{aligned} \text{Let } a^1 &\neq \emptyset \\ (a^1 \neq \emptyset) &= E & (AMX16)(\text{Order Axiom}) \\ (a^1 \neq \emptyset) &\subseteq E & (TM275)(\text{Supersets of the Universal Set}) \\ ((a^1 \neq \emptyset) = E) &\subseteq ((a^1 b^1 = \emptyset) = (a^1 b^1 = \emptyset)) & (TM001)(\text{Universality of } (DM00) \text{ Definition of } =) \\ ((a^1 \neq \emptyset) = E) &\subseteq ((a^1 \subseteq \sim(b^1)) = (a^1 b^1 = \emptyset)) & (TM735)(\subseteq \text{Existential Form}) \\ ((a^1 \neq \emptyset) = E) &\subseteq ((a^1 \not\subseteq b^1) = (a^1 b^1 = \emptyset)) & (TM090)(\text{Inverse Unit Subset Rule}) \\ ((a^1 \neq \emptyset) = E) &\subseteq (\sim(a^1 \subset b^1) \sim(a^1 = b^1) = (a^1 b^1 = \emptyset)) & (TM198)(\text{Eqn1}) \\ (a^1 \neq \emptyset) &\subseteq \sim(a^1 \subset b^1) & (TM348) \\ ((a^1 \neq \emptyset) = E) &\subseteq (\sim(a^1 \subset b^1) = E) & (AMX16)(\text{Order Axiom})(\text{Eqn2}) \\ ((a^1 \neq \emptyset) = E)((a^1 \neq \emptyset) = E) &\subseteq (\sim(a^1 \subset b^1) = E)(\sim(a^1 \subset b^1) \sim(a^1 = b^1) = (a^1 b^1 = \emptyset)) & (TM669)(\text{Combination of Eqns1 \& 2}) \\ ((a^1 \neq \emptyset) = E) &\subseteq (\sim(a^1 \subset b^1) = E)(\sim(a^1 \subset b^1) \sim(a^1 = b^1) = (a^1 b^1 = \emptyset)) & (TM667)(\text{Conjunction Tautology}) \\ ((a^1 \neq \emptyset) = E) &\subseteq (E \sim(a^1 = b^1) = (a^1 b^1 = \emptyset)) & (AMX19)(\text{Substitution}) \\ ((a^1 \neq \emptyset) = E) &\subseteq (\sim(a^1 = b^1) = (a^1 b^1 = \emptyset)) & (AMX11)(\text{Universal Set Conjunction Axiom}) \\ ((a^1 \neq \emptyset) = E) &\subseteq ((a^1 \neq b^1) = (a^1 b^1 = \emptyset)) & (TM260)(\text{Definition of } \neq) \end{aligned}$$

$$(a^1 \neq \emptyset)(a^1 \neq b^1) \subseteq (a^1 \subset \sim b^1) \quad (TM783)$$

Proof of (TM783)

$$\begin{aligned} a^1 &\subseteq E & (TM744)(\text{Universal Set Theorem}) \\ a^1 &\subseteq b^1 + \sim b^1 & (TM671)(\text{Tertium Non Datur}) \\ a^1 &= a^1(b^1 + \sim b^1) & (AMX44)(\subseteq \text{Equality Form}) \\ a^1 &= a^1 b^1 + a^1 \cdot \sim b^1 & (TM407)(\text{Distribution of } \cdot \text{ over } +)(\text{Eqn1}) \\ \text{Let } a^1 &\neq b^1 \\ \text{Let } a^1 &\neq \emptyset \\ a^1 b^1 &= \emptyset & (TM205)(\text{Unit Set Conjunction Rule})(\text{Eqn2}) \\ a^1 &= \emptyset + a^1 \cdot \sim b^1 & (AMX19)(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$a^1 = a^1 \cdot \sim b^1$	(T_{M668})(Null Set Disjunction)
$a^1 \subseteq \sim b^1$	($AMX44$)($\subseteq$ Equality Form)
$(a^1 \subseteq \sim b^1) + (a^1 = \sim b^1)$	(T_{M266})(Definition of $\subseteq$)(Eqn3)
$(a^1 \subseteq \sim b^1) + (a^1 = \sim b^1)E$	($AMX11$)(Universal Set Conjunction Axiom)
$a^1 \neq \sim b^1$	($AMX18$)(2nd Plurality Axiom)
$(a^1 \neq \sim b^1) = E$	($AMX16$)(Order Axiom)(Eqn4)
$(a^1 \subseteq \sim b^1) + (a^1 = \sim b^1)(a^1 \neq \sim b^1)$	($AMX19$)(Substitution of Eqn4 into Eqn3)
$(a^1 \subseteq \sim b^1) + \emptyset$	(T_{M621})(Complement Conjunction)
$a^1 \subseteq \sim b^1$	(T_{M668})(Null Set Disjunction)
Thus $(a^1 \neq \emptyset)(a^1 \neq b^1) \subseteq (a^1 \subseteq \sim b^1)$	($AMX02$)(Conditional Introduction)

$$\emptyset^1 \subseteq \sim E^1 \quad (T_{M781})$$

Proof of (T_{M781})

$(a^1 \neq \emptyset)(a^1 \neq b^1) \subseteq (a^1 \subseteq \sim b^1)$	(T_{M783})(Eqn1)
Let $a^1 = \emptyset^1$	(Eqn2)
Let $b^1 = E^1$	(Eqn3)
$(\emptyset^1 \neq \emptyset)(\emptyset^1 \neq E^1) \subseteq (\emptyset^1 \subseteq \sim E^1)$	($AMX19$)(Substitution of Eqns2&3 into Eqn1)
$E(\emptyset^1 \neq E^1) \subseteq (\emptyset^1 \subseteq \sim E^1)$	(T_{M001})(Universality of (T_{M780}) Existence of $\emptyset^1$)
$(\emptyset^1 \neq E^1) \subseteq (\emptyset^1 \subseteq \sim E^1)$	($AMX11$)(Universal Set Conjunction Axiom)
$E \subseteq (\emptyset^1 \subseteq \sim E^1)$	(T_{M001})(Universality of (T_{M784}))
$E = (\emptyset^1 \subseteq \sim E^1)$	(T_{M275})(Supersets of the Universal Set)
$\emptyset^1 \subseteq \sim E^1$	($AMX16$)(Order Axiom)

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \neq b^1) = (a^1 \subseteq \sim(b^1))) \quad (T_{M351})$$

Proof of (T_{M351})

$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \neq b^1) = (a^1 b^1 = \emptyset))$	(T_{M205})(Unit Set Conjunction Rule)
$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \neq b^1) = (a^1 \subseteq \sim(b^1)))$	(T_{M735})($\subseteq$ Existential Form)

$$E = (a^1 b^1 = (a^1 b^1)^1) + (a^1 b^1 = \emptyset) \quad (T_{M774})$$

Proof of (T_{M774})

Let $a^1 \neq \emptyset$	
$(a^1 \subseteq b) \subseteq ((a^1 b)^1 = a^1 b)$	(T_{M555})(Eqn1)
Let $b = b^1$	(Eqn2)
$(a^1 \subseteq b^1) \subseteq ((a^1 b^1)^1 = a^1 b^1)$	($AMX19$)(Substitution of Eqn2 into Eqn1)
$(a^1 \subseteq b^1) + (a^1 = b^1) \subseteq ((a^1 b^1)^1 = a^1 b^1)$	(T_{M266})(Definition of $\subseteq$)(Eqn3)
$\sim(a^1 \subseteq b^1)$	(T_{M348})
$\sim(a^1 \subseteq b^1) = E$	($AMX16$)(Order Axiom)
$(a^1 \subseteq b^1) = \sim E$	(T_{M008})(Inverse of an Equality)
$(a^1 \subseteq b^1) = \emptyset$	($DM02$)(Definition of $\emptyset$)(Eqn4)
$\emptyset + (a^1 = b^1) \subseteq ((a^1 b^1)^1 = a^1 b^1)$	($AMX19$)(Substitution of Eqn4 into Eqn3)
$(a^1 = b^1) \subseteq ((a^1 b^1)^1 = a^1 b^1)$	(T_{M668})(Null Set Disjunction)(Eqn5)
$(a^1 \neq b^1) = (a^1 b^1 = \emptyset)$	(T_{M205})(Unit Set Conjunction Rule)(Eqn6)
$(a^1 = b^1) + (a^1 \neq b^1) \subseteq ((a^1 b^1)^1 = a^1 b^1) + (a^1 b^1 = \emptyset)$	($AMX21$)(Combination of Eqns5&6)
$E \subseteq ((a^1 b^1)^1 = a^1 b^1) + (a^1 b^1 = \emptyset)$	(T_{M671})(Tertium non Datur)
$E = ((a^1 b^1)^1 = a^1 b^1) + (a^1 b^1 = \emptyset)$	(T_{M275})(Supersets of the Universal Set)
$((a^1 b^1)^1 = a^1 b^1) + (a^1 b^1 = \emptyset)$	($AMX16$)(Order Axiom)
Thus $(a^1 \neq \emptyset) \subseteq (((a^1 b^1)^1 = a^1 b^1) + (a^1 b^1 = \emptyset))$	($AMX02$)(Conditional Introduction)
$(a^1 \neq \emptyset) \subseteq ((a^1 b^1 = (a^1 b^1)^1) + (a^1 b^1 = \emptyset))$	(T_{M197})(Commutativity of $=$)
$(a^1 \neq \emptyset)(a^1 b^1 \neq \emptyset) \subseteq ((a^1 b^1 = (a^1 b^1)^1)$	(T_{M468})(Eqn7)
$(a^1 b^1 \neq \emptyset) \subseteq (a^1 \neq \emptyset)(b^1 \neq \emptyset)$	(T_{M242})
$(a^1 b^1 \neq \emptyset) \subseteq (a^1 \neq \emptyset)$	(T_{M073})(Single Premise Syllogism)(Eqn8)
$(a^1 b^1 \neq \emptyset) \subseteq ((a^1 b^1 = (a^1 b^1)^1)$	($AMX19$)(Substitution of Eqn8 into Eqn7)
$E = (a^1 b^1 = (a^1 b^1)^1) + (a^1 b^1 = \emptyset)$	(T_{M064})($\subseteq$ Universal Form)

Chapter 3 Proofs of Membership Theorems – Part 3

3.1 Theorems of Probability

$((a \subseteq b) = E) \subseteq ((b = E) \subseteq (a \neq \emptyset))$ (U_{M001}) Theorem of Truth Sets

Proof of (U_{M001})

$a \subseteq E$	(T_{M744})(Universal Set Theorem)
$(a \subseteq E) = E$	($AMX16$)(Order Axiom)
$(a \subseteq E) = (a \neq \emptyset) + (a = \emptyset)$	(T_{M671})(Tertium non Datur)(Eqn1)
Let $b = E$	(Eqn2)
$(a \subseteq b) = (a \neq \emptyset) + (a = \emptyset)$	($AMX19$)(Substitution of Eqn2 into Eqn1)
Thus $((b = E) = E) \subseteq ((a \subseteq b) = (a \neq \emptyset) + (a = \emptyset))$	($AMX02$)(Conditional Introduction)
$(a \subseteq b).(b = E) \subseteq (a \neq \emptyset) + (a = \emptyset)$	(T_{M444d})(Biconditional Importation) for truth sets
$((a \subseteq b) = E) \subseteq ((b = E) \subseteq (a \neq \emptyset) + (a = \emptyset))$	($AMX35$)(Exportation/Importation)(Eqn3) for truth sets
$a \neq \emptyset$	(H_{MX01})(Induction Hypothesis)
$(a \neq \emptyset) = E$	($AMX16$)(Order Axiom)
$(a = \emptyset) = \sim E$	(T_{M008})(Inverse of an Equality)
$(a = \emptyset) = \emptyset$	(D_{M02})(Definition of $\emptyset$)(Eqn4)
$((a \subseteq b) = E) \subseteq ((b = E) \subseteq (a \neq \emptyset) + \emptyset)$	($AMX19$)(Substitution of Eqn4 into Eqn3)
$((a \subseteq b) = E) \subseteq ((b = E) \subseteq (a \neq \emptyset))$	(T_{M668})(Null Set Disjunction)

$((a \neq \emptyset) = E) \subseteq ((ab \neq \emptyset) \neq \emptyset)$ (U_{M002})

Proof of (U_{M002})

$ab \subseteq a$	(T_{M083})(Conjunction Elimination)
$(ab \subseteq a) = E$	($AMX16$)(Order Axiom)
$(ab \neq \emptyset) \subseteq (a \neq \emptyset)$	(T_{M096})(Probabilistic Rule)
$((ab \neq \emptyset) \subseteq (a \neq \emptyset)) = E$	($AMX16$)(Order Axiom)
$((a \neq \emptyset) = E) \subseteq ((ab \neq \emptyset) \neq \emptyset)$	(U_{M001})(Theorem of Truth Sets)

$((a \neq \emptyset) = E) \subseteq ((a \subseteq b) \neq E)$ (U_{M003})

Proof of (U_{M003})

Let $a = \emptyset$	(Eqn1)
$\emptyset \subseteq b$	(T_{M670})(Null Set Theorem)(Eqn2)
$a \subseteq b$	($AMX19$)(Substitution of Eqn1 into Eqn2)
Thus $(a = \emptyset) \subseteq (a \subseteq b)$	($AMX02$)(Conditional Introduction)
$((a = \emptyset) \subseteq (a \subseteq b)) = E$	($AMX16$)(Order Axiom)
$((a \subseteq b) = E) \subseteq ((a = \emptyset) \neq \emptyset)$	(U_{M001})(Theorem of Truth Sets)
$((a = \emptyset) = \emptyset) \subseteq ((a \subseteq b) \neq E)$	(T_{M012})(Transposition)
$((a \neq \emptyset) = \sim \emptyset) \subseteq ((a \subseteq b) \neq E)$	(T_{M008})(Inverse of an Equality)
$((a \neq \emptyset) = E) \subseteq ((a \subseteq b) \neq E)$	(T_{M054})(Negative of Null Set)

$((a \not\subseteq b) = E) \subseteq ((a \subseteq b) \neq \emptyset)$ (U_{M004})

Proof of (U_{M004})

$((a \not\subseteq b) = E) \subseteq ((a \subseteq b) \subseteq (a \not\subseteq b))$	(T_{M175})(Negation Introduction)(Eqn1)
$a \neq \emptyset$	(H_{MX01})(Induction Hypothesis)
$((a \neq \emptyset) = E) = E$	($AMX16$)(Order Axiom)(Eqn2)
$E = ((a \subseteq b) \subseteq (a \not\subseteq b))$	($AMX19$)(Substitution of Eqn2 into Eqn1)
$((a \not\subseteq b) = E) \subseteq ((a \subseteq b) \neq \emptyset)$	(U_{M001})(Theorem of Truth Sets)

$((ac \neq \emptyset).(ac \subseteq b) = E) \subseteq ((a \subseteq b) \neq \emptyset)$ (U_{M011}) Induction Theorem

Proof of (U_{M011})

$ac \subseteq a$	(T_{M083})(Conjunction Elimination)
$(ac \subseteq a) = E$	($AMX16$)(Order Axiom)
$(a \subseteq b) \subseteq (ac \subseteq b)$	(T_{M364})(Deductive Rule)
$(ac \not\subseteq b) \subseteq (a \not\subseteq b)$	(T_{M012})(Transposition)(Eqn1)
$((ac \neq \emptyset) = E) \subseteq ((ac \subseteq b) \subseteq (ac \not\subseteq b))$	(T_{M175})(Negation Introduction)(Eqn2)
$((ac \neq \emptyset) = E) \subseteq ((ac \subseteq b) \subseteq (a \not\subseteq b))$	(T_{M139})(Hypothetical Syllogism Eqns1&2)
$(ac \neq \emptyset)(ac \subseteq b) \subseteq (a \not\subseteq b)$	($AMX35$)(Exportation/Importation) for truth sets
$((ac \neq \emptyset)(ac \subseteq b) \subseteq (a \not\subseteq b)) = E$	($AMX16$)(Order Axiom)
$((ac \neq \emptyset)(ac \subseteq b) = E) \subseteq ((a \not\subseteq b) = E)$	(T_{M094})(Probabilistic Rule)(Eqn3)
$((a \not\subseteq b) = E) \subseteq ((a \subseteq b) \neq \emptyset)$	(U_{M004})(Eqn4)
$((ac \neq \emptyset)(ac \subseteq b) = E) \subseteq ((a \subseteq b) \neq \emptyset)$	(T_{M139})(Hypothetical Syllogism Eqns3&4)

$$((a \neq \emptyset) = E) \subseteq ((a = E) \neq \emptyset) \quad (U_{M005})$$

Proof of (U_{M005})

$((a \not\subseteq b) = E) \subseteq ((a \subseteq b) \neq \emptyset)$	(U_{M004})(Eqn1)
Let $a = E$	(Eqn2)
$((E \not\subseteq b) = E) \subseteq ((E \subseteq b) \neq \emptyset)$	($AMX19$)(Substitution of Eqn2 into Eqn1)
$((b \not\subseteq E) = E) \subseteq ((E \subseteq b) \neq \emptyset)$	(T_{M012})(Transposition) & (T_{M008})(Inverse of an Equality)
$((b \not\subseteq \emptyset) = E) \subseteq ((E \subseteq b) \neq \emptyset)$	(D_{M02})(Definition of $\emptyset$)
$((b \neq \emptyset) = E) \subseteq ((E \subseteq b) \neq \emptyset)$	(T_{M052})(Subsets of the Null Set) & (T_{M008})(Inverse of an Equality)
$((b \neq \emptyset) = E) \subseteq ((b = E) \neq \emptyset)$	(T_{M275})(Supersets of the Universal Set) & (T_{M008})(Inverse of an Equality)
$((a \neq \emptyset) = E) \subseteq ((a = E) \neq \emptyset)$	Resymbolising

$$((a \neq E) = E) \subseteq ((a = E) \neq E) \quad (U_{M016})$$

Proof of (U_{M016})

$(a \neq E) = (a \neq E)$	(D_{M00})(Definition of $=$)
$(a \neq E) \subseteq (a \neq E)$	(T_{M425})(Biconditional Elimination)
$((a \neq E) \subseteq (a \neq E)) = E$	($AMX16$)(Order Axiom)
$((a \neq E) = E) \subseteq ((a \neq E) \neq \emptyset)$	(U_{M001})(Theorem of Truth Sets)
$((a \neq E) = E) \subseteq ((a = E) \neq \sim\emptyset)$	(T_{M008})(T_{M008})(Inverse of an Equality)
$((a \neq E) = E) \subseteq ((a = E) \neq E)$	(T_{M054})(Negative of Null Set)

$$((a \neq E) = E) \subseteq ((a = \emptyset) \neq \emptyset) \quad (U_{M006})$$

Proof of (U_{M006})

$((a \neq \emptyset) = E) \subseteq ((a = E) \neq \emptyset)$	(U_{M005})
$((a \neq \emptyset) = E) \subseteq ((a = E) \neq \emptyset)$	Resymbolising
$((a \neq \emptyset) = E) \subseteq ((a = E) \neq \emptyset)$	(T_{M008})(Inverse of an Equality)
$((a \neq \emptyset) = E) \subseteq ((a = \emptyset) \neq \emptyset)$	(D_{M02})(Definition of $\emptyset$)
$((a \neq E) = E) \subseteq ((a = \emptyset) \neq \emptyset)$	(T_{M054})(Negative of Null Set)

$$((a = E) = \emptyset) \subseteq ((a \neq \emptyset) \neq E) \quad (U_{M007})$$

$$((a = \emptyset) = \emptyset) \subseteq ((a \neq E) \neq E) \quad (U_{M008})$$

Example Proof of (U_{M007})

$((a \neq \emptyset) = E) \subseteq ((a = E) \neq \emptyset)$	(U_{M005})
$((a = E) = \emptyset) \subseteq ((a \neq \emptyset) \neq E)$	(T_{M008})(Inverse of an Equality)

$$((a \not\subseteq b) = E) \subseteq ((a \not\subseteq b) \neq E) \quad (U_{M009})$$

Proof of (U_{M009})

$((a \not\subseteq b) = E) \subseteq ((a \subseteq b) \neq \emptyset)$	(U_{M004})(Eqn1)
Let $b = \underline{b}$	(Eqn2)

$$\begin{aligned} ((a \not\subseteq b) = E) &\subseteq ((a \subseteq \underline{b}) \neq \emptyset) \\ ((a \not\subseteq b) = E) &\subseteq ((a \not\subseteq \underline{b}) \neq \emptyset) \\ ((a \not\subseteq b) = E) &\subseteq ((a \not\subseteq \underline{b}) \neq E) \end{aligned}$$

(**AMX19**)(Substitution of Eqn2 into Eqn1)
(**TM008**)(Inverse of an Equality)
(**TM054**)(Negative of Null Set)

$$((a \subseteq b)(b \not\subseteq c) = E) \subseteq ((a \subseteq c) \neq \emptyset) \quad (\mathbf{U_{M010}}) \text{ Probability Syllogism}$$

Proof of (**U_{M010}**)

$$\begin{aligned} (((a \not\subseteq b) = E) &\subseteq ((a \subseteq b) \neq \emptyset)) & (\mathbf{U_{M004}}) \\ ((a \subseteq b) = E)((b \not\subseteq c) = E) &\subseteq ((a \subseteq b) = E)((b \subseteq c) \neq \emptyset) & (\mathbf{TM233})(\text{Deduction}) \\ ((a \subseteq b)(b \not\subseteq c) = E) &\subseteq ((a \subseteq b) = E)((b \subseteq c) \neq \emptyset) & (\mathbf{TM108})(\text{Conjunction Introduction}) \\ ((a \subseteq b)(b \not\subseteq c) = E) &\subseteq ((a \subseteq b)(b \subseteq c) \neq \emptyset) & (\mathbf{TM110})(\text{Probabilistic Conjunction})(\text{Eqn1}) \\ (a \subseteq b)(b \subseteq c) &\subseteq (a \subseteq c) & (\mathbf{TM139})(\text{Hypothetical Syllogism}) \\ ((a \subseteq b)(b \subseteq c) &\subseteq (a \subseteq c)) = E & (\mathbf{AMX16})(\text{Order Axiom}) \\ ((a \subseteq b)(b \subseteq c) \neq \emptyset) &\subseteq ((a \subseteq c) \neq \emptyset) & (\mathbf{TM096})(\text{Probabilistic Rule})(\text{Eqn2}) \\ ((a \subseteq b)(b \not\subseteq c) = E) &\subseteq ((a \subseteq c) \neq \emptyset) & (\mathbf{TM139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$(((a \not\subseteq b) = E) = E) \subseteq (((c \subseteq a) = E) \subseteq ((c \subseteq b) \neq E)) \quad (\mathbf{U_{M017}})$$

$$(((a \not\subseteq b) = E) = E) \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) \neq E)) \quad (\mathbf{U_{M012}})$$

Proof of (**U_{M017}**) & (**U_{M012}**)

$$\begin{aligned} (a \subseteq b)(b \subseteq c) &\subseteq (a \subseteq c) & (\mathbf{TM139})(\text{Hypothetical Syllogism}) \\ ((a \subseteq b) = E) &\subseteq ((b \subseteq c) \subseteq (a \subseteq c)) & (\mathbf{AMX35})(\text{Exportation/Importation}) \text{ for truth sets} \\ ((a \subseteq b) = E) &\subseteq ((a \not\subseteq c) \subseteq (b \not\subseteq c)) & (\mathbf{TM012})(\text{Transposition}) \\ (a \subseteq b)(a \not\subseteq c) &\subseteq (b \not\subseteq c) & (\mathbf{AMX35})(\text{Exportation/Importation}) \text{ for truth sets} \\ (a \subseteq b)(a \not\subseteq c) &\subseteq (a \subseteq b)(b \not\subseteq c) & (\mathbf{TM256})(\text{Conditional Subsets Rule}) \\ ((a \subseteq b)(b \not\subseteq c) = E) &\subseteq ((a \subseteq b)(a \not\subseteq c) \neq \emptyset) & (\mathbf{U_{M007}})(\text{Theorem of Truth Sets})(\text{Eqn1}) \text{ for truth sets} \\ ((a \not\subseteq c) = \emptyset) &\subseteq ((a \subseteq b)(a \not\subseteq c) = \emptyset) & (\mathbf{TM350}) \\ ((a \subseteq b)(a \not\subseteq c) \neq \emptyset) &\subseteq ((a \not\subseteq c) \neq \emptyset) & (\mathbf{TM012})(\text{Transposition})(\text{Eqn2}) \\ ((a \subseteq b)(b \not\subseteq c) = E) &\subseteq ((a \not\subseteq c) \neq \emptyset) & (\mathbf{TM139})(\text{Hypothetical Syllogism Eqns1\&2}) \\ ((a \subseteq b)(b \not\subseteq c) = E) &\subseteq ((a \subseteq c) \neq \sim \emptyset) & (\mathbf{TM008})(\mathbf{TM008})(\text{Inverse of an Equality}) \\ ((a \subseteq b)(b \not\subseteq c) = E) &\subseteq ((a \subseteq c) \neq E) & (\mathbf{TM054})(\text{Negative of Null Set}) \\ ((a \subseteq b) = E)((b \not\subseteq c) = E) &\subseteq ((a \subseteq c) \neq E) & (\mathbf{TM108})(\text{Conjunction Introduction}) \\ (((b \not\subseteq c) = E) = E) &\subseteq (((a \subseteq b) = E) \subseteq ((a \subseteq c) \neq E)) & (\mathbf{AMX35})(\text{Exportation/Importation}) \text{ for truth sets} \\ (((a \not\subseteq b) = E) = E) &\subseteq (((c \subseteq a) = E) \subseteq ((c \subseteq b) \neq E)) & \text{Resymbolising} \\ (((a \not\subseteq b) = E) = E) &\subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) \neq E)) & \text{Resymbolising} \end{aligned}$$

$$(((a \not\subseteq \underline{b}) = E) = E) \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) \neq \emptyset)) \quad (\mathbf{U_{M013}})$$

Proof of (**U_{M013}**)

$$\begin{aligned} (((a \not\subseteq b) = E) = E) &\subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) \neq E)) & (\mathbf{U_{M012}}) \\ (((a \not\subseteq \underline{b}) = E) = E) &\subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq \underline{b}) \neq E)) & \text{Resymbolising} \\ (((a \not\subseteq \underline{b}) = E) = E) &\subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \not\subseteq \underline{b}) \neq \sim E)) & (\mathbf{TM008})(\mathbf{TM008}) \\ (((a \not\subseteq \underline{b}) = E) = E) &\subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \not\subseteq \underline{b}) \neq \emptyset)) & (\mathbf{DM02})(\text{Definition of } \emptyset) \\ (((a \not\subseteq \underline{b}) = E) = E) &\subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) \neq \emptyset)) & (\mathbf{TM090})(\text{Inverse Unit Subset Rule}) \end{aligned}$$

$$(((a \neq E) = E) = E) \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq a) \neq E)) \quad (\mathbf{U_{M014}})$$

Proof of (**U_{M014}**)

$$\begin{aligned} (((a \not\subseteq b) = E) = E) &\subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) \neq E)) & (\mathbf{U_{M012}})(\text{Eqn1}) \\ \text{Let } a = E & & (\text{Eqn2}) \\ (((E \not\subseteq b) = E) = E) &\subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq b) \neq E)) & (\mathbf{AMX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\ (E \subseteq b) = (b = E) & & (\mathbf{TM275})(\text{Supersets of the Universal Set}) \\ (E \not\subseteq b) = (b \neq E) & & (\mathbf{TM012})(\text{Transposition})(\text{Eqn4}) \\ (((b \neq E) = E) = E) &\subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq b) \neq E)) & (\mathbf{AMX19})(\text{Substitution of Eqn4 into Eqn3}) \\ (((a \neq E) = E) = E) &\subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq a) \neq E)) & \text{Resymbolising} \end{aligned}$$

$$(((a \neq \emptyset) = E) = E) \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq a) \neq \emptyset)) \quad (U_{M015})$$

Proof of (U_{M015})

$$\begin{aligned} &(((a \not\subseteq b) = E) = E) \subseteq (((c^1 \subseteq a) = E) \subseteq ((c^1 \subseteq b) \neq \emptyset)) && (U_{M013}) \text{ (Eqn1)} \\ &\text{Let } a = E && \text{(Eqn2)} \\ &(((E \not\subseteq b) = E) = E) \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq b) \neq \emptyset)) && (A_{MX19}) \text{ (Substitution of Eqn2 into Eqn1)(Eqn3)} \\ &(E \subseteq b) = (b = E) && (T_{M275}) \text{ (Supersets of the Universal Set)} \\ &(E \not\subseteq b) = (b \neq E) && (T_{M012}) \text{ (Transposition)} \\ &(E \not\subseteq b) = (b \neq \sim E) && (T_{M008})(T_{M008}) \\ &(E \not\subseteq b) = (b \neq \emptyset) && (D_{M02}) \text{ (Definition of } \emptyset \text{)(Eqn4)} \\ &(((b \neq \emptyset) = E) = E) \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq b) \neq \emptyset)) && (A_{MX19}) \text{ (Substitution of Eqn4 into Eqn3)} \\ &(((a \neq \emptyset) = E) = E) \subseteq (((c^1 \subseteq E) = E) \subseteq ((c^1 \subseteq a) \neq \emptyset)) && \text{Resymbolising} \end{aligned}$$

3.2 Theorems of Series

$$\sim(b_1 \dots b_n) = (\underline{b_1} \dots \underline{b_n}) \quad (T_{M187}) \text{ DeMorgan's Law for Series}$$

$$(\underline{b_1} \dots \underline{b_n}) = \sim(b_1 \dots b_n) \quad (T_{M188}) \text{ DeMorgan's Law for Series}$$

Example Proof of (T_{M187})

$$\begin{aligned} &\text{Let } a_r = (\sim(b_1 \dots b_r) = (\underline{b_1} \dots \underline{b_r})) && \text{(Eqn1)} \\ &\text{Let } r = 1 && \text{(Eqn2)} \\ &a_1 = (\sim(b_1 \dots b_1) = (\underline{b_1} \dots \underline{b_1})) && (A_{MX19}) \text{ (Substitution of Eqn2 into Eqn1)} \\ &a_1 = (\sim b_1 = (\underline{b_1} \dots \underline{b_1})) && (T_{M667}) \text{ (Conjunction Tautology)} \\ &a_1 = (\sim b_1 = \underline{b_1}) && (A_{MX01}) \text{ (Disjunction Tautology)} \\ &a_1 = E && (T_{M001}) \text{ (Universality of } (D_{M03}) \text{ Definition of } \sim \text{)(Eqn3)} \\ &\text{Let } \sim(b_1 \dots b_r) = (\underline{b_1} \dots \underline{b_r}) && \text{(Eqn4)} \\ &\sim(b_1 \dots b_r) + \sim b_{r+1} = (\underline{b_1} \dots \underline{b_r}) + \sim b_{r+1} && (T_{M232}) \text{ (Deduction)} \\ &\sim(b_1 \dots b_r) + \sim b_{r+1} = (\underline{b_1} \dots \underline{b_r}) + \underline{b_{r+1}} && (D_{M03}) \text{ (Definition of } \sim \text{)} \\ &\sim(b_1 \dots b_r) + \sim b_{r+1} = (\underline{b_1} \dots \underline{b_{r+1}}) && \text{Simplified form of expression} \\ &\sim(b_1 \dots b_r \cdot b_{r+1}) = (\underline{b_1} \dots \underline{b_{r+1}}) && (T_{M025}) \text{ (DeMorgan's Law)} \\ &\sim(b_1 \dots b_{r+1}) = (\underline{b_1} \dots \underline{b_{r+1}}) && \text{Simplified form of expression} \\ &\text{Thus } (\sim(b_1 \dots b_r) = (\underline{b_1} \dots \underline{b_r})) \subseteq (\sim(b_1 \dots b_{r+1}) = (\underline{b_1} \dots \underline{b_{r+1}})) && (A_{MX02}) \text{ (Conditional Introduction)(Eqn5)} \\ &a_r \subseteq a_{r+1} && (A_{MX19}) \text{ (Substitution of Eqn1 into Eqn5)(Eqn6)} \\ &\text{Thus } (a_1 = E) \dots (a_r = E) && (A_{MX38}) \text{ (Axiom of Mathematical Induction Eqns3\&6)} \\ &\text{Thus } (a_n = E) && (A_{MX02}) \text{ (Conditional Introduction)(Eqn7)} \\ &(\sim(b_1 \dots b_n) = (\underline{b_1} \dots \underline{b_n})) = E && (A_{MX19}) \text{ (Substitution of Eqn1 into Eqn7)} \\ &\sim(b_1 \dots b_n) = (\underline{b_1} \dots \underline{b_n}) && (A_{MX16}) \text{ (Order Axiom)} \end{aligned}$$

$$b_1 \dots b_n \subseteq b_1 \dots b_{n+r} \quad (T_{M189})$$

$$b_1 \dots b_{n+r} \subseteq b_1 \dots b_n \quad (T_{M190})$$

Example Proof of (T_{M189})

$$\begin{aligned} &a \subseteq a + c && (T_{M082}) \text{ (Disjunction Introduction) (Eqn1)} \\ &\text{Let } b_1 \dots b_n = a && \text{(Eqn2)} \\ &\text{Let } b_{n+1} \dots b_{n+r} = c && \text{(Eqn3)} \\ &b_1 \dots b_n \subseteq b_1 \dots b_n + b_{n+1} \dots b_{n+r} && (A_{MX19}) \text{ (Substitution of Eqns2\&3 into Eqn1)} \\ &b_1 \dots b_n \subseteq b_1 \dots b_{n+r} && \text{Simplified form of expression} \end{aligned}$$

$$(a \subseteq b_1 \dots b_n) = (a \subseteq b_1) \dots (a \subseteq b_n) \quad (T_{M191}) \text{ Composition for a Series}$$

$$(b_1 \dots b_n \subseteq a) = (b_1 \subseteq a) \dots (b_n \subseteq a) \quad (T_{M192}) \text{ Conjunction of Sufficiencies for a Series}$$

Example Proof of (T_{M192})

$$\begin{aligned} &\text{Let } c_r = ((b_1 \dots b_r \subseteq a) = (b_1 \subseteq a) \dots (b_r \subseteq a)) && \text{(Eqn1)} \\ &\text{Let } r = 1 && \text{(Eqn2)} \\ &c_1 = ((b_1 \dots b_1 \subseteq a) = (b_1 \subseteq a) \dots (b_1 \subseteq a)) && (A_{MX19}) \text{ (Substitution of Eqn2 into Eqn1)} \end{aligned}$$

$c_1 = ((b_1+++ b_1 \subseteq a) = (b_1 \subseteq a))$	(T_{M667}) (Conjunction Tautology)
$c_1 = ((b_1 \subseteq a) = (b_1 \subseteq a))$	(A_{MX07}) (Disjunction Tautology)
$c_1 = E$	(T_{M007}) (Universality of (D_{M00}) Definition of $=$)(Eqn3)
Let $(b_1+++ b_r \subseteq a) = (b_1 \subseteq a) \dots (b_r \subseteq a)$	(Eqn4)
$(b_1+++ b_r \subseteq a) \cdot (b_{r+1} \subseteq a) = (b_1 \subseteq a) \dots (b_r \subseteq a) \cdot (b_{r+1} \subseteq a)$	(T_{M233}) (Deduction)
$(b_1+++ b_r \subseteq a) \cdot (b_{r+1} \subseteq a) = (b_1 \subseteq a) \dots (b_{r+1} \subseteq a)$	Simplified form of expression
$(b_1+++ b_r + b_{r+1} \subseteq a) \subseteq a = (b_1 \subseteq a) \dots (b_{r+1} \subseteq a)$	(T_{M131}) (Conjunction of Sufficiencies)
$(b_1+++ b_{r+1} \subseteq a) = (b_1 \subseteq a) \dots (b_{r+1} \subseteq a)$	Simplified form of expression
Thus $((b_1+++ b_r \subseteq a) = (b_1 \subseteq a) \dots (b_r \subseteq a)) \subseteq ((b_1+++ b_{r+1} \subseteq a) = (b_1 \subseteq a) \dots (b_{r+1} \subseteq a))$	(A_{MX02}) (Conditional Introduction)(Eqn5)
$c_r \subseteq c_{r+1}$	(A_{MX19}) (Substitution of Eqn1 into Eqn5)(Eqn6)
Thus $(c_1 = E) \dots (c_E = E)$	(A_{MX38}) (Axiom of Mathematical Induction Eqns3&6)
Thus $(c_n = E)$	(A_{MX02}) (Conditional Introduction)(Eqn7)
$((b_1+++ b_n \subseteq a) = (b_1 \subseteq a) \dots (b_n \subseteq a)) = E$	(A_{MX19}) (Substitution of Eqn1 into Eqn7)
$(b_1+++ b_n \subseteq a) = (b_1 \subseteq a) \dots (b_n \subseteq a)$	(A_{MX16}) (Order Axiom)

$b_1 \dots b_n \subseteq b_1+++ b_n$

(T_{M192a}) Conjunction a Subset of Disjunction for Series

Proof of (T_{M192a})

Let $c_r = (b_1 \dots b_r \subseteq b_1+++ b_r)$	(Eqn1)
Let $r = 1$	(Eqn2)
$c_1 = (b_1 \dots b_1 \subseteq b_1+++ b_1)$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)
$c_1 = (b_1 \dots b_1 \subseteq b_1)$	(A_{MX07}) (Disjunction Tautology)
$c_1 = (b_1 \subseteq b_1)$	(T_{M667}) (Conjunction Tautology)
$c_1 = (b_1 \subseteq b_1) + (b_1 = b_1)$	(T_{M266}) (Definition of $\subseteq$)
$c_1 = (b_1 \subseteq b_1) + E$	(T_{M534})
$c_1 = E$	(T_{M055}) (Universal Set Disjunction)(Eqn3)
Let $b_1 \dots b_r \subseteq b_1+++ b_r$	(Eqn4)
$b_1 \dots b_r \cdot b_{r+1} \subseteq b_1+++ b_r + b_{r+1}$	(T_{M146}) (Conjunction a Subset of Disjunction)
$b_1 \dots b_{r+1} \subseteq b_1+++ b_{r+1}$	Simplified form of expression
Thus $(b_1 \dots b_r \subseteq b_1+++ b_r) \subseteq (b_1 \dots b_{r+1} \subseteq b_1+++ b_{r+1})$	(A_{MX02}) (Conditional Introduction)(Eqn5)
$c_r \subseteq c_{r+1}$	(A_{MX19}) (Substitution of Eqn1 into Eqn5)(Eqn6)
Thus $(c_1 = E) \dots (c_E = E)$	(A_{MX38}) (Axiom of Mathematical Induction Eqns3&6)
Thus $(c_n = E)$	(A_{MX02}) (Conditional Introduction)(Eqn7)
$(b_1 \dots b_n \subseteq b_1+++ b_n) = E$	(A_{MX19}) (Substitution of Eqn1 into Eqn7)
$b_1 \dots b_n \subseteq b_1+++ b_n$	(A_{MX16}) (Order Axiom)

$[a_n+++]_{n=1}^E = [a_n^{1+++}]_{n=1}^E$

(T_{M228})

Proof of (T_{M228})

$[a_n+++]_{n=1}^E = E$	(D_{M10}) (Definition of Universal Set using Sets) (Eqn1)
$[a_n^{1+++}]_{n=1}^E = E$	(D_{M11}) (Definition of Universal Set using Entities) (Eqn2)
$[a_n+++]_{n=1}^E = [a_n^{1+++}]_{n=1}^E$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)

$(a \subseteq b) = [(a = b \cdot c_n)+++]_{n=1}^E$

(T_{M227})

Proof of (T_{M227})

Let $a \subseteq b$	(Eqn1)
$a = ab$	(A_{MX44}) ($\subseteq$ Equality Form)(Eqn2)
Let $\sim[(a = bc_n)+++]_{n=1}^E$	
$\sim[(a = bc_n)E+++]_{n=1}^E$	(A_{MX11}) (Universal Set Conjunction Axiom)
$\sim[(a = bc_n)((c_n = a) + (c_n \neq a))+++]_{n=1}^E$	(T_{M671}) (Tertium non Datur)
$\sim[((a = bc_n)(c_n = a) + (a = bc_n)(c_n \neq a))+++]_{n=1}^E$	(T_{M407}) (Distribution of $\cdot$ over $+$)
$\sim[((a = ba) + (a = bc_n)(c_n \neq a))+++]_{n=1}^E$	(A_{MX19}) (Substitution)
$\sim[((a = ba)+++]_{n=1}^E + [(a = bc_n)(c_n \neq a)+++]_{n=1}^E)$	(T_{M405}) (Associative $+$)
$\sim((a = ba) + [(a = bc_n)(c_n \neq a)+++]_{n=1}^E)$	(A_{MX07}) (Disjunction Tautology)(Eqn3)
$\sim((a = a) + [((a = b + c_n)(c_n \neq a))+++]_{n=1}^E)$	(A_{MX19}) (Substitution of Eqn2 into Eqn3)
$\sim(E + [((a = b + c_n)(c_n \neq a))+++]_{n=1}^E)$	(T_{M007}) (Universality of (D_{M00}) Definition of $=$)

$\sim E$	(T_{M055})(Disjunction with Universal Set)
$\emptyset$	(D_{M02})(Definition of $\emptyset$)
Thus $(a \subseteq b) \cdot \sim[(a = b \cdot c_n)+++]^E_{n=1} = \emptyset$	(A_{MX02})(Conditional Introduction)
$(a \subseteq b) \subseteq [(a = b \cdot c_n)+++]^E_{n=1}$	(T_{M735})($\subseteq$ Existential Form)(Eqn4)
Let $a \not\subseteq b$	
$(a \not\subseteq b) = E$	(A_{MX16})(Order Axiom)
$(a \subseteq b) = \sim E$	(T_{M008})(Inverse of an Equality)
$(a \subseteq b) = \emptyset$	(D_{M02})(Definition of $\emptyset$)(Eqn5)
Let $[(a = b \cdot c_n)+++]^E_{n=1}$	
$[(a \subseteq b)+++]^E_{n=1}$	(T_{M073b})(Single Premise Syllogism)
$(a \subseteq b)$	(A_{MX01})(Disjunction Tautology) (Eqn6)
$\emptyset$	(A_{MX19})(Substitution of Eqn5 into Eqn6)
Thus $(a \not\subseteq b) \cdot [(a = b \cdot c_n)+++]^E_{n=1} = \emptyset$	(A_{MX02})(Conditional Introduction)
$[(a = b \cdot c_n)+++]^E_{n=1} \subseteq (a \subseteq b)$	(T_{M735})($\subseteq$ Existential Form)(Eqn7)
$(a \subseteq b) = [(a = b \cdot c_n)+++]^E_{n=1}$	(T_{M430})(Biconditional Introduction Eqns4&7)

$$(a \supseteq b) = [(a = b + c_n)+++]^E_{n=1} \quad (T_{M226})$$

Proof of (T_{M226})

$(a \subseteq b) = [(a = b \cdot c_n)+++]^E_{n=1}$	(T_{M227})
$(a \subseteq b) = [(\underline{a} = \sim(b \cdot c_n))+++]^E_{n=1}$	(T_{M008})(Inverse of an Equality)
$(a \subseteq b) = [(\underline{a} = \underline{b} + \sim c_n)+++]^E_{n=1}$	(T_{M025})(DeMorgan's Law)
$(\underline{b} \subseteq \underline{a}) = [(\underline{a} = \underline{b} + \sim c_n)+++]^E_{n=1}$	(T_{M012})(Transposition)
$(b \subseteq a) = [(a = b + c_n)+++]^E_{n=1}$	Resymbolising
$(a \supseteq b) = [(a = b + c_n)+++]^E_{n=1}$	(T_{M196})($\subseteq$ Reverse of $\supseteq$)

$$(c^1 \subseteq a) + (c^1 \subseteq b) = (c^1 \subseteq a + b) \quad (T_{M526})$$

Proof of (T_{M526})

$(c^1 \not\subseteq a) + (c^1 \not\subseteq b) = (c^1 \not\subseteq ab)$	(T_{M173})(Disjunction of Unnecessities)
$((c^1 \neq \emptyset) = E) \subseteq ((c^1 \subseteq \underline{a}) + (c^1 \subseteq \underline{b}) = (c^1 \subseteq \sim(ab)))$	(T_{M090})(Inverse Unit Subset Rule)
$((c^1 \neq \emptyset) = E) \subseteq ((c^1 \subseteq \underline{a}) + (c^1 \subseteq \underline{b}) = (c^1 \subseteq \underline{a} + \underline{b}))$	(T_{M025})(DeMorgan's Law)
$((c^1 \neq \emptyset) = E) \subseteq ((c^1 \subseteq a) + (c^1 \subseteq b) = (c^1 \subseteq a + b))$	Resymbolising
$(c^1 \neq \emptyset)((c^1 \subseteq a) + (c^1 \subseteq b)) = (c^1 \neq \emptyset)(c^1 \subseteq a + b)$	(T_{M444b})(Biconditional Exportation/Importation)(Eqn1)
Let $c^1 = \emptyset$	for truth sets
Let $(c^1 \subseteq a) + (c^1 \subseteq b) \neq (c^1 \subseteq a + b)$	(Eqn2)
$(\emptyset \subseteq a) + (\emptyset \subseteq b) \neq (\emptyset \subseteq a + b)$	(Eqn3)
$E + E \neq E$	(A_{MX19})(Substitution of Eqn2 into Eqn3)
$E \neq E$	(T_{M007})(Universality of (T_{M670}) Null Set Theorem)
$\emptyset$	(A_{MX01})(Disjunction Tautology)
Thus $((c^1 = \emptyset) = E) \cdot ((c^1 \subseteq a) + (c^1 \subseteq b) \neq (c^1 \subseteq a + b)) = \emptyset$	(T_{M345})
$((c^1 = \emptyset) = E) \subseteq ((c^1 \subseteq a) + (c^1 \subseteq b) = (c^1 \subseteq a + b))$	(A_{MX02})(Conditional Introduction)
$(c^1 = \emptyset)((c^1 \subseteq a) + (c^1 \subseteq b)) = (c^1 = \emptyset)(c^1 \subseteq a + b)$	(T_{M735})($\subseteq$ Existential Form)
$(c^1 = \emptyset)((c^1 \subseteq a) + (c^1 \subseteq b)) = (c^1 = \emptyset)(c^1 \subseteq a + b)$	(T_{M444b})(Biconditional Exportation/Importation)(Eqn4)
$(c^1 = \emptyset)((c^1 \subseteq a) + (c^1 \subseteq b)) + (c^1 \neq \emptyset)((c^1 \subseteq a) + (c^1 \subseteq b)) = (c^1 = \emptyset)(c^1 \subseteq a + b) + (c^1 \neq \emptyset)(c^1 \subseteq a + b)$	for truth sets
$((c^1 = \emptyset) + (c^1 \neq \emptyset))((c^1 \subseteq a) + (c^1 \subseteq b)) = ((c^1 = \emptyset) + (c^1 \neq \emptyset))(c^1 \subseteq a + b)$	(A_{MX21})(Combination of Eqns1&4)
$E((c^1 \subseteq a) + (c^1 \subseteq b)) = E(c^1 \subseteq a + b)$	(T_{M407})(Distribution of $\cdot$ over $+$)
$(c^1 \subseteq a) + (c^1 \subseteq b) = (c^1 \subseteq a + b)$	(T_{M671})(Tertium non Datur)
	(A_{MX11})(Universal Set Conjunction Axiom)

$$((b = b_1^1 +++ b_n^1) = E) \subseteq ((c^1 \subseteq b) = (c^1 \subseteq b_1^1) +++ (c^1 \subseteq b_n^1)) \quad (T_{M527})$$

Proof of (T_{M527})

Let $b = b_1^1 +++ b_n^1$	(Eqn1)
$(c^1 \subseteq b) = (c^1 \subseteq b)$	(D_{M00})(Definition of $=$)(Eqn2)

$$\begin{aligned}
(c^1 \subseteq b) &= (c^1 \subseteq b_1^1 \text{ +++ } b_n^1) && (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\
(c^1 \subseteq b) &= (c^1 \subseteq b_1^1) \text{ +++ } (c^1 \subseteq b_n^1) && (T_{M526}) \\
\text{Thus } ((b = b_1^1 \text{ +++ } b_n^1) = E) &\subseteq ((c^1 \subseteq b) = (c^1 \subseteq b_1^1) \text{ +++ } (c^1 \subseteq b_n^1)) && (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$(c^1 \neq \emptyset)(c^1 \neq \emptyset^1)(c^1 \subseteq b)(b = b_1^1 \text{ +++ } b_n^1) \subseteq (c^1 = b_1^1) \text{ +++ } (c^1 = b_n^1) \quad (T_{M528})$$

Proof of (T_{M528})

$$\begin{aligned}
\text{Let } c^1 &\subseteq b && (\text{Eqn1}) \\
\text{Let } b &= b_1^1 \text{ +++ } b_n^1 && (\text{Eqn2}) \\
\text{Let } c^1 &\neq \emptyset \\
(c^1 \neq \emptyset) &= E && (A_{MX16})(\text{Order Axiom}) \\
(c^1 = \emptyset) &= \sim E && (T_{M008})(\text{Inverse of an Equality}) \\
(c^1 = \emptyset) &= \emptyset && (D_{M02})(\text{Definition of } \emptyset)(\text{Eqn3}) \\
(c^1 \subseteq b_1^1) \text{ +++ } (c^1 \subseteq b_n^1) &&& (T_{M527}) \\
(c^1 \subseteq b_1^1) \text{ +++ } (c^1 \subseteq b_n^1) + (c^1 = b_1^1) \text{ +++ } (c^1 = b_n^1) &&& (T_{M266})(\text{Definition of } \subseteq)(\text{Eqn4}) \\
(c^1 \subseteq b_r^1) &= (b_r^1 \neq \emptyset)(c^1 = \emptyset) && (A_{MX40})(\text{Proper Subset of Unit Set Axiom}) \\
(c^1 \subseteq b_r^1) &\subseteq (c^1 = \emptyset) && (T_{M073b})(\text{Single Premise Syllogism})(\text{Eqn5}) \\
(c^1 \subseteq b_r^1) &\subseteq \emptyset && (A_{MX19})(\text{Substitution of Eqn3 into Eqn5}) \\
(c^1 \subseteq b_r^1) &= \emptyset && (T_{M052})(\text{Subsets of the Null Set})(\text{Eqn6}) \\
\emptyset \text{ +++ } \emptyset + (c^1 = b_1^1) \text{ +++ } (c^1 = b_n^1) &&& (A_{MX19})(\text{For } r = 1 \text{ to } n \text{ Substitution of Eqn6 into Eqn4}) \\
\emptyset + (c^1 = b_1^1) \text{ +++ } (c^1 = b_n^1) &&& (A_{MX01})(\text{Disjunction Tautology}) \\
(c^1 = b_1^1) \text{ +++ } (c^1 = b_n^1) &&& (T_{M668})(\text{Null Set Disjunction}) \\
\text{Thus } (c^1 \neq \emptyset)(c^1 \subseteq b)(b = b_1^1 \text{ +++ } b_n^1) &\subseteq (c^1 = b_1^1) \text{ +++ } (c^1 = b_n^1) && (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

Chapter 4 Proofs of Relationship Theorems – Part 1 – General and Sub-relationships

4.1 Introduction

This chapter sets out the proofs of theorems associated with:

- (1) General Relationship Theory
- (2) Relationship Symmetry
- (3) Theory of Sub-relationships

The proofs are not however set out in these categories but rather in the order in which they are proven from axioms, definitions, and previous theorems.

4.2 Proofs Using Membership Theory Only

$$(a = b) \subseteq (c * a = c * b) \quad (T_{R522}) \text{ Relational Deduction Theorem}$$

Proof of (T_{R522})

$$\begin{array}{ll} c * b = c * b & (D_{M00})(\text{Definition of } =)(\text{Eqn1}) \\ \text{Let } a = b & (\text{Eqn2}) \\ c * a = c * b & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \text{Thus } (a = b) \subseteq (c * a = c * b) & (A_{MX02})(\text{Conditional Introduction}) \end{array}$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq R_x) = (a^1 \subseteq \sim R_x)) \quad (T_{T701})$$

Proof of (T_{T701})

$$\begin{array}{ll} ((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq b) = (a^1 \subseteq b)) & (T_{M090}) (\text{Inverse Unit Subset Rule})(\text{Eqn1}) \\ \text{Let } b = \sim R_x & (\text{Eqn2}) \\ ((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq R_x) = (a^1 \subseteq \sim R_x)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{array}$$

4.3 Proofs Using the Axioms and Definitions of Relationship Theory Only

$$(c_1^{1+++} c_n^1) * (b_1^1 +++ b_m^1) = (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) +++ (c_1^1 * b_m^1 +++ c_n^1 * b_m^1)$$

(T_{T747}) Definition of the Union Function Operator

Proof of (T_{T747})

$$\begin{array}{ll} c * (b_1^1 +++ b_m^1) = c * b_1^1 +++ c * b_m^1 & (A_{S64})(\text{Distribution of } * \text{ over } +) \\ (c_1^{1+++} c_n^1) * (b_1^1 +++ b_m^1) = (c_1^{1+++} c_n^1) * b_1^1 +++ (c_1^{1+++} c_n^1) * b_m^1 & \text{Resymbolising} \\ (c_1^{1+++} c_n^1) * (b_1^1 +++ b_m^1) = (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) +++ (c_1^1 * b_m^1 +++ c_n^1 * b_m^1) & (A_{S65})(\text{Distribution of } + \text{ over } *) \end{array}$$

$$(c_1^{1+++} c_n^1) \diamond (b_1^1 +++ b_m^1) = (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) \dots (c_1^1 * b_m^1 +++ c_n^1 * b_m^1)$$

(T_{T748}) Definition of the Intersection Function Operator

Proof of (T_{T748})

$$\begin{array}{ll} c \diamond (b_1^1 +++ b_m^1) = c * b_1^1 \dots c * b_m^1 & (D_{S15})(\text{Definition of the Intersection Function Operator}) \\ (c_1^{1+++} c_n^1) \diamond (b_1^1 +++ b_m^1) = (c_1^{1+++} c_n^1) * b_1^1 \dots (c_1^{1+++} c_n^1) * b_m^1 & \text{Resymbolising} \\ (c_1^{1+++} c_n^1) \diamond (b_1^1 +++ b_m^1) = (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) \dots (c_1^1 * b_m^1 +++ c_n^1 * b_m^1) & (A_{S65})(\text{Distribution of } + \text{ over } *) \end{array}$$

$$(c_1^1 +++ c_n^1) * b^1 = c_1^1 * b^1 +++ c_n^1 * b^1 \quad (T_{S284}) \text{ Distribution of } + \text{ over } *$$

Proof of (T_{S284})

$$(c_1^1 +++ c_n^1) * b = c_1^1 * b +++ c_n^1 * b \quad (A_{S65}) \text{ (Distribution of } + \text{ over } *)$$

$$(c_1^1 +++ c_n^1) * b^1 = c_1^1 * b^1 +++ c_n^1 * b^1 \quad \text{Resymbolising}$$

$$c^1 * (b_1^1 +++ b_n^1) = c^1 * b_1^1 +++ c^1 * b_n^1 \quad (T_{S282}) \text{ Distribution of } * \text{ over } +$$

Proof of (T_{S282})

$$c * (b_1^1 +++ b_m^1) = c * b_1^1 +++ c * b_m^1 \quad (A_{S64}) \text{ (Distribution of } * \text{ over } +)$$

$$c^1 * (b_1^1 +++ b_n^1) = c^1 * b_1^1 +++ c^1 * b_n^1 \quad \text{Resymbolising}$$

$$(b \neq \emptyset) = (c \circ b = c * b) \quad (T_{A199})$$

Proof of (T_{A199})

$$(b \neq \emptyset) = (c * b = c \circ b) \quad (A_{R10}) \text{ (Axiom of Complementary Relationship Sets)}$$

$$(b \neq \emptyset) = (c * b = c \circ b) \quad \text{Resymbolising}$$

4.4 Proofs Using The Axioms and Definitions of Membership and Relationship Theory

The following theorems can be derived by extraction from:

$$(X = S, Q)(a = \sim \emptyset^1, \sim E^1): (Xa = E) \quad (A_{S66}) \text{ (Subjects and Objects of } \sim E^1 \text{ and } \sim \emptyset^1)$$

$$S \sim \emptyset^1 = E \quad (T_{T755})$$

$$S \sim E^1 = E \quad (T_{T756})$$

$$Q \sim \emptyset^1 = E \quad (T_{T757})$$

$$Q \sim E^1 = E \quad (T_{T758})$$

The following theorems are proven below.

$$SE^1 = E^1 \quad (T_{T752})$$

$$QE^1 = E^1 \quad (T_{T754})$$

$$S \emptyset^1 = \emptyset^1 \quad (T_{T751})$$

$$Q \emptyset^1 = \emptyset^1 \quad (T_{T753})$$

Example Proof of (T_{T752}) & (T_{T754})

$$(a^1 = E^1, \emptyset^1): (a^1 \subseteq \sim R_R) \quad (A_{S77}) \text{ (Nature of } E^1 \text{ and } \emptyset^1)$$

$$E^1 \subseteq \sim R_R \quad \text{Extraction}$$

$$(E^1 = SE^1). (E^1 = QE^1) \quad (D_{S16}) \text{ (Definition of a Physical Object)}$$

$$(b = b_1^1 +++ b_m^1) \subseteq (c * b = c * b_1^1 +++ c * b_m^1) \quad (T_{R001a}) \text{ Distribution of } * \text{ over } +$$

Proof of (T_{R001a})

$$c * (b_1^1 +++ b_m^1) = c * b_1^1 +++ c * b_m^1 \quad (A_{S64}) \text{ (Distribution of } * \text{ over } +) \text{ (Eqn1)}$$

$$\text{Let } b = b_1^1 +++ b_m^1 \quad \text{(Eqn2)}$$

$$c * b = c * b_1^1 +++ c * b_m^1 \quad (A_{MX19}) \text{ (Substitution of Eqn2 into Eqn1)}$$

$$\text{Thus } (b = b_1^1 +++ b_m^1) \subseteq (c * b = c * b_1^1 +++ c * b_m^1) \quad (A_{MX02}) \text{ (Conditional Introduction)}$$

$$(c = c_1^1 +++ c_n^1) \subseteq (c * b = c_1^1 * b +++ c_n^1 * b) \quad (T_{R002a}) \text{ Distribution of } + \text{ over } *$$

Proof of (T_{R002a})

$$(c_1^1 +++ c_n^1) * b = c_1^1 * b +++ c_n^1 * b \quad (A_{S65}) \text{ (Distribution of } + \text{ over } *) \text{ (Eqn1)}$$

$$\begin{aligned}
&\text{Let } \mathbf{c} = \mathbf{c}_1^1 \vdash \vdash \vdash \mathbf{c}_n^1 && (\text{Eqn2}) \\
&\mathbf{c} * \mathbf{b} = \mathbf{c}_1^1 * \mathbf{b} \vdash \vdash \vdash \mathbf{c}_n^1 * \mathbf{b} && (\mathbf{A}_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\
&\text{Thus } (\mathbf{c} = \mathbf{c}_1^1 \vdash \vdash \vdash \mathbf{c}_n^1) \subseteq (\mathbf{c} * \mathbf{b} = \mathbf{c}_1^1 * \mathbf{b} \vdash \vdash \vdash \mathbf{c}_n^1 * \mathbf{b}) && (\mathbf{A}_{MX02}) (\text{Conditional Introduction})
\end{aligned}$$

$$(\mathbf{b} = \mathbf{b}_1^1 \vdash \vdash \vdash \mathbf{b}_m^1) \subseteq (\mathbf{c}^1 * \mathbf{b} = \mathbf{c}^1 * \mathbf{b}_1^1 \vdash \vdash \vdash \mathbf{c}^1 * \mathbf{b}_m^1) \quad (\mathbf{T}_{R002}) \text{ Distribution of } * \text{ over } \vdash$$

Proof of ($\mathbf{T}_{R002}$)

$$\begin{aligned}
&\text{Let } \mathbf{b} = \mathbf{b}_1^1 \vdash \vdash \vdash \mathbf{b}_m^1 && (\text{Eqn1}) \\
&\text{Let } \mathbf{c} = \mathbf{c}^1 && (\text{Eqn2}) \\
&\mathbf{c} * (\mathbf{b}_1^1 \vdash \vdash \vdash \mathbf{b}_m^1) = \mathbf{c} * \mathbf{b}_1^1 \vdash \vdash \vdash \mathbf{c} * \mathbf{b}_m^1 && (\mathbf{A}_{S64}) (\text{Distribution of } * \text{ over } \vdash) (\text{Eqn3}) \\
&\mathbf{c}^1 * \mathbf{b} = \mathbf{c}^1 * \mathbf{b}_1^1 \vdash \vdash \vdash \mathbf{c}^1 * \mathbf{b}_m^1 && (\mathbf{A}_{MX19}) (\text{Substitution of Eqns1\&2 into Eqn3}) \\
&\text{Thus } (\mathbf{b} = \mathbf{b}_1^1 \vdash \vdash \vdash \mathbf{b}_m^1) \subseteq (\mathbf{c}^1 * \mathbf{b} = \mathbf{c}^1 * \mathbf{b}_1^1 \vdash \vdash \vdash \mathbf{c}^1 * \mathbf{b}_m^1) && (\mathbf{A}_{MX02}) (\text{Conditional Introduction})
\end{aligned}$$

$$(\mathbf{c} = \mathbf{c}_1^1 \vdash \vdash \vdash \mathbf{c}_n^1) \subseteq (\mathbf{c} * \mathbf{b}^1 = \mathbf{c}_1^1 * \mathbf{b}^1 \vdash \vdash \vdash \mathbf{c}_n^1 * \mathbf{b}^1) \quad (\mathbf{T}_{R001}) \text{ Distribution of } \vdash \text{ over } *$$

Proof of ($\mathbf{T}_{R001}$)

$$\begin{aligned}
&\text{Let } \mathbf{c} = \mathbf{c}_1^1 \vdash \vdash \vdash \mathbf{c}_n^1 && (\text{Eqn1}) \\
&\text{Let } \mathbf{b} = \mathbf{b}^1 && (\text{Eqn2}) \\
&(\mathbf{c}_1^1 \vdash \vdash \vdash \mathbf{c}_n^1) * \mathbf{b} = \mathbf{c}_1^1 * \mathbf{b} \vdash \vdash \vdash \mathbf{c}_n^1 * \mathbf{b} && (\mathbf{A}_{S65}) (\text{Distribution of } \vdash \text{ over } *) (\text{Eqn3}) \\
&\mathbf{c} * \mathbf{b} = \mathbf{c}_1^1 * \mathbf{b}^1 \vdash \vdash \vdash \mathbf{c}_n^1 * \mathbf{b}^1 && (\mathbf{A}_{MX19}) (\text{Substitution of Eqns1\&2 into Eqn3}) \\
&\text{Thus } (\mathbf{c} = \mathbf{c}_1^1 \vdash \vdash \vdash \mathbf{c}_n^1) \subseteq (\mathbf{c} * \mathbf{b}^1 = \mathbf{c}_1^1 * \mathbf{b}^1 \vdash \vdash \vdash \mathbf{c}_n^1 * \mathbf{b}^1) && (\mathbf{A}_{MX02}) (\text{Conditional Introduction})
\end{aligned}$$

$$(\mathbf{b} = \mathbf{b}_1^1 \vdash \vdash \vdash \mathbf{b}_m^1) \subseteq (\mathbf{c} \diamond \mathbf{b} = \mathbf{c} * \mathbf{b}_1^1 \dots \mathbf{c} * \mathbf{b}_m^1) \quad (\mathbf{T}_{R233}) \text{ Non Distribution of } \diamond \text{ over } \vdash$$

Proof of ($\mathbf{T}_{R233}$)

$$\begin{aligned}
&\text{Let } \mathbf{b} = \mathbf{b}_1^1 \vdash \vdash \vdash \mathbf{b}_m^1 && (\text{Eqn1}) \\
&\mathbf{c} \diamond (\mathbf{b}_1^1 \vdash \vdash \vdash \mathbf{b}_m^1) = \mathbf{c} * \mathbf{b}_1^1 \dots \mathbf{c} * \mathbf{b}_m^1 && (\mathbf{D}_{S15}) (\text{Definition of the Intersection Function Operator}) (\text{Eqn2}) \\
&\mathbf{c} \diamond \mathbf{b} = \mathbf{c} * \mathbf{b}_1^1 \dots \mathbf{c} * \mathbf{b}_m^1 && (\mathbf{A}_{MX19}) (\text{Substitution of Eqn1 into Eqn2}) \\
&\text{Thus } (\mathbf{b} = \mathbf{b}_1^1 \vdash \vdash \vdash \mathbf{b}_m^1) \subseteq (\mathbf{c} \diamond \mathbf{b} = \mathbf{c} * \mathbf{b}_1^1 \dots \mathbf{c} * \mathbf{b}_m^1) && (\mathbf{A}_{MX02}) (\text{Conditional Introduction})
\end{aligned}$$

$$(\mathbf{c}^1 \subseteq \mathbf{R}_0)(\mathbf{b}^1 \subseteq \mathbf{Q}_c)(\mathbf{b}^1 \neq \emptyset) \subseteq ((\mathbf{c} * \mathbf{b}^1) = (\mathbf{c} * \mathbf{b}^1)^1) \quad (\mathbf{T}_{R533}) \text{ Definition of a One-to-One relationship Set}$$

Proof of ($\mathbf{T}_{R533}$)

$$\begin{aligned}
&((\mathbf{c}^1 \subseteq \mathbf{R}_0) = \mathbf{E}) = ((\mathbf{b}^1 \subseteq \mathbf{Q}_c)(\mathbf{b}^1 \neq \emptyset) \subseteq (\mathbf{c} * \mathbf{b}^1 = (\mathbf{c} * \mathbf{b}^1)^1)) && (\mathbf{D}_{R30}) (\text{Definition of a One-to-One Relationship Set}) \\
&(\mathbf{c}^1 \subseteq \mathbf{R}_0)(\mathbf{b}^1 \subseteq \mathbf{Q}_c)(\mathbf{b}^1 \neq \emptyset) \subseteq ((\mathbf{c} * \mathbf{b}^1) = (\mathbf{c} * \mathbf{b}^1)^1) && (\mathbf{A}_{MX35}) (\text{Exportation/Importation}) \text{ for truth sets}
\end{aligned}$$

$$(\mathbf{c}^1 \subseteq \mathbf{R}_P) = ((\mathbf{c}^1)^1 \subseteq \mathbf{R}_P) \quad (\mathbf{T}_{S517})$$

Proof of ($\mathbf{T}_{S517}$)

$$\begin{aligned}
&(\mathbf{c}^1 \subseteq \mathbf{R}_P) = (\mathbf{c}^1 \subseteq \mathbf{R}_0).((\mathbf{c}^1)^1 \subseteq \mathbf{R}_0) && (\mathbf{D}_{R31}) (\text{Definition of a Pairing Relationship Set}) (\text{Eqn1}) \\
&\text{Let } \mathbf{c} = \mathbf{c}^1 && (\text{Eqn2}) \\
&((\mathbf{c}^1)^1 \subseteq \mathbf{R}_P) = ((\mathbf{c}^1)^1 \subseteq \mathbf{R}_0).(\mathbf{c}^1 \subseteq \mathbf{R}_0) && (\mathbf{A}_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) (\text{Eqn3}) \\
&(\mathbf{c}^1 \subseteq \mathbf{R}_P) = ((\mathbf{c}^1)^1 \subseteq \mathbf{R}_P) && (\mathbf{A}_{MX19}) (\text{Substitution of Eqn3 into Eqn1})
\end{aligned}$$

$$(\mathbf{a}^1 \subseteq \mathbf{D} * \mathbf{c}^1) \subseteq (\mathbf{a} = \mathbf{c} * \mathbf{a} + \mathbf{c} * \mathbf{a}) \quad (\mathbf{T}_{S532})$$

Proof of ($\mathbf{T}_{S532}$)

$$\begin{aligned}
&((\mathbf{a} + \mathbf{b})^1 \subseteq \mathbf{D} * \mathbf{c}^1) = (\mathbf{a} = \mathbf{c} * \mathbf{a})(\mathbf{b} = \mathbf{c} * \mathbf{a}) && (\mathbf{D}_{R57}) (\text{Definition of Closure}) (\text{Eqn1}) \\
&\text{Let } \mathbf{b} = \mathbf{a} && (\text{Eqn2}) \\
&((\mathbf{a} + \mathbf{a})^1 \subseteq \mathbf{D} * \mathbf{c}^1) = (\mathbf{a} = \mathbf{c} * \mathbf{a})(\mathbf{a} = \mathbf{c} * \mathbf{a}) && (\mathbf{A}_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\
&(\mathbf{a}^1 \subseteq \mathbf{D} * \mathbf{c}^1) = (\mathbf{a} = \mathbf{c} * \mathbf{a})(\mathbf{a} = \mathbf{c} * \mathbf{a}) && (\mathbf{A}_{MX01}) (\text{Disjunction Tautology}) \\
&(\mathbf{a}^1 \subseteq \mathbf{D} * \mathbf{c}^1) \subseteq (\mathbf{a} + \mathbf{a} = \mathbf{c} * \mathbf{a} + \mathbf{c} * \mathbf{a}) && (\mathbf{A}_{MX21}) (\text{Combination Axiom}) \\
&(\mathbf{a}^1 \subseteq \mathbf{D} * \mathbf{c}^1) \subseteq (\mathbf{a} = \mathbf{c} * \mathbf{a} + \mathbf{c} * \mathbf{a}) && (\mathbf{A}_{MX01}) (\text{Disjunction Tautology})
\end{aligned}$$

$$(\mathbf{c} = \mathbf{d}) \subseteq (\mathbf{c} * \mathbf{b} = \mathbf{d} * \mathbf{b}) \quad (\mathbf{T}_{R522b}) \text{ Relational Deduction Theorem}$$

Proof of ($\mathbf{T}_{R522b}$)

$c*b = c*b$	(D_{M00})(Definition of =)(Eqn1)
Let $c = d$	(Eqn2)
$c*b = d*b$	($AMX19$)(Substitution of Eqn2 into Eqn1)
Thus $(c = d) \subseteq (c*b = d*b)$	($AMX02$)(Conditional Introduction)

$(a = b)(c = d) \subseteq (c*a = d*b)$ (T_{S287}) Relational Deduction Theorem

Proof of (T_{S287})

$c*b = c*b$	(D_{M00})(Definition of =)(Eqn1)
Let $a = b$	(Eqn2)
Let $c = d$	(Eqn3)
$c*a = d*b$	($AMX19$)(Substitution of Eqns2 & 3 into Eqn1)
Thus $(a = b)(c = d) \subseteq (c*a = d*b)$	($AMX02$)(Conditional Introduction)

$((^v c)^1 \subseteq R_0)(b \subseteq Qc) \subseteq (a = c*b)(b = ^v c*a)$ (T_{S533})

Proof of (T_{S533})

$((^v c)^1 \subseteq R_0)(b \subseteq Qc) \subseteq ((c*b + b)^1 \subseteq D*c^1)$	($AR25$)(One-to-One Relationship Closure Axiom)(Eqn1)
Let $c*b = a$	(Eqn2)
$((^v c)^1 \subseteq R_0)(b \subseteq Qc) \subseteq ((a + b)^1 \subseteq D*c^1)$	($AMX19$)(Substitution of Eqn2 into Eqn1)(Eqn3)
$((a + b)^1 \subseteq D*c^1) = (a = c*b)(b = ^v c*a)$	($DR57$)(Definition of Closure)(Eqn4)
$((^v c)^1 \subseteq R_0)(b \subseteq Qc) \subseteq (a = c*b)(b = ^v c*a)$	($AMX19$)(Substitution of Eqn4 into Eqn3)

$^v v a = a$ (T_{R012k}) Sub-relationship Closure Theorem

Proof of (T_{R012k})

$^v v a = ^v v a$	(D_{M00})(Definition of =)
$^v v a = ^v a$	($AS78$)(Relationship Symmetry Axiom)
$^v a = a$	($AS78$)(Relationship Symmetry Axiom)

$^v c = \sim^v \underline{c}$ (T_{R035})

Proof of (T_{R035})

$c = \sim \underline{c}$	($AMX13$)(Double Complement Axiom)
$^v c = \sim^v \underline{c}$	($AS78$)(Relationship Symmetry Axiom)

$^v E = E$ (T_{T793}) Relationship Opposite of E

Proof of (T_{T793})

$^v a = a$	($AS78$)(Relationship Symmetry Axiom)(Eqn1)
Let $a = E$	(Eqn2)
$^v E = E$	($AMX19$)(Substitution of Eqn2 into Eqn1)

$^v(E^1) = E^1$ (T_{T794})

$^v(\emptyset^1) = \emptyset^1$ (T_{T795})

Example Proof of (T_{T794})

$^v a = a$	($AS78$)(Relationship Symmetry Axiom)(Eqn1)
Let $a = E^1$	(Eqn2)
$^v(E^1) = E^1$	($AMX19$)(Substitution of Eqn2 into Eqn1)

$^v(\sim E^1) = \sim E^1$ (T_{T796})

$^v(\sim \emptyset^1) = \sim \emptyset^1$ (T_{T797})

Example Proof of (T_{T796})

$^v a = a$	($AS78$)(Relationship Symmetry Axiom)(Eqn1)
Let $a = \sim E^1$	(Eqn2)

$\forall (\sim E^1) = \sim E^1$	$(A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$
$\forall Sa^1 = \forall (\forall Qa^1)$	(T_{R508}) Reversibility of Matching Relationships
<u>Proof of (T_{R508})</u> $\forall Sa = \forall (\forall Qa)$ $(A_{R13})(\text{Reversibility of Matching Relationships})(\text{Eqn1})$ Let $a = a^1$ (Eqn2) $\forall Sa^1 = \forall (\forall Qa^1)$ $(A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$	
$\forall (a \neq \Omega) = (a \neq \Omega)$	(T_{T819})
<u>Proof of (T_{T819})</u> $\forall (a \neq \Omega) = \forall (a \neq \Omega)$ $(D_{M00})(\text{Definition of } =)$ $\forall (a \neq \Omega) = (a \neq \Omega)$ $(A_{S78})(\text{Relationship Symmetry Axiom})$	
$\forall (c * b) = c * b$	(T_{T799})
$\forall (c \diamond b) = c \diamond b$	(T_{T800})
<u>Example Proof of (T_{T799})</u> $\forall a = a$ $(A_{S78})(\text{Relationship Symmetry Axiom})(\text{Eqn1})$ Let $a = c * b$ (Eqn2) $\forall (c * b) = c * b$ $(A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$	
$(o^1 \subseteq R_B): \forall (a \langle o \rangle b) = (\forall a \langle o \rangle \forall b)$	(T_{T798}) Relationship Opposite of a Proposition
<u>Proof of (T_{T798})</u> $(o^1 \subseteq R_B):$ $\forall a = a$ $(A_{S78})(\text{Relationship Symmetry Axiom})(\text{Eqn1})$ Let $a = (a \langle o \rangle b)$ (Eqn2) $\forall (a \langle o \rangle b) = (a \langle o \rangle b)$ $(A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$ $\forall (a \langle o \rangle b) = (\forall a \langle o \rangle \forall b)$ $(A_{S78})(\text{Relationship Symmetry Axiom})$	
$\forall V = V$	(T_{S548}) Reverse of the Relationship Symmetry Function
<u>Proof of (T_{S548})</u> $V^1 \subseteq R_P R_S R_U$ $(A_{R28})(\text{Nature of } V)$ $V^1 \subseteq R_S$ Extraction $(V^1 \subseteq R_S) = E$ $(A_{MX16})(\text{Order Axiom})(\text{Eqn1})$ $(V^1 \subseteq R_S) = (\forall V = V)$ $(D_{A94})(\text{Definition of a Symmetrical Relationship Set})(\text{Eqn2})$ $E = (\forall V = V)$ $(A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$ $\forall V = V$ $(A_{MX16})(\text{Order Axiom})$	

4.5 Proofs Using Membership Theory and The Axioms and Definitions of Relationship Theory

$\forall Sa \neq \emptyset^1$	(T_{T761})
$\forall Sa \neq E^1$	(T_{T762})
$\forall Qa \neq \emptyset^1$	(T_{T763})
$\forall Qa \neq E^1$	(T_{T764})
$\forall Sa \neq \sim \emptyset^1$	(T_{T765})
$\forall Sa \neq \sim E^1$	(T_{T766})
$\forall Qa \neq \sim \emptyset^1$	(T_{T767})

$$\forall Qa \neq \sim E^1 \quad (T_{768})$$

Example Proof of (T_{761}), (T_{762}), (T_{765}) & (T_{766})

$$(R_X \subseteq R_R) \subseteq (R_X \neq \Omega). (R_X \neq \emptyset). (R_X \neq E)$$

(A_{559})(Axiom of Relationship Sets)

$$(R_X \subseteq R_R) \subseteq (R_X \neq \Omega)$$

(T_{M073})(Single Premise Syllogism)(Eqn1)

$$\text{Let } R_X = \forall Sa$$

(Eqn1)

$$(\forall Sa \subseteq R_R) \subseteq (\forall Sa \neq \Omega)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)

$$(X = \forall S, \forall Q) \subseteq (Xa \subseteq R_R)$$

(A_{568})(Axiom of Relationships)

$$\forall Sa \subseteq R_R$$

Extraction

$$(\forall Sa \subseteq R_R) = E$$

(A_{MX16})(Order Axiom)(Eqn4)

$$E \subseteq (\forall Sa \neq \Omega)$$

(A_{MX19})(Substitution of Eqn4 into Eqn3)

$$E = (\forall Sa \neq \Omega)$$

(T_{M275})(Supersets of the Universal Set)

$$\forall Sa \neq \Omega$$

(A_{MX16})(Order Axiom)

$$\forall Sa \neq \emptyset^1, \sim \emptyset^1, E^1, \sim E^1$$

(D_{M16})(Definition of Ω)

$$(\forall Sa \neq \emptyset^1). (\forall Sa \neq \sim \emptyset^1). (\forall Sa \neq E^1). (\forall Sa \neq \sim E^1)$$

(T_{M716})

$$c \diamond b^1 = c * b^1$$

(T_{R232}) Interchangeability of $*$ and $\diamond$ for Entities

Proof of (T_{R232})

$$\text{For } r = 1 \text{ to } m, \text{ Let } b_r^1 = b_1^1$$

(Eqns1 to m)

$$c \diamond (b_1^1 + \dots + b_m^1) = c * b_1^1 \dots c * b_m^1$$

(D_{S15})(Definition of the Intersection Function Operator)(Eqn2)

$$c \diamond (b_1^1 + \dots + b_m^1) = c * b_1^1 \dots c * b_m^1$$

(A_{MX19})(Substitution of Eqns 1 to m into Eqn2)

$$c \diamond b_1^1 = c * b_1^1 \dots c * b_1^1$$

(A_{MX01})(Disjunction Tautology)

$$c \diamond b_1^1 = c * b_1^1$$

(T_{M667})(Conjunction Tautology)

$$c \diamond b^1 = c * b^1$$

Resymbolising

$$(b^1 \neq \emptyset) = (c * b^1 = c * b^1)$$

(T_{R245})

Proof of (T_{R245})

$$(b \neq \emptyset) = (c * b = c \diamond b)$$

(A_{R10})(Axiom of Complementary Relationship Sets)(Eqn1)

$$\text{Let } b = b^1$$

(Eqn2)

$$(b^1 \neq \emptyset) = (c * b^1 = c \diamond b^1)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(b^1 \neq \emptyset) = (c * b^1 = c \diamond b^1)$$

(T_{M008})(Inverse of an Equality)

$$(b^1 \neq \emptyset) = (c * b^1 = c * b^1)$$

(T_{R232})(Interchangeability of $*$ and $\diamond$ for Entities)

$$(b^1 \neq \emptyset) = (\sim(c * b^1) = (c * b^1))$$

(D_{R04})(Definition of $*$)

$$(b \neq \emptyset) = (c * b = c \diamond b)$$

(T_{A200})

Proof of (T_{A200})

$$(b \neq \emptyset) = (c * b = c \diamond b)$$

(A_{R10})(Axiom of Complementary Relationship Sets)

$$(b \neq \emptyset) = (c * b = c \diamond b)$$

(T_{M008})(Inverse of an Equality)

$$c * b + c \diamond b = E$$

(T_{R249}) Conditional Negative Theorem

Proof of (T_{R249})

$$(c * b) + (c * b) = E$$

(T_{M671})(Tertium non Datur)(Eqn3)

$$c * b + c \diamond b = E$$

(A_{R10})(Axiom of Complementary Relationship Sets)

$$(b \neq \emptyset) = ((c \diamond b = \sim S(c \vee Qb)))$$

(T_{R260})

Proof of (T_{R260})

$$(b \neq \emptyset) = ((c * b) = \sim(c \diamond b))$$

(A_{R10})(Axiom of Complementary Relationship Sets)

$$(b \neq \emptyset) = ((c * b) = \sim(c \diamond b))$$

Resymbolising

$$(b \neq \emptyset) = (\sim(c * b) = (c \diamond b))$$

(T_{M008})(Inverse of an Equality)(Eqn1)

$$c * b = S(c \vee Qb)$$

(A_{R27})(Function Axiom)

$$\sim(c * b) = \sim S(c \vee Qb)$$

(T_{M008})(Inverse of an Equality)(Eqn2)

$$(b \neq \emptyset) = ((c \diamond b = \sim S(c \vee Qb)))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$R_M = \sim R_O \quad (T_{R586})$$

Proof of (T_{R586})

$$\begin{aligned} R_O &= \sim R_M & (A_{R20}) & \text{(One-to-One Relationships Axiom)} \\ R_M &= \sim R_O & (T_{M008}) & \text{(Inverse of an Equality)} \end{aligned}$$

$$R_P \subseteq R_O \quad (T_{R523})$$

Proof of (T_{R523})

$$\begin{aligned} (c^1 \subseteq R_P) &= (c^1 \subseteq R_O)((^V c)^1 \subseteq R_O) & (D_{R31}) & \text{(Definition of a Pairing Relationship Set)} \\ (c^1 \subseteq R_P) &\subseteq (c^1 \subseteq R_O) & (T_{M073}) & \text{(Single Premise Syllogism)} \\ R_P &\subseteq R_O & (T_{M099}) & \text{(Entity Mode / Set Mode Rule)} \end{aligned}$$

$$V^1 \subseteq R_P \quad (T_{R582})$$

Proof of (T_{R582})

$$\begin{aligned} (X &= (V, S, Q)):(X^1 \subseteq R_O) & (A_{R04}) & \text{(One-to-One Sub-relationship Axiom)} \\ V^1 &\subseteq R_O & & \text{Extraction (Eqn1)} \\ (V^1 &\subseteq R_O) = E & (A_{MX16}) & \text{(Order Axiom)(Eqn2)} \\ ^V V &= V & (T_{S548}) & \text{(Reverse of the Relationship Symmetry Function)} \\ (^V V &= V) = E & (A_{MX16}) & \text{(Order Axiom)(Eqn3)} \\ ((^V V)^1 &= V^1) = (^V V = V) & (T_{M479}) & \text{(Eqn4)} \\ ((^V V)^1 &= V^1) = E & (A_{MX19}) & \text{(Substitution of Eqn3 into Eqn4)} \\ (^V V)^1 &= V^1 & (A_{MX16}) & \text{(Order Axiom)(Eqn5)} \\ (^V V)^1 &\subseteq R_O & (A_{MX19}) & \text{(Substitution of Eqn5 into Eqn1)} \\ ((^V V)^1 &\subseteq R_O) = E & (A_{MX16}) & \text{(Order Axiom)(Eqn6)} \\ (V^1 \subseteq R_P) &= (V^1 \subseteq R_O).(^V V)^1 \subseteq R_O & (D_{R31}) & \text{(Definition of a Pairing Relationship Set)(Eqn7)} \\ (V^1 \subseteq R_P) &= E.E & (A_{MX19}) & \text{(Substitution of Eqns2&6 into Eqn7)} \\ (V^1 \subseteq R_P) &= E & (T_{M667}) & \text{(Conjunction Tautology)} \\ V^1 &\subseteq R_P & (A_{MX16}) & \text{(Order Axiom)} \end{aligned}$$

$$(c^1 \subseteq R_P) \subseteq (c^1 \subseteq R_O) \quad (T_{S530})$$

Proof of (T_{S530})

$$\begin{aligned} (c^1 \subseteq R_P) &= (c^1 \subseteq R_O).(^V c)^1 \subseteq R_O & (D_{R31}) & \text{(Definition of a Pairing Relationship Set)} \\ (c^1 \subseteq R_P) &\subseteq (c^1 \subseteq R_O) & (T_{M073b}) & \text{(Single Premise Syllogism)} \end{aligned}$$

$$(c^1 \not\subseteq R_S) = (^V c \neq c) \quad (T_{R583}) \text{ Definition of a Relationship Set which is not Symmetrical}$$

Proof of (T_{R583})

$$\begin{aligned} (c^1 \subseteq R_S) &= (^V c = c) & (D_{A94}) & \text{(Definition of a Symmetrical Relationship Set)} \\ (c^1 \not\subseteq R_S) &= (^V c \neq c) & (T_{M008}) & \text{(Inverse of an Equality)} \end{aligned}$$

$$(c^1 \not\subseteq R_A) = (^V c \not\subseteq c) \quad (T_{R584}) \text{ Definition of a Relationship Set which is not Asymmetrical}$$

Proof of (T_{R584})

$$\begin{aligned} (c^1 \subseteq R_A) &= (^V c \subseteq c) & (D_{A145}) & \text{(Definition of an Asymmetrical Relationship Set)} \\ (c^1 \not\subseteq R_A) &= (^V c \not\subseteq c) & (T_{M008}) & \text{(Inverse of an Equality)} \end{aligned}$$

$$(c + ^V c)^1 \subseteq D * V^1 \quad (T_{R893}) \text{ Theorem of Closed Reverses}$$

Proof of (T_{R893})

$$\begin{aligned} ((a + b)^1 &\subseteq D * c^1) = (a = c * b)(b = ^V c * a) & (D_{R57}) & \text{(Definition of Closure)(Eqn1)} \\ \text{Let } c &= V & & \text{(Eqn2)} \\ \text{Let } a &= c & & \text{(Eqn3)} \\ \text{Let } b &= ^V c & & \text{(Eqn4)} \\ ((c + ^V c)^1 &\subseteq D * V^1) = (c = ^V V c)(^V c = ^V V c) & (A_{MX19}) & \text{(Substitution of Eqns2,3&4 into Eqn1)} \end{aligned}$$

$((c + {}^Vc)^1 \subseteq D * V^1) = (c = c)({}^Vc = {}^Vc)$	(T_{R012k})(Sub-relationship Closure Theorem)
$((c + {}^Vc)^1 \subseteq D * V^1) = (c = c)({}^Vc = {}^Vc)$	(T_{S548})(Reverse of the Relationship Symmetry Function)
$((c + {}^Vc)^1 \subseteq D * V^1) = EE$	(T_{M007})(Universality of (D_{M00}) Definition of =)
$((c + {}^Vc)^1 \subseteq D * V^1) = E$	($AMX17$)(Universal Set Conjunction Axiom)
$(c + {}^Vc)^1 \subseteq D * V^1$	($AMX16$)(Order Axiom)

$$(((a + b)^1 \subseteq D * c^1) = E) \subseteq ((a = c * b) = (b = {}^Vc * a)) \quad (T_{S531})$$

Proof of (T_{S531})

$((a + b)^1 \subseteq D * c^1) = (a = c * b)(b = {}^Vc * a)$	(D_{R57})(Definition of Closure)
$(((a + b)^1 \subseteq D * c^1) = (a = c * b)(b = {}^Vc * a)) = E$	($AMX16$)(Order Axiom)
$(((a + b)^1 \subseteq D * c^1) = E) = ((a = c * b)(b = {}^Vc * a) = E)$	(T_{M094})(Probabilistic Rule)
$(((a + b)^1 \subseteq D * c^1) = E) = ((a = c * b) = E)((b = {}^Vc * a) = E)$	(T_{M108})(Conjunction Introduction)
$(((a + b)^1 \subseteq D * c^1) = E) \subseteq ((a = c * b) = (b = {}^Vc * a))$	($AMX19$)(Substitution)

$$((a + b)^1 \subseteq D * c^1) = ((a + b)^1 \subseteq D * ({}^Vc)^1) \quad (T_{S525})$$

Proof of (T_{S525})

$((a + b)^1 \subseteq D * c^1) = (a = c * b)(b = {}^Vc * a)$	(D_{R57})(Definition of Closure)(Eqn1)
Let $a = b$	(Eqn2)
Let $c = {}^Vc$	(Eqn3)
$((b + a)^1 \subseteq D * ({}^Vc)^1) = (b = {}^Vc * a)(a = c * b)$	($AMX19$)(Substitution of Eqns2&3 into Eqn1)(Eqn4)
$((a + b)^1 \subseteq D * c^1) = ((b + a)^1 \subseteq D * ({}^Vc)^1)$	($AMX19$)(Substitution of Eqn4 into Eqn1)
$((a + b)^1 \subseteq D * c^1) = ((a + b)^1 \subseteq D * ({}^Vc)^1)$	(T_{M403})(Commutative +)

$$((c * b + b)^1 \subseteq D * c^1) = (b = {}^Vc * c * b) \quad (T_{S526})$$

Proof of (T_{S526})

$((a + b)^1 \subseteq D * c^1) = (a = c * b)(b = {}^Vc * a)$	(D_{R57})(Definition of Closure)(Eqn1)
Let $a = c * b$	(Eqn2)
$((c * b + b)^1 \subseteq D * c^1) = (c * b = c * b)(b = {}^Vc * c * b)$	($AMX19$)(Substitution of Eqns2 into Eqn1)
$((c * b + b)^1 \subseteq D * c^1) = E.(b = {}^Vc * c * b)$	(T_{M007})(Universality of (D_{M00}) Definition of =)
$((c * b + b)^1 \subseteq D * c^1) = (b = {}^Vc * c * b)$	($AMX17$)(Universal Set Conjunction Axiom)

$$(a \subseteq b) \subseteq (c * a \subseteq c * b) \quad (T_{R196}) \text{ Relational Deduction}$$

and, by substitution, the special cases:

$(a \subseteq b) = ({}^VSa \subseteq {}^VSb)$	(T_{R023d}) Sub-Relationship Deduction
$(a \supseteq b) = ({}^VSa \supseteq {}^VSb)$	(T_{R023e}) Sub-Relationship Deduction
$(a \subseteq b) = ({}^VQa \subseteq {}^VQb)$	(T_{R024d}) Sub-Relationship Deduction
$(a \supseteq b) = ({}^VQa \supseteq {}^VQb)$	(T_{R024e}) Sub-Relationship Deduction

Proof of (T_{R196})

$(a \subseteq b)(c \subseteq d) \subseteq (c * a \subseteq d * b)$	(A_{R08})(1 st Relational Deduction Axiom)(Eqn1)
Let $d = c$	(Eqn2)
$(a \subseteq b)(c \subseteq c) \subseteq (c * a \subseteq c * b)$	($AMX19$)(Substitution of Eqn2 into Eqn1)(Eqn3)
$c = c$	(D_{M00})(Definition of =)
$c \subseteq c$	(T_{M425})(Biconditional Elimination)
$(c \subseteq c) = E$	($AMX16$)(Order Axiom)(Eqn4)
$(a \subseteq b)E \subseteq (c * a \subseteq c * b)$	($AMX19$)(Substitution of Eqn4 into Eqn3)
$(a \subseteq b) \subseteq (c * a \subseteq c * b)$	($AMX17$)(Universal Set Conjunction Axiom)

$$(c \subseteq d) \subseteq (c * a \subseteq d * a) \quad (T_{R197}) \text{ Relational Deduction}$$

Proof of (T_{R197})

$(a \subseteq b)(c \subseteq d) \subseteq (c * a \subseteq d * b)$	(A_{R08})(1 st Relational Deduction Axiom)(Eqn 1)
Let $b = a$	(Eqn 2)

$(a \subseteq a)(c \subseteq d) \subseteq (c * a \subseteq d * a)$	(A_{MX19}) (Substitution of Eqn2 into Eqn 1)(Eqn3)
$a = a$	(D_{M00}) (Definition of $=$)
$a \subseteq a$	(T_{M425}) (Biconditional Elimination)
$(a \subseteq a) = E$	(A_{MX16}) (Order Axiom)(Eqn4)
$E (c \subseteq d) \subseteq (c * a \subseteq d * a)$	(A_{MX19}) (Substitution of Eqn4 into Eqn 3)
$(c \subseteq d) \subseteq (c * a \subseteq d * a)$	(A_{MX11}) (Universal Set Conjunction Axiom)

$(c * a \not\subseteq d * b) \subseteq (a \not\subseteq b) + (c \not\subseteq d)$ (T_{R198}) Relational Deduction Theorem

Proof of (T_{R198})

$(a \subseteq b)(c \subseteq d) \subseteq (c * a \subseteq d * b)$	(A_{R08}) (1 st Relational Deduction Axiom)
$(c * a \not\subseteq d * b) \subseteq \sim((a \not\subseteq b)(c \not\subseteq d))$	(T_{M012}) (Transposition)
$(c * a \not\subseteq d * b) \subseteq (a \not\subseteq b) + (c \not\subseteq d)$	(T_{M025}) (DeMorgan's Law)

$(o^1 \subseteq R_B):((c \langle o \rangle d) \subseteq (S c \langle o \rangle S d))$ (T_{R513}) Sub-Relationship Deduction

$(o^1 \subseteq R_B):((c \langle o \rangle d) \subseteq (Q c \langle o \rangle Q d))$ (T_{R514}) Sub-Relationship Deduction

$(X = (S, Q))(o^1 \subseteq R_B):(c \langle o \rangle d) \subseteq (X c \langle o \rangle X d)$ (T_{R516}) Sub-Relationship Deduction

and by substitution the special cases:

$(c = d) \subseteq (S c = S d)$	(T_{R026}) Sub-Relationship Deduction
$(c \subset d) \subseteq (S c \subset S d)$	(T_{R026a}) Sub-Relationship Deduction
$(c \supset d) \subseteq (S c \supset S d)$	(T_{R026c}) Sub-Relationship Deduction
$(c \subseteq d) \subseteq (S c \subseteq S d)$	(T_{R026d}) Sub-Relationship Deduction
$(c \supseteq d) \subseteq (S c \supseteq S d)$	(T_{R026e}) Sub-Relationship Deduction
$(c \parallel d) \subseteq (S c \parallel S d)$	(T_{R026b}) Sub-Relationship Deduction
$(c = d) \subseteq (Q c = Q d)$	(T_{R027}) Sub-Relationship Deduction
$(c \subset d) \subseteq (Q c \subset Q d)$	(T_{R027a}) Sub-Relationship Deduction
$(c \supset d) \subseteq (Q c \supset Q d)$	(T_{R027c}) Sub-Relationship Deduction
$(c \subseteq d) \subseteq (Q c \subseteq Q d)$	(T_{R027d}) Sub-Relationship Deduction
$(c \supseteq d) \subseteq (Q c \supseteq Q d)$	(T_{R027e}) Sub-Relationship Deduction
$(c \parallel d) \subseteq (Q c \parallel Q d)$	(T_{R027b}) Sub-Relationship Deduction

Example Proof of (T_{R513})

$(c \subseteq QX)(d \subseteq QX)(o^1 \subseteq R_B):(c \langle o \rangle d) \subseteq (X * c \langle o \rangle X * d)$	(A_{R38}) (2 nd Relational Deduction Axiom)(Eqn1)
Let $X = S$	(Eqn2)
$(c \subseteq QS)(d \subseteq QS)(o^1 \subseteq R_B):(c \langle o \rangle d) \subseteq (S c \langle o \rangle S d)$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)(Eqn3)
$(X = (S, Q))(Y = (V, S, Q, {}^V S, {}^V Q)): (X * Y = E)$	(A_{R34}) (Universal Sub-relationship Axiom)
$QS = E$	Extraction (Eqn4)
$(c \subseteq E)(d \subseteq E)(o^1 \subseteq R_B):(c \langle o \rangle d) \subseteq (S c \langle o \rangle S d)$	(A_{MX19}) (Substitution of Eqn4 into Eqn3)
$E.E.(o^1 \subseteq R_B):(c \langle o \rangle d) \subseteq (S c \langle o \rangle S d)$	(T_{M001}) (Universality of (T_{M744}) Universal Set Theorem)
$(o^1 \subseteq R_B):(c \langle o \rangle d) \subseteq (S c \langle o \rangle S d)$	(A_{MX11}) (Universal Set Conjunction Axiom)
(T_{R514}) is proven in the same way except that $X = Q$	
(T_{R516}) is a combination of (T_{R513}) and (T_{R514}) .	

$(X = (V, S, Q))(b^1 \neq \emptyset):(X b^1 = (X b^1)^1)$ (T_{R517}) Triad Theorem

and, by substitution, the special cases:

$(c^1 \neq \emptyset) \subseteq ({}^V c^1 = ({}^V c^1)^1)$	(T_{R463}) Triad Theorem
$(c^1 \neq \emptyset) \subseteq (S c^1 = (S c^1)^1)$	(T_{R464}) Triad Theorem
$(c^1 \neq \emptyset) \subseteq (Q c^1 = (Q c^1)^1)$	(T_{R465}) Triad Theorem

Proof of (TR517)

$(X = (V, S, Q)) : (X^1 \subseteq R_o)$	(AR04)(One-to-One Sub-relationship Axiom)
$(X = (V, S, Q)) \subseteq (X^1 \subseteq R_o)$	(DM04)(Definition of :)
$((X = (V, S, Q)) = E) \subseteq ((X^1 \subseteq R_o) = E)$	(TM094)(Probabilistic Rule) (Eqn1)
$((c^1 \subseteq R_o) = E) = ((b^1 \subseteq QX)(b^1 \neq \emptyset) \subseteq (c * b^1 = (c * b^1)^1))$	(DR30)(Definition of a One-to-One Relationship Set)(Eqn2)
Let $c = X$	(Eqn3)
$((X^1 \subseteq R_o) = E) = ((b^1 \subseteq QX)(b^1 \neq \emptyset) \subseteq (Xb^1 = (Xb^1)^1))$	(AMX19)(Substitution of Eqn3 into Eqn2)(Eqn4)
$((X = (V, S, Q)) = E) \subseteq ((b^1 \subseteq QX)(b^1 \neq \emptyset) \subseteq (Xb^1 = (Xb^1)^1))$	(AMX19)(Substitution of Eqn4 into Eqn1)
$(X = (V, S, Q))(b^1 \subseteq QX)(b^1 \neq \emptyset) \subseteq (Xb^1 = (Xb^1)^1)$	(AMX35)(Exportation/Importation)(Eqn5) for truth sets
$b^1 \subseteq E$	(TM744)(Universal Set Theorem)(Eqn6)
$QV = E$	Extraction from (AR34)(Universal Sub-relationship Axiom)(X = Q)
$b^1 \subseteq QV$	(Y = V)(Eqn7)
Let $X = V$	(AMX19)(Substitution of Eqn7 into Eqn6)(Eqn8)
$b^1 \subseteq QX$	(Eqn9)
Thus $(X = V) \subseteq (b^1 \subseteq QX)$	(AMX19)(Substitution of Eqn9 into Eqn8)
$QS = E$	(AMX02)(Conditional Introduction)(Eqn10)
$b^1 \subseteq QS$	Extraction from (AR34)(Universal Sub-relationship Axiom)(X = Q)
Let $X = S$	(Y = S)(Eqn11)
$b^1 \subseteq QX$	(AMX19)(Substitution of Eqn11 into Eqn6)(Eqn12)
Thus $(X = S) \subseteq (b^1 \subseteq QX)$	(Eqn13)
$QQ = E$	(AMX19)(Substitution of Eqn13 into Eqn12)
$b^1 \subseteq QQ$	(AMX02)(Conditional Introduction)(Eqn14)
Let $X = Q$	Extraction from (AR34)(Universal Sub-relationship Axiom)(X = Q)
$b^1 \subseteq QX$	(Y = Q)(Eqn15)
Thus $(X = Q) \subseteq (b^1 \subseteq QX)$	(AMX19)(Substitution of Eqn15 into Eqn6)(Eqn16)
$(X = V) + (X = S) + (X = Q) \subseteq (b^1 \subseteq QX) + (b^1 \subseteq QX) + (b^1 \subseteq QX)$	(Eqn17)
$(X = V) + (X = S) + (X = Q) \subseteq (b^1 \subseteq QX)$	(AMX19)(Substitution of Eqn17 into Eqn16)
$(X = (V, S, Q)) \subseteq (b^1 \subseteq QX)$	(AMX02)(Conditional Introduction)(Eqn18)
$(X = (V, S, Q)) = (X = (V, S, Q))(b^1 \subseteq QX)$	(AMX21)(Combination of Eqns 10, 14 & 18)
$(X = (V, S, Q)) : ((a \neq a^1) = ({}^V X \diamond a = \emptyset))$	(AMX01)(Disjunction Tautology)
$(X = (V, S, Q)) : ((a \neq a^1) = ({}^V X \diamond a = \emptyset))$	(DM15)(Definition of a Variable Set)
$(X = (V, S, Q)) : ((a \neq a^1) = ({}^V X \diamond a = \emptyset))$	(AMX44)(Equality Form)(Eqn19)
$(X = (V, S, Q))(b^1 \neq \emptyset) \subseteq (Xb^1 = (Xb^1)^1)$	(AMX19)(Substitution of Eqn19 into Eqn5)
$(X = (V, S, Q))(b^1 \neq \emptyset) : (Xb^1 = (Xb^1)^1)$	(DM04)(Definition of :)

$$(X = S, Q, V) : ((a \neq a^1) = ({}^V X \diamond a = \emptyset)) \quad (TS184)$$

and by extraction the special cases:

$$(a \neq a^1) = ({}^V S \diamond a = \emptyset) \quad (TR576) \text{ Sub-Relationship Intersection Function Theorem}$$

$$(a \neq a^1) = ({}^V Q \diamond a = \emptyset) \quad (TR577) \text{ Sub-Relationship Intersection Function Theorem}$$

$$(a \neq a^1) = ({}^V V \diamond a = \emptyset) \quad (TR578)$$

Proof of (TS184)

$(({}^V c)^1 \subseteq R_o) : ((b \neq b^1) = (c \diamond b = \emptyset))$	(AR14)(One-to-One Relationship Intersection Function Axiom)
$(X^1 \subseteq R_o) : ((a \neq a^1) = ({}^V X \diamond a = \emptyset))$	Resymbolising
$((X^1 \subseteq R_o) = E) \subseteq ((a \neq a^1) = ({}^V X \diamond a = \emptyset))$	(DM04)(Definition of :)(Eqn1)
$(X = (V, S, Q)) : (X^1 \subseteq R_o)$	(AR04)(One-to-One Sub-relationship Axiom)
$(X = (V, S, Q)) \subseteq (X^1 \subseteq R_o)$	(DM04)(Definition of :)
$((X = (V, S, Q)) = E) \subseteq ((X^1 \subseteq R_o) = E)$	(TM094)(Probabilistic Rule)(Eqn2)
$((X = (V, S, Q)) = E) \subseteq ((a \neq a^1) = ({}^V X \diamond a = \emptyset))$	(TM139)(Hypothetical Syllogism Eqns 1&2)
$(X = (V, S, Q)) : ((a \neq a^1) = ({}^V X \diamond a = \emptyset))$	(DM04)(Definition of :)

$$(b \subseteq Qc). (b \neq \emptyset). (c * b \neq \emptyset) \subseteq (b \subseteq {}^v c * c * b) \quad (TR543)$$

Proof of (TR543)

$$\begin{aligned} (b \subseteq Qc) &\subseteq (b \subseteq {}^v c \diamond c \diamond b) & (AR21)(\text{Intersection Function Axiom})(\text{Eqn1}) \\ (b \neq \emptyset) &= (c * b = c \diamond b) & (AR10)(\text{Axiom of Complementary Relationship Sets})(\text{Eqn2}) \\ (b \neq \emptyset) &= (c * b = c \diamond b) & (TM008)(\text{Inverse of an Equality})(\text{Eqn3}) \\ (c \diamond b \neq \emptyset) &= ({}^v c * c \diamond b = {}^v c \diamond c \diamond b) & (AR10)(\text{Axiom of Complementary Relationship Sets}) \\ (c \diamond b \neq \emptyset) &= ({}^v c * c \diamond b = {}^v c \diamond c \diamond b) & (TM008)(\text{Inverse of an Equality})(\text{Eqn4}) \\ (b \neq \emptyset). (c \diamond b \neq \emptyset) &\subseteq ({}^v c * c * b = {}^v c \diamond c \diamond b) & (AMX19)(\text{Substitution of Eqn3 into Eqn4})(\text{Eqn5}) \\ (b \subseteq Qc). (b \neq \emptyset). (c \diamond b \neq \emptyset) &\subseteq (b \subseteq {}^v c * c * b) & (AMX19)(\text{Substitution of Eqn5 into Eqn1}) \\ (b \subseteq Qc). (b \neq \emptyset). (c \diamond b \neq \emptyset) &\subseteq (b \subseteq {}^v c * c * b) & \text{Resymbolising (Eqn6)} \\ (b \subseteq Qc). (b \neq \emptyset). (c * b \neq \emptyset) &\subseteq (b \subseteq {}^v c * c * b) & (AMX19)(\text{Substitution of Eqn1 into Eqn6}) \end{aligned}$$

$$(b^1 \neq \emptyset) = ({}^v Sb^1. {}^v Qb^1 \neq \emptyset) \quad (TR145a) \text{ Reflexive Relationship Theorem}$$

Proof of (TR145a)

$$\begin{aligned} (a^1 \neq \emptyset). (b^1 \neq \emptyset) &= ({}^v Sa^1. {}^v Qb^1 \neq \emptyset) & (AR11)(\text{Inter-relationship Axiom})(\text{Eqn1}) \\ \text{Let } a^1 &= b^1 & (\text{Eqn2}) \\ (b^1 \neq \emptyset). (b^1 \neq \emptyset) &= ({}^v Sb^1. {}^v Qb^1 \neq \emptyset) & (AMX19)(\text{Substitution of Eqn2 into Eqn1}) \\ (b^1 \neq \emptyset) &= ({}^v Sb^1. {}^v Qb^1 \neq \emptyset) & (TM667)(\text{Conjunction Tautology}) \end{aligned}$$

$$(a^1 = \emptyset) \subseteq ({}^v Sa^1. {}^v Qb^1 = \emptyset) \quad (TR651)$$

$$(b^1 = \emptyset) \subseteq ({}^v Sa^1. {}^v Qb^1 = \emptyset) \quad (TR652)$$

Example Proof of (TR651)

$$\begin{aligned} (a^1 \neq \emptyset). (b^1 \neq \emptyset) &= ({}^v Sa^1. {}^v Qb^1 \neq \emptyset) & (AR11)(\text{Inter-relationship Axiom}) \\ \sim((a^1 \neq \emptyset). (b^1 \neq \emptyset)) &= ({}^v Sa^1. {}^v Qb^1 = \emptyset) & (TM008)(\text{Inverse of an Equality}) \\ \sim(a^1 \neq \emptyset) + \sim(b^1 \neq \emptyset) &= ({}^v Sa^1. {}^v Qb^1 = \emptyset) & (TM025)(\text{DeMorgan's Law}) \\ \sim\sim(a^1 = \emptyset) + \sim\sim(b^1 = \emptyset) &= ({}^v Sa^1. {}^v Qb^1 = \emptyset) & (TM260)(\text{Definition of } \neq) \\ (a^1 = \emptyset) + (b^1 = \emptyset) &= ({}^v Sa^1. {}^v Qb^1 = \emptyset) & (AMX13)(\text{Double Complement Axiom}) \\ (a^1 = \emptyset) &\subseteq ({}^v Sa^1. {}^v Qb^1 = \emptyset) & (TM072b)(\text{Single Premise Syllogism}) \end{aligned}$$

$$(c \neq \emptyset) = (Sc \neq \emptyset) \quad (TR048) \text{ Triad Theorem}$$

$$(c \neq \emptyset) = (Qc \neq \emptyset) \quad (TR047) \text{ Triad Theorem}$$

$$(c \neq \emptyset) = ({}^v c \neq \emptyset) \quad (TR049) \text{ Triad Theorem}$$

Example Proof of (TR048)

$$\begin{aligned} X &= (S, Q, V, {}^v S, {}^v Q): (a^1 \neq \emptyset) = (X * a^1 \neq \emptyset) & (AR42)(\text{Matching Axiom}) \\ (c_s^1 \neq \emptyset) &= (Sc_s^1 \neq \emptyset) & \text{Extraction \& Resymbolising (Eqn1)} \\ (c_1^1 \neq \emptyset) \dots (c_n^1 \neq \emptyset) &= (Sc_1^1 \neq \emptyset) \dots (Sc_n^1 \neq \emptyset) & (TM231)(\text{Combination of Eqn1 for } s=1 \text{ to } n) \\ (c_1^1 +++ c_n^1 \neq \emptyset) &= (Sc_1^1 +++ Sc_n^1 \neq \emptyset) & (TM312)(\text{Probabilistic Conjunction}) \\ (c_1^1 +++ c_n^1 \neq \emptyset) &= (S(c_1^1 +++ c_n^1) \neq \emptyset) & (AS64)(\text{Distribution of } * \text{ over } +)(\text{Eqn2}) \\ \text{Let } c &= c_1^1 +++ c_n^1 & (\text{Eqn3}) \\ (c \neq \emptyset) &= (Sc \neq \emptyset) & (AMX19)(\text{Substitution of Eqn3 into Eqn2}) \end{aligned}$$

$$(c {}^v Qb) * b = c * b \quad (TR098)$$

Proof of (TR098)

$$\begin{aligned} (c * b) &= (c * b) & (DM00)(\text{Definition of } =) \\ (c * b) &= S(c {}^v Qb) & (AR27)(\text{Function Axiom}) \\ (c * b) &= S(c {}^v Qb {}^v Qb) & (TM667)(\text{Conjunction Tautology}) \\ (c * b) &= (c {}^v Qb) * b & (AR27)(\text{Function Axiom}) \end{aligned}$$

$$(a^1 \subseteq c * b) = (a^1 \subseteq S(c. {}^v Qb)) \quad (TR598)$$

Proof of (T_{R598})

$$c * b = S(c \vee Qb)$$

(A_{R27})(Function Axiom)

$$(a^1 \subseteq c * b) = (a^1 \subseteq S(c \vee Qb))$$

(T_{M272})(Deductive Rule)

$$(a \subseteq b)(b^1 \subseteq R_x) \subseteq (a^1 \subseteq R_x)$$

(T_{T534})

Proof of (T_{T534})

$$(a \subseteq b)(b \subseteq R_x) \subseteq (a \subseteq R_x)$$

(T_{M139})(Hypothetical Syllogism)

$$(a \subseteq b)(b^1 \subseteq R_x) \subseteq (a^1 \subseteq R_x)$$

(A_{S41})(Axiom of Relationship Types)

4.6 Higher Tier Proofs

$$\forall V a^1 = a^1$$

(T_{S175})

Proof of (T_{S175})

$$\forall V a = a$$

(T_{R012k})(Sub-Relationship Closure Theorem)

$$\forall V a = a$$

(D_{R27})(Simplification of the Relationship Symmetry Function)(Eqn1)

$$\text{Let } a = a^1$$

(Eqn2)

$$\forall V a^1 = a^1$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$\forall V E = E$$

(T_{S521})

Proof of (T_{S521})

$$\forall E = E$$

(T_{T793})(Relationship Opposite of **E**)

$$VE = E$$

(D_{R27})(Simplification of the Relationship Symmetry Function)(Eqn1)

$$\forall V = V$$

(T_{S548})(Reverse of the Relationship Symmetry Function)(Eqn2)

$$\forall V E = E$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$\forall V V a = a$$

(T_{S173})

$$V V a = a$$

(T_{S174})

$$\forall V V V a = a$$

(T_{S537})

Example Proof of (T_{S174})

$$\forall V a = a$$

(T_{R012k})(Sub-relationship Closure Theorem)

$$V V a = a$$

(D_{R27})(Simplification of the Relationship Symmetry Function)

$$V V V a = a$$

(T_{S548})(Reverse of the Relationship Symmetry Function)

$$\forall V V V a^1 = a^1$$

(T_{S176})

Proof of (T_{S176})

$$\forall V V V a = a$$

(T_{S537})(Eqn1)

$$\text{Let } a = a^1$$

(Eqn2)

$$\forall V V V a^1 = a^1$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$E = VE + \forall V E$$

(T_{S170})

Proof of (T_{S170})

$$\forall E = E$$

(T_{T793})(Relationship Opposite of **E**)

$$VE = E$$

(D_{R27})(Simplification of the Relationship Symmetry Function)(Eqn1)

$$\forall V E = E$$

(T_{S521})(Eqn2)

$$E + E = VE + \forall V E$$

(A_{MX21})(Combination of Eqns1&2)

$$E = VE + \forall V E$$

(A_{MX01})(Disjunction Tautology)

$$\forall V a = a$$

(T_{T838})

Proof of (T_{T838})

$$\begin{array}{ll} \forall \mathbf{a} = \mathbf{a} & (\mathcal{T}_{R012k})(\text{Sub-Relationship Closure Theorem}) \\ \forall \mathbf{V} \mathbf{a} = \mathbf{a} & (\mathcal{D}_{R27})(\text{Simplification of the Relationship Symmetry Function}) \end{array}$$

$$(\mathbf{a} \neq \mathbf{a}^1) = (\mathbf{V} \diamond \mathbf{a} = \emptyset) \quad (\mathcal{T}_{T837})$$

Proof of ($\mathcal{T}_{T837}$)

$$\begin{array}{ll} ((\mathbf{V} \mathbf{c})^1 \subseteq \mathbf{R}_0) = \mathbf{E} \subseteq ((\mathbf{a} \neq \mathbf{a}^1) = (\mathbf{c} \diamond \mathbf{a} = \emptyset)) & (\mathcal{A}_{R14})(\text{One-to-One Relationship Intersection Function Axiom}) \\ & (\text{Eqn1}) \\ \text{Let } \mathbf{c} = \mathbf{V} & (\text{Eqn2}) \\ ((\mathbf{V} \mathbf{V})^1 \subseteq \mathbf{R}_0) = \mathbf{E} \subseteq ((\mathbf{a} \neq \mathbf{a}^1) = (\mathbf{V} \diamond \mathbf{a} = \emptyset)) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ ((\mathbf{V}^1 \subseteq \mathbf{R}_0) = \mathbf{E}) \subseteq ((\mathbf{a} \neq \mathbf{a}^1) = (\mathbf{V} \diamond \mathbf{a} = \emptyset)) & (\mathcal{T}_{S548})(\text{Reverse of the Relationship Symmetry Function})(\text{Eqn3}) \\ (\mathbf{X} = (\mathbf{V}, \mathbf{S}, \mathbf{Q})) : (\mathbf{X}^1 \subseteq \mathbf{R}_0) & (\mathcal{A}_{R04})(\text{One-to-One Sub-relationship Axiom}) \\ \mathbf{V}^1 \subseteq \mathbf{R}_0 & \text{Extraction} \\ (\mathbf{V}^1 \subseteq \mathbf{R}_0) = \mathbf{E} & (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\ (\mathbf{E} = \mathbf{E}) \subseteq ((\mathbf{a} \neq \mathbf{a}^1) = (\mathbf{V} \diamond \mathbf{a} = \emptyset)) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\ \mathbf{E} \subseteq ((\mathbf{a} \neq \mathbf{a}^1) = (\mathbf{V} \diamond \mathbf{a} = \emptyset)) & (\mathcal{T}_{M342}) \\ \mathbf{E} = ((\mathbf{a} \neq \mathbf{a}^1) = (\mathbf{V} \diamond \mathbf{a} = \emptyset)) & (\mathcal{T}_{M275})(\text{Supersets of the Universal Set}) \\ (\mathbf{a} \neq \mathbf{a}^1) = (\mathbf{V} \diamond \mathbf{a} = \emptyset) & (\mathcal{A}_{MX16})(\text{Order Axiom}) \end{array}$$

$$(\mathbf{o}^1 \subseteq \mathbf{R}_B) : \mathbf{V}(\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) \quad (\mathcal{T}_{T802}) \text{ Relationship Opposite of a Proposition}$$

Proof of ($\mathcal{T}_{T802}$)

$$\begin{array}{ll} (\mathbf{o}^1 \subseteq \mathbf{R}_B) : & \\ \mathbf{V} \mathbf{E} = \mathbf{E} & (\mathcal{T}_{T793})(\text{Relationship Opposite of } \mathbf{E})(\text{Eqn1}) \\ \text{Let } \mathbf{a} \langle \mathbf{o} \rangle \mathbf{b} & \\ (\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = \mathbf{E} & (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\ \mathbf{V}(\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{array}$$

$$(\mathbf{o}^1 \subseteq \mathbf{R}_B) : ((\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\sim \mathbf{a} \langle \mathbf{V} \mathbf{o} \rangle \sim \mathbf{b})) \quad (\mathcal{T}_{T773})$$

Proof of ($\mathcal{T}_{T773}$)

$$\begin{array}{ll} (\mathbf{o}^1 \subseteq \mathbf{R}_B) : (\mathbf{V}(\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\sim \mathbf{a} \langle \mathbf{V} \mathbf{o} \rangle \sim \mathbf{b})) & (\mathcal{A}_{S81})(\text{Relationship Opposite of Propositions}) \\ (\mathbf{o}^1 \subseteq \mathbf{R}_B) : ((\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\sim \mathbf{a} \langle \mathbf{V} \mathbf{o} \rangle \sim \mathbf{b})) & (\mathcal{T}_{T802})(\text{Relationship Opposite of a Proposition}) \end{array}$$

$$(\mathbf{o}^1 \subseteq \mathbf{R}_B) : [[\mathbf{V}(\mathbf{a}_r \langle \mathbf{o} \rangle \mathbf{b}_s) + \dots]_{r=1}^{\mathbf{E}} + \dots]_{s=1}^{\mathbf{E}} = [[(\mathbf{a}_r \langle \mathbf{o} \rangle \mathbf{b}_s) + \dots]_{r=1}^{\mathbf{E}} + \dots]_{s=1}^{\mathbf{E}}$$

($\mathcal{T}_{T839}$) Relationship Opposite of a Series of Propositions

Proof of ($\mathcal{T}_{T839}$)

$$\begin{array}{ll} (\mathbf{o}^1 \subseteq \mathbf{R}_B) : & \\ \mathbf{V}(\mathbf{a}_r \langle \mathbf{o} \rangle \mathbf{b}_s) = (\mathbf{a}_r \langle \mathbf{o} \rangle \mathbf{b}_s) & (\mathcal{T}_{T802})(\text{Relationship Opposite of a Proposition})(\text{Eqn1}) \\ [[\mathbf{V}(\mathbf{a}_r \langle \mathbf{o} \rangle \mathbf{b}_s) + \dots]_{r=1}^{\mathbf{E}} + \dots]_{s=1}^{\mathbf{E}} = [[(\mathbf{a}_r \langle \mathbf{o} \rangle \mathbf{b}_s) + \dots]_{r=1}^{\mathbf{E}} + \dots]_{s=1}^{\mathbf{E}} & (\mathcal{A}_{MX27})(\text{Combination of Eqn1 for } r = 1 \text{ to } \mathbf{E} \text{ and } s = 1 \text{ to } \mathbf{E}) \end{array}$$

$$(\mathbf{o}^1 \subseteq \mathbf{R}_B) : (\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\mathbf{V} \mathbf{a} \langle \mathbf{o} \rangle \mathbf{V} \mathbf{b}) \quad (\mathcal{T}_{R510}) \text{ Relationship Opposite of a Proposition}$$

Proof of ($\mathcal{T}_{R510}$)

$$\begin{array}{ll} (\mathbf{o}^1 \subseteq \mathbf{R}_B) : & \\ \mathbf{V}(\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\mathbf{V} \mathbf{a} \langle \mathbf{o} \rangle \mathbf{V} \mathbf{b}) & (\mathcal{T}_{T798})(\text{Relationship Opposite of a Proposition}) \\ (\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\mathbf{V} \mathbf{a} \langle \mathbf{o} \rangle \mathbf{V} \mathbf{b}) & (\mathcal{A}_{S78})(\text{Relationship Symmetry Axiom}) \end{array}$$

$$(\mathbf{o}^1 \subseteq \mathbf{R}_B) : \sim(\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = \sim(\mathbf{V} \mathbf{a} \langle \mathbf{o} \rangle \mathbf{V} \mathbf{b}) \quad (\mathcal{T}_{R029})$$

Proof of ($\mathcal{T}_{R029}$)

$$\begin{array}{ll} (\mathbf{o}^1 \subseteq \mathbf{R}_B) : & \\ (\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\mathbf{V} \mathbf{a} \langle \mathbf{o} \rangle \mathbf{V} \mathbf{b}) & (\mathcal{T}_{R510})(\text{Relationship Opposite of a Proposition}) \\ \sim(\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = \sim(\mathbf{V} \mathbf{a} \langle \mathbf{o} \rangle \mathbf{V} \mathbf{b}) & (\mathcal{T}_{M008})(\text{Inverse of an Equality}) \end{array}$$

$(c = d) = (\forall c = \forall d)$	(T_{R022}) Relationship Opposite of a Proposition
$(c \subset d) = (\forall c \subset \forall d)$	(T_{R022a}) Relationship Opposite of a Proposition
$(c \parallel d) = (\forall c \parallel \forall d)$	(T_{R022b}) Relationship Opposite of a Proposition
$(c \supset d) = (\forall c \supset \forall d)$	(T_{R022c}) Relationship Opposite of a Proposition
$(c \subseteq d) = (\forall c \subseteq \forall d)$	(T_{R022d}) Relationship Opposite of a Proposition
$(c \supseteq d) = (\forall c \supseteq \forall d)$	(T_{R022e}) Relationship Opposite of a Proposition

Example Proof of (T_{R022a})

$(o^1 \subseteq R_B): (a \langle o \rangle b) = (\forall a \langle o \rangle \forall b)$	(T_{R510}) (Relationship Opposite of a Proposition)
$(o = \{=\}, \{\subset\}, \{\supset\}, \{\parallel\}, \{\subseteq\}, \{\supseteq\}): (a \langle o \rangle b) = (\forall a \langle o \rangle \forall b)$	(D_{R94}) (Definition of a Membership Relationship Set)
$(a \subset b) = (\forall a \subset \forall b)$	Extraction
$(c \subset d) = (\forall c \subset \forall d)$	Resymbolising

$$\forall c^1 = c^1 \quad (T_{R012r}) \text{ Sub-relationship Closure Theorem}$$

Proof of (T_{R012r})

$\forall c = c$	(T_{R012k}) (Sub-relationship Closure Theorem) (Eqn1)
Let $c = c^1$	(Eqn2)
$\forall c^1 = c^1$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)

$$(a^1 = b^1) = (\forall a^1 = \forall b^1) \quad (T_{R139a})$$

Proof of (T_{R139a})

$(c = d) = (\forall c = \forall d)$	(T_{R022}) (Relationship Opposite of a Proposition) (Eqn1)
Let $c = a^1$	(Eqn2)
Let $d = b^1$	(Eqn3)
$(a^1 = b^1) = (\forall a^1 = \forall b^1)$	(A_{MX19}) (Substitution of Eqns2&3 into Eqn1)

$$(a \not\subseteq b) = (\forall Sa \not\subseteq \forall Sb) \quad (T_{S168})$$

Proof of (T_{S168})

$(a \subseteq b) = (\forall Sa \subseteq \forall Sb)$	(T_{R023d})
$(a \not\subseteq b) = (\forall Sa \not\subseteq \forall Sb)$	(T_{M008}) (Inverse of an Equality)

$$\forall Qa = \forall (\forall Sa) \quad (T_{R019}) \text{ Reversibility of Matching Relationships}$$

Proof of (T_{R019})

$\forall Sa = \forall (\forall Qa)$	(A_{R13}) (Reversibility of Matching Relationships)
$\forall (\forall Sb) = \forall \forall (\forall Qb)$	(T_{R022}) (Relationship Opposite of a Proposition)
$\forall (\forall Sb) = \forall Qb$	(T_{R012k}) (Sub-Relationship Closure Theorem)

$$\forall Qb^1 = \forall (\forall Sb^1) \quad (T_{R017}) \text{ Reversibility of Matching Relationships}$$

Proof of (T_{R017})

$\forall Qa = \forall (\forall Sa)$	(T_{R019}) (Reversibility of Matching Relationships) (Eqn1)
Let $a = a^1$	(Eqn2)
$\forall Qa^1 = \forall (\forall Sa^1)$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)

$$\forall QE = \forall (\forall SE) \quad (T_{S208})$$

Proof of (T_{S208})

$\forall Qb^1 = \forall (\forall Sb^1)$	(T_{R017}) (Reversibility of Matching Relationships)
$\forall Qb_r^1 = \forall (\forall Sb_r^1)$	Resymbolising
$\forall Qb_1^1 + \dots + \forall Qb_E^1 = \forall (\forall Sb_1^1) + \dots + \forall (\forall Sb_E^1)$	(T_{M230}) (Combination For $r = 1$ to E)
$\forall Q(b_1^1 + \dots + b_E^1) = \forall (\forall Sb_1^1) + \dots + \forall (\forall Sb_E^1)$	(A_{S64}) (Distribution of $*$ over $+$)

$$\begin{aligned}
{}^vQ(b_1^1+++b_{E^1}) &= {}^v({}^vSb_1^1+++{}^vSb_{E^1}) & (A_{S64})(\text{Distribution of } * \text{ over } +) \\
{}^vQ(b_1^1+++b_{E^1}) &= {}^v({}^vS(b_1^1+++b_{E^1})) & (A_{S64})(\text{Distribution of } * \text{ over } +) \\
{}^vQE &= {}^v({}^vSE) & (D_{M17})(\text{Definition of Universal Set using Entities})
\end{aligned}$$

$${}^vSE = {}^v({}^vQE) \quad (T_{S297})$$

Proof of (T_{S297})

$$\begin{aligned}
{}^vQE &= {}^v({}^vSE) & (T_{S208}) \\
{}^v({}^vQE) &= {}^{vv}({}^vSE) & (T_{R022})(\text{Relationship Opposite of a Proposition}) \\
{}^v({}^vQE) &= {}^vSE & (T_{R012k})(\text{Sub-Relationship Closure Theorem})
\end{aligned}$$

$$(a \neq \emptyset) = ({}^v({}^vSa) \diamond a = E) \quad (T_{R269})$$

Proof of (T_{R269})

$$\begin{aligned}
(a \neq \emptyset) &= ({}^vQa \diamond a = E) & (A_{R09})(\text{Universality of Inter-relationships Axiom}) \text{ Resymbolised} \\
(a \neq \emptyset) &= ({}^v({}^vSa) \diamond a = E) & (T_{R019})(\text{Reversibility of Matching Relationships})
\end{aligned}$$

$$(b = b_1^1+++b_m^1).(c = c_1^1+++c_n^1) \subseteq (c * b = (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) +++ (c_1^1 * b_m^1 +++ c_n^1 * b_m^1))$$

(T_{T749}) Definition of the Union Function Operator

Proof of (T_{T749})

$$\begin{aligned}
\text{Let } c &= c_1^1+++c_n^1 & (\text{Eqn1}) \\
\text{Let } b &= b_1^1+++b_m^1 & (\text{Eqn2}) \\
(c_1^1+++c_n^1) * (b_1^1+++b_m^1) &= (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) +++ (c_1^1 * b_m^1 +++ c_n^1 * b_m^1) & (T_{T747})(\text{Definition of the Union Function Operator})(\text{Eqn3}) \\
c * b &= (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) +++ (c_1^1 * b_m^1 +++ c_n^1 * b_m^1) & (A_{MX19})(\text{Substitution of Eqns1\&2 into Eqn3}) \\
\text{Thus } (b = b_1^1+++b_m^1).(c = c_1^1+++c_n^1) &\subseteq (c * b = (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) +++ (c_1^1 * b_m^1 +++ c_n^1 * b_m^1)) & (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$(b = b_1^1+++b_m^1).(c = c_1^1+++c_n^1) \subseteq (c * b = (c_1^1 * b_1^1 +++ c_1^1 * b_m^1) +++ (c_n^1 * b_1^1 +++ c_n^1 * b_m^1))$$

(T_{R003}) Alternative Definition of the Union Function Operator

Proof of (T_{R003})

$$\begin{aligned}
(b = b_1^1+++b_m^1).(c = c_1^1+++c_n^1) &\subseteq (c * b = (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) +++ (c_1^1 * b_m^1 +++ c_n^1 * b_m^1)) & (T_{T749})(\text{Definition of the Union Function Operator}) \\
(b = b_1^1+++b_m^1).(c = c_1^1+++c_n^1) &\subseteq (c * b = (c_1^1 * b_1^1 +++ c_1^1 * b_m^1) +++ (c_n^1 * b_1^1 +++ c_n^1 * b_m^1)) & (T_{M403})(\text{Commutative } +)
\end{aligned}$$

$$(b = b_1^1 +++ b_m^1).(c = c_1^1 +++ c_n^1) \subseteq (c \diamond b = (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) \dots (c_1^1 * b_m^1 +++ c_n^1 * b_m^1)) \text{ or}$$

(T_{T750}) Definition of the Intersection Function Operator

Proof of (T_{T750})

$$\begin{aligned}
\text{Let } c &= c_1^1+++c_n^1 & (\text{Eqn1}) \\
\text{Let } b &= b_1^1+++b_m^1 & (\text{Eqn2}) \\
(c_1^1+++c_n^1) \diamond (b_1^1+++b_m^1) &= (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) \dots (c_1^1 * b_m^1 +++ c_n^1 * b_m^1) \\
(c_1^1+++c_n^1) \diamond (b_1^1+++b_m^1) &= (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) \dots (c_1^1 * b_m^1 +++ c_n^1 * b_m^1) & (T_{T748})(\text{Definition of the Intersection Function Operator})(\text{Eqn3}) \\
c \diamond b &= (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) \dots (c_1^1 * b_m^1 +++ c_n^1 * b_m^1) & (A_{MX19})(\text{Substitution of Eqns1\&2 into Eqn3}) \\
\text{Thus } (b = b_1^1 +++ b_m^1).(c = c_1^1 +++ c_n^1) &\subseteq (c \diamond b = (c_1^1 * b_1^1 +++ c_n^1 * b_1^1) \dots (c_1^1 * b_m^1 +++ c_n^1 * b_m^1)) & (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$(c = c_1^1+++c_n^1) \subseteq (c \diamond b^1 = c_1^1 * b^1 +++ c_n^1 * b^1) \quad (T_{R231}) \text{ Distribution of } + \text{ over } \diamond \text{ for Entities}$$

Proof of (T_{R231})

$$\text{Let } c = c_1^1+++c_n^1 \quad (\text{Eqn1})$$

$$\begin{aligned}
c \diamond b^1 &= c * b^1 & (T_{R232}) (\text{Interchangeability of } * \text{ and } \diamond \text{ for Entities}) (\text{Eqn2}) \\
c \diamond b^1 &= (c_1^1 \vdash \vdash \vdash c_n^1) * b^1 & (A_{MX19}) (\text{Substitution of Eqn1 into Eqn2}) \\
c \diamond b^1 &= c_1^1 * b^1 \vdash \vdash \vdash c_n^1 * b^1 & (A_{S65}) (\text{Distribution of } \vdash \text{ over } *) \\
\text{Thus } (c = c_1^1 \vdash \vdash \vdash c_n^1) &\subseteq (c \diamond b^1 = c_1^1 * b^1 \vdash \vdash \vdash c_n^1 * b^1) & (A_{MX02}) (\text{Conditional Introduction})
\end{aligned}$$

$$c * (b + h) = c * b + c * h \quad (T_{R002b}) \text{ Distribution of } * \text{ over } +$$

Proof of (T_{R002b})

$$\begin{aligned}
\text{Let } b &= b_1^1 \vdash \vdash \vdash b_m^1 & (\text{Eqn1}) \\
\text{And } h &= h_1^1 \vdash \vdash \vdash h_n^1 & (\text{Eqn2}) \\
c * (b + h) &= c * (b + h) & (D_{M00}) (\text{Definition of } =) (\text{Eqn3}) \\
c * (b + h) &= c * (b_1^1 \vdash \vdash \vdash b_m^1 + h_1^1 \vdash \vdash \vdash h_n^1) & (A_{MX19}) (\text{Substitution of Eqns1\&2 into Eqn3}) \\
c * (b + h) &= c * b_1^1 \vdash \vdash \vdash c * b_m^1 + c * h_1^1 \vdash \vdash \vdash c * h_n^1 & (T_{R001a}) (\text{Distribution of } * \text{ over } \vdash) \\
c * (b + h) &= c * (b_1^1 \vdash \vdash \vdash b_m^1) + c * (h_1^1 \vdash \vdash \vdash h_n^1) & (T_{R001a}) (\text{Distribution of } * \text{ over } \vdash) (\text{Eqn4}) \\
c * (b + h) &= c * b + c * h & (A_{MX19}) (\text{Substitution of Eqns1\&2 into Eqn4})
\end{aligned}$$

The theorems:

$$\forall (a + b) = \forall a + \forall b \quad (T_{S288}) \text{ Distribution of } \forall * \text{ over } +$$

$$\forall S(a + b) = \forall Sa + \forall Sb \quad (T_{S289}) \text{ Distribution of } \forall S * \text{ over } +$$

$$\forall Q(a + b) = \forall Qa + \forall Qb \quad (T_{S290}) \text{ Distribution of } \forall Q * \text{ over } +$$

$$S(c + d) = Sc + Sd \quad (T_{S291}) \text{ Distribution of } S * \text{ over } +$$

$$Q(c + d) = Qc + Qd \quad (T_{S292}) \text{ Distribution of } Q * \text{ over } +$$

are derived by substituting the appropriate sub-relationship for c in (T_{R002b}) Distribution of $*$ over $+$

$$(c + d) * b = c * b + d * b \quad (T_{R002c}) \text{ Distribution of } + \text{ over } *$$

Proof of (T_{R002c})

$$\begin{aligned}
\text{Let } c &= c_1^1 \vdash \vdash \vdash c_m^1 & (\text{Eqn1}) \\
\text{And } d &= d_1^1 \vdash \vdash \vdash d_n^1 & (\text{Eqn2}) \\
(c + d) * b &= (c + d) * b & (D_{M00}) (\text{Definition of } =) (\text{Eqn3}) \\
(c + d) * b &= (c_1^1 \vdash \vdash \vdash c_m^1 + d_1^1 \vdash \vdash \vdash d_n^1) * b & (A_{MX19}) (\text{Substitution of Eqns1\&2 into Eqn3}) \\
(c + d) * b &= c_1^1 * b \vdash \vdash \vdash c_m^1 * b + d_1^1 * b \vdash \vdash \vdash d_n^1 * b & (T_{R002a}) (\text{Distribution of } + \text{ over } *) \\
(c + d) * b &= (c_1^1 \vdash \vdash \vdash c_m^1) * b + (d_1^1 \vdash \vdash \vdash d_n^1) * b & (T_{R002a}) (\text{Distribution of } + \text{ over } *) (\text{Eqn4}) \\
(c + d) * b &= c * b + d * b & (A_{MX19}) (\text{Substitution of Eqns1\&2 into Eqn4})
\end{aligned}$$

$$c \diamond b = (c \diamond b)(d \diamond b) \quad (T_{R237}) \text{ Distribution of } . \text{ over } \diamond$$

Proof of (T_{R237})

$$\begin{aligned}
c \diamond b &= c \diamond b & (D_{M00}) (\text{Definition of } =) \\
c \diamond b &= \sim (cd) * b & (A_{R10}) (\text{Axiom of Complementary Relationship Sets}) \\
c \diamond b &= (\underline{c} + \underline{d}) * b & (T_{M025}) (\text{DeMorgan's Law}) \\
c \diamond b &= \sim ((\underline{c} + \underline{d}) * b) & (D_{R04}) (\text{Definition of } *) \\
c \diamond b &= \sim (\underline{c} * b + \underline{d} * b) & (T_{R002c}) (\text{Distribution of } + \text{ over } *) \\
c \diamond b &= (\underline{c} * b)(\underline{d} * b) & (T_{M026}) (\text{DeMorgan's Law}) \\
c \diamond b &= (c \diamond b)(d \diamond b) & (A_{R10}) (\text{Axiom of Complementary Relationship Sets}) \\
\text{N.B. true whether or not } b &= \emptyset
\end{aligned}$$

$$(b \neq \emptyset)(h \neq \emptyset): c \diamond (b + h) = (c \diamond b)(c \diamond h) \quad (T_{R235}) \text{ Non Distribution of } \diamond \text{ over } +$$

Proof of (T_{R235})

$$\begin{aligned}
\text{Let } b &= b_1^1 \vdash \vdash \vdash b_m^1 & (\text{Eqn1}) \\
\text{And } h &= h_1^1 \vdash \vdash \vdash h_n^1 & (\text{Eqn2}) \\
c \diamond (b + h) &= c \diamond (b + h) & (D_{M00}) (\text{Definition of } =) (\text{Eqn3}) \\
c \diamond (b + h) &= c \diamond (b_1^1 \vdash \vdash \vdash b_m^1 + h_1^1 \vdash \vdash \vdash h_n^1) & (A_{MX19}) (\text{Substitution of Eqns1 \& 2 into Eqn3}) \\
c \diamond (b + h) &= c * b_1^1 \dots c * b_m^1 . c * h_1^1 \dots c * h_n^1 & (T_{R233}) (\text{Non Distribution of } \diamond \text{ over } +)
\end{aligned}$$

$c \diamond (b + h) = (c \diamond (b_1^1 + \dots + b_m^1))(c \diamond (h_1^1 + \dots + h_n^1))$ (T_{R233})(Non Distribution of $\diamond$ over $+$)(Eqn4)
 $c \diamond (b + h) = (c \diamond b)(c \diamond h)$ ($AMX19$)(Substitution of Eqns1 & 2 into Eqn4)
 However, this is conditional on $(b \neq \emptyset)$ and $(h \neq \emptyset)$ because if, for example, $(h = \emptyset)$ then the above expression would be $c \diamond (b + \emptyset) = (c \diamond b)(c \diamond \emptyset) = (c \diamond b)\emptyset = \emptyset$ which is false.

$(b \neq \emptyset)(h \neq \emptyset): c \diamond (b + h) = c \diamond b + c \diamond h$ (T_{S285}) Distribution of $\diamond$ over $+$

Proof of (T_{S285})

$(b \neq \emptyset)(h \neq \emptyset): c \diamond (b + h) = (c \diamond b)(c \diamond h)$ (T_{R233})(Non Distribution of $\diamond$ over $+$)
 $(b \neq \emptyset)(h \neq \emptyset): c \diamond (b + h) = \sim((c \diamond b)(c \diamond h))$ (T_{M008})(Inverse of an Equality)
 $(b \neq \emptyset)(h \neq \emptyset): c \diamond (b + h) = (c \diamond b) + (c \diamond h)$ (T_{M025})(DeMorgan's Law)

$(b \neq \emptyset): ((c \diamond b) = \emptyset) \subseteq (c \diamond (b + h) = \emptyset)$ (T_{R255})

Proof of (T_{R255})

$(b \neq \emptyset)(h \neq \emptyset): c \diamond (b + h) = (c \diamond b)(c \diamond h)$ (T_{R233})(Non Distribution of $\diamond$ over $+$)(Eqn1)
 Let $(c \diamond b) = \emptyset$ (Eqn2)
 $c \diamond (b + h) = \emptyset(c \diamond h)$ ($AMX19$)(Substitution of Eqn2 into Eqn1)
 $c \diamond (b + h) = \emptyset$ (T_{M050})(Conjunction with the Null Set)
 Thus $(b \neq \emptyset)(h \neq \emptyset): ((c \diamond b) = \emptyset) \subseteq ((c \diamond (b + h)) = \emptyset)$ ($AMX02$)(Conditional Introduction)
 $(b \neq \emptyset)(h \neq \emptyset): ((c \diamond b) = \emptyset) \subseteq ((c \diamond (b + h)) = \emptyset)$ ($AMX35$)(Exportation/Importation) for truth sets (Eqn3)
 Let $h = \emptyset$ (Eqn4)
 Let $(c \diamond b) = \emptyset$
 $(c \diamond (b + \emptyset)) = \emptyset$ (T_{M668})(Null Set Disjunction)(Eqn5)
 $(c \diamond (b + h)) = \emptyset$ ($AMX19$)(Substitution of Eqn4 into Eqn5)
 Thus $(h = \emptyset): ((c \diamond b) = \emptyset) \subseteq (c \diamond (b + h) = \emptyset)$ ($AMX02$)(Conditional Introduction)(Eqn6)
 $(b \neq \emptyset)(h \neq \emptyset): ((c \diamond b) = \emptyset) + (h = \emptyset): ((c \diamond b) = \emptyset) \subseteq ((c \diamond (b + h)) = \emptyset) + ((c \diamond (b + h)) = \emptyset)$ ($AMX21$)(Combination of Eqns3&6)
 $((b \neq \emptyset)(h \neq \emptyset) + (h = \emptyset)): ((c \diamond b) = \emptyset) \subseteq ((c \diamond (b + h)) = \emptyset) + ((c \diamond (b + h)) = \emptyset)$ (T_{M407})(Distribution of $.$ over $+$)
 $((b \neq \emptyset)(h \neq \emptyset) + (h = \emptyset)): ((c \diamond b) = \emptyset) \subseteq ((c \diamond (b + h)) = \emptyset)$ ($AMX01$)(Disjunction Tautology)
 $((b \neq \emptyset) + (h = \emptyset)): ((c \diamond b) = \emptyset) \subseteq ((c \diamond (b + h)) = \emptyset)$ (T_{M612})(Eqn7)
 $(b \neq \emptyset) \subseteq (b \neq \emptyset) + (h = \emptyset)$ (T_{M082})(Disjunction Introduction)
 $(b \neq \emptyset): ((c \diamond b) = \emptyset) \subseteq ((b \neq \emptyset) + (h = \emptyset)): ((c \diamond b) = \emptyset)$ (T_{M029})(Deduction)(Eqn8)
 $(b \neq \emptyset): ((c \diamond b) = \emptyset) \subseteq ((c \diamond (b + h)) = \emptyset)$ (T_{M139})(Hypothetical Syllogism)
 $(b \neq \emptyset): ((c \diamond b) = \emptyset) \subseteq ((c \diamond (b + h)) = \emptyset)$ ($AMX35$)(Exportation/Importation) for truth sets

$c \diamond (b_1^1 + \dots + b_n^1) = c \diamond b_1^1 + \dots + c \diamond b_n^1$ (T_{R646}) Distribution of $\diamond$ over $+$

Proof of (T_{R646})

$c \diamond (b_1^1 + \dots + b_n^1) = c \diamond b_1^1 \dots c \diamond b_n^1$ (D_{S15})(Definition of the Intersection Function Operator)
 $c \diamond (b_1^1 + \dots + b_n^1) = \sim(c \diamond b_1^1 \dots c \diamond b_n^1)$ (T_{M008})(Inverse of an Equality)
 $c \diamond (b_1^1 + \dots + b_n^1) = c \diamond b_1^1 + \dots + c \diamond b_n^1$ (T_{M187})(DeMorgan's Law for Series)

$(b = b_1^1 + \dots + b_m^1) \subseteq (c^1 \diamond b = c^1 \diamond b_1^1 \dots c^1 \diamond b_m^1)$ (T_{R234}) Non Distribution of $\diamond$ over $+$

Proof of (T_{R234})

$(b = b_1^1 + \dots + b_m^1) \subseteq (c \diamond b = c \diamond b_1^1 \dots c \diamond b_m^1)$ (T_{R233}) (Non Distribution of $\diamond$ over $+$)
 $(b = b_1^1 + \dots + b_m^1) \subseteq (c^1 \diamond b = c^1 \diamond b_1^1 \dots c^1 \diamond b_m^1)$ Resymbolising

$c \diamond b \subseteq c \diamond b$ (T_{R241})

Proof of (T_{R241})

$(b = b_1^1 + \dots + b_m^1). (c = c_1^1 + \dots + c_n^1) \subseteq (c \diamond b = (c_1^1 \diamond b_1^1 + \dots + c_n^1 \diamond b_1^1) + \dots + (c_1^1 \diamond b_m^1 + \dots + c_n^1 \diamond b_m^1))$ (T_{749})(Definition of the Union Function Operator)(Eqn1)
 $(b = b_1^1 + \dots + b_m^1). (c = c_1^1 + \dots + c_n^1) \subseteq (c \diamond b = (c_1^1 \diamond b_1^1 + \dots + c_n^1 \diamond b_1^1) \dots (c_1^1 \diamond b_m^1 + \dots + c_n^1 \diamond b_m^1))$

$$\begin{aligned}
& \text{Let } (b = b_1^1 \text{+++} b_m^1). (c = c_1^1 \text{+++} c_n^1) = E \quad (T_{750}) \text{(Definition of the Intersection Function Operator)} \text{(Eqn2)} \\
& E \subseteq (c * b = (c_1^1 * b_1^1 \text{+++} c_n^1 * b_1^1) \text{+++} (c_1^1 * b_m^1 \text{+++} c_n^1 * b_m^1)) \quad (\text{Eqns 3}) \\
& E = (c * b = (c_1^1 * b_1^1 \text{+++} c_n^1 * b_1^1) \text{+++} (c_1^1 * b_m^1 \text{+++} c_n^1 * b_m^1)) \quad (A_{MX19}) \text{(Substitution of Eqns 3 into Eqn 1)} \\
& c * b = (c_1^1 * b_1^1 \text{+++} c_n^1 * b_1^1) \text{+++} (c_1^1 * b_m^1 \text{+++} c_n^1 * b_m^1) \quad (T_{M275}) \text{(Supersets of the Universal Set)} \\
& E \subseteq (c \diamond b = (c_1^1 * b_1^1 \text{+++} c_n^1 * b_1^1) \dots (c_1^1 * b_m^1 \text{+++} c_n^1 * b_m^1)) \quad (A_{MX16}) \text{(Order Axiom)} \text{(Eqn4)} \\
& E = (c \diamond b = (c_1^1 * b_1^1 \text{+++} c_n^1 * b_1^1) \dots (c_1^1 * b_m^1 \text{+++} c_n^1 * b_m^1)) \quad (A_{MX19}) \text{(Substitution of Eqns 3 into Eqn 2)} \\
& c \diamond b = (c_1^1 * b_1^1 \text{+++} c_n^1 * b_1^1) \dots (c_1^1 * b_m^1 \text{+++} c_n^1 * b_m^1) \quad (T_{M275}) \text{(Supersets of the Universal Set)} \\
& \text{For } s = 1 \text{ to } m \text{ let } (c_1^1 * b_s^1 \text{+++} c_n^1 * b_s^1) = a_s^1 \quad (A_{MX16}) \text{(Order Axiom)} \text{(Eqn5)} \\
& c * b = a_1^1 \text{+++} a_m^1 \quad (\text{Eqns6}) \\
& c \diamond b = a_1^1 \dots a_m^1 \quad (A_{MX19}) \text{(Substitution of Eqns6 into Eqn 4)} \text{(Eqn7)} \\
& a_1^1 \dots a_m^1 \subseteq a_1^1 \text{+++} a_m^1 \quad (A_{MX19}) \text{(Substitution of Eqns6 into Eqn 5)} \text{(Eqn8)} \\
& c \diamond b \subseteq c * b \quad (T_{M192a}) \text{(Conjunction a Subset of Disjunction for Series)} \text{(Eqn9)} \\
& \quad \quad \quad (A_{MX19}) \text{(Substitution of Eqns 7 \& 8 into Eqn 9)}
\end{aligned}$$

$$\sim(c * b) \subseteq \sim(c \diamond b) \quad (T_{R242})$$

Proof of (T_{R242})

$$\begin{aligned}
c \diamond b & \subseteq c * b & (T_{R241}) \\
\sim(c * b) & \subseteq \sim(c \diamond b) & (T_{M012}) \text{(Transposition)}
\end{aligned}$$

$$(b \neq \emptyset): (\underline{c * b} \subseteq c * b) \quad (T_{T840})$$

Proof of (T_{T840})

$$\begin{aligned}
b & \neq \emptyset: \\
c \diamond b & \subseteq c * b & (T_{R241}) \text{(Eqn1)} \\
\underline{c * b} & = c \diamond b & (A_{R10}) \text{(Axiom of Complementary Relationship Sets)} \\
\underline{c * b} & = c \diamond b & (T_{M008}) \text{(Inverse of an Equality)} \text{(Eqn2)} \\
\underline{c * b} & \subseteq c * b & (A_{MX19}) \text{(Substitution of Eqn2 into Eqn1)}
\end{aligned}$$

$$(b \neq \emptyset): (c \diamond b \subseteq \underline{c \diamond b}) \quad (T_{T841})$$

Proof of (T_{T841})

$$\begin{aligned}
b & \neq \emptyset: \\
c \diamond b & \subseteq c * b & (T_{R241}) \text{(Eqn1)} \\
\underline{c * b} & = c \diamond b & (A_{R10}) \text{(Axiom of Complementary Relationship Sets)} \\
c * b & = \underline{c \diamond b} & \text{Resymbolising (Eqn2)} \\
c \diamond b & \subseteq \underline{c \diamond b} & (A_{MX19}) \text{(Substitution of Eqn2 into Eqn1)}
\end{aligned}$$

$$(\underline{c * b})(\underline{c * b}) = \emptyset \quad (T_{R842}) \text{ Conditional Negative Theorem}$$

Proof of (T_{R842})

$$\begin{aligned}
(\underline{c * b}) & \subseteq (c * b) & (T_{T840}) \\
(\underline{c * b})(\underline{c * b}) & = \emptyset & (T_{M735}) (\subseteq \text{Existential Form})
\end{aligned}$$

$$(c \diamond b)(\underline{c \diamond b}) = \emptyset \quad (T_{R246}) \text{ Conditional Negative Theorem}$$

Proof of (T_{R246})

$$\begin{aligned}
(c \diamond b) & \subseteq (\underline{c \diamond b}) & (T_{T841}) \\
(c \diamond b)(\underline{c \diamond b}) & = \emptyset & (T_{M735}) (\subseteq \text{Existential Form})
\end{aligned}$$

$$(c * b = \emptyset) \subseteq (c \diamond b = \emptyset) \quad (T_{S246})$$

Proof of (Ts246)

Let $c * b = \emptyset$ (Eqn1)
 $c \diamond b \subseteq c * b$ (TR_{241})(Eqn2)
 $c \diamond b \subseteq \emptyset$ (AMX_{19})(Substitution of Eqn2 into Eqn1)
 $c \diamond b = \emptyset$ (TM_{052})(Subsets of the Null Set)
Thus $(c * b = \emptyset) \subseteq (c \diamond b = \emptyset)$ (AMX_{02})(Conditional Introduction)

$(b \neq \emptyset) = (\underline{c} \diamond b = \underline{c} * b)$ (TA_{201})

Proof of (TA201)

$(b \neq \emptyset) = (c * b = \underline{c} \diamond b)$ (TA_{199})
 $(b \neq \emptyset) = (\underline{c} \diamond b = c * b)$ (TM_{008})(Inverse of an Equality)

$(c * a \not\subseteq c * b) \subseteq (a \not\subseteq b)$ (TR_{199}) Relational Deduction Theorem

Proof of (TR199)

$(a \subseteq b) \subseteq (c * a \subseteq c * b)$ (TR_{196})(Relational Deduction)
 $(c * a \not\subseteq c * b) \subseteq (a \not\subseteq b)$ (TM_{012})(Transposition)

$(c * a \not\subseteq d * a) \subseteq (c \not\subseteq d)$ (TR_{200}) Relational Deduction Theorem

Proof of (TR200)

$(c \subseteq d) \subseteq (c * a \subseteq d * a)$ (TR_{197})(Relational Deduction)
 $(c * a \not\subseteq d * a) \subseteq (c \not\subseteq d)$ (TM_{012})(Transposition)

$((\forall c)^1 \subseteq R_0)(b \subseteq Qc) = E \subseteq ((b = \forall c * a) = (a = c * b))$ (TS_{528})

Proof of (TS528)

$((\forall c)^1 \subseteq R_0)(b \subseteq Qc) \subseteq (a = c * b)(b = \forall c * a)$ (TS_{533})
 $((\forall c)^1 \subseteq R_0)(b \subseteq Qc) \subseteq (a = c * b)(b = \forall c * a) = E$ (AMX_{16})(Order Axiom)
 $((\forall c)^1 \subseteq R_0)(b \subseteq Qc) = E \subseteq ((a = c * b)(b = \forall c * a) = E)$ (TM_{094})(Probabilistic Rule)
 $((\forall c)^1 \subseteq R_0)(b \subseteq Qc) = E \subseteq ((a = c * b) = E)((b = \forall c * a) = E)$ (TM_{108})(Conjunction Introduction)
 $((\forall c)^1 \subseteq R_0)(b \subseteq Qc) = E \subseteq ((b = \forall c * a) = (a = c * b))$ (AMX_{19})(Substitution)

$((\forall c)^1 \subseteq R_0)(b^1 \subseteq Qc) = E \subseteq ((b^1 = \forall c * a) = (a = c * b^1))$ (TS_{520})

Proof of (TS520)

$((\forall c)^1 \subseteq R_0)(b \subseteq Qc) = E \subseteq ((b = \forall c * a) = (a = c * b))$ (TS_{528})(Eqn1)
Let $b = b^1$ (Eqn2)
 $((\forall c)^1 \subseteq R_0)(b^1 \subseteq Qc) = E \subseteq ((b^1 = \forall c * a) = (a = c * b^1))$ (AMX_{19})(Substitution of Eqn2 into Eqn1)

$((\forall c)^1 \subseteq R_0)(b \subseteq Qc) \subseteq (b = \forall c * c * b)$ (TS_{529})

Proof of (TS529)

$((\forall c)^1 \subseteq R_0)(b \subseteq Qc) \subseteq (a = c * b)(b = \forall c * a)$ (TS_{533})
 $((\forall c)^1 \subseteq R_0)(b \subseteq Qc) \subseteq (b = \forall c * c * b)$ (AMX_{19})(Substitution)

$((\forall c)^1 \subseteq R_0)(b^1 \subseteq Qc) \subseteq (b^1 = \forall c * c * b^1)$ (TS_{519})

Proof of (TS519)

$((\forall c)^1 \subseteq R_0)(b \subseteq Qc) \subseteq (b = \forall c * c * b)$ (TS_{529})(Eqn1)
Let $b = b^1$ (Eqn2)
 $((\forall c)^1 \subseteq R_0)(b^1 \subseteq Qc) \subseteq (b^1 = \forall c * c * b^1)$ (AMX_{19})(Substitution of Eqn2 into Eqn1)

$((\forall c)^1 \subseteq R_0)(a^1 \subseteq Qc)(b^1 \subseteq Qc) \subseteq ((a^1 = b^1) = (c * a^1 = c * b^1))$ (TR_{856})

Proof of (TR856)

$(a = b) \subseteq (c * a = c * b)$	$(TR_{522})(Eqn1)$
$(a = b) \subseteq (\forall c * a = \forall c * b)$	$(Eqn1 \text{ Resymbolised}) (Eqn2)$
$(a^1 = b^1) \subseteq (c * a^1 = c * b^1)$	$(Eqn1 \text{ Resymbolised}) (Eqn3)$
$(c * a^1 = c * b^1) \subseteq (\forall c * c * a^1 = \forall c * c * b^1)$	$(Eqn2 \text{ Resymbolised}) (Eqn4)$
Let $(\forall c)^1 \subseteq R_o$	
Let $a^1 \subseteq Qc$	
Let $b^1 \subseteq Qc$	
$a^1 = \forall c * c * a^1$	$(TS_{519})(Eqn5)$
$b^1 = \forall c * c * b^1$	$(TS_{519})(Eqn6)$
$(c * a^1 = c * b^1) \subseteq (a^1 = b^1)$	$(AMX_{19})(\text{Substitution of Eqns5 \& 6 into Eqn4})(Eqn7)$
$(a^1 = b^1) = (c * a^1 = c * b^1)$	$(TM_{430})(\text{Biconditional Introduction Eqns3\&7})$
Thus $((\forall c)^1 \subseteq R_o)(a^1 \subseteq Qc)(b^1 \subseteq Qc) \subseteq ((a^1 = b^1) = (c * a^1 = c * b^1))$	$(AMX_{02})(\text{Conditional Introduction})$

$(a \subseteq b)(a \neq \emptyset) \subseteq (c \diamond b \subseteq c \diamond a) \quad (TR_{262}) \text{ Relational Deduction}$

Proof of (TR_{262})

$(a \subseteq b) \subseteq (c * a \subseteq c * b)$	$(TR_{196})(\text{Relational Deduction})$
$(a \subseteq b) \subseteq (\neg(c * b) \subseteq \neg(c * a))$	$(TM_{012})(\text{Transposition})(Eqn1)$
$(a \neq \emptyset) = (c * a = c \diamond a)$	$(AR_{10})(\text{Axiom of Complementary Relationship Sets})$
$(a \neq \emptyset) = (\neg(c * a) = c \diamond a)$	$(TM_{008})(\text{Inverse of an Equality})(Eqn2)$
$(b \neq \emptyset) = (\neg(c * b) = c \diamond b)$	Resymbolising (Eqn3)
$(a \subseteq b)(a \neq \emptyset)(b \neq \emptyset) \subseteq (c \diamond b \subseteq c \diamond a)$	$(AMX_{19})(\text{Substitution of Eqns2\&3 into Eqn1})(Eqn4)$
$((a \subseteq b) = E) = ((a \neq \emptyset) \subseteq (b \neq \emptyset))$	$(TM_{096})(\text{Probabilistic Rule})$
$(a \subseteq b)(a \neq \emptyset) \subseteq (b \neq \emptyset)$	$(AMX_{35})(\text{Exportation/Importation}) \text{ for truth sets}$
$(a \subseteq b)(a \neq \emptyset) = (a \subseteq b)(a \neq \emptyset)(b \neq \emptyset)$	$(AMX_{44})(\subseteq \text{Equality Form})(Eqn5)$
$(a \subseteq b)(a \neq \emptyset) \subseteq (c \diamond b \subseteq c \diamond a)$	$(AMX_{19})(\text{Substitution of Eqn5 into Eqn4})$

$(o^1 \subseteq R_B):(\neg(Sc \langle o \rangle Sd) \subseteq \neg(c \langle o \rangle d)) \quad (TR_{033}) \text{ Sub-Relationship Deduction}$

$(o^1 \subseteq R_B):(\neg(Qc \langle o \rangle Qd) \subseteq \neg(c \langle o \rangle d)) \quad (TR_{034}) \text{ Sub-Relationship Deduction}$

$(X = S, Q) (o^1 \subseteq R_B):(\neg(Xc \langle o \rangle Xd) \subseteq \neg(c \langle o \rangle d)) \quad (TR_{032}) \text{ Sub-Relationship Deduction}$

Example Proof of (TR_{033})

$(o^1 \subseteq R_B):((c \langle o \rangle d) \subseteq (Sc \langle o \rangle Sd))$	$(TR_{513})(\text{Sub-Relationship Deduction})$
$(o^1 \subseteq R_B):(\neg(Sc \langle o \rangle Sd) \subseteq \neg(c \langle o \rangle d))$	$(TM_{012})(\text{Transposition})$

$(o^1 \subseteq R_B):(c \langle o \rangle d) \subseteq (Sc \langle o \rangle Sd)(Qc \langle o \rangle Qd) \quad (TR_{193}) \text{ Sub-Relationship Deduction}$

Proof of (TR_{193})

$(o^1 \subseteq R_B):((c \langle o \rangle d) \subseteq (Sc \langle o \rangle Sd))$	$(TR_{513})(\text{Sub-Relationship Deduction})(Eqn1)$
$(o^1 \subseteq R_B):((c \langle o \rangle d) \subseteq (Sc \langle o \rangle Sd))$	$(TR_{514})(\text{Sub-Relationship Deduction})(Eqn2)$
$(o^1 \subseteq R_B):(c \langle o \rangle d)(c \langle o \rangle d) \subseteq (Sc \langle o \rangle Sd)(Qc \langle o \rangle Qd)$	$(TM_{669})(\text{Combination of Eqns1 \& 2})$
$(o^1 \subseteq R_B):(c \langle o \rangle d) \subseteq (Sc \langle o \rangle Sd)(Qc \langle o \rangle Qd)$	$(TM_{667})(\text{Conjunction Tautology})$

$(o^1 \subseteq R_B):(\neg(Sc \langle o \rangle Sd) + \neg(Qc \langle o \rangle Qd) \subseteq \neg(c \langle o \rangle d)) \quad (TR_{194}) \text{ Sub-Relationship Deduction}$

Proof of (TR_{194})

$(o^1 \subseteq R_B):(c \langle o \rangle d) \subseteq (Sc \langle o \rangle Sd)(Qc \langle o \rangle Qd)$	$(TR_{193})(\text{Sub-Relationship Deduction})$
$(o^1 \subseteq R_B):(\neg(Sc \langle o \rangle Sd)(Qc \langle o \rangle Qd) \subseteq \neg(c \langle o \rangle d))$	$(TM_{012})(\text{Transposition})$
$(o^1 \subseteq R_B):(\neg(Sc \langle o \rangle Sd) + \neg(Qc \langle o \rangle Qd) \subseteq \neg(c \langle o \rangle d))$	$(TM_{025})(\text{DeMorgan's Law})$

$\forall(a.b) = \forall a.\forall b \quad (TR_{012a}) \text{ Distribution of } \forall * \text{ over } .$

$\forall S(ab) = \forall Sa.\forall Sb \quad (TR_{012b}) \text{ Distribution of } \forall S * \text{ over } .$

$\forall Q(ab) = \forall Qa.\forall Qb \quad (TR_{012c}) \text{ Distribution of } \forall Q * \text{ over } .$

Example Proof of (TR_{012a})

$$(X = (V, S, Q)): (X^1 \subseteq R_0)$$

$$V^1 \subseteq R_0$$

$$(V^1 \subseteq R_0) = E$$

$$((^Vc)^1 \subseteq R_0): (c * ab) = (c * a)(c * b)$$

$$\text{Let } c = V$$

$$((^VV)^1 \subseteq R_0): (V * ab) = (V * a)(V * b)$$

$$((^VV)^1 \subseteq R_0): ^V(ab) = ^Va.^Vb$$

$$(V^1 \subseteq R_0): ^V(ab) = ^Va.^Vb$$

$$E: ^V(ab) = ^Va.^Vb$$

$$E \subseteq (^V(ab) = ^Va.^Vb)$$

$$E = (^V(ab) = ^Va.^Vb)$$

$$^V(ab) = ^Va.^Vb$$

$$(A_{R04}) \text{ (One-to-One Sub-relationship Axiom)}$$

Extraction

$$(A_{MX16}) \text{ (Order Axiom) (Eqn1)}$$

$$(A_{R35}) \text{ (One-to-One Relationship Distribution Axiom) (Eqn2) (Eqn3)}$$

$$(A_{MX19}) \text{ (Substitution of Eqn3 into Eqn2)}$$

$$(D_{R27}) \text{ (Simplification of the Relationship Symmetry Function)}$$

$$(T_{S548}) \text{ (Reverse of the Relationship Symmetry Function) (Eqn4)}$$

$$(A_{MX19}) \text{ (Substitution of Eqn1 into Eqn4)}$$

$$(D_{M04}) \text{ (Definition of :)}$$

$$(T_{M275}) \text{ (Supersets of the Universal Set)}$$

$$(A_{MX16}) \text{ (Order Axiom)}$$

(T_{R012b}) and (T_{R012c}) are proven in the same way except that Eqn3 is $c = ^VS$ or $c = ^VQ$ respectively.

$$(b \subseteq Qc) \subseteq (b \subseteq ^Vc * c \diamond b)$$

$$(T_{R544})$$

Proof of (T_{R544})

$$(b \subseteq Qc) \subseteq (b \subseteq ^Vc \diamond c \diamond b)$$

$$^Vc \diamond c \diamond b \subseteq ^Vc * c \diamond b$$

$$(b \subseteq Qc) \subseteq (b \subseteq ^Vc * c \diamond b)$$

$$(A_{R21}) \text{ (Intersection Function Axiom) (Eqn1)}$$

$$(T_{R241}) \text{ (Eqn 2)}$$

$$(T_{M139}) \text{ (Hypothetical Syllogism Eqns1 & 2)}$$

$$(b \subseteq Qc) \subseteq (b \subseteq ^Vc * c * b)$$

$$(T_{R527})$$

Proof of (T_{R527})

$$(b \subseteq Qc) \subseteq (b \subseteq ^Vc * c \diamond b)$$

$$c \diamond b \subseteq c * b$$

$$^Vc * c \diamond b \subseteq ^Vc * c * b$$

$$(b \subseteq Qc) \subseteq (b \subseteq ^Vc * c * b)$$

$$(T_{R544}) \text{ (Eqn1)}$$

$$(T_{R241})$$

$$(T_{R196}) \text{ (Relational Deduction) (Eqn2)}$$

$$(T_{M139}) \text{ (Hypothetical Syllogism Eqns1 & 2)}$$

$$(b^1 \subseteq Qc) \subseteq (b^1 \subseteq ^Vc * c * b^1)$$

$$(T_{R526})$$

Proof of (T_{R526})

$$(b \subseteq Qc) \subseteq (b \subseteq ^Vc * c * b)$$

$$\text{Let } b = b^1$$

$$(b^1 \subseteq Qc) \subseteq (b^1 \subseteq ^Vc * c * b^1)$$

$$(T_{R527}) \text{ (Eqn1)}$$

$$\text{ (Eqn2)}$$

$$(A_{MX19}) \text{ (Substitution of Eqn2 into Eqn1)}$$

$$(b \subseteq Qc). (b \neq \emptyset). (c \diamond b \neq \emptyset) \subseteq (b \subseteq ^Vc \diamond c \diamond b)$$

$$(T_{R545})$$

Proof of (T_{R545})

$$(b \subseteq Qc) \subseteq (b \subseteq ^Vc * c * b)$$

$$(b \neq \emptyset) = ((c * b) = (c \diamond b))$$

$$(b \subseteq Qc)(b \neq \emptyset) \subseteq (b \subseteq ^Vc * c \diamond b)$$

$$(c \diamond b \neq \emptyset) \subseteq (^Vc * c \diamond b = ^Vc \diamond c \diamond b)$$

$$(b \subseteq Qc)(b \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (b \subseteq ^Vc \diamond c \diamond b)$$

$$(b \subseteq Qc)(b \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (b \subseteq ^Vc \diamond c \diamond b)$$

$$(T_{R527}) \text{ (Eqn1)}$$

$$(A_{R10}) \text{ (Axiom of Complementary Relationship Sets) (Eqn2)}$$

$$(A_{MX19}) \text{ (Substitution of Eqn2 into Eqn1) (Eqn3)}$$

$$(A_{R10}) \text{ (Axiom of Complementary Relationship Sets) (Eqn4)}$$

$$(A_{MX19}) \text{ (Substitution of Eqn4 into Eqn3)}$$

Resymbolising

$$(b^1 \subseteq Qc)(b^1 \subseteq ^Vc * c * b^1) \subseteq ((^Vc)^1 \subseteq \sim R_0)$$

$$(T_{R547})$$

Proof of (T_{R547})

$$(b^1 \subseteq Qc)((^Vc)^1 \subseteq R_0) \subseteq (b^1 = ^Vc * c * b^1)$$

$$((b^1 \subseteq Qc) = E) \subseteq (((^Vc)^1 \subseteq R_0) \subseteq (b^1 = ^Vc * c * b^1))$$

$$((b^1 \subseteq Qc) = E) \subseteq ((b^1 \neq ^Vc * c * b^1) \subseteq ((^Vc)^1 \not\subseteq R_0))$$

$$((^Vc)^1 \not\subseteq R_0) \subseteq ((^Vc)^1 \subseteq \sim R_0)$$

$$((b^1 \subseteq Qc) = E) \subseteq ((b^1 \neq ^Vc * c * b^1) \subseteq ((^Vc)^1 \subseteq \sim R_0))$$

$$((b^1 \subseteq Qc) = E) \subseteq (E.(b^1 \neq ^Vc * c * b^1) \subseteq ((^Vc)^1 \subseteq \sim R_0))$$

$$(b^1 \subseteq Qc) \subseteq (b^1 \subseteq ^Vc * c * b^1)$$

$$((b^1 \subseteq Qc) = E) \subseteq ((b^1 \subseteq ^Vc * c * b^1) = E)$$

$$(T_{S519})$$

$$(A_{MX35}) \text{ (Exportation/Importation) for truth sets}$$

$$(T_{M012}) \text{ (Transposition) (Eqn1)}$$

$$(T_{M604}) \text{ (Eqn2)}$$

$$(T_{M139}) \text{ (Hypothetical Syllogism Eqns1&2)}$$

$$(A_{MX11}) \text{ (Universal Set Conjunction Axiom) (Eqn3)}$$

$$(T_{R526})$$

$$(A_{MX16}) \text{ (Order Axiom)}$$

$$\begin{aligned}
((b^1 \subseteq Qc) = E) &\subseteq ((b^1 \subset \vee c * c * b^1) + (b^1 = \vee c * c * b^1)) = E && (T_{M266})(\text{Definition of } \subseteq) \text{ (Eqn4)} \\
((b^1 \subseteq Qc) = E) &\subseteq (((b^1 \subset \vee c * c * b^1) + (b^1 = \vee c * c * b^1)).(b^1 \neq \vee c * c * b^1) \subseteq ((\vee c)^1 \subseteq \sim R_0)) && (A_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\
((b^1 \subseteq Qc) = E) &\subseteq ((b^1 \subset \vee c * c * b^1).(b^1 \neq \vee c * c * b^1) + (b^1 = \vee c * c * b^1).(b^1 \neq \vee c * c * b^1) \subseteq ((\vee c)^1 \subseteq \sim R_0)) && (T_{M407})(\text{Distribution of } . \text{ over } +) \\
((b^1 \subseteq Qc) = E) &\subseteq ((b^1 \subset \vee c * c * b^1).(b^1 \neq \vee c * c * b^1) + \emptyset \subseteq ((\vee c)^1 \subseteq \sim R_0)) && (T_{M627})(\text{Complement Conjunction}) \\
((b^1 \subseteq Qc) = E) &\subseteq ((b^1 \subset \vee c * c * b^1).(b^1 \neq \vee c * c * b^1) \subseteq ((\vee c)^1 \subseteq \sim R_0)) && (T_{M668})(\text{Null Set Disjunction})(\text{Eqn5}) \\
(b^1 \subset \vee c * c * b^1).(b^1 = \vee c * c * b^1) &= \emptyset && (T_{M037})(\text{Conjunction of Binary Relations}) \\
(b^1 \subset \vee c * c * b^1) &\subseteq (b^1 \neq \vee c * c * b^1) && (T_{M735})(\subseteq \text{ Existential Form}) \\
(b^1 \subset \vee c * c * b^1) &= (b^1 \subset \vee c * c * b^1).(b^1 \neq \vee c * c * b^1) && (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn6}) \\
((b^1 \subseteq Qc) = E) &\subseteq ((b^1 \subset \vee c * c * b^1) \subseteq ((\vee c)^1 \subseteq \sim R_0)) && (A_{MX19})(\text{Substitution of Eqn6 into Eqn5}) \\
(b^1 \subseteq Qc)(b^1 \subset \vee c * c * b^1) &\subseteq ((\vee c)^1 \subseteq \sim R_0) && (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets}
\end{aligned}$$

$$((c \diamond b) = \emptyset) \subseteq ((cd \diamond b) = \emptyset) \quad (T_{R254})$$

Proof of (T_{R254})

$$\begin{aligned}
cd \diamond b &= (c \diamond b)(d \diamond b) && (T_{R237})(\text{Distribution of } . \text{ over } \diamond)(\text{Eqn1}) \\
\text{Let } (c \diamond b) &= \emptyset && (\text{Eqn2}) \\
cd \diamond b &= \emptyset(d \diamond b) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
cd \diamond b &= \emptyset && (T_{M050})(\text{Conjunction with the Null Set}) \\
\text{Thus } ((c \diamond b) = \emptyset) &\subseteq ((cd \diamond b) = \emptyset) && (A_{MX02})(\text{Conditional Introduction}) \\
\text{NB It can also be shown that } ((d \diamond b) = \emptyset) &\subseteq ((cd \diamond b) = \emptyset)
\end{aligned}$$

$$(bh \neq \emptyset):(c \diamond b)(c \diamond h) \subseteq c \diamond bh \quad (T_{R238}) \text{ Non Distribution of } \diamond \text{ over } .$$

Proof of (T_{R238})

$$\begin{aligned}
c \diamond (b + h) &= c \diamond (b + h) && (D_{M00})(\text{Definition of } =) \\
c \diamond (b + h) &= c \diamond (b + bh + h) && (T_{M063})(\text{Partial Redundancy}) \\
(b \neq \emptyset)(bh \neq \emptyset)(h \neq \emptyset): c \diamond (b + h) &= (c \diamond b)(c \diamond bh)(c \diamond h) && (T_{R235})(\text{Non Distribution of } \diamond \text{ over } +)(\text{Eqn1}) \\
(c \diamond b)(c \diamond bh)(c \diamond h) &\subseteq (c \diamond bh) && (T_{M083})(\text{Conjunction Elimination})(\text{Eqn2}) \\
(b \neq \emptyset)(bh \neq \emptyset)(h \neq \emptyset): c \diamond (b + h) &\subseteq (c \diamond bh) && (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})(\text{Eqn3}) \\
(b \neq \emptyset)(h \neq \emptyset): c \diamond (b + h) &= (c \diamond b)(c \diamond h) && (T_{R235})(\text{Non Distribution of } \diamond \text{ over } +)(\text{Eqn4}) \\
(b \neq \emptyset)(bh \neq \emptyset)(h \neq \emptyset): (c \diamond b)(c \diamond h) &\subseteq c \diamond bh && (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})(\text{Eqn5}) \\
(bh \neq \emptyset) &\subseteq (b \neq \emptyset)(h \neq \emptyset) && (T_{M242}) \\
(bh \neq \emptyset) &= (bh \neq \emptyset)(b \neq \emptyset)(h \neq \emptyset) && (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn6}) \\
(bh \neq \emptyset):(c \diamond b)(c \diamond h) &\subseteq c \diamond bh && (A_{MX19})(\text{Substitution of Eqn6 into Eqn5})
\end{aligned}$$

$$(b \neq \emptyset)(h \neq \emptyset): cd \diamond (b + h) = (c \diamond b)(d \diamond b)(c \diamond h)(d \diamond h) \quad (T_{R253})$$

Proof of (T_{R253})

$$\begin{aligned}
cd \diamond b &= (c \diamond b)(d \diamond b) && (T_{R237})(\text{Distribution of } . \text{ over } \diamond) \\
cd \diamond (b + h) &= (c \diamond (b + h))(d \diamond (b + h)) && \text{Resymbolising } b = b + h \\
(b \neq \emptyset)(h \neq \emptyset): cd \diamond (b + h) &= (c \diamond b)(d \diamond b)(c \diamond h)(d \diamond h) && (T_{R235})(\text{Non Distribution of } \diamond \text{ over } +)
\end{aligned}$$

$$(cd) * b^1 = (c * b^1)(d * b^1) \quad (T_{R239}) \text{ Distribution of } . \text{ over } * \text{ for Entities}$$

Proof of (T_{R239})

$$\begin{aligned}
(cd) \diamond b &= (c \diamond b)(d \diamond b) && (T_{R237})(\text{Distribution of } . \text{ over } \diamond)(\text{Eqn1}) \\
\text{Let } b &= b^1 && (\text{Eqn2}) \\
(cd) \diamond b^1 &= (c \diamond b^1)(d \diamond b^1) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
(cd) * b^1 &= (c * b^1)(d * b^1) && (T_{R232})(\text{Interchangeability of } * \text{ and } \diamond \text{ for Entities})
\end{aligned}$$

$$c \diamond b + d \diamond b \subseteq (c + d) \diamond b \quad (T_{R236}) \text{ Non Distribution of } + \text{ over } \diamond$$

Proof of (TR236)

Let $b = (b_1^1 + \dots + b_m^1)$	(Eqn1)
$(c + d) \diamond b = (c + d) \diamond b$	(DM00)(Definition of =)(Eqn2)
$(c + d) \diamond b = (c + d) \diamond (b_1^1 + \dots + b_m^1)$	(AMX19)(Substitution of Eqn1 into Eqn2)
$(c + d) \diamond b = (c + d) * b_1^1 \dots (c + d) * b_m^1$	(TR233)(Non Distribution of $\diamond$ over $+$)
$(c + d) \diamond b = (c * b_1^1 + d * b_1^1) \dots (c * b_m^1 + d * b_m^1)$	(TR002c)(Distribution of $+$ over $*$)
$(c + d) \diamond b = (c * b_1^1 \dots c * b_m^1) + a + (d * b_1^1 \dots d * b_m^1)$	(TM407)(Distribution of $.$ over $+$)
where a represents all the other terms in the expansion.	
$(c + d) \diamond b = c \diamond (b_1^1 + \dots + b_m^1) + a + d \diamond (b_1^1 + \dots + b_m^1)$	(TR233)(Non Distribution of $\diamond$ over $+$)(Eqn3)
$(c + d) \diamond b = c \diamond b + a + d \diamond b$	(AMX19)(Substitution of Eqn1 into Eqn3)(Eqn4)
$c \diamond b + d \diamond b \subseteq c \diamond b + a + d \diamond b$	(TM082)(Disjunction Introduction)(Eqn5)
$c \diamond b + d \diamond b \subseteq (c + d) \diamond b$	(AMX19)(Substitution of Eqn4 into Eqn5)

$$((c \diamond b) = E) \subseteq ((c + d) \diamond b) = E$$

(TR256)

Proof of (TR256)

$c \diamond b + d \diamond b \subseteq (c + d) \diamond b$	(TR236)(Non Distribution of $+$ over $\diamond$)(Eqn1)
Let $(c \diamond b) = E$	(Eqn2)
$E + d \diamond b \subseteq (c + d) \diamond b$	(AMX19)(Substitution of Eqn2 into Eqn1)
$E \subseteq (c + d) \diamond b$	(TM055)(Universal Set Disjunction)
$E = (c + d) \diamond b$	(TM275)(Supersets of the Universal Set)
Thus $((c \diamond b) = E) \subseteq ((c + d) \diamond b) = E$	(AMX02)(Conditional Introduction)

$$(c + d) * (b + h) = c * b + d * b + c * h + d * h$$

(TR002d) Distribution of $*$ and $+$ Proof of (TR002d)

$c * (b + h) = c * b + c * h$	(TR002b)(Distribution of $*$ over $+$)(Eqn 1)
Let $c = (c + d)$	(Eqn 2)
$(c + d) * (b + h) = (c + d) * b + (c + d) * h$	(AMX19)(Substitution of Eqn 2 into Eqn 1)
$(c + d) * (b + h) = c * b + d * b + c * h + d * h$	(TR002c)(Distribution of $+$ over $*$)

$$(a \subseteq b) \subseteq (c * b.c * a = c * b.a.c * a)$$

(TS502)

Proof of (TS502)

Let $a \subseteq b$	
$a = ab$	(AMX44)($\subseteq$ Equality Form)(Eqn1)
$c * b.c * a = c * b.c * a$	(DM00)(Definition of =)
$c * b.c * a = c * b.E.c * a$	(AMX11)(Universal Set Conjunction Axiom)
$c * b.c * a = c * (b.(a + a)).c * a$	(TM671)(Tertium non Datur)
$c * b.c * a = c * (b.a + ba).c * a$	(TM407)(Distribution of $.$ over $+$)(Eqn2)
$c * b.c * a = c * (b.a + a).c * a$	(AMX19)(Substitution of Eqn1 into Eqn2)
$c * b.c * a = (c * b.a + c * a).c * a$	(TR002b)(Distribution of $*$ over $+$)
$c * b.c * a = c * b.a.c * a + c * a.c * a$	(TM407)(Distribution of $.$ over $+$)
$c * b.c * a = c * b.a.c * a + \emptyset$	(TM621)(Complement Conjunction)
$c * b.c * a = c * b.a.c * a$	(TM668)(Null Set Disjunction)
Thus $(a \subseteq b) \subseteq (c * b.c * a = c * b.a.c * a)$	(AMX02)(Conditional Introduction)

$$(c * bh) \subseteq (c * b)(c * h)$$

(TR219) Conditional Distribution of $*$ over $.$ Proof of (TR219)

$c * b = c * b.(h + \underline{h})$	(TM006)(Universal Pair Introduction)
$c * b = c * (bh + \underline{bh})$	(TM407)(Distribution of $.$ over $+$)
$c * b = c * bh + c * \underline{bh}$	(TR002b)(Distribution of $*$ over $+$)
$(c * b)(c * h) = (c * bh + c * \underline{bh})(c * h)$	(TM233)(Deduction)(Eqn1)
$c * h = c * (b + \underline{b})h$	(TM006)(Universal Pair Introduction)
$c * h = c * (bh + \underline{bh})$	(TM407)(Distribution of $.$ over $+$)
$c * h = c * bh + c * \underline{bh}$	(TR002b)(Distribution of $*$ over $+$)(Eqn2)
$(c * b)(c * h) = (c * bh + c * \underline{bh})(c * bh + c * \underline{bh})$	(AMX19)(Substitution of Eqn2 into Eqn1)

$$(c*b)(c*h) = (c*bh)(c*bh) + (c*bh)(c*bh) + (c*bh)(c*bh) + (c*bh)(c*bh)$$

(T_{M407}) (Distribution of . over +)

$$(c*bh) \subseteq (c*b)(c*h)$$

(T_{M072})

$$(c*bh = \emptyset)(c*bh = \emptyset) \subseteq ((c*b)(c*h) = (c*bh)) \quad (T_{R221}) \text{ Conditional Distribution of } * \text{ over } .$$

Proof of (T_{R221})

$$c*b = c*b.(h + h) \quad (T_{M006}) \text{ (Universal Pair Introduction)}$$

$$c*b = c*(bh + bh) \quad (T_{M407}) \text{ (Distribution of . over +)}$$

$$c*b = c*bh + c*bh \quad (T_{R002b}) \text{ (Distribution of * over +)}$$

$$(c*b)(c*h) = (c*bh + c*bh)(c*h) \quad (T_{M233}) \text{ (Deduction) (Eqn 1)}$$

$$c*h = c*(b + b)h \quad (T_{M006}) \text{ (Universal Pair Introduction)}$$

$$c*h = c*(bh + bh) \quad (T_{M407}) \text{ (Distribution of . over +)}$$

$$c*h = c*bh + c*bh \quad (T_{R002b}) \text{ (Distribution of * over +) (Eqn 2)}$$

$$(c*b)(c*h) = (c*bh + c*bh)(c*bh + c*bh) \quad (A_{MX19}) \text{ (Substitution of Eqn 2 into Eqn 1) (Eqn 3)}$$

Let $c*bh = \emptyset$ (Eqn 4)

And $c*bh = \emptyset$ (Eqn 5)

$$(c*b)(c*h) = (c*bh + \emptyset)(c*bh + \emptyset) \quad (A_{MX19}) \text{ (Substitution of Eqns 4 & 5 into Eqn 3)}$$

$$(c*b)(c*h) = (c*bh)(c*bh) \quad (T_{M668}) \text{ (Null Set Disjunction)}$$

$$(c*b)(c*h) = (c*bh) \quad (T_{M667}) \text{ (Conjunction Tautology)}$$

Thus $(c*bh = \emptyset)(c*bh = \emptyset) \subseteq ((c*b)(c*h) = (c*bh)) \quad (A_{MX02}) \text{ (Conditional Introduction)}$

$$(cd*b) \subseteq (c*b)(d*b) \quad (T_{R220}) \text{ Conditional Distribution of . over } *$$

Proof of (T_{R220})

$$c*b = (c(d + d))*b \quad (T_{M006}) \text{ (Universal Pair Introduction)}$$

$$c*b = (cd + cd)*b \quad (T_{M407}) \text{ (Distribution of . over +)}$$

$$c*b = cd*b + cd*b \quad (T_{R002c}) \text{ (Distribution of + over *)}$$

$$(c*b)(d*b) = (cd*b + cd*b)(d*b) \quad (T_{M233}) \text{ (Deduction) (Eqn 1)}$$

$$d*b = ((c + c)d)*b \quad (T_{M006}) \text{ (Universal Pair Introduction)}$$

$$d*b = (cd + cd)*b \quad (T_{M407}) \text{ (Distribution of . over +)}$$

$$d*b = cd*b + cd*b \quad (T_{R002c}) \text{ (Distribution of + over *) (Eqn 2)}$$

$$(c*b)(d*b) = (cd*b + cd*b)(cd*b + cd*b) \quad (A_{MX19}) \text{ (Substitution of Eqn 2 into Eqn 1)}$$

$$(c*b)(d*b) = (cd*b)(cd*b) + (cd*b)(cd*b) + (cd*b)(cd*b) + (cd*b)(cd*b) \quad (T_{M407}) \text{ (Distribution of . over +)}$$

$$(cd*b) \subseteq (c*b)(d*b) \quad (T_{M072})$$

$$(cd*b = \emptyset)(cd*b = \emptyset) \subseteq ((c*b)(d*b) = (cd*b)) \quad (T_{R222}) \text{ Conditional Distribution of . over } *$$

Proof of (T_{R222})

$$c*b = (c(d + d))*b \quad (T_{M006}) \text{ (Universal Pair Introduction)}$$

$$c*b = (cd + cd)*b \quad (T_{M407}) \text{ (Distribution of . over +)}$$

$$c*b = cd*b + cd*b \quad (T_{R002c}) \text{ (Distribution of + over *)}$$

$$(c*b)(d*b) = (cd*b + cd*b)(d*b) \quad (T_{M233}) \text{ (Deduction) (Eqn 1)}$$

$$d*b = ((c + c)d)*b \quad (T_{M006}) \text{ (Universal Pair Introduction)}$$

$$d*b = (cd + cd)*b \quad (T_{M407}) \text{ (Distribution of . over +)}$$

$$d*b = cd*b + cd*b \quad (T_{R002c}) \text{ (Distribution of + over *) (Eqn 2)}$$

$$(c*b)(d*b) = (cd*b + cd*b)(cd*b + cd*b) \quad (A_{MX19}) \text{ (Substitution of Eqn 2 into Eqn 1) (Eqn 3)}$$

Let $cd*b = \emptyset$ (Eqn 4)

And $cd*b = \emptyset$ (Eqn 5)

$$(c*b)(d*b) = (cd*b + \emptyset)(cd*b + \emptyset) \quad (A_{MX19}) \text{ (Substitution of Eqns 4 & 5 into Eqn 3)}$$

$$(c*b)(d*b) = (cd*b)(cd*b) \quad (T_{M668}) \text{ (Null Set Disjunction)}$$

$$(c*b)(d*b) = (cd*b) \quad (T_{M667}) \text{ (Conjunction Tautology)}$$

Thus $(cd*b = \emptyset)(cd*b = \emptyset) \subseteq ((c*b)(d*b) = (cd*b)) \quad (A_{MX02}) \text{ (Conditional Introduction)}$

$$(c*b = \emptyset) \subseteq (cd*b = \emptyset) \quad (T_{R164})$$

Proof of (T_{R164})

$cd \subseteq c$	(T_{M083}) (Conjunction Elimination)
$cd*b \subseteq c*b$	(T_{R197}) (Relational Deduction)
$(cd*b \subseteq c*b) = E$	(A_{MX16}) (Order Axiom)
$(c*b = \emptyset) \subseteq (cd*b = \emptyset)$	(T_{M097}) (Probabilistic Rule)

$(c*b = \emptyset) \subseteq (c*bh = \emptyset) \quad (T_{R165})$

Proof of (T_{R165})

$bh \subseteq b$	(T_{M083}) (Conjunction Elimination)
$c*bh \subseteq c*b$	(T_{R196}) (Relational Deduction)
$(c*bh \subseteq c*b) = E$	(A_{MX16}) (Order Axiom)
$(c*b = \emptyset) \subseteq (c*bh = \emptyset)$	(T_{M097}) (Probabilistic Rule)

$(c*b = E) \subseteq ((c + d)*b = E) \quad (T_{R166})$

Proof of (T_{R166})

$c \subseteq c + d$	(T_{M082}) (Disjunction Introduction)
$c*b \subseteq (c + d)*b$	(T_{R197}) (Relational Deduction)
$(c*b \subseteq (c + d)*b) = E$	(A_{MX16}) (Order Axiom)
$(c*b = E) \subseteq ((c + d)*b = E)$	(T_{M094}) (Probabilistic Rule)

$(c*b = E) \subseteq ((c*(b + h) = E) \quad (T_{R167})$

Proof of (T_{R167})

$b \subseteq b + h$	(T_{M082}) (Disjunction Introduction)
$c*b \subseteq c*(b + h)$	(T_{R196}) (Relational Deduction)
$(c*b \subseteq c*(b + h)) = E$	(A_{MX16}) (Order Axiom)
$(c*b = E) \subseteq ((c*(b + h) = E)$	(T_{M094}) (Probabilistic Rule)

$c*b \subseteq c*E \quad (T_{R223})$

Proof of (T_{R223})

$b \subseteq E$	(T_{M744}) (Universal Set Theorem)
$c*b \subseteq c*E$	(T_{R196}) (Relational Deduction)

$((a \subseteq Q^v c) = E) \subseteq ((^v c*a \subseteq b) \subseteq (a \subseteq c*b)) \quad (T_{388})$

$((a \subseteq Qc) = E) \subseteq ((c*a \subseteq b) \subseteq (a \subseteq ^v c*b)) \quad (T_{389})$

Example Proof of (T_{389})

Let $a \subseteq Qc$	
Let $c*a \subseteq b$	
$^v c*c*a \subseteq ^v c*b$	(T_{R196}) (Relational Deduction)(Eqn1)
$a \subseteq ^v c*c*a$	(T_{R527}) (Eqn2)
$a \subseteq ^v c*b$	(T_{M139}) (Hypothetical Syllogism Eqns2&1)
Thus $(c*a \subseteq b) \subseteq (a \subseteq ^v c*b)$	(A_{MX02}) (Conditional Introduction)
Thus $((a \subseteq Qc) = E) \subseteq ((c*a \subseteq b) \subseteq (a \subseteq ^v c*b))$	(A_{MX02}) (Conditional Introduction)

$X = (S, Q, V, ^v S, ^v Q): (a \neq \emptyset) = (Xa \neq \emptyset) \quad (T_{S200})$

Proof of (T_{S200})

$X = (S, Q, V, ^v S, ^v Q): (a_1^1 \neq \emptyset) = (Xa_1^1 \neq \emptyset)$	(A_{R42}) (Matching Axiom)(Eqn1)
...	
$X = (S, Q, V, ^v S, ^v Q): (a_n^1 \neq \emptyset) = (Xa_n^1 \neq \emptyset)$	(A_{R42}) (Matching Axiom)(Eqnn)
$X = (S, Q, V, ^v S, ^v Q): (a_1^1 \neq \emptyset)+++ (a_n^1 \neq \emptyset) = (Xa_1^1 \neq \emptyset)+++ (Xa_n^1 \neq \emptyset)$	(T_{M230}) (Combination of Eqns1 to n)
$X = (S, Q, V, ^v S, ^v Q): (a_1^1+++a_n^1 \neq \emptyset) = (Xa_1^1+++Xa_n^1 \neq \emptyset)$	(T_{M238}) (Probabilistic Disjunction)
$X = (S, Q, V, ^v S, ^v Q): (a_1^1+++a_n^1 \neq \emptyset) = (X(a_1^1+++a_n^1) \neq \emptyset)$	(T_{R002b}) (Distribution of * over +)(Eqn2)

Let $a = a_1^{1+++} a_n^1$

$$X = (S, Q, V, {}^vS, {}^vQ): (a \neq \emptyset) = (Xa \neq \emptyset)$$

(Eqn3)

(A_{MX19})(Substitution of Eqn3 into Eqn2)

$${}^vSE = E \quad (T_{R115}) \text{ Sub-Relationships with Universal Set}$$

$${}^vQE = E \quad (T_{R116}) \text{ Sub-Relationships with Universal Set}$$

$$SE = E \quad (T_{R124}) \text{ Sub-Relationships with Universal Set}$$

$$QE = E \quad (T_{R127}) \text{ Sub-Relationships with Universal Set}$$

Example Proof of (T_{R115})

$$X = (S, Q, V, {}^vS, {}^vQ): (a \neq \emptyset) = (X*a \neq \emptyset) \quad (T_{S200})$$

$$X = (S, Q, V, {}^vS, {}^vQ): (a = \emptyset) = (X*a = \emptyset) \quad (T_{M008}) \text{ (Inverse of an Equality)}$$

$$X = (S, Q, V, {}^vS, {}^vQ): (\underline{a} = \sim\emptyset) = (X*\underline{a} = \sim\emptyset) \quad (T_{M008}) \text{ (Inverse of an Equality)}$$

$$X = (S, Q, V, {}^vS, {}^vQ): (\underline{a} = E) = (X*\underline{a} = E) \quad (T_{M054}) \text{ (Negative of Null Set)}$$

$$X = (S, Q, V, {}^vS, {}^vQ): \underline{a} = X*a \quad (T_{M094b}) \text{ (Probabilistic Rule)}$$

$$X = (S, Q, V, {}^vS, {}^vQ): a = X*a \quad (T_{M008}) \text{ (Inverse of an Equality)(Eqn1)}$$

$$\text{Let } a = E \quad (\text{Eqn2})$$

$$X = (S, Q, V, {}^vS, {}^vQ): E = XE \quad (A_{MX19}) \text{ (Substitution of Eqn2 into Eqn1)}$$

$$E = {}^vSE \quad (A_{MX19}) \text{ (Substitution)}$$

$$c*\emptyset = \emptyset \quad (T_{S181}) \text{ Null Object Theorem}$$

Proof of (T_{S181})

$$c*b = S(c^vQb) \quad (A_{R27}) \text{ (Function Axiom)(Eqn1)}$$

$$\text{Let } b = \emptyset \quad (\text{Eqn2})$$

$$c*\emptyset = S(c^vQ\emptyset) \quad (A_{MX19}) \text{ (Substitution of Eqn2 into Eqn1)(Eqn3)}$$

$$X = (S, Q, V, {}^vS, {}^vQ): (a \neq \emptyset) = (X*a \neq \emptyset) \quad (T_{S200}) \text{ (Eqn4)}$$

$$\text{Let } X = {}^vQ \quad (\text{Eqn5})$$

$$(a \neq \emptyset) = ({}^vQa \neq \emptyset) \quad (A_{MX19}) \text{ (Substitution of Eqn5 into Eqn4)}$$

$$(a = \emptyset) = ({}^vQa = \emptyset) \quad (T_{M008}) \text{ (Inverse of an Equality)(Eqn6)}$$

$$\text{Let } a = \emptyset \quad (\text{Eqn7})$$

$$(\emptyset = \emptyset) = ({}^vQ\emptyset = \emptyset) \quad (A_{MX19}) \text{ (Substitution of Eqn7 into Eqn6)}$$

$$E = ({}^vQ\emptyset = \emptyset) \quad (T_{M343})$$

$${}^vQ\emptyset = \emptyset \quad (A_{MX16}) \text{ (Order Axiom)(Eqn8)}$$

$$c*\emptyset = S(c.\emptyset) \quad (A_{MX19}) \text{ (Substitution of Eqn8 into Eqn3)}$$

$$c*\emptyset = S\emptyset \quad (T_{M050}) \text{ (Null Set Conjunction)(Eqn9)}$$

$$X = (S, Q, V, {}^vS, {}^vQ): (a \neq \emptyset) = (X*a \neq \emptyset) \quad (T_{S200}) \text{ (Eqn10)}$$

$$\text{Let } X = S \quad (\text{Eqn11})$$

$$(a \neq \emptyset) = (Sa \neq \emptyset) \quad (A_{MX19}) \text{ (Substitution of Eqn11 into Eqn10)}$$

$$(a = \emptyset) = (Sa = \emptyset) \quad (T_{M008}) \text{ (Inverse of an Equality)(Eqn12)}$$

$$\text{Let } a = \emptyset \quad (\text{Eqn13})$$

$$(\emptyset = \emptyset) = (S\emptyset = \emptyset) \quad (A_{MX19}) \text{ (Substitution of Eqn13 into Eqn12)}$$

$$E = (S\emptyset = \emptyset) \quad (T_{M343})$$

$$S\emptyset = \emptyset \quad (A_{MX16}) \text{ (Order Axiom)(Eqn13)}$$

$$c*\emptyset = \emptyset \quad (A_{MX19}) \text{ (Substitution of Eqn13 into Eqn9)}$$

$$S\emptyset = \emptyset$$

(T_{R066}) Null Object Theorem

$${}^vS\emptyset = \emptyset$$

(T_{R067}) Null Object Theorem

$$Q\emptyset = \emptyset$$

(T_{R068}) Null Object Theorem

$${}^vQ\emptyset = \emptyset$$

(T_{R069}) Null Object Theorem

Example Proof of (T_{R066})

$$c*\emptyset = \emptyset \quad (T_{S181}) \text{ (Null Object Theorem)(Eqn1)}$$

$$\text{Let } c = S \quad (\text{Eqn2})$$

$$S*\emptyset = \emptyset \quad (A_{MX19}) \text{ (Substitution of Eqn2 into Eqn1)}$$

$$\sim^v Qb * b = \emptyset \quad (T_{R154})$$

Proof of (T_{R154})

$$\begin{aligned} \sim^v Qb. \sim^v Qb &= \emptyset & (T_{M621}) (\text{Complement Conjunction}) \\ S(\sim^v Qb. \sim^v Qb) &= S\emptyset & (T_{R026}) \\ S(\sim^v Qb. \sim^v Qb) &= \emptyset & (T_{R066}) (\text{Null Object Theorem}) \\ \sim^v Qb * b &= \emptyset & (A_{R27}) (\text{Function Axiom}) \end{aligned}$$

$$\sim^v Qb \diamond b = \emptyset \quad (T_{R280})$$

Proof of (T_{R280})

$$\begin{aligned} \sim^v Qb * b &= \emptyset & (T_{R154}) \\ \sim^v Qb \diamond b &= \emptyset & (T_{S246}) \end{aligned}$$

$$(c \sim^v Qb) * b = \emptyset \quad (T_{R102})$$

$$^v(c \sim^v Sa) * a = \emptyset \quad (T_{R104})$$

Example Proof of (T_{R102})

$$\begin{aligned} c. \sim^v Qb &\subseteq \sim^v Qb & (T_{M083}) (\text{Conjunction Elimination}) \\ (c. \sim^v Qb) * b &\subseteq \sim^v Qb * b & (T_{R197}) (\text{Relational Deduction}) \\ (c. \sim^v Qb) * b &\subseteq \emptyset & (T_{R154}) \\ (c. \sim^v Qb) * b &= \emptyset & (T_{M052}) (\text{Subsets of the Null Set}) \end{aligned}$$

$$(c \sim^v Qb) \diamond b = \emptyset \quad (T_{R270})$$

Proof of (T_{R270})

$$\begin{aligned} (c \sim^v Qb) * b &= \emptyset & (T_{R102}) \\ (c \sim^v Qb) \diamond b &= \emptyset & (T_{S246}) \end{aligned}$$

$$^v(c \sim^v Sa) \diamond a = \emptyset \quad (T_{R272})$$

Proof of (T_{R272})

$$\begin{aligned} ^v(c \sim^v Sa) * a &= \emptyset & (T_{R104}) \\ ^v(c \sim^v Sa) \diamond a &= \emptyset & (T_{S246}) \end{aligned}$$

$$^v S E^1 \neq \emptyset \quad (T_{T853})$$

$$^v Q E^1 \neq \emptyset \quad (T_{T854})$$

$$^v S \emptyset^1 \neq \emptyset \quad (T_{T855})$$

$$^v Q \emptyset^1 \neq \emptyset \quad (T_{T856})$$

Example Proof of (T_{T853})

$$\begin{aligned} ((X = \emptyset^1, E^1) = E) &\subseteq ((a = X) \subseteq (a \neq \emptyset)) & (A_{MX49}) (\text{Existence of } E^1 \text{ and } \emptyset^1) \\ (a = E^1) &\subseteq (a \neq \emptyset) & \text{Extraction} \\ E^1 &\neq \emptyset & (A_{MX19}) (\text{Substitution}) \\ ^v S E^1 &\neq \emptyset & (T_{S200}) (\text{Matching Theorem } X = ^v S) \end{aligned}$$

$$^v S \sim \emptyset^1 = \sim^v S \emptyset^1 \quad (T_{T869})$$

$$^v Q \sim \emptyset^1 = \sim^v Q \emptyset^1 \quad (T_{T870})$$

$$^v S \sim E^1 = \sim^v S E^1 \quad (T_{T871})$$

$$^v Q \sim E^1 = \sim^v Q E^1 \quad (T_{T872})$$

Example Proof of (T_{T869})

$\emptyset^1 + \sim\emptyset^1 = E$	(T_{M671})(Tertium non Datur)
$\forall S(\emptyset^1 + \sim\emptyset^1) = \forall SE$	(T_{R023d})(Sub-Relationship Deduction)
$\forall S(\emptyset^1 + \sim\emptyset^1) = E$	(T_{R115})
$\forall S\emptyset^1 + \forall S\sim\emptyset^1 = E$	(T_{S289})(Distribution of $\forall S^*$ over $+$)
$\sim\forall S\emptyset^1 \subseteq \forall S\sim\emptyset^1$	(T_{M064})($\subseteq$ Universal Form)(Eqn1)
$\emptyset^1 \cdot \sim\emptyset^1 = \emptyset$	(T_{M621})(Complement Conjunction)
$\forall S(\emptyset^1 \cdot \sim\emptyset^1) = \forall S\emptyset$	(T_{R023d})(Sub-Relationship Deduction)
$\forall S(\emptyset^1 \cdot \sim\emptyset^1) = \emptyset$	(T_{R067})(Null Object Theorem)
$\forall S\emptyset^1 \cdot \forall S\sim\emptyset^1 = \emptyset$	(T_{R012b})(Distribution of $\forall S^*$ over $\cdot$)
$\forall S\sim\emptyset^1 \subseteq \sim\forall S\emptyset^1$	(T_{M735})($\subseteq$ Existential Form)(Eqn2)
$\forall S\sim\emptyset^1 = \sim\forall S\emptyset^1$	(T_{M430})(Biconditional Introduction Eqns1&2)

$$\forall \emptyset = \emptyset \quad (T_{R070}) \text{ Null Object Theorem}$$

Proof of (T_{R070})

$c * \emptyset = \emptyset$	(T_{S187})(Null Object Theorem)(Eqn1)
Let $c = V$	(Eqn2)
$V * \emptyset = \emptyset$	(A_{MX19})(Substitution of Eqn2 into Eqn1)
$\forall \emptyset = \emptyset$	(D_{R27})(Simplification of the Relationship Symmetry Function)

$$(b = \emptyset) \subseteq (c * b = \emptyset) \quad (T_{S304})$$

Proof of (T_{S304})

Let $b = \emptyset$	
$c * b = c * \emptyset$	(T_{R196})(Relational Deduction)
$c * b = \emptyset$	(T_{S187})(Null Object Theorem)
thus $(b = \emptyset) \subseteq (c * b = \emptyset)$	(A_{MX02})(Conditional Introduction)

$$(c * b \neq \emptyset) \subseteq (b \neq \emptyset) \quad (T_{R170})$$

Proof of (T_{R170})

$(b = \emptyset) \subseteq (c * b = \emptyset)$	(T_{S304})
$(c * b \neq \emptyset) \subseteq (b \neq \emptyset)$	(T_{M012})(Transposition)

$$S(cd) \subseteq Sc.Sd \quad (T_{R012d}) \text{ Distribution of } S^* \text{ over } \cdot$$

$$Q(cd) \subseteq Qc.Qd \quad (T_{R012e}) \text{ Distribution of } Q^* \text{ over } \cdot$$

Example Proof of (T_{R012d})

$(c * bh) \subseteq (c * b)(c * h)$	(T_{R219}) (Conditional Distribution of $*$ over $\cdot$)(Eqn1)
Let $c = S$	(Eqn2)
$S * (bh) \subseteq Sb.Sh$	(A_{MX19})(Substitution of Eqn2 into Eqn1)
$S * (cd) \subseteq Sc.Sd$	Resymbolising
(T_{R012e}) is proven in the same way except that $c = Q$.	

$$(a \neq \emptyset) \cdot (b \neq \emptyset) = (\forall Sa. \forall Qb. a \neq \emptyset) \quad (T_{R147})$$

Proof of (T_{R147})

Let $a = a_1^1 +++ a_n^1$	(Eqn1)
Let $b = b_1^1 +++ b_m^1$	(Eqn2)
$(a_r^1 \neq \emptyset)(b_s^1 \neq \emptyset) = (\forall Sa_r^1 \forall Qb_s^1. a_r^1 \neq \emptyset)$	(A_{R11})(Inter-relationship Axiom)(Eqn3)
$(a_r^1 \neq \emptyset)(b_1^1 \neq \emptyset) +++ (a_r^1 \neq \emptyset)(b_m^1 \neq \emptyset) = (\forall Sa_r^1 \forall Qb_1^1. a_r^1 \neq \emptyset) +++ (\forall Sa_r^1 \forall Qb_m^1. a_r^1 \neq \emptyset)$	(A_{MX21})(Combination of Eqn3 for $s = 1$ to m)
$(a_r^1 \neq \emptyset)((b_1^1 \neq \emptyset) +++ (b_m^1 \neq \emptyset)) = (\forall Sa_r^1 \forall Qb_1^1. a_r^1 \neq \emptyset) +++ (\forall Sa_r^1 \forall Qb_m^1. a_r^1 \neq \emptyset)$	(T_{M407})(Distribution of $\cdot$ over $+$)
$(a_r^1 \neq \emptyset)(b_1^1 +++ b_m^1 \neq \emptyset) = (\forall Sa_r^1 \forall Qb_1^1 +++ \forall Sa_r^1 \forall Qb_m^1. a_r^1 \neq \emptyset)$	(T_{M086})&(T_{M012})

$$\begin{aligned}
(a_r^1 \neq \emptyset)(b_1^1+++b_m^1 \neq \emptyset) &= (\forall Sa_r^1(\forall Qb_1^1+++ \forall Qb_m^1) \neq \emptyset) & (T_{M407})(\text{Distribution of } \cdot \text{ over } +) \\
(a_r^1 \neq \emptyset)(b_1^1+++b_m^1 \neq \emptyset) &= (\forall Sa_r^1 \forall Q(b_1^1+++b_m^1) \neq \emptyset) & (T_{R001a})(\text{Distribution of } * \text{ over } +)(\text{Eqn4}) \\
(a_r^1 \neq \emptyset)(b \neq \emptyset) &= (\forall Sa_r^1 \forall Qb \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn4})(\text{Eqn5}) \\
(a_1^1 \neq \emptyset)(b \neq \emptyset)+++ (a_n^1 \neq \emptyset)(b \neq \emptyset) &= (\forall Sa_1^1 \forall Qb \neq \emptyset)+++ (\forall Sa_n^1 \forall Qb \neq \emptyset) & (A_{MX21})(\text{Combination of Eqn 5 for } r=1 \text{ to } n) \\
((a_1^1 \neq \emptyset)+++ (a_n^1 \neq \emptyset))(b \neq \emptyset) &= (\forall Sa_1^1 \forall Qb \neq \emptyset)+++ (\forall Sa_n^1 \forall Qb \neq \emptyset) & (T_{M407})(\text{Distribution of } \cdot \text{ over } +) \\
(a_1^1+++a_n^1 \neq \emptyset)(b \neq \emptyset) &= (\forall Sa_1^1 \forall Qb+++ \forall Sa_n^1 \forall Qb \neq \emptyset) & (T_{M086}) \& (T_{M012}) \\
(a_1^1+++a_n^1 \neq \emptyset)(b \neq \emptyset) &= ((\forall Sa_1^1+++ \forall Sa_n^1) \forall Qb \neq \emptyset) & (T_{M407})(\text{Distribution of } \cdot \text{ over } +) \\
(a_1^1+++a_n^1 \neq \emptyset)(b \neq \emptyset) &= (\forall S(a_1^1+++a_n^1) \forall Qb \neq \emptyset) & (T_{R001a})(\text{Distribution of } * \text{ over } +)(\text{Eqn6}) \\
(a \neq \emptyset)(b \neq \emptyset) &= (\forall Sa \forall Qb \neq \emptyset) & (A_{MX19})(\text{Substituting Eqn1 into Eqn6})
\end{aligned}$$

$$(b \neq \emptyset) = (\forall Sb \forall Qb \neq \emptyset) \quad (T_{R065})$$

Proof of (T_{R065})

$$\begin{aligned}
(a \neq \emptyset).(b \neq \emptyset) &= (\forall Sa. \forall Qb \neq \emptyset) & (T_{R147}) \\
\text{Let } a = b & & (\text{Eqn2}) \\
(b \neq \emptyset).(b \neq \emptyset) &= (\forall Sb. \forall Qb \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
(b \neq \emptyset) &= (\forall Sb. \forall Qb \neq \emptyset) & (T_{M667})(\text{Conjunction Tautology})
\end{aligned}$$

$$(a = \emptyset) \subseteq (\forall Sa. \forall Qb = \emptyset) \quad (T_{S238})$$

$$(b = \emptyset) \subseteq (\forall Sa. \forall Qb = \emptyset) \quad (T_{S239})$$

Example Proof of (T_{R238})

$$\begin{aligned}
(a \neq \emptyset).(b \neq \emptyset) &= (\forall Sa. \forall Qb \neq \emptyset) & (T_{R147}) \\
\sim((a \neq \emptyset).(b \neq \emptyset)) &= (\forall Sa. \forall Qb = \emptyset) & (T_{M008})(\text{Inverse of an Equality}) \\
\sim(a \neq \emptyset) + \sim(b \neq \emptyset) &= (\forall Sa. \forall Qb = \emptyset) & (T_{M025})(\text{DeMorgan's Law}) \\
\sim\sim(a = \emptyset) + \sim\sim(b = \emptyset) &= (\forall Sa. \forall Qb = \emptyset) & (T_{M260})(\text{Definition of } \neq) \\
(a = \emptyset) + (b = \emptyset) &= (\forall Sa. \forall Qb = \emptyset) & (A_{MX13})(\text{Double Complement Axiom}) \\
(a = \emptyset) \subseteq (\forall Sa. \forall Qb = \emptyset) & & (T_{M072b})(\text{Single Premise Syllogism})
\end{aligned}$$

$$E = [[\forall Sb_r^1. \forall Qb_s^1+++]_{r=1}^E+++]_{s=1}^E \quad (T_{R114}) \text{ Definition of Universal set using Relationships}$$

Proof of (T_{R114})

$$\begin{aligned}
E &= EE & (T_{M667})(\text{Conjunction Tautology}) (\text{Eqn1}) \\
\forall SE &= E & (T_{R115})(\text{Sub-Relationships with Universal Set})(\text{Eqn2}) \\
\forall QE &= E & (T_{R116})(\text{Sub-Relationships with Universal Set})(\text{Eqn3}) \\
E &= \forall SE. \forall QE & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn4}) \\
E &= [b_r^1+++]_{r=1}^E & (D_{M11})(\text{Definition of the Universal Set using entities})(\text{Eqn5}) \\
E &= [b_s^1+++]_{s=1}^E & \text{Resymbolising (Eqn6)} \\
E &= \forall S[b_r^1+++]_{r=1}^E. \forall Q[b_s^1+++]_{s=1}^E & (A_{MX19})(\text{Substitution of Eqns5\&6 into Eqn4}) \\
E &= [\forall Sb_r^1+++]_{r=1}^E. [\forall Qb_s^1+++]_{s=1}^E & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\
E &= [[\forall Sb_r^1. \forall Qb_s^1+++]_{r=1}^E+++]_{s=1}^E & (T_{M407})(\text{Distribution of } \cdot \text{ over } +)
\end{aligned}$$

$$\sim Sc \subseteq Sc \quad (T_{R168}) \text{ Sub-Relationship Negative Theorem}$$

$$\sim Qc \subseteq Qc \quad (T_{R169}) \text{ Sub-Relationship Negative Theorem}$$

Example Proof of (T_{R168})

$$\begin{aligned}
E &= SE & (T_{R124})(\text{Sub-Relationships with Universal Set})(\text{Eqn1}) \\
E &= c + \underline{c} & (T_{M671})(\text{Tertium non Datur})(\text{Eqn2}) \\
E &= S(c + \underline{c}) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
E &= Sc + S\underline{c} & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\
\sim Sc &\subseteq S\underline{c} & (T_{M064})(\subseteq \text{ Universal Form}) \\
\sim S\underline{c} &\subseteq Sc & \text{Resymbolising}
\end{aligned}$$

$$Sa + S\underline{a} = E \quad (T_{S115})$$

$$Qa + Q\bar{a} = E \quad (T_{S295})$$

Example Proof of (T_{S115})

$$\begin{aligned} E &= SE & (T_{R124})(\text{Sub-Relationships with Universal Set})(\text{Eqn1}) \\ E &= a + \bar{a} & (T_{M671})(\text{Tertium non Datur})(\text{Eqn2}) \\ E &= S(a + \bar{a}) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ E &= Sa + S\bar{a} & (T_{R002b})(\text{Distribution of } * \text{ over } +) \end{aligned}$$

$$\sim Sa. \sim S\bar{a} = \emptyset \quad (T_{S114})$$

$$\sim Qa. \sim Q\bar{a} = \emptyset \quad (T_{S296})$$

Example Proof of (T_{S114})

$$\begin{aligned} \sim Sa \subseteq S\bar{a} & \quad (T_{R168})(\text{Sub-Relationship Negative Theorem}) \\ \sim Sa. \sim S\bar{a} = \emptyset & \quad (T_{M735})(\subseteq \text{ Existential Form}) \end{aligned}$$

$$(b \subseteq \sim Sa) \subseteq (b \subseteq S\bar{a}) \quad (T_{S110})$$

$$(b \not\subseteq S\bar{a}) \subseteq (b \not\subseteq \sim Sa) \quad (T_{S111})$$

Proofs of (T_{S110}) & (T_{S111})

$$\begin{aligned} \sim Sa \subseteq S\bar{a} & \quad (T_{R168})(\text{Sub-Relationship Negative Theorem}) \\ (b \subseteq \sim Sa) \subseteq (b \subseteq S\bar{a}) & \quad (T_{M272})(\text{Deductive Rule}) \\ (b \not\subseteq S\bar{a}) \subseteq (b \not\subseteq \sim Sa) & \quad (T_{M012})(\text{Transposition}) \end{aligned}$$

$$(b \subseteq \sim S\bar{a}) \subseteq (b \subseteq Sa) \quad (T_{S112})$$

$$(b \not\subseteq Sa) \subseteq (b \not\subseteq \sim S\bar{a}) \quad (T_{S113})$$

Proofs of (T_{S112}) & (T_{S113})

$$\begin{aligned} \sim S\bar{a} \subseteq Sa & \quad (T_{R168})(\text{Sub-Relationship Negative Theorem}) \\ (b \subseteq \sim S\bar{a}) \subseteq (b \subseteq Sa) & \quad (T_{M272})(\text{Deductive Rule}) \\ (b \not\subseteq Sa) \subseteq (b \not\subseteq \sim S\bar{a}) & \quad (T_{M012})(\text{Transposition}) \end{aligned}$$

$$c \subseteq {}^vQQc \quad (T_{R012n}) \text{ Sub-relationship Non-Closure Theorem}$$

$$c \subseteq {}^vSSc \quad (T_{R012o}) \text{ Sub-relationship Non-Closure Theorem}$$

$$c^1 \subseteq {}^vQQc^1 \quad (T_{R012i}) \text{ Sub-relationship Non-Closure Theorem}$$

$$c^1 \subseteq {}^vSSc^1 \quad (T_{R012j}) \text{ Sub-relationship Non-Closure Theorem}$$

Example Proof of (T_{R012n})

$$\begin{aligned} (c \subseteq QQ) &\subseteq (c \subseteq {}^vQQc) & (T_{R527})(\text{Eqn1}) \\ (X = (S, Q))(Y = (V, S, Q, {}^vS, {}^vQ)) : (X * Y = E) & \quad (A_{R34})(\text{Universal Sub-relationship Axiom}) \\ QQ = E & \quad \text{Extraction (Eqn2)} \\ (c \subseteq E) &\subseteq (c \subseteq {}^vQQc) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ E \subseteq (c \subseteq {}^vQQc) & \quad (T_{M007})(\text{Universality of (T}_{M744}\text{) Universal Set Theorem}) \\ E = (c \subseteq {}^vQQc) & \quad (T_{M275})(\text{Supersets of the Universal Set}) \\ c \subseteq {}^vQQc & \quad (A_{MX16})(\text{Order Axiom}) \\ (T_{R012o}) & \text{ is proven in a similar manner. (T}_{R012i}\text{) and (T}_{R012j}\text{) have } c^1 \text{ substituted for } c. \end{aligned}$$

$$c^1 * Qc^1 = Sc^1 \quad (T_{R462})$$

$${}^v c^1 * Sc^1 = Qc^1 \quad (T_{R463a})$$

Example Proof of (T_{R462})

$$\begin{aligned} c^1 * b^1 &= S(c^1 {}^vQb^1) & (A_{R27})(\text{Function Axiom})(\text{Eqn1}) \\ \text{Let } b^1 &= Qc^1 & (\text{Eqn2}) \\ c^1 * Qc^1 &= S(c^1 {}^vQQc^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\ c^1 \subseteq {}^vQQc^1 & \quad (T_{R012i})(\text{Sub-Relationship Non-Closure Theorem}) \\ c^1 &= c^1 {}^vQQc^1 & (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn 4}) \end{aligned}$$

$$c^1 * Qc^1 = Sc^1$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn4 into Eqn3)(Eqn5)

$$Sc = c * Qc$$

($\mathbf{T}_{R020a}$) Function Theorem

$$Qc = {}^v c * Sc$$

($\mathbf{T}_{R020b}$) Function Theorem

Example Proof of ($\mathbf{T}_{R020a}$)

$$c * b = S(c {}^v Qb)$$

($\mathbf{A}_{R27}$)(Function Axiom)(Eqn1)

$$\text{Let } b = Qc$$

(Eqn2)

$$c * Qc = S(c {}^v QQc)$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)(Eqn3)

$$c \subseteq {}^v QQc$$

($\mathbf{T}_{R012n}$)(Sub-Relationship Non-Closure Theorem)

$$c = c {}^v QQc$$

($\mathbf{A}_{MX44}$)($\subseteq$ Equality Form)(Eqn4)

$$c * Qc = Sc$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn4 into Eqn3)

$$Qc = {}^v c * c * Qc$$

($\mathbf{T}_{R020c}$) Matching Object Closure Theorem

$$Sc = c * {}^v c * Sc$$

($\mathbf{T}_{R020d}$) Matching Object Closure Theorem

Proof of ($\mathbf{T}_{R020c}$) & ($\mathbf{T}_{R020d}$)

$$c * Qc = Sc$$

($\mathbf{T}_{R020a}$)(Function Theorem)(Eqn1)

$${}^v c * Sc = Qc$$

($\mathbf{T}_{R020b}$)(Function Theorem)(Eqn2)

$$c * {}^v c * Sc = Sc$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$${}^v c * c * Qc = Qc$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn1 into Eqn2)

$$(c^1 \subseteq Rs) \subseteq (Qc = c * c * Qc)$$

($\mathbf{T}_{S183}$) Symmetrical Relationships Theorem

Proof of ($\mathbf{T}_{S183}$)

$$Sc = c * Qc$$

($\mathbf{T}_{R020a}$)(Eqn1)

$$Qc = {}^v c * Sc$$

($\mathbf{T}_{R020b}$)(Eqn2)

$$Qc = {}^v c * c * Qc$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn1 into Eqn2)

$$E = (Qc = {}^v c * c * Qc)$$

($\mathbf{A}_{MX16}$)(Order Axiom)(Eqn3)

$$(c^1 \subseteq Rs) = ({}^v c = c)$$

($\mathbf{D}_{A94}$)(Definition of a Symmetrical Relationship Set)(Eqn4)

$$(c^1 \subseteq Rs).E = ({}^v c = c).(Qc = {}^v c * c * Qc)$$

($\mathbf{T}_{M237}$)(Combination of Eqns3 & 4)

$$(c^1 \subseteq Rs) = ({}^v c = c).(Qc = {}^v c * c * Qc)$$

($\mathbf{A}_{MX17}$)(Universal Set Conjunction Axiom)

$$(c^1 \subseteq Rs) = ({}^v c = c).(Qc = c * c * Qc)$$

($\mathbf{A}_{MX19}$)(Substitution)(Eqn5)

$$({}^v c = c).(Qc = c * c * Qc) \subseteq (Qc = c * c * Qc)$$

($\mathbf{T}_{M083}$)(Conjunction Elimination)(Eqn6)

$$(c^1 \subseteq Rs) \subseteq (Qc = c * c * Qc)$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn5 into Eqn6)

$$(c = \emptyset) = ({}^v c = \emptyset)$$

($\mathbf{T}_{R053}$)

$$(c = \emptyset) = (Sc = \emptyset)$$

($\mathbf{T}_{R052}$)

$$(c = \emptyset) = (Qc = \emptyset)$$

($\mathbf{T}_{R051}$)

Example Proof of ($\mathbf{T}_{R052}$)

$$(c \neq \emptyset) = (Sc \neq \emptyset)$$

($\mathbf{T}_{R048}$)(Triad Theorem)

$$(c = \emptyset) = (Sc = \emptyset)$$

($\mathbf{T}_{M008}$)(Inverse of an Equality)

$$E = SV$$

($\mathbf{T}_{S515}$)

Proof of ($\mathbf{T}_{S515}$)

$$SV = V * QV$$

($\mathbf{T}_{R020a}$)(Function Theorem)(Eqn1)

$$(X = (S, Q))(Y = (V, S, Q, {}^v S, {}^v Q)) : (X * Y = E)$$

($\mathbf{A}_{R34}$)(Universal Sub-relationship Axiom)

$$QV = E$$

Extraction (Eqn2)

$$SV = VE$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$SV = E$$

($\mathbf{T}_{7793}$)(Relationship Opposite of E)

$$a \subseteq SV$$

($\mathbf{T}_{S516}$)

Proof of ($\mathbf{T}_{S516}$)

$$\begin{array}{ll}
SV = E & (T_{S515})(\text{Eqn1}) \\
a \subseteq E & (T_{M744})(\text{Universal Set Theorem})(\text{Eqn2}) \\
a \subseteq SV & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})
\end{array}$$

$$E = QV \quad (T_{S536})$$

Proof of (T_{S536})

$$\begin{array}{ll}
QV = {}^vV * SV & (T_{R020b})(\text{Function Theorem}) \\
QV = {}^vV * E & (T_{S515}) \\
QV = E & (T_{S521})
\end{array}$$

$$(a^1 \neq \emptyset) \subseteq ((Va^1) = (Va^1)^1) \quad (T_{S171})$$

Proof of (T_{S171})

$$\begin{array}{ll}
(V^1 \subseteq R_0)(a^1 \subseteq QV)(a^1 \neq \emptyset) \subseteq ((Va^1) = (Va^1)^1) & (T_{R533})(\text{Definition of a One-to-One relationship set})(\text{Eqn1}) \\
(V^1 \subseteq R_P) = (V^1 \subseteq R_0).({}^vV)^1 \subseteq R_0 & (D_{R31})(\text{Definition of a Pairing Relationship Set}) \\
(V^1 \subseteq R_P) \subseteq (V^1 \subseteq R_0) & (T_{M073b})(\text{Single Premise Syllogism})(\text{Eqn2}) \\
V^1 \subseteq R_P R_S R_U & (A_{R28})(\text{Nature of } V) \\
V^1 \subseteq R_P & \text{Extraction} \\
(V^1 \subseteq R_P) = E & (A_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\
E \subseteq (V^1 \subseteq R_0) & (A_{MX19})(\text{Substitution of Eqn3 into Eqn2}) \\
E = (V^1 \subseteq R_0) & (T_{M275})(\text{Supersets of the Universal Set})(\text{Eqn4}) \\
E(a^1 \subseteq QV)(a^1 \neq \emptyset) \subseteq ((Va^1) = (Va^1)^1) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn1}) \\
(a^1 \subseteq QV)(a^1 \neq \emptyset) \subseteq ((Va^1) = (Va^1)^1) & (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\
(a^1 \subseteq E)(a^1 \neq \emptyset) \subseteq ((Va^1) = (Va^1)^1) & (T_{S536}) \\
E(a^1 \neq \emptyset) \subseteq ((Va^1) = (Va^1)^1) & (T_{M001})(\text{Universality of (T}_{M744}\text{) Universal Set Theorem}) \\
(a^1 \neq \emptyset) \subseteq ((Va^1) = (Va^1)^1) & (A_{MX11})(\text{Universal Set Conjunction Axiom})
\end{array}$$

$$(a^1 \neq \emptyset) \subseteq ({}^vVa^1 = ({}^vVa^1)^1) \quad (T_{S172})$$

Proof of (T_{S172})

$$\begin{array}{ll}
(a^1 \neq \emptyset) \subseteq (Va^1 = (Va^1)^1) & (T_{S171}) \\
(a^1 \neq \emptyset) \subseteq ({}^vVa^1 = ({}^vVa^1)^1) & (T_{S548})(\text{Reverse of the Relationship Symmetry Function})(\text{Eqn2})
\end{array}$$

$$a \subseteq QV \quad (T_{S523})$$

Proof of (T_{S523})

$$\begin{array}{ll}
E = QV & (T_{S536})(\text{Eqn1}) \\
a \subseteq E & (T_{M744})(\text{Universal Set Theorem})(\text{Eqn2}) \\
a \subseteq QV & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})
\end{array}$$

$$E = SV + QV \quad (T_{S535})$$

Proof of (T_{S535})

$$\begin{array}{ll}
E = SV & (T_{S515})(\text{Eqn1}) \\
E = QV & (T_{S536})(\text{Eqn2}) \\
E + E = SV + QV & (A_{MX21})(\text{Combination of Eqns1\&2}) \\
E = SV + QV & (A_{MX01})(\text{Disjunction Tautology})
\end{array}$$

$$\emptyset * b = \emptyset \quad (T_{R072}) \text{ Null Function Theorem}$$

Proof of (T_{R072})

$$\begin{array}{ll}
c * b = S(c^vQb) & (A_{R27})(\text{Function Axiom})(\text{Eqn1}) \\
\text{Let } c = \emptyset & (\text{Eqn2}) \\
\emptyset * b = S(\emptyset^vQb) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
\emptyset * b = S\emptyset & (T_{M050})(\text{Conjunction with Null Set})
\end{array}$$

$$\emptyset * \mathbf{b} = \emptyset$$

($\mathcal{T}_{R066}$)(Null Object Theorem)

$$\emptyset * \emptyset = \emptyset$$

($\mathcal{T}_{R073}$) Null Function Theorem

Proof of ($\mathcal{T}_{R073}$)

$$\emptyset * \mathbf{b} = \emptyset$$

($\mathcal{T}_{R072}$)(Null Function Theorem)(Eqn1)

$$\text{Let } \mathbf{b} = \emptyset$$

(Eqn2)

$$\emptyset * \emptyset = \emptyset$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$\emptyset \diamond \mathbf{b} = \emptyset$$

($\mathcal{T}_{R173}$) Null Function Theorem

Proof of ($\mathcal{T}_{R173}$)

$$\mathbf{c} \diamond \mathbf{b} \subseteq \mathbf{c} * \mathbf{b}$$

($\mathcal{T}_{R241}$)(Eqn1)

$$\text{Let } \mathbf{c} = \emptyset$$

(Eqn2)

$$\emptyset \diamond \mathbf{b} \subseteq \emptyset * \mathbf{b}$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$\emptyset \diamond \mathbf{b} \subseteq \emptyset$$

($\mathcal{T}_{R072}$)(Null Function Theorem)

$$\emptyset \diamond \mathbf{b} = \emptyset$$

($\mathcal{T}_{M052}$)(Subsets of the Null Set)

$$\mathbf{c} \diamond \emptyset = \emptyset$$

($\mathcal{T}_{R174}$) Null Object Theorem

Proof of ($\mathcal{T}_{R174}$)

$$\mathbf{c} \diamond \mathbf{b} \subseteq \mathbf{c} * \mathbf{b}$$

($\mathcal{T}_{R241}$)(Eqn1)

$$\text{Let } \mathbf{b} = \emptyset$$

(Eqn2)

$$\mathbf{c} \diamond \emptyset \subseteq \mathbf{c} * \emptyset$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$\mathbf{c} \diamond \emptyset \subseteq \emptyset$$

($\mathcal{T}_{S181}$)(Null Object Theorem)

$$\mathbf{c} \diamond \emptyset = \emptyset$$

($\mathcal{T}_{M052}$)(Subsets of the Null Set)

$$(\mathbf{c} = \emptyset) = (\mathbf{c} * \mathbf{Qc} = \emptyset)$$

($\mathcal{T}_{T843}$)

Proof of ($\mathcal{T}_{T843}$)

$$\emptyset * \emptyset = \emptyset$$

($\mathcal{T}_{R073}$)(Null Object Theorem)(Eqn1)

$$\mathbf{Q}\emptyset = \emptyset$$

($\mathcal{T}_{R068}$)(Null Object Theorem)(Eqn2)

$$\emptyset * \mathbf{Q}\emptyset = \emptyset$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)(Eqn3)

$$\text{Let } \mathbf{c} = \emptyset$$

(Eqn4)

$$\mathbf{c} * \mathbf{Qc} = \emptyset$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn4 into Eqn3)

$$\text{Thus } (\mathbf{c} = \emptyset) \subseteq (\mathbf{c} * \mathbf{Qc} = \emptyset)$$

($\mathcal{A}_{MX02}$)(Conditional Introduction)(Eqn5)

$$\text{Let } \mathbf{c} * \mathbf{Qc} = \emptyset$$

$$\mathbf{S}(\mathbf{c}. \forall \mathbf{Q}\mathbf{Qc}) = \emptyset$$

($\mathcal{A}_{R27}$)(Function Axiom)(Eqn6)

$$\mathbf{c} \subseteq \forall \mathbf{Q}\mathbf{Qc}$$

($\mathcal{T}_{R012n}$)(Sub-Relationship Non-Closure Theorem)

$$\mathbf{c} = \mathbf{c}. \forall \mathbf{Q}\mathbf{Qc}$$

($\mathcal{A}_{MX44}$)($\subseteq$ Equality Form)(Eqn7)

$$\mathbf{Sc} = \emptyset$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn7 into Eqn6)

$$\mathbf{c} = \emptyset$$

($\mathcal{T}_{R066}$)(Null Object Theorem)

$$\text{Thus } (\mathbf{c} * \mathbf{Qc} = \emptyset) \subseteq (\mathbf{c} = \emptyset)$$

($\mathcal{A}_{MX02}$)(Conditional Introduction)(Eqn8)

$$(\mathbf{c} = \emptyset) = (\mathbf{c} * \mathbf{Qc} = \emptyset)$$

($\mathcal{T}_{M430}$)(Biconditional Introduction Eqns5&8)

$$(\mathbf{c} \neq \emptyset) = (\mathbf{c} * \mathbf{Qc} \neq \emptyset)$$

($\mathcal{T}_{R159}$)

Proof of ($\mathcal{T}_{R159}$)

$$(\mathbf{c} = \emptyset) = (\mathbf{c} * \mathbf{Qc} = \emptyset)$$

($\mathcal{T}_{T843}$)

$$(\mathbf{c} \neq \emptyset) = (\mathbf{c} * \mathbf{Qc} \neq \emptyset)$$

($\mathcal{T}_{M008}$)(Inverse of an Equality)

$$(\mathbf{c} = \emptyset) \subseteq (\mathbf{c} * \mathbf{b} = \emptyset)$$

($\mathcal{T}_{S303}$)

Proof of ($\mathcal{T}_{S303}$)

$$\emptyset * \mathbf{b} = \emptyset$$

($\mathcal{T}_{R072}$)(Null Function Theorem)(Eqn1)

$$\text{Let } \mathbf{c} = \emptyset$$

(Eqn2)

$$\begin{array}{ll} \mathbf{c} * \mathbf{b} = \emptyset & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \text{Thus } (\mathbf{c} = \emptyset) \subseteq (\mathbf{c} * \mathbf{b} = \emptyset) & (\mathbf{A}_{MX02})(\text{Conditional Introduction}) \end{array}$$

$$(\mathbf{c} * \mathbf{b} \neq \emptyset) \subseteq (\mathbf{c} \neq \emptyset) \quad (\mathbf{T}_{R171b})$$

$$\begin{array}{ll} \text{Proof of } (\mathbf{T}_{R171b}) & \\ (\mathbf{c} = \emptyset) \subseteq (\mathbf{c} * \mathbf{b} = \emptyset) & (\mathbf{T}_{S303}) \\ (\mathbf{c} * \mathbf{b} \neq \emptyset) \subseteq (\mathbf{c} \neq \emptyset) & (\mathbf{T}_{M012})(\text{Transposition}) \end{array}$$

$$\mathbf{c} * \sim \mathbf{Qc} = \emptyset \quad (\mathbf{T}_{R160})$$

$$\begin{array}{ll} \text{Proof of } (\mathbf{T}_{R160}) & \\ \mathbf{Qc} . \sim \mathbf{Qc} = \emptyset & (\mathbf{T}_{M621})(\text{Complement Conjunction}) \\ \forall \mathbf{Q}(\mathbf{Qc} . \sim \mathbf{Qc}) = \emptyset & (\mathbf{T}_{S304}) \\ \forall \mathbf{Q} \mathbf{Qc} . \forall \mathbf{Q} \sim \mathbf{Qc} = \emptyset & (\mathbf{T}_{R012c})(\text{Distribution of } \forall \mathbf{Q} * \text{ over } .) \\ \mathbf{c} . \forall \mathbf{Q} \mathbf{Qc} . \forall \mathbf{Q} \sim \mathbf{Qc} = \emptyset & (\mathbf{T}_{M350})(\text{Eqn1}) \\ \mathbf{c} \subseteq \forall \mathbf{Q} \mathbf{Qc} & (\mathbf{T}_{R012n})(\text{Sub-relationship Non-Closure Theorem}) \\ \mathbf{c} = \mathbf{c} . \forall \mathbf{Q} \mathbf{Qc} & (\mathbf{A}_{MX44})(\subseteq \text{Equality Form})(\text{Eqn2}) \\ \mathbf{c} . \forall \mathbf{Q} \sim \mathbf{Qc} = \emptyset & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \mathbf{S}(\mathbf{c} . \forall \mathbf{Q} \sim \mathbf{Qc}) = \emptyset & (\mathbf{T}_{S303}) \\ \mathbf{c} * \sim \mathbf{Qc} = \emptyset & (\mathbf{A}_{R27})(\text{Function Axiom}) \end{array}$$

$$\mathbf{Sc} = \mathbf{c} * \mathbf{E} \quad (\mathbf{T}_{R210})$$

$$\begin{array}{ll} \text{Proof of } (\mathbf{T}_{R210}) & \\ \mathbf{c} * \mathbf{E} = \mathbf{c} * \mathbf{E} & (\mathbf{D}_{M00})(\text{Definition of } =) \\ \mathbf{c} * \mathbf{E} = \mathbf{c} * (\mathbf{Qc} + \sim \mathbf{Qc}) & (\mathbf{T}_{M671})(\text{Tertium non Datur}) \\ \mathbf{c} * \mathbf{E} = \mathbf{c} * \mathbf{Qc} + \mathbf{c} * \sim \mathbf{Qc} & (\mathbf{T}_{R002b})(\text{Distribution of } * \text{ over } +)(\text{Eqn1}) \\ \mathbf{Sc} = \mathbf{c} * \mathbf{Qc} & (\mathbf{T}_{R020a})(\text{Function Theorem})(\text{Eqn2}) \\ \mathbf{c} * \sim \mathbf{Qc} = \emptyset & (\mathbf{T}_{R160})(\text{Eqn3}) \\ \mathbf{c} * \mathbf{E} = \mathbf{Sc} + \emptyset & (\mathbf{A}_{MX19})(\text{Substitution of Eqns2 \& 3 into Eqn1}) \\ \mathbf{c} * \mathbf{E} = \mathbf{Sc} & (\mathbf{T}_{M668})(\text{Null Set Disjunction}) \end{array}$$

$$\mathbf{Sc} = \mathbf{c} * \mathbf{b} + \mathbf{c} * \underline{\mathbf{b}} \quad (\mathbf{T}_{R215})$$

$$\begin{array}{ll} \text{Proof of } (\mathbf{T}_{R215}) & \\ \mathbf{Sc} = \mathbf{c} * \mathbf{E} & (\mathbf{T}_{R210}) \\ \mathbf{Sc} = \mathbf{c} * (\mathbf{b} + \underline{\mathbf{b}}) & (\mathbf{T}_{M671})(\text{Tertium non Datur}) \\ \mathbf{Sc} = \mathbf{c} * \mathbf{b} + \mathbf{c} * \underline{\mathbf{b}} & (\mathbf{T}_{R002b})(\text{Distribution of } * \text{ over } +) \end{array}$$

$$\mathbf{c} * \mathbf{E} \subseteq \underline{\mathbf{c}} * \mathbf{E} \quad (\mathbf{T}_{R497})$$

$$\begin{array}{ll} \text{Proof of } (\mathbf{T}_{R497}) & \\ \sim \mathbf{Sc} \subseteq \mathbf{Sc} & (\mathbf{T}_{R168})(\text{Sub-Relationship Negative Theorem}) \\ \sim \mathbf{Sc} \subseteq \underline{\mathbf{Sc}} & \text{Resymbolising (Eqn1)} \\ \mathbf{Sc} = \mathbf{c} * \mathbf{E} & (\mathbf{T}_{R210})(\text{Eqn2}) \\ \sim \mathbf{Sc} = \sim(\mathbf{c} * \mathbf{E}) & (\mathbf{T}_{M008})(\text{Inverse of an Equality}) \\ \sim \mathbf{Sc} = \underline{\mathbf{c}} * \mathbf{E} & (\mathbf{D}_{R04})(\text{Definition of } \underline{*})(\text{Eqn3}) \\ \underline{\mathbf{Sc}} = \underline{\mathbf{c}} * \mathbf{E} & (\text{Resymbolising Eqn2})(\text{Eqn4}) \\ \underline{\mathbf{c}} * \mathbf{E} \subseteq \underline{\mathbf{c}} * \mathbf{E} & (\mathbf{A}_{MX19})(\text{Substitution of Eqns3 \& 4 into Eqn1}) \end{array}$$

$$\mathbf{c} * \mathbf{b} \subseteq \mathbf{Sc} \quad (\mathbf{T}_{R528})$$

$$\forall \mathbf{c} * \mathbf{b} \subseteq \mathbf{Qc} \quad (\mathbf{T}_{R529})$$

$$\begin{array}{ll} \text{Example Proof of } (\mathbf{T}_{R528}) & \\ \mathbf{c} * \mathbf{b} \subseteq \mathbf{c} * \mathbf{E} & (\mathbf{T}_{R223})(\text{Eqn1}) \\ \mathbf{Sc} = \mathbf{c} * \mathbf{E} & (\mathbf{T}_{R210})(\text{Eqn2}) \end{array}$$

$$c * b \subseteq Sc \quad (A_{MX19})(\text{Substitution of Eqns2 into Eqn1})$$

$$(a \subseteq c * b) \subseteq (a \subseteq Sc) \quad (T_{R530})$$

$$(a \subseteq {}^v c * b) \subseteq (a \subseteq Qc) \quad (T_{R531})$$

Example Proof of (T_{R530})

$$\begin{aligned} c * b &\subseteq Sc & (T_{R528}) \\ (c * b \subseteq Sc) &= E & (A_{MX16})(\text{Order Axiom}) \\ (a \subseteq c * b) &\subseteq (a \subseteq Sc) & (T_{M272})(\text{Deductive Rule}) \end{aligned}$$

$$c * (b \sim Qc) = \emptyset \quad (T_{R103})$$

$${}^v c * (a \sim Sc) = \emptyset \quad (T_{R105})$$

Example Proof of (T_{R103})

$$\begin{aligned} c * \sim Qc &= \emptyset & (T_{R160}) \\ c * (b \sim Qc) &= \emptyset & (T_{R165}) \end{aligned}$$

$$c * (bQc) = c * b \quad (T_{R099})$$

Proof of (T_{R099})

$$\begin{aligned} c * b &= c * b & (D_{M00})(\text{Definition of } =) \\ c * b &= c * b(Qc + \sim Qc) & (T_{M006})(\text{Universal Pair Introduction}) \\ c * b &= c * (bQc + b \sim Qc) & (T_{M407})(\text{Distribution of } \cdot \text{ over } +) \\ c * b &= c * (bQc) + c * (b \sim Qc) & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\ c * b &= c * (bQc) + \emptyset & (T_{R103}) \\ c * b &= c * (bQc) & (T_{M668})(\text{Null Set Disjunction}) \end{aligned}$$

$${}^v c * (aSc) = {}^v c * a \quad (T_{R101})$$

Proof of (T_{R101})

$$\begin{aligned} {}^v c * b &= {}^v c * b & (D_{M00})(\text{Definition of } =) \\ {}^v c * b &= {}^v c * b(Sc + \sim Sc) & (T_{M006})(\text{Universal Pair Introduction}) \\ {}^v c * b &= {}^v c * (bSc + b \sim Sc) & (T_{M407})(\text{Distribution of } \cdot \text{ over } +) \\ {}^v c * b &= {}^v c * (bSc) + {}^v c * (b \sim Sc) & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\ {}^v c * b &= {}^v c * (bQc) + \emptyset & (T_{R105}) \\ {}^v c * b &= {}^v c * (bQc) & (T_{M668})(\text{Null Set Disjunction}) \end{aligned}$$

$${}^v c * c * E = {}^v c * E \quad (T_{T506})$$

Proof of (T_{T506})

$$\begin{aligned} {}^v c * (a.Sc) &= {}^v c * a & (T_{R101})(\text{Eqn1}) \\ \text{Let } a &= E & (\text{Eqn2}) \\ {}^v c * (E.Sc) &= {}^v c * E & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ {}^v c * Sc &= {}^v c * E & (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\ {}^v c * c * E &= {}^v c * E & (T_{R210}) \end{aligned}$$

$$c * b = (c {}^v Qb) * (bQc) \quad (T_{R106})$$

$${}^v c * a = {}^v (c {}^v Sa) * (aSc) \quad (T_{R107})$$

Proof of (T_{R106})

$$\begin{aligned} c * b &= c * b & (D_{M00})(\text{Definition of } =) \\ (c * b) &= c({}^v Qb + \sim {}^v Qb) * b(Qc + \sim Qc) & (T_{M006})(\text{Universal Pair Introduction}) \\ (c * b) &= (c {}^v Qb + c \sim {}^v Qb) * (bQc + b \sim Qc) & (T_{M407})(\text{Distribution of } \cdot \text{ over } +) \\ (c * b) &= (c {}^v Qb + c \sim {}^v Qb) * (bQc) + (c {}^v Qb + c \sim {}^v Qb) * (b \sim Qc) & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\ (c * b) &= (c {}^v Qb) * (bQc) + (c \sim {}^v Qb) * (bQc) + (c {}^v Qb) * (b \sim Qc) + (c \sim {}^v Qb) * (b \sim Qc) & (T_{R002c})(\text{Distribution of } + \text{ over } *) \\ (c * b) &= (c {}^v Qb) * (bQc) + (c \sim {}^v Qb) * ((bQc) + (b \sim Qc)) + (c {}^v Qb) * (b \sim Qc) \end{aligned}$$

$$\begin{aligned}
(c * b) &= (c^V Qb) * (b Qc) + (c^{\sim V} Qb) * (b(Qc + \sim Qc)) + (c^V Qb) * (b \sim Qc) & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\
(c * b) &= (c^V Qb) * (b Qc) + (c^{\sim V} Qb) * b + (c^V Qb) * (b \sim Qc) & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
(c * b) &= (c^V Qb) * (b Qc) + \emptyset + (c^V Qb) * (b \sim Qc) & (T_{M006})(\text{Universal Pair Elimination}) \\
(c * b) &= (c^V Qb) * (b Qc) + (c^V Qb) * (b \sim Qc) & (T_{R102}) \\
(c * (b \sim Qc) = \emptyset) &\subseteq ((c^V Qb) * (b \sim Qc) = \emptyset) & (T_{M668})(\text{Null Set Disjunction})(\text{Eqn1}) \\
E &\subseteq ((c^V Qb) * (b \sim Qc) = \emptyset) & (T_{R164}) \\
E &= ((c^V Qb) * (b \sim Qc) = \emptyset) & (T_{M001})(\text{Universality of } (T_{R103})) \\
(c^V Qb) * (b \sim Qc) &= \emptyset & (T_{M275})(\text{Supersets of the Universal Set}) \\
(c * b) &= (c^V Qb) * (b Qc) + \emptyset & (A_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\
(c * b) &= (c^V Qb) * (b Qc) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
(c * b) &= (c^V Qb) * (b Qc) & (T_{M668})(\text{Null Set Disjunction})
\end{aligned}$$

$$c^1 * \sim Qc^1 = \emptyset \quad (T_{R579})$$

Proof of (T_{R579})

$$\begin{aligned}
c * \sim Qc &= \emptyset & (T_{R160})(\text{Eqn1}) \\
\text{Let } c &= c^1 & (\text{Eqn2}) \\
Qc^1 &\neq \Omega: & \\
c^1 * \sim Qc^1 &= \emptyset & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
c^1 * \sim Qc^1 &= \emptyset & (A_{MX11})(\text{Universal Set Conjunction Axiom})
\end{aligned}$$

$$c \diamond \sim Qc = \emptyset \quad (T_{R285})$$

Proof of (T_{R285})

$$\begin{aligned}
c * \sim Qc &= \emptyset & (T_{R160}) \\
c \diamond \sim Qc &= \emptyset & (T_{S246})
\end{aligned}$$

$$c \diamond (b \sim Qc) = \emptyset \quad (T_{R271})$$

Proof of (T_{R271})

$$\begin{aligned}
c * (b \sim Qc) &= \emptyset & (T_{R103}) \\
c \diamond (b \sim Qc) &= \emptyset & (T_{S246})
\end{aligned}$$

$${}^V c \diamond (a \sim Sc) = \emptyset \quad (T_{R273})$$

Proof of (T_{R273})

$$\begin{aligned}
{}^V c * (a \sim Sc) &= \emptyset & (T_{R105}) \\
{}^V c \diamond (a \sim Sc) &= \emptyset & (T_{S246})
\end{aligned}$$

$$(a = b) \subseteq (c \diamond a = c \diamond b) \quad (T_{S475}) \text{ Relational Deduction}$$

Proof of (T_{S475})

$$\begin{aligned}
(a \neq \emptyset)(a \subseteq b) &\subseteq (c \diamond b \subseteq c \diamond a) & (T_{R262})(\text{Relational Deduction})(\text{Eqn1}) \\
(b \neq \emptyset)(b \subseteq a) &\subseteq (c \diamond a \subseteq c \diamond b) & \text{Resymbolising (Eqn2)} \\
(a \neq \emptyset)(a \subseteq b)(b \neq \emptyset)(b \subseteq a) &\subseteq (c \diamond b \subseteq c \diamond a)(c \diamond a \subseteq c \diamond b) & (T_{M669})(\text{Combination of Eqns1\&2}) \\
(a \neq \emptyset)(b \neq \emptyset)(a = b) &\subseteq (c \diamond a = c \diamond b) & (T_{M430})(\text{Biconditional Introduction})(\text{Eqn3}) \\
(a \neq \emptyset)(a = b) &\subseteq (b \neq \emptyset) & (T_{M538}) \\
(a \neq \emptyset)(a = b) &= (a \neq \emptyset)(b \neq \emptyset)(a = b) & (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn4}) \\
(a \neq \emptyset)(a = b) &\subseteq (c \diamond a = c \diamond b) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})(\text{Eqn4}) \\
\text{Let } a &= \emptyset & (\text{Eqn5}) \\
\text{Let } a &= b & (\text{Eqn6}) \\
\text{Let } c \diamond a &\neq c \diamond b & (\text{Eqn7}) \\
b &= \emptyset & (A_{MX19})(\text{Substitution of Eqn5 into Eqn6})(\text{Eqn8}) \\
c \diamond \emptyset &\neq c \diamond \emptyset & (A_{MX19})(\text{Substitution of Eqns5\&8 into Eqn7}) \\
\emptyset &\neq \emptyset & (T_{R174})(\text{Null Object Theorem})
\end{aligned}$$

$\emptyset$	(T_{M346})
Thus $(a = \emptyset)(a = b)(c \diamond a \neq c \diamond b) = \emptyset$	($AMX02$)(Conditional Introduction)
$(a = \emptyset)(a = b) \subseteq (c \diamond a = c \diamond b)$	(T_{M735})($\subseteq$ Existential Form)(Eqn9)
$(a \neq \emptyset)(a = b) + (a = \emptyset)(a = b) \subseteq (c \diamond a = c \diamond b) + (c \diamond a = c \diamond b)$	($AMX21$)(Combination of Eqns4&9)
$(a \neq \emptyset)(a = b) + (a = \emptyset)(a = b) \subseteq (c \diamond a = c \diamond b)$	($AMX01$)(Disjunction Tautology)
$((a \neq \emptyset) + (a = \emptyset))(a = b) \subseteq (c \diamond a = c \diamond b)$	(T_{M407})(Distribution of . over +)
$E(a = b) \subseteq (c \diamond a = c \diamond b)$	(T_{M671})(Tertium non Datur)
$(a = b) \subseteq (c \diamond a = c \diamond b)$	($AMX11$)(Universal Set Conjunction Axiom)

$$(b = \emptyset) \subseteq (c \diamond b = \emptyset) \quad (T_{S546})$$

$$(c \diamond b \neq \emptyset) \subseteq (b \neq \emptyset) \quad (T_{S547})$$

Proof of (T_{S546}) & (T_{S547})

Let $b = \emptyset$	(Eqn1)
$c \diamond \emptyset = \emptyset$	(T_{R174})(Null Object Theorem)(Eqn2)
$c \diamond b = \emptyset$	($AMX19$)(Substitution of Eqn1 into Eqn2)
Thus $(b = \emptyset) \subseteq (c \diamond b = \emptyset)$	($AMX02$)(Conditional Introduction)
$(c \diamond b \neq \emptyset) \subseteq (b \neq \emptyset)$	(T_{M012})(Transposition)

$$(b \neq \emptyset) = (E \diamond b = E) \quad (T_{S245}) \text{ Universality of Inter-relationships Theorem}$$

Proof of (T_{S245})

$(b \neq \emptyset) = (\forall Qb \diamond b = E)$	($AR09$)(Universality of Inter-relationships Axiom)(Eqn1)
$(\forall Qb \diamond b = E) \subseteq ((\forall Qb + \sim \forall Qb) \diamond b = E)$	(T_{R256})
$(\forall Qb \diamond b = E) \subseteq (E \diamond b = E)$	(T_{M671})(Tertium non Datur)(Eqn2)
$(b \neq \emptyset) \subseteq (E \diamond b = E)$	($AMX19$)(Substitution of Eqn1 into Eqn2)(Eqn3)
Let $b = \emptyset$	(Eqn4)
Let $E \diamond b = E$	(Eqn5)
$E \diamond \emptyset = E$	($AMX19$)(Substitution of Eqn4 into Eqn5)
$\emptyset = E$	(T_{R174})(Null Object Theorem)
$\emptyset$	(T_{M347})
Thus $(b = \emptyset)(E \diamond b = E) = \emptyset$	($AMX02$)(Conditional Introduction)
$(E \diamond b = E) \subseteq (b \neq \emptyset)$	(T_{M735})($\subseteq$ Existential Form)(Eqn6)
$(b \neq \emptyset) = (E \diamond b = E)$	(T_{M430})(Biconditional Introduction Eqns3&6)

$$(b^1 \neq \emptyset) = (E * b^1 = E) \quad (T_{S243}) \text{ Universality of Inter-relationships Theorem}$$

Proof of (T_{S243})

$(b \neq \emptyset) = (E \diamond b = E)$	(T_{S245})(Universality of Inter-relationships Theorem)(Eqn1)
Let $b = b^1$	(Eqn2)
$(b^1 \neq \emptyset) = (E \diamond b^1 = E)$	($AMX19$)(Substitution of Eqn2 into Eqn1)
$(b^1 \neq \emptyset) = (E * b^1 = E)$	(T_{R232})(Interchangeability of * and $\diamond$ for Entities)

$$(b \neq \emptyset) = (E * b = E) \quad (T_{S244})$$

Proof of (T_{S244})

$(b \neq \emptyset) = (E \diamond b = E)$	(T_{S245})(Universality of Inter-relationships Theorem)
$(b \neq \emptyset) = (E \subseteq E \diamond b)$	(T_{M275})(Supersets of the Universal Set)(Eqn1)
$E \diamond b \subseteq E * b$	(T_{R241})(Eqn2)
$(b \neq \emptyset) \subseteq (E \subseteq E * b)$	(T_{M139})(Hypothetical Syllogism Eqns1&2)
$(b \neq \emptyset) \subseteq (E * b = E)$	(T_{M275})(Supersets of the Universal Set)(Eqn3)
Let $b = \emptyset$	(Eqn4)
Let $E * b = E$	(Eqn5)
$E * \emptyset = E$	($AMX19$)(Substitution of Eqn4 into Eqn5)
$\emptyset = E$	(T_{S187})(Null Object Theorem)

$\emptyset$	(T_{M347})
Thus $(b = \emptyset)(E * b = E) = \emptyset$	($AMX02$)(Conditional Introduction)
$(E * b = E) \subseteq (b \neq \emptyset)$	(T_{M735})($\subseteq$ Existential Form)(Eqn6)
$(b \neq \emptyset) = (E * b = E)$	(T_{M430})(Biconditional Introduction Eqns3&6)

$c * b + \underline{c} * b = E$ (T_{R248}) Conditional Negative Theorem

Proof of (T_{R248})

$E * b = E$	(T_{S244})
$(c + \underline{c}) * b = E$	(T_{M671})(Tertium non Datur)
$(c * b) + (\underline{c} * b) = E$	(T_{R002c})(Distribution of $+$ over $*$)

$c * b \subseteq E * b$ (T_{R224})

Proof of (T_{R224})

Let $b \neq \emptyset$	
$c * b \subseteq E$	(T_{M744})(Universal Set Theorem)
$c * b \subseteq E * b$	(T_{S244})
Thus $(b \neq \emptyset) \subseteq (c * b \subseteq E * b)$	($AMX02$)(Conditional Introduction)(Eqn1)
Let $b = \emptyset$	(Eqn2)
Let $c * b \not\subseteq E * b$	(Eqn3)
$c * \emptyset \not\subseteq E * \emptyset$	($AMX19$)(Substitution of Eqn2 into Eqn3)
$\emptyset \not\subseteq E * \emptyset$	(T_{S181})(Null Object Theorem)
$\emptyset$	(T_{M024})(Non-existence of Anti- (T_{M670}) Null Set Theorem)
Thus $(b = \emptyset).(c * b \not\subseteq E * b) = \emptyset$	($AMX02$)(Conditional Introduction)
$(b = \emptyset) \subseteq (c * b \subseteq E * b)$	(T_{M735})($\subseteq$ Existential Form)(Eqn4)
$(b = \emptyset) + (b \neq \emptyset) \subseteq (c * b \subseteq E * b) + (c * b \subseteq E * b)$	($AMX21$)(Combination of Eqns1&4)
$(b = \emptyset) + (b \neq \emptyset) \subseteq (c * b \subseteq E * b)$	($AMX01$)(Disjunction Tautology)
$E \subseteq (c * b \subseteq E * b)$	(T_{M671})(Tertium non Datur)
$E = (c * b \subseteq E * b)$	(T_{M275})(Supersets of the Universal Set)
$c * b \subseteq E * b$	($AMX16$)(Order Axiom)

$(c \subseteq d) \subseteq (c \diamond b \subseteq d \diamond b)$ (T_{R263}) Relational Deduction

Proof of (T_{R263})

$(\underline{d} \subseteq \underline{c}) \subseteq (\underline{d} * b \subseteq \underline{c} * b)$	(T_{R197})(Relational Deduction)
$(\underline{d} \subseteq \underline{c}) \subseteq (\sim(\underline{c} * b) \subseteq \sim(\underline{d} * b))$	(T_{M012})(Transposition)(Eqn1)
$(b \neq \emptyset) = (\underline{c} * b = \underline{c} \diamond b)$	($AR10$)(Axiom of Complementary Relationship Sets)
$(b \neq \emptyset) = (\sim(\underline{c} * b) = \underline{c} \diamond b)$	(T_{M008})(Inverse of an Equality)(Eqn2)
$(b \neq \emptyset) = (\sim(\underline{d} * b) = \underline{d} \diamond b)$	Resymbolising (Eqn3)
$(\underline{d} \subseteq \underline{c})(b \neq \emptyset) \subseteq (\underline{c} \diamond b \subseteq \underline{d} \diamond b)$	($AMX19$)(Substitution of Eqns2&3 into Eqn1)
$(c \subseteq d)(b \neq \emptyset) \subseteq (c \diamond b \subseteq d \diamond b)$	(T_{M012})(Transposition)(Eqn4)
Let $b = \emptyset$	(Eqn5)
Let $(c \subseteq d) \not\subseteq (c \diamond b \subseteq d \diamond b)$	(Eqn6)
$(c \subseteq d) \not\subseteq (c \diamond \emptyset \subseteq d \diamond \emptyset)$	($AMX19$)(Substitution of Eqn5 into Eqn6)
$(c \subseteq d) \not\subseteq (\emptyset \subseteq \emptyset)$	(T_{R174})(Null Object Theorem)
$(c \subseteq d) \not\subseteq (\emptyset \subset \emptyset) + (\emptyset = \emptyset)$	(T_{M266})(Definition of $\subseteq$)
$(c \subseteq d) \not\subseteq \emptyset + (\emptyset = \emptyset)$	(T_{M053})(Proper Subset of the Null Set)
$(c \subseteq d) \not\subseteq \emptyset + E$	(T_{M001})(Universality of (D_{M00}) (Definition of $=$))
$(c \subseteq d) \not\subseteq E$	(T_{M668})(Null Set Disjunction)
$\sim E$	(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)
$\emptyset$	(D_{M02})(Definition of $\emptyset$)
Thus $(b = \emptyset) = E.((c \subseteq d) \not\subseteq (c \diamond b \subseteq d \diamond b)) = \emptyset$	($AMX02$)(Conditional Introduction)
$((b = \emptyset) = E) \subseteq ((c \subseteq d) \subseteq (c \diamond b \subseteq d \diamond b))$	(T_{M735})($\subseteq$ Existential Form)
$(c \subseteq d)(b = \emptyset) \subseteq (c \diamond b \subseteq d \diamond b)$	($AMX35$)(Exportation/Importation) for truth sets (Eqn7)
$(c \subseteq d)(b \neq \emptyset) + (c \subseteq d)(b = \emptyset) \subseteq (c \diamond b \subseteq d \diamond b) + (c \diamond b \subseteq d \diamond b)$	

$(c \subseteq d)(b \neq \emptyset) + (c \subseteq d)(b = \emptyset) \subseteq (c \diamond b \subseteq d \diamond b)$	$(\mathbf{A}_{MX27})$ (Combination of Eqns4&7)
$(c \subseteq d)((b \neq \emptyset) + (b = \emptyset)) \subseteq (c \diamond b \subseteq d \diamond b)$	$(\mathbf{A}_{MX07})$ (Disjunction Tautology)
$(c \subseteq d)E \subseteq (c \diamond b \subseteq d \diamond b)$	$(\mathbf{T}_{M407})$ (Distribution of $\cdot$ over $+$)
$(c \subseteq d) \subseteq (c \diamond b \subseteq d \diamond b)$	$(\mathbf{T}_{M671})$ (Tertium non Datur)
	$(\mathbf{A}_{MX17})$ (Universal Set Conjunction Axiom)

$$(a \neq \emptyset)(a \subseteq b)(c \subseteq d) \subseteq (c \diamond b \subseteq d \diamond a) \quad (\mathbf{T}_{R264}) \text{ Relational Deduction}$$

Proof of $(\mathbf{T}_{R264})$

$(a \subseteq b)(a \neq \emptyset) \subseteq (c \diamond b \subseteq c \diamond a)$	$(\mathbf{T}_{R262})$ (Relational Deduction)(Eqn1)
$(c \subseteq d) \subseteq (c \diamond a \subseteq d \diamond a)$	$(\mathbf{T}_{R263})$ (Relational Deduction)(Eqn2)
$(a \neq \emptyset)(a \subseteq b)(c \subseteq d) \subseteq (c \diamond b \subseteq c \diamond a)(c \diamond a \subseteq d \diamond a)$	$(\mathbf{T}_{M669})$ (Combination of Eqns1&2)
$(a \neq \emptyset)(a \subseteq b)(c \subseteq d) \subseteq (c \diamond b \subseteq d \diamond a)$	$(\mathbf{T}_{M139})$ (Hypothetical Syllogism)

$$(b \neq \emptyset)(c * b = \emptyset) \subseteq (\underline{c} * b = E) \quad (\mathbf{T}_{R089})$$

Proof of $(\mathbf{T}_{R089})$

$(b \neq \emptyset) = (E * b = E)$	$(\mathbf{T}_{S244})$ (Eqn1)
$c + \underline{c} = E$	$(\mathbf{T}_{M671})$ (Tertium non Datur)(Eqn2)
$(b \neq \emptyset) = ((c + \underline{c}) * b = E)$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn2 into Eqn1)
$(b \neq \emptyset) = (c * b + \underline{c} * b = E)$	$(\mathbf{T}_{R002c})$ (Distribution of $+$ over $*$)(Eqn3)
Let $c * b = \emptyset$	(Eqn4)
$(b \neq \emptyset) = (\emptyset + \underline{c} * b = E)$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn4 into Eqn3)
$(b \neq \emptyset) = (\underline{c} * b = E)$	$(\mathbf{T}_{M668})$ (Null Set Disjunction)
Thus $((c * b = \emptyset) = E) \subseteq ((b \neq \emptyset) = (\underline{c} * b = E))$	$(\mathbf{A}_{MX02})$ (Conditional Introduction)
$(b \neq \emptyset)(c * b = \emptyset) \subseteq (\underline{c} * b = E)$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets

$$(\forall Qb \subseteq \underline{c}) = (c * b = \emptyset) \quad (\mathbf{T}_{R079})$$

$$(\forall Sa \subseteq \underline{c}) = (\forall c * a = \emptyset) \quad (\mathbf{T}_{R084})$$

Example Proof of $(\mathbf{T}_{R079})$

$c * b = S(c^V Qb)$	$(\mathbf{A}_{R27})$ (Function Axiom)(Eqn1)
Let $c * b = \emptyset$	(Eqn2)
$S(c^V Qb) = \emptyset$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn2 into Eqn1)
$c^V Qb = \emptyset$	$(\mathbf{T}_{R066})$ (Null Object Theorem)
$c \subseteq \sim^V Qb$	$(\mathbf{T}_{M735})$ ($\subseteq$ Existential Form)
$\forall Qb \subseteq \underline{c}$	$(\mathbf{T}_{M012})$ (Transposition)
Thus $(c * b = \emptyset) = (\forall Qb \subseteq \underline{c})$	$(\mathbf{A}_{MX02})$ (Conditional Introduction) since this proof is reversible

$$\forall (\sim^V Sa) * a = \emptyset \quad (\mathbf{T}_{R157})$$

Proof of $(\mathbf{T}_{R157})$

$(c \subseteq \sim^V Sa) = (\forall c * a = \emptyset)$	$(\mathbf{T}_{R084})$ (Eqn1)
Let $c = \sim^V Sa$	(Eqn2)
$(\sim^V Sa \subseteq \sim^V Sa) = (\forall (\sim^V Sa) * a = \emptyset)$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn2 into Eqn1)
$(\sim^V Sa^V Sa = \emptyset) = (\forall (\sim^V Sa) * a = \emptyset)$	$(\mathbf{T}_{M735})$ ($\subseteq$ Existential Form)
$E = (\forall (\sim^V Sa) * a = \emptyset)$	$(\mathbf{T}_{M001})$ (Universality of $(\mathbf{T}_{M621})$)($\sim^V Sa^V Sa = \emptyset$)
$\forall (\sim^V Sa) * a = \emptyset$	$(\mathbf{A}_{MX16})$ (Order Axiom)

$$\forall (\sim^V Sa) \diamond a = \emptyset \quad (\mathbf{T}_{R283})$$

Proof of $(\mathbf{T}_{R283})$

$\forall (\sim^V Sa) * a = \emptyset$	$(\mathbf{T}_{R157})$
$\forall (\sim^V Sa) \diamond a = \emptyset$	$(\mathbf{T}_{S246})$

$$\sim^v(\forall Sa) * a = \emptyset \quad (T_{R158})$$

Proof of (T_{R158})

$$\forall(\sim^v Sa) * a = \emptyset \quad (T_{R157})$$

$$\sim^v(\forall Sa) * a = \emptyset \quad (T_{R035})$$

$$\sim^v(\forall Sa) \diamond a = \emptyset \quad (T_{R284})$$

Proof of (T_{R284})

$$\sim^v(\forall Sa) * a = \emptyset \quad (T_{R158})$$

$$\forall(\sim^v Sa) \diamond a = \emptyset \quad (T_{S246})$$

$$(b \neq \emptyset)(\forall Qb \subseteq c) \subseteq (c * b = E) \quad (T_{R080})$$

$$(a \neq \emptyset)(\forall Sa \subseteq c) \subseteq (\forall c * a = E) \quad (T_{R085})$$

Example Proof of (T_{R080})

$$c * b = S(c \forall Qb) \quad (A_{R27})(\text{Function Axiom Resymbolised})(\text{Eqn1})$$

$$\text{Let } c * b = \emptyset \quad (\text{Eqn2})$$

$$S(c \forall Qb) = \emptyset \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$c \forall Qb = \emptyset \quad (T_{R066})(\text{Null Object Theorem})$$

$$\forall Qb \subseteq c \quad (T_{M735})(\subseteq \text{ Existential Form})$$

$$\text{Thus } (c * b = \emptyset) = (\forall Qb \subseteq c) \quad (A_{MX02})(\text{Conditional Introduction})(\text{Eqn3})$$

$$(b \neq \emptyset)(c * b = \emptyset) \subseteq (c * b = E) \quad (T_{R089})(\text{Resymbolised})(\text{Eqn4})$$

$$(b \neq \emptyset)(\forall Qb \subseteq c) \subseteq (c * b = E) \quad (A_{MX19})(\text{Substitution of Eqn3 into Eqn4})$$

$$b^1 = Q^v Qb^1 \quad (T_{R012g}) \text{ Sub-relationship Closure Theorem}$$

$$a^1 = S^v Sa^1 \quad (T_{R012h}) \text{ Sub-relationship Closure Theorem}$$

Example Proof of (T_{R012h})

$$((\forall c)^1 \subseteq R_0)(b^1 \subseteq Qc) \subseteq (b^1 = \forall c * c * b^1) \quad (T_{S519})(\text{Eqn1})$$

$$\text{Let } c = \forall S \quad (\text{Eqn2})$$

$$((\forall S)^1 \subseteq R_0)(b^1 \subseteq Q^v S) \subseteq (b^1 = \forall S * \forall S * b^1) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(S^1 \subseteq R_0)(b^1 \subseteq Q^v S) \subseteq (b^1 = S * \forall S * b^1) \quad (T_{R012h})(\text{Sub-relationship Closure Theorem})(\text{Eqn3})$$

$$(X = (V, S, Q))(X^1 \subseteq R_0) \quad (A_{R04})(\text{One-to-One Sub-relationship Axiom})$$

$$S^1 \subseteq R_0 \quad \text{Extraction}$$

$$(S^1 \subseteq R_0) = E \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn4})$$

$$E(b^1 \subseteq Q^v S) \subseteq (b^1 = S * \forall S * b^1) \quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})$$

$$(b^1 \subseteq Q^v S) \subseteq (b^1 = S * \forall S * b^1) \quad (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn5})$$

$$(X = (S, Q))(Y = (V, S, Q, \forall S, \forall Q)): (X * Y = E) \quad (A_{R34})(\text{Universal Sub-relationship Axiom})$$

$$Q^v S = E \quad \text{Extraction (Eqn6)}$$

$$b^1 \subseteq E \quad (T_{M744})(\text{Universal Set Theorem})(\text{Eqn7})$$

$$b^1 \subseteq Q^v S \quad (A_{MX19})(\text{Substitution of Eqn6 into Eqn7})$$

$$(b^1 \subseteq Q^v S) = E \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn8})$$

$$E \subseteq (b^1 = S * \forall S * b^1) \quad (A_{MX19})(\text{Substitution of Eqn8 into Eqn5})$$

$$E = (b^1 = S * \forall S * b^1) \quad (T_{M275})(\text{Supersets of the Universal Set})$$

$$b^1 = S * \forall S * b^1 \quad (A_{MX16})(\text{Order Axiom})$$

$$b = Q^v Qb \quad (T_{R012l}) \text{ Sub-relationship Closure Theorem}$$

$$a = S^v Sa \quad (T_{R012m}) \text{ Sub-relationship Closure Theorem}$$

Example Proof of (T_{R012l})

$$\text{Let } a = (a_1^1 +++ a_n^1) \quad (\text{Eqn1})$$

$$S^v Sa = S^v Sa \quad (D_{M00})(\text{Definition of } =)(\text{Eqn2})$$

$$S^v Sa = S^v S(a_1^1 +++ a_n^1) \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$$

$$S^v Sa = S(\forall Sa_1^1 +++ \forall Sa_n^1) \quad (T_{R001a})(\text{Distribution of } * \text{ over } +)$$

$$S^v Sa = (S^v Sa_1^1 +++ S^v Sa_n^1) \quad (T_{R001a})(\text{Distribution of } * \text{ over } +)$$

$$\begin{aligned} S^{\vee}Sa &= (a_1^{1+++}a_n^1) \\ S^{\vee}Sa &= a \end{aligned}$$

(T_{R012h})(Sub-relationship Closure Theorem)(Eqn3)
($AMX19$)(Substitution of Eqn1 into Eqn3)

$$(a = b) = (\vee Sa = \vee Sb) \quad (T_{R023}) \text{ Sub-Relationship Deduction}$$

$$(a = b) = (\vee Qa = \vee Qb) \quad (T_{R024}) \text{ Sub-Relationship Deduction}$$

Example Proof of (T_{R023})

$$\begin{aligned} (a = b) &\subseteq (c * a = c * b) & (T_{R522}) \text{ (Relational Deduction Theorem) (Eqn1)} \\ \text{Let } c &= \vee S & (\text{Eqn2}) \\ (a = b) &\subseteq (\vee Sa = \vee Sb) & (AMX19) \text{ (Substitution of Eqn2 into Eqn1) (Eqn3)} \\ \text{Let } \vee Sa &= \vee Sb \\ S^{\vee}Sa &= S^{\vee}Sb & (T_{R026}) \text{ (Sub-Relationship Deduction)} \\ a &= b & (T_{R012m}) \text{ (Sub-relationship Closure Theorem)} \\ \text{Thus } (\vee Sa = \vee Sb) &\subseteq (a = b) & (AMX02) \text{ (Conditional Introduction) (Eqn4)} \\ (a = b) &= (\vee Sa = \vee Sb) & (T_{M430}) \text{ (Biconditional Introduction Eqns3\&4)} \end{aligned}$$

$$(a^1 = b^1) = (\vee Sa^1 = \vee Sb^1) \quad (T_{R138})$$

$$(a^1 = b^1) = (\vee Qa^1 = \vee Qb^1) \quad (T_{R139})$$

Example Proof of (T_{R138})

$$\begin{aligned} (a^1 = b^1) &= (a^1 = b^1) & (D_{M00}) \text{ (Definition of } =) \\ (a^1 = b^1) &= (\vee Sa^1 = \vee Sb^1) & (T_{R023}) \text{ (Sub-Relationship Deduction)} \end{aligned}$$

$$(a(c * b) = \emptyset) = (\vee Sa.c. \vee Qb = \emptyset) \quad (T_{R228})$$

Proof of (T_{R228})

$$\begin{aligned} \text{Let } \vee Sa.c. \vee Qb &= \emptyset \\ S(\vee Sa.c. \vee Qb) &= S\emptyset & (T_{R026}) \\ S(\vee Sa.c. \vee Qb) &= \emptyset & (T_{R066}) \text{ (Null Object Theorem)} \\ S^{\vee}Sa. S(c. \vee Qb) &= \emptyset & (T_{R012d}) \text{ (Distribution of } S^* \text{ over } .) \\ a.S(c. \vee Qb) &= \emptyset & (T_{R012m}) \text{ (Sub-Relationship Closure Theorem)} \\ a.(c * b) &= \emptyset & (AR27) \text{ (Function Axiom)} \\ \text{Thus } (\vee Sa.c. \vee Qb = \emptyset) &\subseteq (a(c * b) = \emptyset) & (AMX02) \text{ (Conditional Introduction) (Eqn1)} \\ \text{Let } a(c * b) &= \emptyset \\ a.S(c. \vee Qb) &= \emptyset & (AR27) \text{ (Function Axiom)} \\ \vee S(a.S(c. \vee Qb)) &= \vee S\emptyset & (T_{R023}) \text{ (Sub-Relationship Deduction)} \\ \vee S(a.S(c. \vee Qb)) &= \emptyset & (T_{R067}) \text{ (Null Object Theorem)} \\ \vee Sa. \vee SS(c. \vee Qb) &= \emptyset & (T_{R012b}) \text{ (Distribution of } \vee S^* \text{ over } .) \text{ (Eqn2)} \\ c. \vee Qb &\subseteq \vee SS(c. \vee Qb) & (T_{R012o}) \text{ (Sub-Relationship Non-Closure Theorem)} \\ \vee Sa.c. \vee Qb &\subseteq \vee Sa. \vee SS(c. \vee Qb) & (T_{M029}) \text{ (Deduction) (Eqn3)} \\ \vee Sa.c. \vee Qb &\subseteq \emptyset & (AMX19) \text{ (Substitution of Eqn2 into Eqn3)} \\ \vee Sa.c. \vee Qb &\subseteq \emptyset & (T_{M052}) \text{ (Subsets of the Null Set)} \\ \text{Thus } (a(c * b) = \emptyset) &\subseteq (\vee Sa.c. \vee Qb = \emptyset) & (AMX02) \text{ (Conditional Introduction) (Eqn4)} \\ (a(c * b) = \emptyset) &= (\vee Sa.c. \vee Qb = \emptyset) & (T_{M430}) \text{ (Biconditional Introduction Eqns1\&4)} \end{aligned}$$

$$(a \subseteq c * b) = (\vee Sa.c. \vee Qb = \emptyset) \quad (T_{S082})$$

Proof of (T_{S082})

$$\begin{aligned} (a(c * b) = \emptyset) &= (\vee Sa.c. \vee Qb = \emptyset) & (T_{R228}) \\ (a \subseteq c * b) &= (\vee Sa.c. \vee Qb = \emptyset) & (T_{M735}) \text{ (} \subseteq \text{ Existential Form)} \end{aligned}$$

$$(a \not\subseteq c * b) = (\vee Sa.c. \vee Qb \neq \emptyset) \quad (T_{S081})$$

Proof of (T_{S081})

$$\begin{aligned} (a \subseteq c * b) &= (\vee Sa.c. \vee Qb = \emptyset) & (T_{S082}) \\ (a \not\subseteq c * b) &= (\vee Sa.c. \vee Qb \neq \emptyset) & (T_{M008}) \text{ (Inverse of an Equality)} \end{aligned}$$

$$(a(c*b) = \emptyset) = (b(^v c*a) = \emptyset)$$

(T_{R230}) Inverse Relational Propositions

Proof of (T_{R230})

Let $a(c*b) = \emptyset$

$^v Sa.c.^v Qb = \emptyset$

(T_{R228})

$^v(^v Sa.c.^v Qb) = ^v \emptyset$

(T_{R022})(Relationship Opposite of a Proposition)

$^v(^v Sa.c.^v Qb) = \emptyset$

(T_{R070})(Null Object Theorem)

$^v(^v Sa).^v c.^v(^v Qb) = \emptyset$

(T_{R012a})(Distribution of V^* over $\cdot$)

$^v Qa.^v c.^v Sb = \emptyset$

(A_{R13}) and (T_{R019}) Reversibility of Matching Relationships)

$b(^v c*a) = \emptyset$

(T_{R228})

Thus $(a(c*b) = \emptyset) = (b(^v c*a) = \emptyset)$

($AMX02$)(Conditional Introduction)

$$(a(c*b) \neq \emptyset) = (b(^v c*a) \neq \emptyset)$$

(T_{R229}) Inverse Relational Propositions

Proof of (T_{R229})

$(a(c*b) = \emptyset) = (b(^v c*a) = \emptyset)$

(T_{R230})(Inverse Relational Propositions)

$(a(c*b) \neq \emptyset) = (b(^v c*a) \neq \emptyset)$

(T_{M008})(Inverse of an Equality)

$$(a \subseteq \sim(c*b)) = (b \subseteq \sim(^v c*a))$$

(T_{R457}) Active/Passive Theorem

Proof of (T_{R457})

$(a(c*b) = \emptyset) = (b(^v c*a) = \emptyset)$

(T_{R230})(Inverse Relational Propositions)

$(a \subseteq \sim(c*b)) = (b \subseteq \sim(^v c*a))$

(T_{M735})($\subseteq$ Existential Form)

$$((b \neq \emptyset)(a \neq \emptyset) = E) + ((a = \emptyset)(b = \emptyset) = E) \subseteq ((a \subseteq c \diamond b) = (b \subseteq ^v c \diamond a))$$

(T_{R293}) Active/Passive Theorem

Proof of (T_{R293})

Let $(a \neq \emptyset)(b \neq \emptyset)$

$(a \subseteq c \diamond b) = (b \subseteq ^v c \diamond a)$

(T_{R457})(Active/Passive Theorem)(Eqn1)

$c \diamond b = c \diamond b$

(A_{R10})(Axiom of Complementary Relationship Sets)

$c \diamond b = c \diamond b$

(T_{M008})(Inverse of an Equality)

$c \diamond b = c \diamond b$

Resymbolising(Eqn2)

$^v c \diamond a = ^v c \diamond a$

Resymbolising(Eqn3)

$(a \subseteq c \diamond b) = (b \subseteq ^v c \diamond a)$

($AMX19$)(Substitution of Eqns2&3 into Eqn1)

$(a \subseteq c \diamond b) = (b \subseteq ^v c \diamond a)$

Resymbolising

Thus $((a \neq \emptyset)(b \neq \emptyset) = E) \subseteq ((a \subseteq c \diamond b) = (b \subseteq ^v c \diamond a))$

($AMX02$)(Conditional Introduction)(Eqn4)

Let $a = \emptyset$

(Eqn5)

Let $b = \emptyset$

(Eqn6)

Let $(a \subseteq c \diamond b) \neq (b \subseteq ^v c \diamond a)$

(Eqn7)

$(\emptyset \subseteq c \diamond \emptyset) \neq (\emptyset \subseteq ^v c \diamond \emptyset)$

($AMX19$)(Substitution of Eqns5&6 into Eqn7)

$E \neq E$

(T_{M001})(Universality of (T_{M670}) Null Set Theorem)

$\emptyset$

(T_{M345})

Thus $((a = \emptyset) = E).((b = \emptyset) = E).((a \subseteq c \diamond b) \neq (b \subseteq ^v c \diamond a)) = \emptyset$

($AMX02$)(Conditional Introduction)

$((a = \emptyset) = E).((b = \emptyset) = E) \subseteq ((a \subseteq c \diamond b) = (b \subseteq ^v c \diamond a))$

(T_{M735})($\subseteq$ Existential Form)

$((a = \emptyset).(b = \emptyset) = E) \subseteq ((a \subseteq c \diamond b) = (b \subseteq ^v c \diamond a))$

(T_{M108})(Conjunction Introduction)(Eqn8)

$((b \neq \emptyset)(a \neq \emptyset) = E) + ((a = \emptyset)(b = \emptyset) = E) \subseteq ((a \subseteq c \diamond b) = (b \subseteq ^v c \diamond a)) + ((a \subseteq c \diamond b) = (b \subseteq ^v c \diamond a))$

($AMX21$)(Combination of Eqns4&8)

$((b \neq \emptyset)(a \neq \emptyset) = E) + ((a = \emptyset)(b = \emptyset) = E) \subseteq ((a \subseteq c \diamond b) = (b \subseteq ^v c \diamond a))$

($AMX01$)(Disjunction Tautology)

$$((b^1 \neq \emptyset)(a \neq \emptyset) = E) + ((a = \emptyset)(b^1 = \emptyset) = E) \subseteq ((a \subseteq c \diamond b^1) = (b^1 \subseteq ^v c \diamond a))$$

(T_{R521}) Active/Passive Theorem

Proof of (T_{R521})

$((b \neq \emptyset)(a \neq \emptyset) = E) + ((a = \emptyset)(b = \emptyset) = E) \subseteq ((a \subseteq c \diamond b) = (b \subseteq ^v c \diamond a))$

Let $\mathbf{b} = \mathbf{b}^1$ (T_{R293})(Active/Passive Theorem) (Eqn1)
(Eqn2)
 $((\mathbf{b}^1 \neq \emptyset)(\mathbf{a} \neq \emptyset) = \mathbf{E}) + ((\mathbf{a} = \emptyset)(\mathbf{b}^1 = \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \subseteq \mathbf{c} \diamond \mathbf{b}^1) = (\mathbf{b}^1 \subseteq \mathbf{v} \mathbf{c} \diamond \mathbf{a}))$
(A_{MX19})(Substitution of Eqn2 into Eqn1)
 $((\mathbf{b}^1 \neq \emptyset)(\mathbf{a} \neq \emptyset) = \mathbf{E}) + ((\mathbf{a} = \emptyset)(\mathbf{b}^1 = \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \subseteq \mathbf{c} * \mathbf{b}^1) = (\mathbf{b}^1 \subseteq \mathbf{v} \mathbf{c} \diamond \mathbf{a}))$
(T_{R232})(Interchangeability of $*$ and $\diamond$ for Entities)

$$((b^1 \neq \emptyset)(a \neq \emptyset) = E) + ((a = \emptyset)(b^1 = \emptyset) = E) \subseteq ((a \not\subseteq c * b^1) = (b^1 \not\subseteq {}^v c \diamond a)) \quad (T_{S286}) \text{ Active/Passive Theorem}$$

Proof of ($\mathcal{T}_{S286}$)
 $((b^1 \neq \emptyset)(a \neq \emptyset) = E) + ((a = \emptyset)(b^1 = \emptyset) = E) \subseteq ((a \subseteq c * b^1) = (b^1 \subseteq {}^v c \diamond a))$
 $(\mathcal{T}_{R521})(\text{Active/Passive Theorem})$
 $((b^1 \neq \emptyset)(a \neq \emptyset) = E) + ((a = \emptyset)(b^1 = \emptyset) = E) \subseteq ((a \not\subseteq c * b^1) = (b^1 \not\subseteq {}^v c \diamond a))$
 $(\mathcal{T}_{M008})(\text{Inverse of an Equality})$

$$((b \neq \emptyset)(a \neq \emptyset) = E) + ((a = \emptyset)(b = \emptyset) = E) \subseteq ((a \not\subseteq c \diamond b) = (b \not\subseteq^v c \diamond a)) \quad (T_{R691}) \text{ Active/Passive Theorem}$$

$$\begin{aligned} & \text{Proof of } (\mathbf{T}_{R691}) \\ & ((\mathbf{b} \neq \emptyset)(\mathbf{a} \neq \emptyset) = \mathbf{E}) + ((\mathbf{a} = \emptyset)(\mathbf{b} = \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \subseteq \mathbf{c} \diamond \mathbf{b}) = (\mathbf{b} \subseteq {}^v\mathbf{c} \diamond \mathbf{a})) \\ & \hspace{20em} (\mathbf{T}_{R293})(\text{Active/Passive Theorem})(\text{Eqn1}) \\ & ((\mathbf{b} \neq \emptyset)(\mathbf{a} \neq \emptyset) = \mathbf{E}) + ((\mathbf{a} = \emptyset)(\mathbf{b} = \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \not\subseteq \mathbf{c} \diamond \mathbf{b}) = (\mathbf{b} \not\subseteq {}^v\mathbf{c} \diamond \mathbf{a})) \\ & \hspace{20em} (\mathbf{T}_{M008})(\text{Inverse of an Equality}) \end{aligned}$$

$$((\mathbf{b} \neq \emptyset)(\mathbf{a}^1 \neq \emptyset) = \mathbf{E}) + ((\mathbf{a}^1 = \emptyset)(\mathbf{b} = \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a}^1 \subseteq \mathbf{c} \diamond \mathbf{b}) = (\mathbf{b} \subseteq {}^v \mathbf{c} * \mathbf{a}^1)) \quad (\mathcal{T}_{S028}) \text{ Active/Passive Theorem}$$

Proof of (T₅₀₂₈)

$$((b \neq \emptyset)(a \neq \emptyset) = E) + ((a = \emptyset)(b = \emptyset) = E) \subseteq ((a \subseteq c \diamond b) = (b \subseteq {}^v c \diamond a))$$

(T_{R293})(Active/Passive Theorem)(Eqn1)

Let $a = a^1$

(Eqn2)

$$((b \neq \emptyset)(a^1 \neq \emptyset) = E) + ((a^1 = \emptyset)(b = \emptyset) = E) \subseteq ((a^1 \subseteq c \diamond b) = (b \subseteq {}^v c \diamond a^1))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$((b \neq \emptyset)(a^1 \neq \emptyset) = E) + ((a^1 = \emptyset)(b = \emptyset) = E) \subseteq ((a^1 \subseteq c \diamond b) = (b \subseteq {}^v c * a^1))$$

(T_{R232})(Interchangeability of * and $\diamond$ for Entities)

$$((\mathbf{b} \neq \emptyset)(\mathbf{a}^1 \neq \emptyset) = \mathbf{E}) + ((\mathbf{a}^1 = \emptyset)(\mathbf{b} = \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a}^1 \not\subseteq \mathbf{c} \diamond \mathbf{b}) = (\mathbf{b} \not\subseteq^{\mathbf{v}} \mathbf{c} * \mathbf{a}^1)) \quad (T_{S029}) \text{ Active/Passive Theorem}$$

$$\begin{aligned} & \text{Proof of } (T_{5029}) \\ & ((b \neq \emptyset)(a^1 \neq \emptyset) = E) + ((a^1 = \emptyset)(b = \emptyset) = E) \subseteq ((a^1 \subseteq c \diamond b) = (b \subseteq {}^v c * a^1)) \\ & \hspace{20em} (T_{5028}) \text{ (Active/Passive Theorem)} \\ & ((b \neq \emptyset)(a^1 \neq \emptyset) = E) + ((a^1 = \emptyset)(b = \emptyset) = E) \subseteq ((a^1 \not\subseteq c \diamond b) = (b \not\subseteq {}^v c * a^1)). \\ & \hspace{20em} (T_{1003}) \text{ (Inverse of an Equality)} \end{aligned}$$

$$((b \neq \emptyset) = E) \subseteq ((a \subseteq c \diamond b) \subseteq (b \not\subseteq^v c \diamond a)) \quad (T_{R541})$$

<u>Proof of</u> ($\mathcal{T}_{R541}$)	
$((a \neq \emptyset) = E) \subseteq ((a \subseteq b) \subseteq (a \not\subseteq b))$	($\mathcal{T}_{M175}$)(Negation Introduction)
$((a \neq \emptyset) = E) \subseteq ((a \subseteq c \diamond b) \subseteq (a \not\subseteq c \diamond b))$	Resymbolising
$((a \neq \emptyset) = E) \subseteq ((a \subseteq c \diamond b) \subseteq (a \not\subseteq c \diamond b))$	($\mathcal{T}_{M012}$)(Transposition)(Eqn1)
$(b \neq \emptyset)(a \neq \emptyset) \subseteq (a \neq \emptyset)$	($\mathcal{T}_{M083}$)(Conjunction Elimination)
$((b \neq \emptyset)(a \neq \emptyset) = E) \subseteq ((a \neq \emptyset) = E)$	($\mathcal{T}_{M094}$)(Probabilistic Rule)(Eqn2)
$((b \neq \emptyset)(a \neq \emptyset) = E) \subseteq ((a \subseteq c \diamond b) \subseteq (a \not\subseteq c \diamond b))$	($\mathcal{T}_{M139}$)(Hypothetical Syllogism Eqns1&2)(Eqn3)
$((b \neq \emptyset)(a \neq \emptyset) = E) + ((a = \emptyset)(b = \emptyset) = E) \subseteq ((a \subseteq c \diamond b) = (b \subseteq {}^v c \diamond a))$	($\mathcal{T}_{R293}$)(Active/Passive Theorem)
$((b \neq \emptyset)(a \neq \emptyset) = E) \subseteq ((a \subseteq c \diamond b) = (b \subseteq {}^v c \diamond a))$	($\mathcal{T}_{M072}$)(Single Premise Syllogism)
$((b \neq \emptyset)(a \neq \emptyset) = E) \subseteq ((a \not\subseteq c \diamond b) = (b \not\subseteq {}^v c \diamond a))$	($\mathcal{T}_{M008}$)(Inverse of an Equality)(Eqn4)
$((b \neq \emptyset)(a \neq \emptyset) = E) \subseteq ((a \subseteq c \diamond b) \subseteq (b \not\subseteq {}^v c \diamond a))$	($\mathcal{A}_{MX19}$)(Substitution of Eqn4 into Eqn3)

$$(b \neq \emptyset)(a \neq \emptyset)(a \subseteq c \diamond b) \subseteq (b \not\subseteq {}^v c \diamond a)$$

Let $a = \emptyset$

Let $b \neq \emptyset$

$$(b \neq \emptyset) = E$$

Let $(a \subseteq c \diamond b) \not\subseteq (b \not\subseteq {}^v c \diamond a)$

$$(\emptyset \subseteq c \diamond b) \not\subseteq (b \not\subseteq {}^v c \diamond \emptyset)$$

$$E \not\subseteq (b \not\subseteq {}^v c \diamond \emptyset)$$

$$E \not\subseteq (b \not\subseteq \emptyset)$$

$$E \not\subseteq (b \sim \emptyset)(b \neq \emptyset)$$

$$E \not\subseteq E(b \neq \emptyset)$$

$$E \not\subseteq (b \neq \emptyset)$$

$$(E \sim \emptyset)(b \neq \emptyset)(E \neq (b \neq \emptyset))$$

$$E(E \neq (b \neq \emptyset))$$

$$E \neq (b \neq \emptyset)$$

$$E \neq E$$

$$\emptyset$$

Thus $((b \neq \emptyset)(a = \emptyset) = E)((a \subseteq c \diamond b) \not\subseteq (b \not\subseteq {}^v c \diamond a)) = \emptyset$

$$((b \neq \emptyset)(a = \emptyset) = E) \subseteq ((a \subseteq c \diamond b) \subseteq (b \not\subseteq {}^v c \diamond a))$$

$$(b \neq \emptyset)(a = \emptyset)(a \subseteq c \diamond b) \subseteq (b \not\subseteq {}^v c \diamond a)$$

$$(b \neq \emptyset)(a \neq \emptyset)(a \subseteq c \diamond b) + (b \neq \emptyset)(a = \emptyset)(a \subseteq c \diamond b) \subseteq (b \not\subseteq {}^v c \diamond a) + (b \not\subseteq {}^v c \diamond a)$$

$$(b \neq \emptyset)(a \neq \emptyset)(a \subseteq c \diamond b) + (b \neq \emptyset)(a = \emptyset)(a \subseteq c \diamond b) \subseteq (b \not\subseteq {}^v c \diamond a)$$

$$(b \neq \emptyset)(a \subseteq c \diamond b)((a \neq \emptyset) + (a = \emptyset)) \subseteq (b \not\subseteq {}^v c \diamond a)$$

$$(b \neq \emptyset)(a \subseteq c \diamond b)E \subseteq (b \not\subseteq {}^v c \diamond a)$$

$$(b \neq \emptyset)(a \subseteq c \diamond b) \subseteq (b \not\subseteq {}^v c \diamond a)$$

$$((b \neq \emptyset) = E) \subseteq ((a \subseteq c \diamond b) \subseteq (b \not\subseteq {}^v c \diamond a))$$

($\mathbf{A}_{MX35}$)(Exportation/Importation)(Eqn5)

for truth sets

(Eqn6)

($\mathbf{A}_{MX16}$)(Order Axiom)(Eqn7)

(Eqn8)

($\mathbf{A}_{MX19}$)(Substitution of Eqn6 into Eqn8)

($\mathbf{T}_{M001}$)(Universality of ($\mathbf{T}_{M670}$)Null Set Theorem)

($\mathbf{T}_{R174}$)(Null Object Theorem)

($\mathbf{T}_{M198}$)

($\mathbf{T}_{M053b}$)

($\mathbf{A}_{MX11}$)(Universal Set Conjunction Axiom)

($\mathbf{T}_{M198}$)

($\mathbf{T}_{M098d}$)

($\mathbf{A}_{MX11}$)(Universal Set Conjunction Axiom)(Eqn9)

($\mathbf{A}_{MX19}$)(Substitution of Eqn7 into Eqn9)

($\mathbf{T}_{M345}$)

($\mathbf{A}_{MX02}$)(Conditional Introduction)

($\mathbf{T}_{M735}$)($\subseteq$ Existential Form)

($\mathbf{A}_{MX35}$)(Exportation/Importation)(Eqn10)

for truth sets

($\mathbf{A}_{MX21}$)(Combination of Eqns5 & 10)

($\mathbf{A}_{MX01}$)(Disjunction Tautology)

($\mathbf{T}_{M407}$)(Distribution of . over +)

($\mathbf{T}_{M671}$)(Tertium non Datur)

($\mathbf{A}_{MX11}$)(Universal Set Conjunction Axiom)

($\mathbf{A}_{MX35}$)(Exportation/Importation) for truth sets

$$b \subseteq {}^v c * c * b$$

($\mathbf{T}_{T348}$)

Proof of ($\mathbf{T}_{T348}$)

$$(a \subseteq c * b) = (b \subseteq {}^v c * a)$$

Let $a = c * b$

$$(c * b \subseteq c * b) = (b \subseteq {}^v c * c * b)$$

$$(c * b \subseteq c * b) + (c * b = c * b) = (b \subseteq {}^v c * c * b)$$

$$(c * b \subseteq c * b) + E = (b \subseteq {}^v c * c * b)$$

$$E = (b \subseteq {}^v c * c * b)$$

$$b \subseteq {}^v c * c * b$$

($\mathbf{T}_{R457}$)(Active/Passive Theorem)(Eqn1)

(Eqn2)

($\mathbf{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

($\mathbf{T}_{M266}$)(Definition of $\subseteq$)

($\mathbf{T}_{M001}$)(Universality of ($\mathbf{D}_{M00}$) Definition of $=$)

($\mathbf{T}_{M055}$)(Universal Set Disjunction)

($\mathbf{A}_{MX16}$)(Order Axiom)

$$(a \not\subseteq \sim(c * b)) = (b \not\subseteq \sim({}^v c * a))$$

($\mathbf{T}_{R458}$) Active/Passive Theorem

Proof of ($\mathbf{T}_{R458}$)

$$(a \subseteq \sim(c * b)) = (b \subseteq \sim({}^v c * a))$$

$$(a \not\subseteq \sim(c * b)) = (b \not\subseteq \sim({}^v c * a))$$

($\mathbf{T}_{R457}$)(Active/Passive Theorem)

($\mathbf{T}_{M008}$)(Inverse of an Equality)

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq c * b) \subseteq (b \not\subseteq {}^v c * a))$$

($\mathbf{T}_{R540}$)

Proof of ($\mathbf{T}_{R540}$)

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq b) \subseteq (a \not\subseteq b))$$

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq c * b) \subseteq (a \not\subseteq c * b))$$

$$(a \not\subseteq c * b) = (b \not\subseteq {}^v c * a)$$

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq c * b) \subseteq (b \not\subseteq {}^v c * a))$$

($\mathbf{T}_{M175}$)(Negation Introduction)

Resymbolising (Eqn1)

($\mathbf{T}_{R458}$)(Active/Passive Theorem)(Eqn2)

($\mathbf{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq c * b) \subseteq (b \not\subseteq {}^v c * a))$$

($\mathbf{T}_{T385}$)

Proof of (T₇₃₈₅)

$$\begin{aligned} ((a \neq \emptyset) = E) &\subseteq ((a \subseteq c*b) \subseteq (a \not\subseteq c*b)) & (T_{M175})(\text{Negation Introduction})(\text{Eqn1}) \\ (a \not\subseteq c*b) &\subseteq (b \not\subseteq {}^v c*a) & (T_{R458})(\text{Eqn2}) \\ ((a \neq \emptyset) \subseteq E) &\subseteq ((a \subseteq c*b) \subseteq (b \not\subseteq {}^v c*a)) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$((c*b) \subseteq a) = (b \subseteq \sim({}^v c*a)) \quad (T_{R459}) \text{ Active/Passive Theorem}$$

Proof of (T_{R459})

$$\begin{aligned} (a \subseteq \sim(c*b)) &= (b \subseteq \sim({}^v c*a)) & (T_{R457})(\text{Active/Passive Theorem}) \\ ((c*b) \subseteq a) &= (b \subseteq \sim({}^v c*a)) & (T_{M012})(\text{Transposition}) \\ ((c*b) \subseteq a) &= (b \subseteq \sim({}^v c*a)) & \text{Resymbolising} \end{aligned}$$

$$((c*b) \not\subseteq a) = (b \not\subseteq \sim({}^v c*a)) \quad (T_{R460}) \text{ Active/Passive Theorem}$$

Proof of (T_{R460})

$$\begin{aligned} ((c*b) \subseteq a) &= (b \subseteq \sim({}^v c*a)) & (T_{R459})(\text{Active/Passive Theorem}) \\ ((c*b) \not\subseteq a) &= (b \not\subseteq \sim({}^v c*a)) & (T_{M008})(\text{Inverse of an Equality}) \end{aligned}$$

$$(a^1 \subseteq c*b) = (b \subseteq {}^v c*a^1) \quad (T_{S030}) \text{ Active/Passive Theorem}$$

Proof of (T_{S030})

$$\begin{aligned} (a \subseteq c*b) &= (b \subseteq {}^v c*a) & (T_{R457})(\text{Active/Passive Theorem})(\text{Eqn1}) \\ \text{Let } a &= a^1 & (\text{Eqn2}) \\ (a^1 \subseteq c*b) &= (b \subseteq {}^v c*a^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(a^1 \not\subseteq c*b) = (b \not\subseteq {}^v c*a^1) \quad (T_{S031}) \text{ Active/Passive Theorem}$$

Proof of (T_{S031})

$$\begin{aligned} (a^1 \subseteq c*b) &= (b \subseteq {}^v c*a^1) & (T_{S030}) \text{ (Active/Passive Theorem)} \\ (a^1 \not\subseteq c*b) &= (b \not\subseteq {}^v c*a^1) & (T_{M008})(\text{Inverse of an Equality}) \end{aligned}$$

$$(c*b^1 \neq \emptyset) \subseteq (b^1 \subseteq Qc) \quad (T_{S410b})$$

Proof of (T_{S410b})

$$\begin{aligned} \text{Let } c*b^1 &\neq \emptyset \\ b^1 &\neq \emptyset & (T_{R171b}) \\ S(c.{}^v Qb^1) &\neq \emptyset & (A_{R27})(\text{Function Axiom}) \\ c.{}^v Qb^1 &\neq \emptyset & (T_{R048})(\text{Triad Theorem}) \\ c \not\subseteq {}^v Q*b^1 & & (T_{M735})(\subseteq \text{Existential Form}) \& (T_{M008})(\text{Inverse of an Equality}) \\ b^1 &\not\subseteq Q*c & (T_{S031})(\text{Active/Passive Theorem}) \\ b^1 &\subseteq Qc & (T_{M090})(\text{Inverse Unit Subset Rule}) \text{ since } b^1 \neq \emptyset \\ \text{Thus } (c*b^1 \neq \emptyset) &\subseteq (b^1 \subseteq Qc) & (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(b^1 \subseteq Qc)(b^1 \neq \emptyset) \subseteq (c*b^1 \neq \emptyset) \quad (T_{S410c})$$

Proof of (T_{S410c})

$$\begin{aligned} \text{Let } b^1 &\subseteq Qc \\ \text{Let } b^1 &\neq \emptyset \\ b^1 &\not\subseteq Q*c & (T_{M090})(\text{Inverse Unit Subset Rule}) \text{ since } b^1 \neq \emptyset \\ c \not\subseteq {}^v Q*b^1 & & (T_{S031})(\text{Active/Passive Theorem}) \\ c.{}^v Qb^1 &\neq \emptyset & (T_{M735})(\subseteq \text{Existential Form}) \& (T_{M008})(\text{Inverse of an Equality}) \\ S(c.{}^v Qb^1) &\neq \emptyset & (T_{R048})(\text{Triad Theorem}) \\ c*b^1 &\neq \emptyset & (A_{R27})(\text{Function Axiom}) \\ \text{Thus } (b^1 \subseteq Qc)(b^1 \neq \emptyset) &\subseteq (c*b^1 \neq \emptyset) & (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(b^1 \neq \emptyset)(c*b^1 \neq \emptyset) \subseteq (b^1 \subseteq Qc) \quad (T_{S410})$$

Proof of (T_{S410})

$$\begin{aligned}
 (c * b^1 \neq \emptyset) &\subseteq (b^1 \subseteq Qc) & (T_{S410b})(Eqn1) \\
 (c * b^1 \neq \emptyset) &\subseteq (b^1 \neq \emptyset) & (T_{R171b}) \\
 (c * b^1 \neq \emptyset) &= (c * b^1 \neq \emptyset)(b^1 \neq \emptyset) & (A_{MX44})(\subseteq \text{Equality Form})(Eqn2) \\
 (b^1 \neq \emptyset)(c * b^1 \neq \emptyset) &\subseteq (b^1 \subseteq Qc) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
 \end{aligned}$$

$$((b^1 \neq \emptyset) = E) \subseteq ((b^1 \subseteq Qc) = (c * b^1 \neq \emptyset)) \quad (T_{T702})$$

Proof of (T_{T702})

$$\begin{aligned}
 (b^1 \neq \emptyset)(c * b^1 \neq \emptyset) &\subseteq (b^1 \subseteq Qc) & (T_{S410}) \\
 ((b^1 \neq \emptyset) = E) &\subseteq ((c * b^1 \neq \emptyset) \subseteq (b^1 \subseteq Qc)) & (A_{MX35})(\text{Exportation/Importation})(Eqn1) \\
 (b^1 \subseteq Qc)(b^1 \neq \emptyset) &\subseteq (c * b^1 \neq \emptyset) & (T_{S410c}) \\
 ((b^1 \neq \emptyset) = E) &\subseteq ((b^1 \subseteq Qc) \subseteq (c * b^1 \neq \emptyset)) & (A_{MX35})(\text{Exportation/Importation})(Eqn2) \\
 ((b^1 \neq \emptyset) = E)((b^1 \neq \emptyset) = E) &\subseteq ((c * b^1 \neq \emptyset) \subseteq (b^1 \subseteq Qc))((b^1 \subseteq Qc) \subseteq (c * b^1 \neq \emptyset)) & (A_{MX27})(\text{Combination of Eqns1\&2}) \\
 ((b^1 \neq \emptyset) = E) &\subseteq ((c * b^1 \neq \emptyset) \subseteq (b^1 \subseteq Qc))((b^1 \subseteq Qc) \subseteq (c * b^1 \neq \emptyset)) & (T_{M667})(\text{Conjunction Tautology}) \\
 ((b^1 \neq \emptyset) = E) &\subseteq ((c * b^1 \neq \emptyset) = (b^1 \subseteq Qc)) & (T_{M430})(\text{Biconditional Introduction})
 \end{aligned}$$

$$(b \subseteq \sim Qc) \subseteq (c * b = \emptyset) \quad (T_{T970})$$

Proof of (T_{T970})

$$\begin{aligned}
 \text{Let } b &= b_1^1 + \dots + b_n^1 & (Eqn1) \\
 \text{Let } (b_1^1 \neq \emptyset) \dots (b_n^1 \neq \emptyset) & & \\
 (c * b_r^1 \neq \emptyset) &\subseteq (b_r^1 \subseteq Qc) & (T_{S410b}) \\
 (c * b_r^1 \neq \emptyset) &\subseteq (b_r^1 \not\subseteq \sim Qc) & (T_{M090})(\text{Inverse Unit Subset Rule}) \\
 (b_r^1 \subseteq \sim Qc) &\subseteq (c * b_r^1 = \emptyset) & (T_{M012})(\text{Transposition})(Eqn2) \\
 (b_1^1 \subseteq \sim Qc) \dots (b_n^1 \subseteq \sim Qc) &\subseteq (c * b_1^1 = \emptyset) \dots (c * b_n^1 = \emptyset) & (T_{M669})(\text{Combination of Eqn2 for } r = 1 \text{ to } n) \\
 (b_1^1 \subseteq \sim Qc) \dots (b_n^1 \subseteq \sim Qc) &\subseteq (c * b_1^1 \subseteq \emptyset) \dots (c * b_n^1 \subseteq \emptyset) & (T_{M052})(\text{Subsets of the Null Set}) \\
 (b_1^1 + \dots + b_n^1 \subseteq \sim Qc) &\subseteq (c * b_1^1 + \dots + c * b_n^1 \subseteq \emptyset) & (T_{M131b})(\text{Conjunction of Sufficiencies})(Eqn2) \\
 (b_1^1 + \dots + b_n^1 \subseteq \sim Qc) &\subseteq (c * b_1^1 + \dots + c * b_n^1 = \emptyset) & (T_{M052})(\text{Subsets of the Null Set}) \\
 (b_1^1 + \dots + b_n^1 \subseteq \sim Qc) &\subseteq (c * (b_1^1 + \dots + b_n^1) = \emptyset) & (A_{S64})(\text{Distribution of } * \text{ over } +) \\
 (b \subseteq \sim Qc) &\subseteq (c * b = \emptyset) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})
 \end{aligned}$$

$$((a^1 \neq \emptyset) = E) \subseteq ((b \not\subseteq {}^V c * a^1) = (a^1 \subseteq c * b)) \quad (T_{T387})$$

Proof of (T_{S387})

$$\begin{aligned}
 (b \not\subseteq {}^V c * a^1) &= (a^1 \not\subseteq c * b) & (T_{S037})(\text{Active/Passive Theorem}) \\
 ((a^1 \neq \emptyset) = E) &\subseteq ((b \not\subseteq {}^V c * a^1) = (a^1 \subseteq c * b)) & (T_{M090})(\text{Inverse Unit Subset Rule})
 \end{aligned}$$

$$(a^1 \subseteq c * b^1) = (b^1 \subseteq {}^V c * a^1) \quad (T_{R519}) \text{ Active/Passive Theorem}$$

Proof of (T_{R519})

$$\begin{aligned}
 (a \subseteq \sim(c * b)) &= (b \subseteq \sim({}^V c * a)) & (T_{R457})(\text{Active/Passive Theorem}) \\
 (a \not\subseteq \sim(c * b)) &= (b \not\subseteq \sim({}^V c * a)) & (T_{M008})(\text{Inverse of an Equality})(Eqn1) \\
 \text{Let } a &= a^1 \text{ and } b = b^1 & (\text{Eqns2 \& 3}) \\
 (a^1 \not\subseteq \sim(c * b^1)) &= (b^1 \not\subseteq \sim({}^V c * a^1)) & (A_{MX19})(\text{Substitution of Eqns2 \& 3 into Eqn1}) \\
 (a^1 \subseteq c * b^1) &= (b^1 \subseteq {}^V c * a^1) & (T_{M090})(\text{Inverse Unit Subset Rule})
 \end{aligned}$$

$$(a^1 \not\subseteq c * b^1) = (b^1 \not\subseteq {}^V c * a^1) \quad (T_{R612}) \text{ Active/Passive Theorem}$$

Proof of (T_{R612})

$$\begin{aligned}
 (a^1 \subseteq c * b^1) &= (b^1 \subseteq {}^V c * a^1) & (T_{R519})(\text{Active/Passive Theorem}) \\
 (a^1 \not\subseteq c * b^1) &= (b^1 \not\subseteq {}^V c * a^1) & (T_{M008})(\text{Inverse of an Equality})
 \end{aligned}$$

$$(a^1 \not\subseteq c * b^1) = (b^1 \not\subseteq {}^V c * a^1) \quad (T_{R613}) \text{ Active/Passive Theorem}$$

Proof of (TR613)

$$\begin{aligned} (a^1 \subseteq c * b^1) &= (b^1 \subseteq {}^v c * a^1) & (TR519) \text{ (Active/Passive Theorem)} \\ (a^1 \not\subseteq c * b^1) &= (b^1 \not\subseteq {}^v c * a^1) & (TM090) \text{ (Inverse Unit Subset Rule)} \end{aligned}$$

$$(a^1 \subseteq c * b^1) = (b^1 \subseteq {}^v c * a^1) \quad (TR614) \text{ Active/Passive Theorem}$$

Proof of (TR614)

$$\begin{aligned} (a^1 \not\subseteq c * b^1) &= (b^1 \not\subseteq {}^v c * a^1) & (TR612) \text{ (Active/Passive Theorem)} \\ (a^1 \subseteq c * b^1) &= (b^1 \subseteq {}^v c * a^1) & (TM090) \text{ (Inverse Unit Subset Rule)} \end{aligned}$$

$$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 = c^1 * b^1) = (b^1 = {}^v c^1 * a^1)) \quad (TS247) \text{ Active/Passive Theorem}$$

Proof of (TS247)

$$\begin{aligned} \text{Let } (a^1 \neq \emptyset) & \\ (a^1 \neq \emptyset) &= E & (AMX16) \text{ (Order Axiom)} \\ (a^1 = \emptyset) &= \sim E & (TM008) \text{ (Inverse of an Equality)} \\ (a^1 = \emptyset) &= \emptyset & (DM02) \text{ (Definition of } \emptyset \text{) (Eqn1)} \\ (a^1 \subseteq c^1 * b^1) &= (c^1 * b^1 \neq \emptyset)(a^1 = \emptyset) & (AMX40) \text{ (Proper Subset of Unit Set Axiom) (Eqn2)} \\ (a^1 \subseteq c^1 * b^1) &= (c^1 * b^1 \neq \emptyset) \emptyset & (AMX19) \text{ (Substitution of Eqn1 into Eqn2)} \\ (a^1 \subseteq c^1 * b^1) &= \emptyset & (TM050) \text{ (Null Set Conjunction) (Eqn3)} \\ \text{Let } (b^1 \neq \emptyset) & \\ (b^1 \neq \emptyset) &= E & (AMX16) \text{ (Order Axiom)} \\ (b^1 = \emptyset) &= \sim E & (TM008) \text{ (Inverse of an Equality)} \\ (b^1 = \emptyset) &= \emptyset & (DM02) \text{ (Definition of } \emptyset \text{) (Eqn4)} \\ (b^1 \subseteq {}^v c^1 * a^1) &= ({}^v c^1 * a^1 \neq \emptyset)(b^1 = \emptyset) & (AMX40) \text{ (Proper Subset of Unit Set Axiom) (Eqn5)} \\ (b^1 \subseteq {}^v c^1 * a^1) &= ({}^v c^1 * a^1 \neq \emptyset) \emptyset & (AMX19) \text{ (Substitution of Eqn4 into Eqn5)} \\ (b^1 \subseteq {}^v c^1 * a^1) &= \emptyset & (TM050) \text{ (Null Set Conjunction) (Eqn6)} \\ (a^1 \subseteq c * b^1) &= (b^1 \subseteq {}^v c * a^1) & (TR519) \text{ (Active/Passive Theorem) (Eqn7)} \\ \text{Let } c &= c^1 & (Eqn8) \\ {}^v c &= {}^v c^1 & (TR022) \text{ (Relationship Opposite of a Proposition) (Eqn9)} \\ (a^1 \subseteq c^1 * b^1) &= (b^1 \subseteq {}^v c^1 * a^1) & (AMX19) \text{ (Substitution of Eqns8 \& 9 into Eqn7)} \\ (a^1 \subseteq c^1 * b^1) + (a^1 = c^1 * b^1) &= (b^1 \subseteq {}^v c^1 * a^1) + (b^1 = {}^v c^1 * a^1) & (TM266) \text{ (Definition of } \subseteq \text{) (Eqn10)} \\ \emptyset + (a^1 = c^1 * b^1) &= \emptyset + (b^1 = {}^v c^1 * a^1) & (AMX19) \text{ (Substitution of Eqns3 \& 6 into Eqn10)} \\ (a^1 = c^1 * b^1) &= (b^1 = {}^v c^1 * a^1) & (TM668) \text{ (Null Set Disjunction)} \\ \text{Thus } ((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) &\subseteq ((a^1 = c^1 * b^1) = (b^1 = {}^v c^1 * a^1)) & (AMX02) \text{ (Conditional Introduction)} \end{aligned}$$

$$((b \subseteq Qc) = E) \subseteq ((a \not\subseteq c * b) \subseteq (b \subseteq {}^v c \diamond a)) \quad (TR542)$$

Proof of (TR542)

$$\begin{aligned} a &= a.E & (AMX11) \text{ (Universal Set Conjunction Axiom) (Eqn1)} \\ E &= {}^v c * b + {}^v c * b & (TM671) \text{ (Tertium non Datur) (Eqn2)} \\ a &= a.({}^v c * b + {}^v c * b) & (AMX19) \text{ (Substitution of Eqn2 into Eqn1)} \\ a &= a.({}^v c * b) + a.({}^v c * b) & (TM407) \text{ (Distribution of } \cdot \text{ over } + \text{) (Eqn3)} \\ b.(c \diamond a) &= b.(c \diamond a) & (DM00) \text{ (Definition of } = \text{) (Eqn4)} \\ b.(c \diamond a) &= b.(c \diamond (a.({}^v c * b) + a.({}^v c * b))) & (AMX19) \text{ (Substitution of Eqn3 into Eqn4)} \\ (a.({}^v c * b) \neq \emptyset)(a.({}^v c * b) \neq \emptyset): b.c \diamond a &= b.(c \diamond (a.({}^v c * b))).(c \diamond (a.({}^v c * b))) & (TR235) \text{ (Non Distribution of } \diamond \text{ over } + \text{) (Eqn5)} \\ (b \subseteq Q^v c). (b \neq \emptyset). ({}^v c * b \neq \emptyset) &\subseteq (b \subseteq c * {}^v c * b) & (TR543) \\ (b \subseteq Q^v c). (b \neq \emptyset). ({}^v c * b \neq \emptyset) &\subseteq (c * {}^v c * b \subseteq b) & (TM012) \text{ (Transposition) (Eqn6)} \\ a.({}^v c * b) &\subseteq {}^v c * b & (TM083) \text{ (Conjunction Elimination)} \\ c * (a.({}^v c * b)) &\subseteq c * {}^v c * b & (TR196) \text{ (Relational Deduction) (Eqn7)} \\ (b \subseteq Q^v c). (b \neq \emptyset). ({}^v c * b \neq \emptyset) &\subseteq (c * (a.({}^v c * b)) \subseteq b) & (TM139) \text{ (Hypothetical Syllogism Eqns6\&7) (Eqn8)} \\ c \diamond (a.({}^v c * b)) &\subseteq c * (a.({}^v c * b)) & (TR241) \text{ (Eqn9)} \\ (b \subseteq Q^v c). (b \neq \emptyset). ({}^v c * b \neq \emptyset) &\subseteq (c \diamond (a.({}^v c * b)) \subseteq b) & (TM139) \text{ (Hypothetical Syllogism Eqns8\&9)} \\ (b \subseteq Q^v c). (b \neq \emptyset). ({}^v c * b \neq \emptyset) &\subseteq (c \diamond (a.({}^v c * b)) = b.(c \diamond (a.({}^v c * b)))) & \end{aligned}$$

$(b \subseteq Q^V c). (b \neq \emptyset). (\forall c * b \neq \emptyset). (a. (\forall c * b) \neq \emptyset) (a. (\forall c * b) \neq \emptyset) \subseteq$	$(\mathbf{A}_{MX44})(\subseteq \text{Equality Form})(\text{Eqn10})$
$(b. c \diamond a = b. (c \diamond (a. (\forall c * b)))) . b. (c \diamond (a. (\forall c * b))))$	$(\mathbf{A}_{MX19})(\text{Substitution of Eqn10 into Eqn5})(\text{Eqn11})$
$\forall c * b \subseteq a + \forall c * b$	$(\mathbf{T}_{M082})(\text{Disjunction Introduction})$
$(\forall c * b = E) \subseteq (a + (\forall c * b) = E)$	$(\mathbf{A}_{MX16})(\text{Order Axiom})$
$(\forall c * b = \sim E) \subseteq (\sim(a + (\forall c * b)) = \sim E)$	$(\mathbf{T}_{M008})(\text{Inverse of an Equality})$
$(\forall c * b = \sim E) \subseteq (a. (\forall c * b) = \sim E)$	$(\mathbf{T}_{M026})(\text{DeMorgan's Law})$
$(\forall c * b = \emptyset) \subseteq (a. (\forall c * b) = \emptyset)$	$(\mathbf{D}_{M02})(\text{Definition of } \emptyset)$
$(a. (\forall c * b) \neq \emptyset) \subseteq (\forall c * b \neq \emptyset)$	$(\mathbf{T}_{M012})(\text{Transposition})$
$(a. (\forall c * b) \neq \emptyset) = (a. (\forall c * b) \neq \emptyset). (\forall c * b \neq \emptyset)$	$(\mathbf{A}_{MX44})(\subseteq \text{Equality Form})(\text{Eqn12})$
$(a. (\forall c * b) \neq \emptyset) \subseteq (\forall c * b \neq \emptyset)$	$(\mathbf{T}_{M350})(\mathbf{T}_{M012})(\text{Eqn13})$
$(\forall c * b \neq \emptyset) \subseteq (b \neq \emptyset)$	$(\mathbf{T}_{R170})(\text{Eqn14})$
$(a. (\forall c * b) \neq \emptyset) \subseteq (b \neq \emptyset)$	$(\mathbf{T}_{M139})(\text{Hypothetical Syllogism Eqns13 \& 14})$
$(a. (\forall c * b) \neq \emptyset) = (a. (\forall c * b) \neq \emptyset). (b \neq \emptyset)$	$(\mathbf{A}_{MX44})(\subseteq \text{Equality Form})(\text{Eqn15})$
$(b \subseteq Q^V c). (a. (\forall c * b) \neq \emptyset) (a. (\forall c * b) \neq \emptyset) \subseteq (b. c \diamond a = b. (c \diamond (a. (\forall c * b)))) . b. (c \diamond (a. (\forall c * b))))$	$(\mathbf{A}_{MX19})(\text{Substitution of Eqns12 \& 15 into Eqn11})(\text{Eqn16})$
$(b \subseteq Q^V c). (a. (\forall c * b) \neq \emptyset) (a. (\forall c * b) \neq \emptyset) \subseteq (b. c \diamond a = \emptyset. (c \diamond (a. (\forall c * b)))) . (c \diamond (a. (\forall c * b))))$	$(\mathbf{T}_{M621})(\text{Complement Conjunction})$
$(b \subseteq Q^V c). (a. (\forall c * b) \neq \emptyset) (a. (\forall c * b) \neq \emptyset) \subseteq (b. c \diamond a = \emptyset)$	$(\mathbf{T}_{M050})(\text{Conjunction with the Null Set})(\text{Eqn17})$
$(a. (\forall c * b) = \emptyset) = (a. (\forall c * b) = \emptyset)$	$(\mathbf{D}_{M00})(\text{Definition of } =)$
$(a. (\forall c * b) = \emptyset) = (a \subseteq (\forall c * b))$	$(\mathbf{T}_{M735})(\subseteq \text{Existential Form})$
$(a. (\forall c * b) = \emptyset) = (b \subseteq (c * a))$	$(\mathbf{T}_{R457})(\text{Active/Passive Theorem})(\text{Eqn18})$
$(c \diamond a) \subseteq (c * a)$	$(\mathbf{T}_{R241})$
$(c * a) \subseteq (c \diamond a)$	$(\mathbf{T}_{M012})(\text{Transposition})$
$(b \subseteq (c * a)) \subseteq (b \subseteq (c \diamond a))$	$(\mathbf{T}_{M272})(\text{Deductive Rule})(\text{Eqn19})$
$(a. (\forall c * b) = \emptyset) \subseteq (b \subseteq (c \diamond a))$	$(\mathbf{A}_{MX19})(\text{Substitution of Eqn18 into Eqn19})$
$(a. (\forall c * b) = \emptyset) \subseteq (b. (c \diamond a) = \emptyset)$	$(\mathbf{T}_{M735})(\subseteq \text{Existential Form})(\text{Eqn20})$
$(a. (\forall c * b) = \emptyset). (a. (\forall c * b) \neq \emptyset) \subseteq (a. (\forall c * b) = \emptyset)$	$(\mathbf{T}_{M083})(\text{Conjunction Elimination})(\text{Eqn21})$
$(a. (\forall c * b) = \emptyset). (a. (\forall c * b) \neq \emptyset) \subseteq (b. (c \diamond a) = \emptyset)$	$(\mathbf{T}_{M139})(\text{Hypothetical Syllogism Eqns21\&20})(\text{Eqn22})$
$(b \subseteq Q^V c). (a. (\forall c * b) = \emptyset). (a. (\forall c * b) \neq \emptyset) \subseteq (a. (\forall c * b) = \emptyset). (a. (\forall c * b) \neq \emptyset)$	$(\mathbf{T}_{M083})(\text{Conjunction Elimination})(\text{Eqn23})$
$(b \subseteq Q^V c). (a. (\forall c * b) = \emptyset). (a. (\forall c * b) \neq \emptyset) \subseteq (b. (c \diamond a) = \emptyset)$	$(\mathbf{T}_{M139})(\text{Hypothetical Syllogism Eqns23\&22})(\text{Eqn24})$
$(b \subseteq Q^V c). (a. (\forall c * b) \neq \emptyset) (a. (\forall c * b) \neq \emptyset) + (b \subseteq Q^V c). (a. (\forall c * b) = \emptyset). (a. (\forall c * b) \neq \emptyset) \subseteq (b. (c \diamond a) = \emptyset)$	$(\mathbf{T}_{M131b})(\text{Conjunction of Sufficiencies Eqns17\&24})$
$(b \subseteq Q^V c). ((a. (\forall c * b) \neq \emptyset) + (a. (\forall c * b) = \emptyset)). (a. (\forall c * b) \neq \emptyset) \subseteq (b. (c \diamond a) = \emptyset)$	$(\mathbf{T}_{M407})(\text{Distribution of } . \text{ over } +)$
$(b \subseteq Q^V c). E. (a. (\forall c * b) \neq \emptyset) \subseteq (b. (c \diamond a) = \emptyset)$	$(\mathbf{T}_{M671})(\text{Tertium non Datur})$
$(b \subseteq Q^V c). (a. (\forall c * b) \neq \emptyset) \subseteq (b. (c \diamond a) = \emptyset)$	$(\mathbf{A}_{MX11})(\text{Universal Set Conjunction Axiom})$
$(b \subseteq Q^V c). (a \not\subseteq \forall c * b) \subseteq (b. (c \diamond a) = \emptyset)$	$(\mathbf{T}_{M735})(\subseteq \text{Existential Form})(\mathbf{T}_{M008})$
$(b \subseteq Q^V c). (a \not\subseteq \forall c * b) \subseteq (b \subseteq c \diamond a)$	$(\mathbf{T}_{M735})(\subseteq \text{Existential Form})$
$(b \subseteq Q^V c). (a \not\subseteq \forall c * b) \subseteq (b \subseteq \forall c \diamond a)$	Resymbolising
$((b \subseteq Q^V c) = E) \subseteq ((a \not\subseteq \forall c * b) \subseteq (b \subseteq \forall c \diamond a))$	$(\mathbf{A}_{MX35})(\text{Exportation/Importation}) \text{ for truth sets}$

$$(\forall Qb \subseteq c) \subseteq (b \subseteq Qc) \quad (\mathbf{T}_{R096})$$

$$(\forall Sa \subseteq c) \subseteq (a \subseteq Sc) \quad (\mathbf{T}_{R097})$$

Example Proof of $(\mathbf{T}_{R096})$

$$(\forall Qb \subseteq c) \subseteq (Q^V Qb \subseteq Qc) \quad (\mathbf{T}_{R027d})(\text{Sub-Relationship Deduction})$$

$$(\forall Qb \subseteq c) \subseteq (b \subseteq Qc) \quad (\mathbf{T}_{R012})(\text{Sub-Relationship Closure Theorem})$$

$$(c = \forall Qb) \subseteq (b = Qc) \quad (\mathbf{T}_{R094})$$

$$(c = \forall Sa) \subseteq (a = Sc) \quad (\mathbf{T}_{R095})$$

Example Proof of $(\mathbf{T}_{R094})$

$$(c = {}^vQb) \subseteq (Qc = Q{}^vQb)$$

$$(c = {}^vQb) \subseteq (Qc = b)$$

(T_{R027}) (Sub-Relationship Deduction)
 (T_{R012l}) (Sub-Relationship Closure Theorem)

$${}^vSa{}^{1v}S(\sim a^1) = \emptyset \quad (T_{R134})$$

$${}^vQb{}^{1v}Q(\sim b^1) = \emptyset \quad (T_{R135})$$

$${}^v c^{1v}(\sim c^1) = \emptyset \quad (T_{R135a})$$

Example Proof of (T_{R134})

$$a^1(\sim a^1) = \emptyset \quad (T_{M621}) \text{ (Complement Conjunction)}$$

$${}^vS(a^1(\sim a^1)) = {}^vS\emptyset \quad (T_{R023}) \text{ (Sub-Relationship Deduction)}$$

$${}^vSa{}^{1v}S(\sim a^1) = {}^vS\emptyset \quad (T_{R012b}) \text{ (Distribution of } {}^vS^* \text{ over } .)$$

$${}^vSa{}^{1v}S(\sim a^1) = \emptyset \quad (T_{R067}) \text{ (Null Object Theorem)}$$

$${}^vSa{}^vSa = \emptyset \quad (T_{S300})$$

$${}^vQb{}^vQb = \emptyset \quad (T_{S301})$$

$${}^v c^v c = \emptyset \quad (T_{S302})$$

Example Proof of (T_{S300})

$$a.a = \emptyset \quad (T_{M621}) \text{ (Complement Conjunction)}$$

$${}^vS(a.a) = {}^vS\emptyset \quad (T_{R023}) \text{ (Sub-Relationship Deduction)}$$

$${}^vSa{}^vSa = {}^vS\emptyset \quad (T_{R012b}) \text{ (Distribution of } {}^vS^* \text{ over } .)$$

$${}^vSa{}^vSa = \emptyset \quad (T_{R067}) \text{ (Null Object Theorem)}$$

$$(a^1 \neq b^1) = ({}^vSa{}^{1v}Sb^1 = \emptyset) \quad (T_{R136})$$

$$(a^1 \neq b^1) = ({}^vQa{}^{1v}Qb^1 = \emptyset) \quad (T_{R137})$$

$$(a^1 \neq b^1) = ({}^v a^{1v} b^1 = \emptyset) \quad (T_{R137a})$$

Example Proof of (T_{R136})

Let $a^1 \neq b^1$

$$a^1 \subseteq \sim(b^1) \quad (T_{M090}) \text{ (Inverse Unit Subset Rule)}$$

$$a^1 b^1 = \emptyset \quad (T_{M735}) \text{ (} \subseteq \text{ Existential Form)}$$

$${}^vS(a^1 b^1) = {}^vS\emptyset \quad (T_{R023}) \text{ (Sub-Relationship Deduction)}$$

$${}^vSa{}^{1v}Sb^1 = {}^vS\emptyset \quad (T_{R012b}) \text{ (Distribution of } {}^vS^* \text{ over } .)$$

$${}^vSa{}^{1v}Sb^1 = \emptyset \quad (T_{R067}) \text{ (Null Object Theorem)}$$

Thus $(a^1 \neq b^1) = ({}^vSa{}^{1v}Sb^1 = \emptyset) \quad (A_{MX02}) \text{ (Conditional Introduction) since this proof is reversible}$

$$Sc = Q{}^v c \quad (T_{R015}) \text{ Reversibility of Matching Entities}$$

Proof of (T_{R015})

$${}^vSa = {}^v({}^vQa) \quad (A_{R13}) \text{ (Reversibility of Matching Relationships)(Eqn1)}$$

Let ${}^vSa = c$ (Eqn2)

$$S{}^vSa = Sc \quad (T_{R026})$$

$$a = Sc \quad (T_{R012m}) \text{ (Eqn3)}$$

$$c = {}^v({}^vQa) \quad (A_{MX19}) \text{ (Substitution of Eqn2 into Eqn1)}$$

$$V^*c = V^*{}^v({}^vQa) \quad (T_{R522}) \text{ (Relational Deduction)}$$

$${}^v c = {}^{vv}({}^vQa) \quad (D_{R27}) \text{ (Simplification of the Relationship Symmetry Function)}$$

$${}^v c = {}^vQa \quad (T_{R012k})$$

$$Q{}^v c = Q{}^vQa \quad (T_{R027})$$

$$Q{}^v c = a \quad (T_{R012l}) \text{ (Sub-relationship Closure Theorem)(Eqn4)}$$

$$Sc = Q{}^v c \quad (A_{MX19}) \text{ (Substitution of Eqn4 into Eqn3)}$$

$$Sc^1 = Q{}^v c^1 \quad (T_{R015a}) \text{ Reversibility of Matching Entities}$$

Proof of (T_{R015a})

$Sc = Q^Vc$	(T_{R015}) (Reversibility of Matching Entities)(Eqn1)
Let $c = c^1$	(Eqn2)
$Sc^1 = Q^Vc^1$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)

$$S^Vc = Qc \quad (T_{R016}) \text{ Reversibility of Matching Entities}$$

<u>Proof of</u> (T_{R016})	
$Sc = Q^Vc$	(T_{R015}) (Reversibility of Matching Entities)
$S^Vc = Q^{VV}c$	Resymbolising
$S^Vc = Qc$	(T_{R012k}) (Sub-Relationship Closure Theorem)

$$S^Vc^1 = Qc^1 \quad (T_{R014}) \text{ Reversibility of Matching Entities}$$

<u>Proof of</u> (T_{R014})	
$S^Vc = Qc$	(T_{R016}) (Reversibility of Matching Entities)(Eqn1)
Let $c = c^1$	(Eqn2)
$S^Vc^1 = Qc^1$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)

$$c \subseteq {}^V S Q^Vc \quad (T_{R217}) \text{ Reversibility of Matching Entities}$$

<u>Proof of</u> (T_{R217})	
$Sc = Q^Vc$	(T_{R015})
${}^V S S c = {}^V S Q^Vc$	(T_{R023}) (Sub-Relationship Deduction)
$c \subseteq {}^V S S c$	(T_{R012a}) (Sub-Relationship Non-Closure Theorem)
$c \subseteq {}^V S Q^Vc$	(T_{M139}) (Hypothetical Syllogism)

$${}^V c * a = Q(c^V S a) \quad (T_{R020}) \text{ Function Theorem}$$

<u>Proof of</u> (T_{R020})	
$c * b = S(c^V Q b)$	(A_{R27}) (Function Axiom)
$c * b = Q^V(c^V Q b)$	(T_{R015})
$c * b = Q^V(c^V ({}^V S b))$	(T_{R019})
${}^V c * b = Q^V({}^V c^V ({}^V S b))$	Resymbolising
${}^V c * b = Q(c^V S b)$	(T_{R012k}) (Sub-Relationship Closure Theorem)
${}^V c * a = Q(c^V S a)$	Resymbolising

$${}^V (c^V S a) * a = {}^V c * a \quad (T_{R100})$$

<u>Proof of</u> (T_{R100})	
$({}^V c * b) = ({}^V c * b)$	(D_{M00}) (Definition of $=$)
$({}^V c * b) = Q(c^V S b)$	(T_{R020}) (Function Theorem)
$({}^V c * b) = Q(c^V S b^V S b)$	(T_{M667}) (Conjunction Tautology)
$({}^V c * b) = (c^V S b) * b$	(T_{R020}) (Function Theorem)

$$(c = \emptyset) = ({}^V c * S c = \emptyset) \quad (T_{844})$$

<u>Proof of</u> (T_{844})	
$\emptyset * \emptyset = \emptyset$	(T_{R073}) (Null Object Theorem)(Eqn1)
$S \emptyset = \emptyset$	(T_{R066}) (Null Object Theorem)(Eqn2)
$\emptyset * S \emptyset = \emptyset$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)(Eqn3)
Let $c = \emptyset$	(Eqn4)
${}^V c = {}^V \emptyset$	(T_{R022}) (Relationship Opposite of a Proposition)
${}^V c = \emptyset$	(T_{S181}) (Null Object Theorem)(Eqn5)
${}^V c * S c = \emptyset$	(A_{MX19}) (Substitution of Eqns4&5 into Eqn3)
Thus $(c = \emptyset) \subseteq ({}^V c * S c = \emptyset)$	(A_{MX02}) (Conditional Introduction)(Eqn6)
Let ${}^V c * S c = \emptyset$	
$Q(c. {}^V S S c) = \emptyset$	(T_{R020}) (Function Theorem) (Eqn7)

$c \subseteq {}^V SSc$	(T_{R012a})(Sub-Relationship Non-Closure Theorem)
$c = c. {}^V SSc$	($AMX44$)($\subseteq$ Equality Form)(Eqn8)
$Qc = \emptyset$	($AMX19$)(Substitution of Eqn8 into Eqn7)
$c = \emptyset$	(T_{R068})(Null Object Theorem)
Thus $({}^V c * Sc = \emptyset) \subseteq (c = \emptyset)$	($AMX02$)(Conditional Introduction)(Eqn9)
$(c = \emptyset) = ({}^V c * Sc = \emptyset)$	($TM430$)(Biconditional Introduction Eqns6&9)

$$(c \neq \emptyset) = ({}^V c * Sc \neq \emptyset) \quad (T_{R161})$$

Proof of (T_{R161})

$(c = \emptyset) = ({}^V c * Sc = \emptyset)$	(T_{T844})
$(c \neq \emptyset) = ({}^V c * Sc \neq \emptyset)$	($TM008$)(Inverse of an Equality)

$${}^V c * \sim Sc = \emptyset \quad (T_{R162})$$

Proof of (T_{R162})

$Sc. \sim Sc = \emptyset$	($TM621$)(Complement Conjunction)
${}^V S(Sc. \sim Sc) = \emptyset$	($TS304$)
${}^V SSc. {}^V S \sim Sc = \emptyset$	(T_{R012b})(Distribution of ${}^V S*$ over .)
$c. {}^V SSc. {}^V S \sim Sc = \emptyset$	($TM350$)(Eqn1)
$c \subseteq {}^V SSc$	(T_{R012a})(Sub-relationship Non-Closure Theorem)
$c = c. {}^V SSc$	($AMX44$)($\subseteq$ Equality Form)(Eqn2)
$c. {}^V S \sim Sc = \emptyset$	($AMX19$)(Substitution of Eqn2 into Eqn1)
$Q(c. {}^V S \sim Sc) = \emptyset$	($TS303$)
${}^V c * \sim Sc = \emptyset$	(T_{R020})(Function Theorem)

$${}^V c \diamond \sim Sc = \emptyset \quad (T_{R286})$$

Proof of (T_{R286})

${}^V c * \sim Sc = \emptyset$	(T_{R162})
${}^V c \diamond \sim Sc = \emptyset$	($TS246$)

$$Qc = {}^V c * E \quad (T_{R211})$$

Proof of (T_{R211})

${}^V c * E = {}^V c * E$	($DM00$)(Definition of =)
${}^V c * E = {}^V c * (Sc + \sim Sc)$	($TM671$)(Tertium non Datur)
${}^V c * E = {}^V c * Sc + {}^V c * \sim Sc$	(T_{R002b})(Distribution of $*$ over +)(Eqn1)
$Qc = {}^V c * Sc$	(T_{R020b})(Function Theorem)(Eqn2)
${}^V c * \sim Sc = \emptyset$	(T_{R162})(Eqn3)
${}^V c * E = Qc + \emptyset$	($AMX19$)(Substitution of Eqns2 & 3 into Eqn1)
${}^V c * E = Qc$	($TM668$)(Null Set Disjunction)

$$Qc = {}^V c * a + {}^V c * \underline{a} \quad (T_{R216})$$

Proof of (T_{R216})

$Qc = {}^V c * E$	(T_{R211})
$Qc = {}^V c * (a + \underline{a})$	($TM671$)(Tertium non Datur)
$Qc = {}^V c * a + {}^V c * \underline{a}$	(T_{R002b})(Distribution of $*$ over +)

$$(b^1 \neq \emptyset) = (S({}^V Qb^1) = E) \quad (T_{R055})$$

Proof of (T_{R055})

$c * b^1 = S(c {}^V Qb^1)$	($AR27$)(Function Axiom)(Eqn1)
Let $c = E$	(Eqn2)
$E * b^1 = S(E {}^V Qb^1)$	($AMX19$)(Substitution of Eqn2 into Eqn1)
$E * b^1 = S({}^V Qb^1)$	($AMX11$)(Universal Set Conjunction Axiom)(Eqn3)

$$(b^1 \neq \emptyset) = (E * b^1 = E)$$

$$(b^1 \neq \emptyset) = (S(\forall Qb^1) = E)$$

(T_{S243})(Universality of Inter-relationships Theorem)(Eqn4)
 (A_{MX19})(Substitution of Eqn3 into Eqn4)

$$(b^1 \neq \emptyset) = (Q(\forall Sb^1) = E)$$

(T_{R057})

Proof of (T_{R057})

$$\forall c * b^1 = Q(\forall c \forall Sb^1)$$

(T_{R020})(Function Theorem)(Eqn1)

$$\text{Let } \forall c = E$$

(Eqn2)

$$E * b^1 = Q(E \forall Sb^1)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$E * b^1 = Q(\forall Sb^1)$$

(A_{MX11})(Universal Set Conjunction Axiom)(Eqn3)

$$(b^1 \neq \emptyset) = (E * b^1 = E)$$

(T_{S243})(Universality of Inter-relationships Theorem)(Eqn4)

$$(b^1 \neq \emptyset) = (Q(\forall Sb^1) = E)$$

(A_{MX19})(Substitution of Eqn3 into Eqn4)

$$(b^1 \neq \emptyset) = (\forall Qb^1 * b^1 = E)$$

(T_{R056})

Proof of (T_{R056})

$$c * b^1 = S(c \forall Qb^1)$$

(A_{R27})(Function Axiom)(Eqn1)

$$\text{Let } c = \forall Qb^1$$

(Eqn2)

$$\forall Qb^1 * b^1 = S(\forall Qb^1 \forall Qb^1)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$\forall Qb^1 * b^1 = S(\forall Qb^1)$$

(T_{M667})(Conjunction Tautology)(Eqn3)

$$(b^1 \neq \emptyset) = (S(\forall Qb^1) = E)$$

(T_{R055})(Eqn4)

$$(b^1 \neq \emptyset) = (\forall Qb^1 * b^1 = E)$$

(A_{MX19})(Substitution of Eqn3 into Eqn4)

$$(b^1 \neq \emptyset) = (\forall (\forall Sb^1) * b^1 = E)$$

(T_{R058})

Proof of (T_{R058})

$$(b^1 \neq \emptyset) = (\forall Qb^1 * b^1 = E)$$

(T_{R056})(Eqn1)

$$\forall Qb^1 = \forall (\forall Sb^1)$$

(T_{R017})(Reversibility of Matching Relationships)(Eqn2)

$$(b^1 \neq \emptyset) = (\forall (\forall Sb^1) * b^1 = E)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(b \neq \emptyset) = (S \forall Qb = E)$$

(T_{R060})

Proof of (T_{R060})

$$(b \neq \emptyset) = (E * b = E)$$

(T_{S244})

$$(b \neq \emptyset) = (S(E \forall Qb) = E)$$

(A_{R27})(Function Axiom)

$$(b \neq \emptyset) = (S \forall Qb = E)$$

(A_{MX11})(Universal Set Conjunction Axiom)

$$(b \neq \emptyset) = (Q \forall Sb = E)$$

(T_{R062})

Proof of (T_{R062})

$$(b \neq \emptyset) = (E * b = E)$$

(T_{S244})

$$(b \neq \emptyset) = (Q(\forall E \forall Sb) = E)$$

(T_{R020})(Function Theorem)

$$(b \neq \emptyset) = (Q(E \forall Sb) = E)$$

(T_{T793})(Relationship Opposite of E)

$$(b \neq \emptyset) = (Q \forall Sb = E)$$

(A_{MX11})(Universal Set Conjunction Axiom)

$$(b \neq \emptyset) = (\forall Qb * b = E)$$

(T_{R061})

Proof of (T_{R061})

$$c * b = S(c \forall Qb)$$

(A_{R27})(Function Axiom)(Eqn1)

$$\text{Let } c = \forall Qb$$

(Eqn2)

$$\forall Qb * b = S(\forall Qb \forall Qb)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$\forall Qb * b = S(\forall Qb)$$

(T_{M667})(Conjunction Tautology)(Eqn3)

$$S(\forall Qb) = E$$

(T_{R060})(Eqn4)

$$\forall Qb * b = E$$

(A_{MX19})(Substitution of Eqn4 into Eqn3)

$$(b \neq \emptyset) = (\forall (\forall Sb) * b = E)$$

(T_{R063})

Proof of (TR063)

$$\begin{aligned} (b \neq \emptyset) &= (\vee Qb * b = E) & (TR061)(Eqn1) \\ \vee(\vee Sb) &= \vee Qb & (TR019)(\text{Reversibility of Matching Relationships})(Eqn2) \\ (b \neq \emptyset) &= (\vee(\vee Sb) * b = E) & (AMX19)(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(b^1 \neq \emptyset) + (a^1 = \emptyset) = (S(\vee Sa^1 \vee Qb^1) = a^1) \quad (TR489)$$

$$(a^1 \neq \emptyset) + (b^1 = \emptyset) = (Q(\vee Sa^1 \vee Qb^1) = b^1) \quad (TR490)$$

Example Proof of (TR489)

$$\begin{aligned} S(\vee Sa^1 \vee Qb^1) &\subseteq (S(\vee Sa^1).(S \vee Qb^1)) & (TR012d)(\text{Distribution of } S * \text{ over } .)(Eqn1) \\ \text{Let } b^1 \neq \emptyset & & \\ S(\vee Qb^1) &= E & (TR055)(Eqn2) \\ S(\vee Sa^1 \vee Qb^1) &\subseteq (S(\vee Sa^1).E) & (AMX19)(\text{Substitution of Eqn2 into Eqn1}) \\ S(\vee Sa^1 \vee Qb^1) &\subseteq (S(\vee Sa^1)) & (AMX11)(\text{Universal Set Conjunction Axiom}) \\ S(\vee Sa^1 \vee Qb^1) &\subseteq a^1 & (TR012h)(\text{Sub-Relationship Closure Theorem}) \\ (S(\vee Sa^1 \vee Qb^1) \subset a^1) + (S(\vee Sa^1 \vee Qb^1) = a^1) & & (TM266)(\text{Definition of } \subseteq)(Eqn3) \\ (S(\vee Sa^1 \vee Qb^1) \subset a^1) = (a^1 \neq \emptyset)(S(\vee Sa^1 \vee Qb^1) = \emptyset) & & (AMX40)(\text{Proper Subset of Unit Set Axiom})(Eqn4) \\ \text{Let } a^1 \neq \emptyset & & \\ (a^1 \neq \emptyset) &= E & (AMX16)(\text{Order Axiom})(Eqn5) \\ (S(\vee Sa^1 \vee Qb^1) \subset a^1) = E.(S(\vee Sa^1 \vee Qb^1) = \emptyset) & & (AMX19)(\text{Substitution of Eqn5 into Eqn4}) \\ (S(\vee Sa^1 \vee Qb^1) \subset a^1) = (S(\vee Sa^1 \vee Qb^1) = \emptyset) & & (AMX11)(\text{Universal Set Conjunction Axiom})(Eqn6) \\ (S(\vee Sa^1 \vee Qb^1) = \emptyset) + (S(\vee Sa^1 \vee Qb^1) = a^1) & & (AMX19)(\text{Substitution of Eqn6 into Eqn3}) \\ (S(\vee Sa^1 \vee Qb^1) = \emptyset).E + (S(\vee Sa^1 \vee Qb^1) = a^1) & & (AMX11)(\text{Universal Set Conjunction Axiom})(Eqn7) \\ \vee Sa^1 \vee Qb^1 \neq \emptyset & & (TR147) \text{ since } a \neq \emptyset \text{ and } b \neq \emptyset \\ (\vee Sa^1 \vee Qb^1 \neq \emptyset) &= E & (AMX16)(\text{Order Axiom})(Eqn8) \\ (S(\vee Sa^1 \vee Qb^1) = \emptyset).(\vee Sa^1 \vee Qb^1 \neq \emptyset) + (S(\vee Sa^1 \vee Qb^1) = a^1) & & (AMX19)(\text{Substitution of Eqn8 into Eqn7}) \\ \emptyset + (S(\vee Sa^1 \vee Qb^1) = a^1) & & (TM621)(\text{Complement Conjunction}) \\ S(\vee Sa^1 \vee Qb^1) &= a^1 & (TM668)(\text{Null Set Disjunction}) \\ \text{Thus } (b^1 \neq \emptyset)(a^1 \neq \emptyset) &\subseteq (S(\vee Sa^1 \vee Qb^1) = a^1) & (AMX02)(\text{Conditional Introduction})(Eqn9) \\ \text{Let } a^1 = \emptyset & & (Eqn10) \\ \text{Let } S(\vee Sa^1 \vee Qb^1) \neq a^1 & & (Eqn11) \\ S(\vee S\emptyset \vee Qb^1) &\neq \emptyset & (AMX19)(\text{Substitution of Eqn10 into Eqn11}) \\ S(\emptyset \vee Qb^1) &\neq \emptyset & (TS181)(\text{Null Object Theorem}) \\ S\emptyset &\neq \emptyset & (TM050)(\text{Null Set Conjunction}) \\ \emptyset &\neq \emptyset & (TS181)(\text{Null Object Theorem}) \\ \emptyset & & (TM346) \\ \text{Thus } (a^1 = \emptyset)(S(\vee Sa^1 \vee Qb^1) \neq a^1) &= \emptyset & (AMX02)(\text{Conditional Introduction}) \\ (a^1 = \emptyset) &\subseteq (S(\vee Sa^1 \vee Qb^1) = a^1) & (TM735)(\subseteq \text{Existential Form})(Eqn12) \\ (b^1 \neq \emptyset)(a^1 \neq \emptyset) + (a^1 = \emptyset) &\subseteq (S(\vee Sa^1 \vee Qb^1) = a^1) + (S(\vee Sa^1 \vee Qb^1) = a^1) & (AMX21)(\text{Combination of Eqns9 \& 12}) \\ (b^1 \neq \emptyset)(a^1 \neq \emptyset) + (a^1 = \emptyset) &\subseteq (S(\vee Sa^1 \vee Qb^1) = a^1) & (AMX01)(\text{Disjunction Tautology}) \\ (b^1 \neq \emptyset)(a^1 \neq \emptyset) + (b^1 \neq \emptyset)(a^1 = \emptyset) + (a^1 = \emptyset) &\subseteq (S(\vee Sa^1 \vee Qb^1) = a^1) & (TM063)(\text{Partial Redundancy}) \\ (b^1 \neq \emptyset)((a^1 \neq \emptyset) + (a^1 = \emptyset)) + (a^1 = \emptyset) &\subseteq (S(\vee Sa^1 \vee Qb^1) = a^1) & (TM407)(\text{Distribution of } . \text{ over } +) \\ (b^1 \neq \emptyset)E + (a^1 = \emptyset) &\subseteq (S(\vee Sa^1 \vee Qb^1) = a^1) & (TM671)(\text{Tertium non Datur}) \\ (b^1 \neq \emptyset) + (a^1 = \emptyset) &\subseteq (S(\vee Sa^1 \vee Qb^1) = a^1) & (AMX11)(\text{Universal Set Conjunction Axiom})(Eqn13) \\ \text{Let } S(\vee Sa^1 \vee Qb^1) &= a^1 & (Eqn14) \\ \text{Let } b^1 &= \emptyset & (Eqn15) \\ \text{Let } a^1 &\neq \emptyset & (Eqn16) \\ S(\vee Sa^1 \vee Qb^1) &\neq \emptyset & (AMX19)(\text{Substitution of Eqn14 into Eqn16})(Eqn17) \\ S(\vee Sa^1 \vee Q\emptyset) &\neq \emptyset & (AMX19)(\text{Substitution of Eqn15 into Eqn17}) \\ S(\vee Sa^1 \emptyset) &\neq \emptyset & (TS181)(\text{Null Object Theorem}) \end{aligned}$$

$$S\emptyset \neq \emptyset$$

$$\emptyset \neq \emptyset$$

$$\emptyset$$

$$\text{Thus } (S(\vee Sa^1 \vee Qb^1) = a^1)(b^1 = \emptyset)(a^1 \neq \emptyset) = \emptyset$$

$$(S(\vee Sa^1 \vee Qb^1) = a^1) \subseteq \sim((b^1 = \emptyset)(a^1 \neq \emptyset))$$

$$(S(\vee Sa^1 \vee Qb^1) = a^1) \subseteq \sim(b^1 = \emptyset) + \sim(a^1 \neq \emptyset)$$

$$(S(\vee Sa^1 \vee Qb^1) = a^1) \subseteq (b^1 \neq \emptyset) + (a^1 = \emptyset)$$

$$(b^1 \neq \emptyset) + (a^1 = \emptyset) = (S(\vee Sa^1 \vee Qb^1) = a^1)$$

(T_{M050})(Null Set Conjunction)

(T_{S181})(Null Object Theorem)

(T_{M346})

($AMX02$)(Conditional Introduction)

(T_{M735})($\subseteq$ Existential Form)

(T_{M025})(DeMorgan's Law)

(T_{M260})(Definition of $\neq$)(Eqn18)

(T_{M430})(Biconditional Introduction Eqns13&18)

$$(b \neq \emptyset) + (a = \emptyset) = (S(\vee Sa \vee Qb) = a) \quad (T_{S248})$$

$$(a \neq \emptyset) + (b = \emptyset) = (Q(\vee Sa \vee Qb) = b) \quad (T_{S249})$$

Example Proof of (T_{S248})

For $r = 1$ to n

For $s = 1$ to m

$$(b_s^1 \neq \emptyset) + (a_r^1 = \emptyset) = (S(\vee Sa_r^1 \vee Qb_s^1) = a_r^1) \quad (T_{R489})(\text{Eqn1})$$

$$(b_1^1 \neq \emptyset) + \dots + (b_m^1 \neq \emptyset) + (a_1^1 = \emptyset) + \dots + (a_n^1 = \emptyset) =$$

$$((S(\vee Sa_1^1 \vee Qb_1^1) + \dots + S(\vee Sa_1^1 \vee Qb_m^1)) + \dots + (S(\vee Sa_n^1 \vee Qb_1^1) + \dots + S(\vee Sa_n^1 \vee Qb_m^1))) =$$

$$(a_1^1 + \dots + a_1^1) + \dots + (a_n^1 + \dots + a_n^1) \quad (T_{M230})(\text{Combination of Eqns1})$$

$$(b_1^1 \neq \emptyset) + \dots + (b_m^1 \neq \emptyset) + (a_1^1 = \emptyset) + \dots + (a_n^1 = \emptyset) =$$

$$((S(\vee Sa_1^1 \vee Qb_1^1) + \dots + S(\vee Sa_1^1 \vee Qb_m^1)) + \dots + (S(\vee Sa_n^1 \vee Qb_1^1) + \dots + S(\vee Sa_n^1 \vee Qb_m^1))) = a_1^1 + \dots + a_n^1$$

($AMX01$)(Disjunction Tautology)

$$(b_1^1 \neq \emptyset) + \dots + (b_m^1 \neq \emptyset) \subseteq$$

$$((S(\vee Sa_1^1 \vee Qb_1^1) + \dots + S(\vee Sa_1^1 \vee Qb_m^1)) + \dots + (S(\vee Sa_n^1 \vee Qb_1^1) + \dots + S(\vee Sa_n^1 \vee Qb_m^1))) = a_1^1 + \dots + a_n^1$$

(T_{M072b})(Single Premise Syllogism)

$$(b_1^1 \neq \emptyset) + \dots + (b_m^1 \neq \emptyset) \subseteq$$

$$(S((\vee Sa_1^1 \vee Qb_1^1) + \dots + (\vee Sa_1^1 \vee Qb_m^1)) + \dots + S((\vee Sa_n^1 \vee Qb_1^1) + \dots + (\vee Sa_n^1 \vee Qb_m^1))) = a_1^1 + \dots + a_n^1$$

(T_{R002b})(Distribution of $\vee$ over $+$)

$$(b_1^1 \neq \emptyset) + \dots + (b_m^1 \neq \emptyset) \subseteq (S(\vee Sa_1^1 (\vee Qb_1^1 + \dots + \vee Qb_m^1)) + \dots + S(\vee Sa_n^1 (\vee Qb_1^1 + \dots + \vee Qb_m^1))) = a_1^1 + \dots + a_n^1$$

(T_{M407})(Distribution of $\vee$ over $+$)

$$(b_1^1 \neq \emptyset) + \dots + (b_m^1 \neq \emptyset) \subseteq (S(\vee Sa_1^1 \vee Q(b_1^1 + \dots + b_m^1)) + \dots + S(\vee Sa_n^1 \vee Q(b_1^1 + \dots + b_m^1))) = a_1^1 + \dots + a_n^1$$

(T_{R002b})(Distribution of $\vee$ over $+$)(Eqn2)

$$\text{Let } b = b_1^1 + \dots + b_m^1$$

(Eqn3)

$$(b \neq \emptyset) = (b_1^1 \neq \emptyset) + \dots + (b_m^1 \neq \emptyset)$$

(T_{M524})(Eqn4)

$$(b \neq \emptyset) \subseteq (S(\vee Sa_1^1 \vee Qb) + \dots + S(\vee Sa_n^1 \vee Qb)) = a_1^1 + \dots + a_n^1$$

($AMX19$)(Substitution of Eqns3 & 4 into Eqn2)

$$(b \neq \emptyset) \subseteq (S((\vee Sa_1^1 \vee Qb) + \dots + (\vee Sa_n^1 \vee Qb))) = a_1^1 + \dots + a_n^1$$

(T_{R002b})(Distribution of $\vee$ over $+$)

$$(b \neq \emptyset) \subseteq (S(\vee Sa_1^1 + \dots + \vee Sa_n^1 \vee Qb)) = a_1^1 + \dots + a_n^1$$

(T_{M407})(Distribution of $\vee$ over $+$)

$$(b \neq \emptyset) \subseteq (S(\vee S(a_1^1 + \dots + a_n^1) \vee Qb)) = a_1^1 + \dots + a_n^1$$

(T_{R002b})(Distribution of $\vee$ over $+$)(Eqn5)

$$\text{Let } a = a_1^1 + \dots + a_n^1$$

(Eqn6)

$$(b \neq \emptyset) \subseteq (S(\vee Sa \vee Qb) = a)$$

($AMX19$)(Substitution of Eqn6 into Eqn5)(Eqn7)

$$\text{Let } a = \emptyset$$

(Eqn8)

$$\text{Let } S(\vee Sa \vee Qb) \neq a$$

(Eqn9)

$$S(\vee S\emptyset \vee Qb) \neq \emptyset$$

($AMX19$)(Substitution of Eqn8 into Eqn9)

$$S(\emptyset \vee Qb) \neq \emptyset$$

(T_{S181})(Null Object Theorem)

$$S\emptyset \neq \emptyset$$

(T_{M050})(Null Set Conjunction)

$$\emptyset \neq \emptyset$$

(T_{S181})(Null Object Theorem)

$$\emptyset$$

(T_{M346})

$$\text{Thus } (a = \emptyset)(S(\vee Sa \vee Qb) \neq a) = \emptyset$$

($AMX02$)(Conditional Introduction)

$$(a = \emptyset) \subseteq (S(\vee Sa \vee Qb) = a)$$

(T_{M735})($\subseteq$ Existential Form)(Eqn10)

$$(b \neq \emptyset) + (a = \emptyset) \subseteq (S(\vee Sa \vee Qb) = a) + (S(\vee Sa \vee Qb) = a)$$

($AMX21$)(Combination of Eqns7 & 10)

$$(b \neq \emptyset) + (a = \emptyset) \subseteq (S(\vee Sa \vee Qb) = a)$$

($AMX01$)(Disjunction Tautology)(Eqn11)

$$\text{Let } b = \emptyset$$

(Eqn12)

Let $a \neq \emptyset$	(Eqn13)
Let $S(\vee Sa \vee Qb) = a$	(Eqn14)
$S(\vee Sa \vee Qb) \neq \emptyset$	($AMX19$)(Substitution of Eqn14 into Eqn13)(Eqn15)
$S(\vee Sa \vee Q\emptyset) \neq \emptyset$	($AMX19$)(Substitution of Eqn12 into Eqn15)
$S(\vee Sa \emptyset) \neq \emptyset$	($TS181$)(Null Object Theorem)
$S\emptyset \neq \emptyset$	($TM050$)(Null Set Conjunction)
$\emptyset \neq \emptyset$	($TS181$)(Null Object Theorem)
$\emptyset$	($TM346$)
Thus $(b = \emptyset)(a \neq \emptyset)(S(\vee Sa \vee Qb) = a) = \emptyset$	($AMX02$)(Conditional Introduction)
$(S(\vee Sa \vee Qb) = a) \subseteq \sim((b = \emptyset)(a \neq \emptyset))$	($TM735$)($\subseteq$ Existential Form)
$(S(\vee Sa \vee Qb) = a) \subseteq \sim(b = \emptyset) + \sim(a \neq \emptyset)$	($TM025$)(DeMorgan's Law)
$(S(\vee Sa \vee Qb) = a) \subseteq (b \neq \emptyset) + (a = \emptyset)$	($TM260$)(Definition of $\neq$)(Eqn16)
$(b \neq \emptyset) + (a = \emptyset) = (S(\vee Sa \vee Qb) = a)$	($TM430$)(Biconditional Introduction Eqns11&16)

$$((c^1 \neq \emptyset) = E) \subseteq ((c^1 \subseteq \vee Sa^1 \vee Qb^1) = (a^1 = Sc^1)(b^1 = Qc^1)) \quad (TR498)$$

Proof of ($TR498$)

Let $a^1 = Sc^1$	(Eqn1)
$\vee Sa^1 = \vee SSc^1$	($TR522$)(Relational Deduction)(Eqn2)
$c^1 \subseteq \vee SSc^1$	($TR012i$)(Sub-Relationship Non-Closure Theorem)(Eqn3)
$c^1 \subseteq \vee Sa^1$	($AMX19$)(Substitution of Eqn2 into Eqn3)(Eqn4)
Let $b^1 = Qc^1$	(Eqn5)
$\vee Qb^1 = \vee QQc^1$	($TR522$)(Relational Deduction)(Eqn6)
$c^1 \subseteq \vee QQc^1$	($TR012i$)(Sub-Relationship Non-Closure Theorem)(Eqn7)
$c^1 \subseteq \vee Qb^1$	($AMX19$)(Substitution of Eqn5 into Eqn7)(Eqn8)
$c^1 \subseteq \vee Sa^1 \vee Qb^1$	($TM132$)(Composition of Eqns4&8)
Thus $(a^1 = Sc^1)(b^1 = Qc^1) \subseteq (c^1 \subseteq \vee Sa^1 \vee Qb^1)$	($AMX02$)(Conditional Introduction)(Eqn9)
$(c^1 \neq \emptyset)(a^1 = Sc^1)(b^1 = Qc^1) \subseteq (a^1 = Sc^1)(b^1 = Qc^1)$	($TM083$)(Conjunction Elimination)(Eqn10)
$(c^1 \neq \emptyset)(a^1 = Sc^1)(b^1 = Qc^1) \subseteq (c^1 \subseteq \vee Sa^1 \vee Qb^1)$	($TM139$)(Hypothetical Syllogism Eqns9&10)
$((c^1 \neq \emptyset) = E) \subseteq ((a^1 = Sc^1)(b^1 = Qc^1) \subseteq (c^1 \subseteq \vee Sa^1 \vee Qb^1))$	($AMX35$)(Exportation/Importation)(Eqn11) for truth sets
Let $c^1 \neq \emptyset$	(Eqn12)
$Sc^1 \neq \emptyset$	($TR048$)(Triad Theorem)(Eqn13)
$a^1 \neq \emptyset$	($AMX19$)(Substitution of Eqn1 into Eqn13)
$Qc^1 \neq \emptyset$	($TR047$)(Triad Theorem)(Eqn14)
$b^1 \neq \emptyset$	($AMX19$)(Substitution of Eqn5 into Eqn14)
Thus $(c^1 \neq \emptyset) \subseteq (a^1 \neq \emptyset)(b^1 \neq \emptyset)$	($AMX02$)(Conditional Introduction)(Eqn15)
$(Sc^1 \neq \emptyset) = E$	($AMX16$)(Order Axiom)
$(Sc^1 = \emptyset) = \sim E$	($TM008$)(Inverse of an Equality)
$(Sc^1 = \emptyset) = \emptyset$	($DM02$)(Definition of $\emptyset$)(Eqn16)
$(Qc^1 \neq \emptyset) = E$	($AMX16$)(Order Axiom)
$(Qc^1 = \emptyset) = \sim E$	($TM008$)(Inverse of an Equality)
$(Qc^1 = \emptyset) = \emptyset$	($DM02$)(Definition of $\emptyset$)(Eqn17)
Let $c^1 \subseteq \vee Sa^1 \vee Qb^1$	
$(c^1 \subseteq \vee Sa^1 \vee Qb^1) = E$	($AMX16$)(Order Axiom)
$(c^1 \neq \emptyset) \subseteq (\vee Sa^1 \vee Qb^1 \neq \emptyset)$	($TM096$)(Probabilistic Rule)
$(c^1 \neq \emptyset) \subseteq (a^1 \neq \emptyset)(b^1 \neq \emptyset)$	($AR11$)(Inter-relationship Axiom)(Eqn18)
$Sc^1 \subseteq S(\vee Sa^1 \vee Qb^1)$	($TR522$)(Relational Deduction)
$Sc^1 \subseteq S\vee Sa^1 S\vee Qb^1$	($TR012d$)(Distribution of S^* over $\cdot$)
$Sc^1 \subseteq a^1.S\vee Qb^1$	($TR012h$)(Sub-relationship Closure Theorem)
$Sc^1 \subseteq a^1.E$	($TR055$) N.B. conditional on $b^1 \neq \emptyset$
$Sc^1 \subseteq a^1$	($AMX11$)(Universal Set Conjunction Axiom)
$(Sc^1 \subseteq a^1) + (Sc^1 = a^1)$	($TM266$)(Definition of $\subseteq$)
$(Sc^1 = \emptyset) + (Sc^1 = a^1)$	($AMX40$)(Proper Subset of Unit Set Axiom)(Eqn19)
$\emptyset + (Sc^1 = a^1)$	($AMX19$)(Substitution of Eqn16 into Eqn19)

$Sc^1 = a^1$	(T_{M668})(Null Set Disjunction)
$Qc^1 \subseteq Q(\vee Sa^{1v}Qb^1)$	(T_{R522})(Relational Deduction)
$Qc^1 \subseteq Q\vee Sa^1.Q^vQb^1$	(T_{R012e})(Distribution of Q^* over $\vee$)
$Qc^1 \subseteq Q\vee Sa^1.b^1$	(T_{R012g})(Sub-relationship Closure Theorem)
$Qc^1 \subseteq E.b^1$	(T_{R060}) N.B. conditional on $a^1 \neq \emptyset$
$Qc^1 \subseteq b^1$	($AMX11$)(Universal Set Conjunction Axiom)
$(Qc^1 \subset b^1) + (Qc^1 = b^1)$	(T_{M266})(Definition of $\subseteq$)
$(Qc^1 = \emptyset) + (Qc^1 = b^1)$	($AMX40$)(Proper Subset of Unit Set Axiom)(Eqn20)
$\emptyset + (Qc^1 = b^1)$	($AMX19$)(Substitution of Eqn17 into Eqn20)
$Qc^1 = b^1$	(T_{M668})(Null Set Disjunction)
Thus $(c^1 \neq \emptyset).(c^1 \subseteq \vee Sa^{1v}Qb^1) \subseteq (Sc^1 = a^1).(Qc^1 = b^1)$	($AMX02$)(Conditional Introduction)
$(c^1 \neq \emptyset).(c^1 \subseteq \vee Sa^{1v}Qb^1) \subseteq (a^1 = Sc^1).(a^1 = Qc^1)$	(T_{M197})(Commutativity of $=$)
$((c^1 \neq \emptyset) = E) \subseteq ((c^1 \subseteq \vee Sa^{1v}Qb^1) \subseteq (a^1 = Sc^1).(a^1 = Qc^1))$	($AMX35$)(Exportation/Importation)(Eqn21) for truth sets
$((c^1 \neq \emptyset) = E)((c^1 \neq \emptyset) = E) \subseteq$	
$((a^1 = Sc^1)(b^1 = Qc^1) \subseteq (c^1 \subseteq \vee Sa^{1v}Qb^1)) ((c^1 \subseteq \vee Sa^{1v}Qb^1) \subseteq (a^1 = Sc^1).(a^1 = Qc^1))$	(T_{M669})(Combination of Eqns11&21)
$((c^1 \neq \emptyset) = E) \subseteq ((a^1 = Sc^1)(b^1 = Qc^1) \subseteq (c^1 \subseteq \vee Sa^{1v}Qb^1)) ((c^1 \subseteq \vee Sa^{1v}Qb^1) \subseteq (a^1 = Sc^1).(a^1 = Qc^1))$	(T_{M667})(Conjunction Tautology)
$((c^1 \neq \emptyset) = E) \subseteq ((c^1 \subseteq \vee Sa^{1v}Qb^1) = (a^1 = Sc^1).(a^1 = Qc^1))$	(T_{M430})(Biconditional Introduction)

$$((c^1 \neq \emptyset) = E) \subseteq ((c^1 \subseteq \vee Sa^{1v}Qb^1) = (a^1 = c^1 * b^1)) \quad (T_{R499})$$

Proof of (T_{R499})

Let $b^1 = Qc^1$	
$\vee Qb^1 = \vee QQc^1$	(T_{R024})(Sub-relationship Deduction)(Eqn1)
$c^1 \subseteq \vee QQc^1$	(T_{R012i})(Sub-relationship Non-closure Theorem)(Eqn2)
$c^1 \subseteq \vee Qb^1$	($AMX19$)(Substitution of Eqn1 into Eqn2)
$c^1 = c^1 \vee Qb^1$	($AMX44$)($\subseteq$ Equality Form)(Eqn3)
Let $a^1 = Sc^1$	(Eqn4)
$a^1 = S(c^1 \vee Qb^1)$	($AMX19$)(Substitution of Eqn3 into Eqn4)
$a^1 = c^1 * b^1$	($AR27$)(Function Axiom)
Thus $(a^1 = Sc^1)(b^1 = Qc^1) \subseteq (a^1 = c^1 * b^1)$	($AMX02$)(Conditional Introduction)
$E = ((a^1 = Sc^1)(b^1 = Qc^1) \subseteq (a^1 = c^1 * b^1))$	($AMX16$)(Order Axiom)(Eqn5)
Let $a^1 \neq \emptyset$	
$(a^1 \neq \emptyset) = E$	($AMX16$)(Order Axiom)
$(a^1 = \emptyset) = \sim E$	(T_{M008})(Inverse of an Equality)
$(a^1 = \emptyset) = \emptyset$	($DM02$)(Definition of $\emptyset$)(Eqn6)
Let $a^1 = c^1 * b^1$	
$a^1 = S(c^1 \vee Qb^1)$	($AR27$)(Function Axiom)
$a^1 \subseteq Sc^1.S^vQb^1$	(T_{R012d})(Distribution of S^* over $\vee$)
$a^1 \subseteq Sc^1$	(T_{M073})(Single Premise Syllogism)
$(a^1 \subset Sc^1) + (a^1 = Sc^1)$	(T_{M266})(Definition of $\subseteq$)(Eqn7)
$(a^1 \subset Sc^1) = (Sc^1 \neq \emptyset)(a^1 = \emptyset)$	($AMX40$)(Proper Subset of Unit Set Axiom)(Eqn8)
$(Sc^1 \neq \emptyset)(a^1 = \emptyset) + (a^1 = Sc^1)$	($AMX19$)(Substitution of Eqn8 into Eqn7)(Eqn9)
$(Sc^1 \neq \emptyset)\emptyset + (a^1 = Sc^1)$	($AMX19$)(Substitution of Eqn6 into Eqn9)
$\emptyset + (a^1 = Sc^1)$	(T_{M050})(Null Set Conjunction)
$a^1 = Sc^1$	(T_{M668})(Null Set Disjunction)
Thus $(a^1 \neq \emptyset)(a^1 = c^1 * b^1) \subseteq (a^1 = Sc^1)$	($AMX02$)(Conditional Introduction)(Eqn10)
Let $b^1 \neq \emptyset$	
$(b^1 \neq \emptyset) = E$	($AMX16$)(Order Axiom)
$(b^1 = \emptyset) = \sim E$	(T_{M008})(Inverse of an Equality)
$(b^1 = \emptyset) = \emptyset$	($DM02$)(Definition of $\emptyset$)(Eqn11)
Let $a^1 = c^1 * b^1$	
$b^1 = \vee c^1 * a^1$	(T_{S247})(Active/Passive Theorem)

$b^1 = S^{\vee}c^1 \vee Qa^1$	(A_{R27})(Function Axiom)
$b^1 \subseteq S^{\vee}c^1 \cdot S^{\vee}Qa^1$	(T_{R012d})(Distribution of S^* over $\cdot$)
$b^1 \subseteq S^{\vee}c^1$	(T_{M073})(Single Premise Syllogism)
$b^1 \subseteq Qc^1$	(T_{R016})(Reversibility of Matching Entities)
$(b^1 \subset Qc^1) + (b^1 = Qc^1)$	(T_{M266})(Definition of $\subseteq$)(Eqn12)
$(b^1 \subset Qc^1) = (Qc^1 \neq \emptyset)(b^1 = \emptyset)$	(A_{MX40})(Proper Subset of Unit Set Axiom)(Eqn13)
$(Qc^1 \neq \emptyset)(b^1 = \emptyset) + (b^1 = Qc^1)$	(A_{MX19})(Substitution of Eqn13 into Eqn12)(Eqn14)
$(Qc^1 \neq \emptyset)\emptyset + (b^1 = Qc^1)$	(A_{MX19})(Substitution of Eqn11 into Eqn14)
$\emptyset + (b^1 = Qc^1)$	(T_{M050})(Null Set Conjunction)
$b^1 = Qc^1$	(T_{M668})(Null Set Disjunction)
Thus $(b^1 \neq \emptyset)(a^1 = c^1 * b^1) \subseteq (b^1 = Qc^1)$	(A_{MX02})(Conditional Introduction)(Eqn15)
$(a^1 \neq \emptyset)(a^1 = c^1 * b^1)(b^1 \neq \emptyset)(a^1 = c^1 * b^1) \subseteq (a^1 = Sc^1)(b^1 = Qc^1)$	(T_{M669})(Combination of Eqns10 & 15)
$(a^1 \neq \emptyset)(b^1 \neq \emptyset)(a^1 = c^1 * b^1) \subseteq (a^1 = Sc^1)(b^1 = Qc^1)$	(T_{M667})(Conjunction Tautology)
$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 = c^1 * b^1) \subseteq (a^1 = Sc^1)(b^1 = Qc^1))$	(A_{MX35})(Exportation/Importation)(Eqn16) for truth sets
$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \cdot E \subseteq ((a^1 = c^1 * b^1) \subseteq (a^1 = Sc^1)(b^1 = Qc^1))((a^1 = Sc^1)(b^1 = Qc^1) \subseteq (a^1 = c^1 * b^1))$	(T_{M669})(Combination of Eqns5 & 16)
$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 = c^1 * b^1) \subseteq (a^1 = Sc^1)(b^1 = Qc^1))((a^1 = Sc^1)(b^1 = Qc^1) \subseteq (a^1 = c^1 * b^1))$	(A_{MX17})(Universal Set Conjunction Axiom)
$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 = Sc^1)(b^1 = Qc^1) = (a^1 = c^1 * b^1))$	(T_{M430})(Biconditional Introduction)(Eqn17)
$((c^1 \neq \emptyset) = E) \subseteq ((c^1 \subseteq \vee Sa^1 \vee Qb^1) = (a^1 = Sc^1)(b^1 = Qc^1))$	(T_{R498})(Eqn18)
$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E)((c^1 \neq \emptyset) = E) \subseteq$ $((a^1 = Sc^1)(b^1 = Qc^1) = (a^1 = c^1 * b^1))((c^1 \subseteq \vee Sa^1 \vee Qb^1) = (a^1 = Sc^1)(b^1 = Qc^1))$	(T_{M669})(Combination of Eqns17 & 18)
$((a^1 \neq \emptyset)(b^1 \neq \emptyset)(c^1 \neq \emptyset) = E) \subseteq$ $((a^1 = Sc^1)(b^1 = Qc^1) = (a^1 = c^1 * b^1))((c^1 \subseteq \vee Sa^1 \vee Qb^1) = (a^1 = Sc^1)(b^1 = Qc^1))$	(T_{M108})(Conjunction Introduction)
$((a^1 \neq \emptyset)(b^1 \neq \emptyset)(c^1 \neq \emptyset) = E) \subseteq ((c^1 \subseteq \vee Sa^1 \vee Qb^1) = (a^1 = c^1 * b^1))$	(A_{MX19})(Substitution)
$(a^1 \neq \emptyset)(b^1 \neq \emptyset)(c^1 \neq \emptyset)(c^1 \subseteq \vee Sa^1 \vee Qb^1) = (a^1 = c^1 * b^1)$	(A_{MX35})(Exportation/Importation)(Eqn19) for truth sets
Let $c^1 \neq \emptyset$	
$(c^1 \neq \emptyset) = E$	(A_{MX16})(Order Axiom)(Eqn20)
Let $c^1 \subseteq \vee Sa^1 \vee Qb^1$	
$(c^1 \subseteq \vee Sa^1 \vee Qb^1) = E$	(A_{MX16})(Order Axiom)
$(c^1 \neq \emptyset) \subseteq (\vee Sa^1 \vee Qb^1 \neq \emptyset)$	(T_{M096})(Probabilistic Rule)(Eqn21)
$E \subseteq (\vee Sa^1 \vee Qb^1 \neq \emptyset)$	(A_{MX19})(Substitution of Eqn20 into Eqn21)
$E = (\vee Sa^1 \vee Qb^1 \neq \emptyset)$	(T_{M275})(Supersets of the Universal Set)
$\vee Sa^1 \vee Qb^1 \neq \emptyset$	(A_{MX16})(Order Axiom)
$\vee Sa^1 \neq \emptyset$	(T_{M350}) & (T_{M012})(Transposition)
$a^1 \neq \emptyset$	(T_{R170})
$\vee Qb^1 \neq \emptyset$	(T_{M350}) & (T_{M012})(Transposition)
$b^1 \neq \emptyset$	(T_{R170})
Thus $(c^1 \neq \emptyset)(c^1 \subseteq \vee Sa^1 \vee Qb^1) \subseteq (a^1 \neq \emptyset)(b^1 \neq \emptyset)$	(A_{MX02})(Conditional Introduction)
$(c^1 \neq \emptyset)(c^1 \subseteq \vee Sa^1 \vee Qb^1) = (a^1 \neq \emptyset)(b^1 \neq \emptyset)(c^1 \neq \emptyset)(c^1 \subseteq \vee Sa^1 \vee Qb^1)$	(A_{MX44})($\subseteq$ Equality Form)(Eqn22)
$(c^1 \neq \emptyset)(c^1 \subseteq \vee Sa^1 \vee Qb^1) = (a^1 = c^1 * b^1)$	(A_{MX19})(Substitution of Eqn22 into Eqn19)
$((c^1 \neq \emptyset) = E) \subseteq ((c^1 \subseteq \vee Sa^1 \vee Qb^1) = (a^1 = c^1 * b^1))$	(A_{MX35})(Exportation/Importation) for truth sets

$$((b \neq \emptyset) = E) \subseteq ((\vee Sa^1 \vee Qb \subseteq c) \subseteq (a^1 \subseteq c * b))$$

(T_{7456})

Proof of (T₇₄₅₆)Let $\mathbf{b} \neq \emptyset$ Let $\forall \mathbf{Sa}^1. \forall \mathbf{Qb} \subseteq \mathbf{c}$ $\forall \mathbf{Sa}^1. \forall \mathbf{Qb}. \forall \mathbf{Qb} \subseteq \mathbf{c}. \forall \mathbf{Qb}$ (T_{M029})(Deduction) $\forall \mathbf{Sa}^1. \forall \mathbf{Qb} \subseteq \mathbf{c}. \forall \mathbf{Qb}$ (T_{M667})(Conjunction Tautology) $\mathbf{S}(\forall \mathbf{Sa}^1. \forall \mathbf{Qb}) \subseteq \mathbf{S}(\mathbf{c}. \forall \mathbf{Qb})$ (T_{R196})(Relational Deduction) $\mathbf{a}^1 \subseteq \mathbf{S}(\mathbf{c}. \forall \mathbf{Qb})$ (T_{S248}) $\mathbf{a}^1 \subseteq \mathbf{c} * \mathbf{b}$ (T_{R598})Thus $(\mathbf{b} \neq \emptyset)(\forall \mathbf{Sa}^1. \forall \mathbf{Qb} \subseteq \mathbf{c}) \subseteq (\mathbf{a}^1 \subseteq \mathbf{c} * \mathbf{b})$ (A_{MX02})(Conditional Introduction) $((\mathbf{b} \neq \emptyset) = \mathbf{E}) \subseteq ((\forall \mathbf{Sa}^1. \forall \mathbf{Qb} \subseteq \mathbf{c}) \subseteq (\mathbf{a}^1 \subseteq \mathbf{c} * \mathbf{b}))$ (A_{MX35})(Exportation/Importation) for truth sets

$$(\mathbf{b} \neq \emptyset) + (\mathbf{a} = \emptyset) = ((\forall \mathbf{Sa}. \forall \mathbf{Qb}) * \mathbf{b} = \mathbf{a})$$

(T_{S828})Proof of (T_{S828}) $\mathbf{S}^v \mathbf{Sa} = \mathbf{a}$ (T_{R012m})(Sub-Relationship Closure Theorem) $(\mathbf{b} \neq \emptyset) + (\mathbf{a} = \emptyset) = (\mathbf{S}(\forall \mathbf{Sa}^v \mathbf{Qb}) = \mathbf{a})$ (T_{S248}) $(\mathbf{b} \neq \emptyset) + (\mathbf{a} = \emptyset) = (\mathbf{S}(\forall \mathbf{Sa}. \forall \mathbf{Qb}. \forall \mathbf{Qb}) = \mathbf{a})$ (T_{M667})(Conjunction Tautology) $(\mathbf{b} \neq \emptyset) + (\mathbf{a} = \emptyset) = ((\forall \mathbf{Sa}. \forall \mathbf{Qb}) * \mathbf{b} = \mathbf{a})$ (A_{R27})(Function Axiom)

$$(\mathbf{b} \neq \emptyset) + (\mathbf{a} = \emptyset) = (\forall \mathbf{Sa} * \mathbf{b} = \mathbf{a})$$

(T_{S250})Proof of (T_{S250}) $(\mathbf{b} \neq \emptyset) + (\mathbf{a} = \emptyset) = (\mathbf{S}(\forall \mathbf{Sa}^v \mathbf{Qb}) = \mathbf{a})$ (T_{S248}) $(\mathbf{b} \neq \emptyset) + (\mathbf{a} = \emptyset) = (\forall \mathbf{Sa} * \mathbf{b} = \mathbf{a})$ (A_{R27})(Function Axiom)

$$(\mathbf{b} \neq \emptyset) + (\mathbf{a} = \emptyset) = (\forall \mathbf{Sa} \diamond \mathbf{b} = \mathbf{a})$$

(T_{S251})Proof of (T_{S251})For $\mathbf{r} = 1$ to $\mathbf{n}$ $(\mathbf{b}_r^1 \neq \emptyset) + (\mathbf{a} = \emptyset) = (\forall \mathbf{Sa} * \mathbf{b}_r^1 = \mathbf{a})$ (T_{S250})(Eqn1) $((\mathbf{b}_1^1 \neq \emptyset) + (\mathbf{a} = \emptyset)) \dots ((\mathbf{b}_n^1 \neq \emptyset) + (\mathbf{a} = \emptyset)) = (\forall \mathbf{Sa} * \mathbf{b}_1^1 = \mathbf{a}) \dots (\forall \mathbf{Sa} * \mathbf{b}_n^1 = \mathbf{a})$ (T_{M237})(Combination of Eqns1) $((\mathbf{b}_1^1 \neq \emptyset) + (\mathbf{a} = \emptyset)) \dots ((\mathbf{b}_n^1 \neq \emptyset) + (\mathbf{a} = \emptyset)) \subseteq ((\forall \mathbf{Sa} * \mathbf{b}_1^1) \dots (\forall \mathbf{Sa} * \mathbf{b}_n^1) = \mathbf{a} \dots \mathbf{a})$ (T_{M237})(Combination) $((\mathbf{b}_1^1 \neq \emptyset) + (\mathbf{a} = \emptyset)) \dots ((\mathbf{b}_n^1 \neq \emptyset) + (\mathbf{a} = \emptyset)) \subseteq ((\forall \mathbf{Sa} * \mathbf{b}_1^1) \dots (\forall \mathbf{Sa} * \mathbf{b}_n^1) = \mathbf{a})$ (T_{M667})(Conjunction Tautology) $((\mathbf{b}_1^1 \neq \emptyset) + (\mathbf{a} = \emptyset)) \dots ((\mathbf{b}_n^1 \neq \emptyset) + (\mathbf{a} = \emptyset)) \subseteq (\forall \mathbf{Sa} \diamond (\mathbf{b}_1^1 + \dots + \mathbf{b}_n^1) = \mathbf{a})$ (T_{T750})(Definition of the Intersection Function Operator)(Eqn2) $((\mathbf{b}_1^1 \neq \emptyset) \dots (\mathbf{b}_n^1 \neq \emptyset)) + (\mathbf{a} = \emptyset) \subseteq ((\mathbf{b}_1^1 \neq \emptyset) + (\mathbf{a} = \emptyset)) \dots ((\mathbf{b}_n^1 \neq \emptyset) + (\mathbf{a} = \emptyset))$ (T_{M522})(Eqn3) $((\mathbf{b}_1^1 \neq \emptyset) \dots (\mathbf{b}_n^1 \neq \emptyset)) + (\mathbf{a} = \emptyset) \subseteq (\forall \mathbf{Sa} \diamond (\mathbf{b}_1^1 + \dots + \mathbf{b}_n^1) = \mathbf{a})$ (T_{M139})(Hypothetical Syllogism Eqns2 & 3)(Eqn4)Let $\mathbf{b} = \mathbf{b}_1^1 + \dots + \mathbf{b}_n^1$

(Eqn5)

Let $((\mathbf{b}_1^1 \neq \emptyset) \dots (\mathbf{b}_n^1 \neq \emptyset)) = (\mathbf{b} \neq \emptyset)$ (Eqn6) It is inherent in Eqn5 that if $\mathbf{b}$ exists then all of $\mathbf{b}_1^1$ to $\mathbf{b}_n^1$ exist. $(\mathbf{b} \neq \emptyset) + (\mathbf{a} = \emptyset) \subseteq (\forall \mathbf{Sa} \diamond \mathbf{b} = \mathbf{a})$ (A_{MX19})(Substitution of Eqns5 & 6 into Eqn4)(Eqn7)Let $\mathbf{b} = \emptyset$

(Eqn8)

Let $\mathbf{a} \neq \emptyset$

(Eqn9)

Let $\forall \mathbf{Sa} \diamond \mathbf{b} = \mathbf{a}$

(Eqn10)

 $\forall \mathbf{Sa} \diamond \emptyset = \mathbf{a}$ (A_{MX19})(Substitution of Eqn8 into Eqn10) $\emptyset = \mathbf{a}$ (T_{R174})(Null Object Theorem)(Eqn11) $\emptyset \neq \emptyset$ (A_{MX19})(Substitution of Eqn11 into Eqn9) $\emptyset$ (T_{M346})Thus $(\mathbf{b} = \emptyset)(\mathbf{a} \neq \emptyset)(\forall \mathbf{Sa} \diamond \mathbf{b} = \mathbf{a}) = \emptyset$ (A_{MX02})(Conditional Introduction) $(\forall \mathbf{Sa} \diamond \mathbf{b} = \mathbf{a}) \subseteq \sim((\mathbf{b} = \emptyset)(\mathbf{a} \neq \emptyset))$ (T_{M735})($\subseteq$ Existential Form) $(\forall \mathbf{Sa} \diamond \mathbf{b} = \mathbf{a}) \subseteq \sim(\mathbf{b} = \emptyset) + \sim(\mathbf{a} \neq \emptyset)$ (T_{M025})(DeMorgan's Law) $(\forall \mathbf{Sa} \diamond \mathbf{b} = \mathbf{a}) \subseteq (\mathbf{b} \neq \emptyset) + (\mathbf{a} = \emptyset)$ (T_{M260})(Definition of $\neq$)(Eqn12)

$$(\forall \text{Sa} \diamond \text{b} = \text{a}) = (\text{b} \neq \emptyset) + (\text{a} = \emptyset)$$

($\mathcal{T}_{M430}$)(Biconditional Introduction Eqns7&12)

$$(\text{b} \neq \emptyset) + (\text{a} = \emptyset) = ((\forall \text{Sa} \forall \text{Qb}) \diamond \text{b} = \text{a})$$

($\mathcal{T}_{S252}$)

Proof of ($\mathcal{T}_{S252}$)

$$(\text{b} \neq \emptyset) + (\text{a} = \emptyset) = (\forall \text{Sa} \diamond \text{b} = \text{a})$$

($\mathcal{T}_{S251}$)

$$(\text{b} \neq \emptyset) \subseteq (\forall \text{Sa} \diamond \text{b} = \text{a})$$

($\mathcal{T}_{M072b}$)(Single Premise Syllogism)(Eqn1)

$$(\text{b} \neq \emptyset) = (\forall \text{Qb} \diamond \text{b} = \text{E})$$

($\mathcal{A}_{R09}$)(Universality of Inter-relationships Axiom)

$$(\text{b} \neq \emptyset) \subseteq (\forall \text{Qb} \diamond \text{b} = \text{E})$$

($\mathcal{T}_{M425}$)(Biconditional Elimination)(Eqn2)

$$(\text{b} \neq \emptyset) \subseteq (\forall \text{Sa} \diamond \text{b} = \text{a})(\forall \text{Qb} \diamond \text{b} = \text{E})$$

($\mathcal{T}_{M132}$)(Composition of Eqns1 & 2)(Eqn3)

$$(\forall \text{Sa} \diamond \text{b} = \text{a})(\forall \text{Qb} \diamond \text{b} = \text{E}) \subseteq ((\forall \text{Sa} \diamond \text{b})(\forall \text{Qb} \diamond \text{b}) = \text{aE})$$

($\mathcal{T}_{M669}$)(Combination)(Eqn4)

$$(\text{b} \neq \emptyset) \subseteq ((\forall \text{Sa} \diamond \text{b})(\forall \text{Qb} \diamond \text{b}) = \text{aE})$$

($\mathcal{T}_{M139}$)(Hypothetical Syllogism Eqns3 & 4)

$$(\text{b} \neq \emptyset) \subseteq ((\forall \text{Sa} \diamond \text{b})(\forall \text{Qb} \diamond \text{b}) = \text{a})$$

($\mathcal{A}_{MX11}$)(Universal Set Conjunction Axiom)

$$(\text{b} \neq \emptyset) \subseteq ((\forall \text{Sa} \forall \text{Qb}) \diamond \text{b} = \text{a})$$

($\mathcal{T}_{R237}$)(Distribution of $\cdot$ over $\diamond$)(Eqn5)

$$\text{Let } \text{a} = \emptyset$$

(Eqn6)

$$\text{Let } (\forall \text{Sa} \forall \text{Qb}) \diamond \text{b} \neq \text{a}$$

(Eqn7)

$$(\forall \text{S} \emptyset \forall \text{Qb}) \diamond \text{b} \neq \emptyset$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn6 into Eqn7)

$$(\emptyset \forall \text{Qb}) \diamond \text{b} \neq \emptyset$$

($\mathcal{T}_{S181}$)(Null Object Theorem)

$$\emptyset \diamond \text{b} \neq \emptyset$$

($\mathcal{T}_{M050}$)(Null Set Conjunction)

$$\emptyset \neq \emptyset$$

($\mathcal{T}_{R173}$)(Null Function Theorem)

$$\emptyset$$

($\mathcal{T}_{M346}$)

$$\text{Thus } (\text{a} = \emptyset)((\forall \text{Sa} \forall \text{Qb}) \diamond \text{b} \neq \text{a}) = \emptyset$$

($\mathcal{A}_{MX02}$)(Conditional Introduction)

$$(\text{a} = \emptyset) \subseteq ((\forall \text{Sa} \forall \text{Qb}) \diamond \text{b} = \text{a})$$

($\mathcal{T}_{M735}$)($\subseteq$ Existential Form)(Eqn8)

$$(\text{b} \neq \emptyset) + (\text{a} = \emptyset) \subseteq ((\forall \text{Sa} \forall \text{Qb}) \diamond \text{b} = \text{a}) + ((\forall \text{Sa} \forall \text{Qb}) \diamond \text{b} = \text{a})$$

($\mathcal{A}_{MX21}$)(Combination Axiom)

$$(\text{b} \neq \emptyset) + (\text{a} = \emptyset) \subseteq ((\forall \text{Sa} \forall \text{Qb}) \diamond \text{b} = \text{a})$$

($\mathcal{A}_{MX01}$)(Disjunction Tautology)(Eqn9)

$$\text{Let } \text{b} = \emptyset$$

(Eqn10)

$$\text{Let } (\forall \text{Sa} \forall \text{Qb}) \diamond \text{b} = \text{a}$$

(Eqn11)

$$(\forall \text{Sa} \forall \text{Qb}) \diamond \emptyset = \text{a}$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn10 into Eqn11)

$$\emptyset = \text{a}$$

($\mathcal{T}_{R174}$)(Null Object Theorem)

$$\text{Thus } (\text{b} = \emptyset)((\forall \text{Sa} \forall \text{Qb}) \diamond \text{b} = \text{a}) \subseteq (\text{a} = \emptyset)$$

($\mathcal{A}_{MX02}$)(Conditional Introduction)

$$((\forall \text{Sa} \forall \text{Qb}) \diamond \text{b} = \text{a}) \subseteq (\text{b} \neq \emptyset) + (\text{a} = \emptyset)$$

($\mathcal{T}_{M468}$)(Eqn12)

$$(\text{b} \neq \emptyset) + (\text{a} = \emptyset) = ((\forall \text{Sa} \forall \text{Qb}) \diamond \text{b} = \text{a})$$

($\mathcal{T}_{M430}$)(Biconditional Introduction Eqns9 & 12)

$$((\text{b} \neq \emptyset) + (\text{a} = \emptyset) = \text{E}) \subseteq ((\forall \text{Sa} \forall \text{Qb} = \text{c}) \subseteq (\text{c} \diamond \text{b} = \text{a}))$$

($\mathcal{T}_{R503}$)

Proof of ($\mathcal{T}_{R503}$)

$$(\text{b} \neq \emptyset) + (\text{a} = \emptyset) = ((\forall \text{Sa} \forall \text{Qb}) \diamond \text{b} = \text{a})$$

($\mathcal{T}_{S252}$)(Eqn1)

$$\text{Let } \forall \text{Sa} \forall \text{Qb} = \text{c}$$

(Eqn2)

$$(\text{b} \neq \emptyset) + (\text{a} = \emptyset) = (\text{c} \diamond \text{b} = \text{a})$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$\text{Thus } ((\forall \text{Sa} \forall \text{Qb} = \text{c}) = \text{E}) \subseteq ((\text{b} \neq \emptyset) + (\text{a} = \emptyset) = (\text{c} \diamond \text{b} = \text{a}))$$

($\mathcal{A}_{MX02}$)(Conditional Introduction)(Eqn3)

$$((\text{b} \neq \emptyset) + (\text{a} = \emptyset) = (\text{c} \diamond \text{b} = \text{a})) \subseteq ((\text{b} \neq \emptyset) + (\text{a} = \emptyset) = ((\text{b} \neq \emptyset) + (\text{a} = \emptyset))(\text{c} \diamond \text{b} = \text{a}))$$

($\mathcal{T}_{M772}$)(Eqn4)

$$((\forall \text{Sa} \forall \text{Qb} = \text{c}) = \text{E}) \subseteq ((\text{b} \neq \emptyset) + (\text{a} = \emptyset) = ((\text{b} \neq \emptyset) + (\text{a} = \emptyset))(\text{c} \diamond \text{b} = \text{a}))$$

($\mathcal{T}_{M139}$)(Hypothetical Syllogism)

$$((\forall \text{Sa} \forall \text{Qb} = \text{c})((\text{b} \neq \emptyset) + (\text{a} = \emptyset))) = (\forall \text{Sa} \forall \text{Qb} = \text{c})((\text{b} \neq \emptyset) + (\text{a} = \emptyset))(\text{c} \diamond \text{b} = \text{a}) = \text{E}$$

($\mathcal{T}_{M139}$)(Hypothetical Syllogism)

$$((\forall \text{Sa} \forall \text{Qb} = \text{c})((\text{b} \neq \emptyset) + (\text{a} = \emptyset))).\text{E} = (\forall \text{Sa} \forall \text{Qb} = \text{c})((\text{b} \neq \emptyset) + (\text{a} = \emptyset))(\text{c} \diamond \text{b} = \text{a}) = \text{E}$$

($\mathcal{T}_{M444b}$)(Exportation from a Biconditional)

$$((\forall \text{Sa} \forall \text{Qb} = \text{c})((\text{b} \neq \emptyset) + (\text{a} = \emptyset))).\text{E} = (\forall \text{Sa} \forall \text{Qb} = \text{c})((\text{b} \neq \emptyset) + (\text{a} = \emptyset))(\text{c} \diamond \text{b} = \text{a}) = \text{E}$$

($\mathcal{A}_{MX11}$)(Universal Set Conjunction Axiom)

$$((\forall \text{Sa} \forall \text{Qb} = \text{c})((\text{b} \neq \emptyset) + (\text{a} = \emptyset))) = \text{E} \subseteq (\text{E} = (\text{c} \diamond \text{b} = \text{a}))$$

($\mathcal{T}_{M444b}$)(Exportation from a Biconditional)

$$((\forall \text{Sa} \forall \text{Qb} = \text{c})((\text{b} \neq \emptyset) + (\text{a} = \emptyset))) \subseteq (\text{c} \diamond \text{b} = \text{a}) = \text{E}$$

($\mathcal{T}_{M094}$)(Probabilistic Rule)

$$(((\text{b} \neq \emptyset) + (\text{a} = \emptyset)) = \text{E}) \subseteq ((\forall \text{Sa} \forall \text{Qb} = \text{c}) \subseteq (\text{c} \diamond \text{b} = \text{a}))$$

($\mathcal{T}_{M444b}$)(Exportation from a Biconditional)

$$\forall \text{Sa} = \sim \forall \text{Sa}$$

($\mathcal{T}_{R036}$) Sub-Relationship Negative Theorem

$$\forall \text{Qb} = \sim \forall \text{Qb}$$

($\mathcal{T}_{R037}$) Sub-Relationship Negative Theorem

Example Proof of (T_{R036})

$((^Vc)^1 \subseteq R_o) : (c * b)(c * \underline{b}) = \emptyset$	(A _{R36})(One-to-One Relationship Negation Axiom)(Eqn1)
Let $c = ^VS$	(Eqn2)
$((^VS)^1 \subseteq R_o) : (^VS * b)(^VS * \underline{b}) = \emptyset$	(A _{MX19})(Substitution of Eqn1 into Eqn2)
$(S^1 \subseteq R_o) : (^VS * b)(^VS * \underline{b}) = \emptyset$	(T _{R012k})(Sub-Relationship Closure Theorem)(Eqn3)
$(X = (V, S, Q)) : (X^1 \subseteq R_o)$	(A _{R04})(One-to-One Sub-relationship Axiom)
$S^1 \subseteq R_o$	Extraction
$(S^1 \subseteq R_o) = E$	(A _{MX16})(Order Axiom)(Eqn4)
$E : (^VS * b)(^VS * \underline{b}) = \emptyset$	(A _{MX19})(Substitution of Eqn4 into Eqn3)
$E \subseteq ((^VS * b)(^VS * \underline{b}) = \emptyset)$	(D _{M04})(Definition of :)
$E = ((^VS * b)(^VS * \underline{b}) = \emptyset)$	(T _{M275})(Supersets of the Universal Set)
$(^VS * b)(^VS * \underline{b}) = \emptyset$	(A _{MX16})(Order Axiom)
$(^VS * \underline{b}) \subseteq (^VS * b)$	(T _{M735})($\subseteq$ Existential Form)
$^VSb \subseteq \sim^VSb$	(D _{R04})(Definition of $\sim$)(Eqn5)
$^VSE = E$	(T _{R115})
$^VS(b + \underline{b}) = E$	(T _{M671})(Tertium non Datur)
$^VSb + ^VSb = E$	(T _{S289})(Distribution of $^VS*$ over $+$)
$(^VSb + ^VSb) \sim^VSb = E \sim^VSb$	(T _{M233})(Deduction Equality Form)
$(^VSb + ^VSb) \sim^VSb = \sim^VSb$	(A _{MX11})(Universal Set Conjunction Axiom)
$\sim^VSb^VSb + \sim^VSb^VSb = \sim^VSb$	(T _{M407})(Distribution of $\cdot$ over $+$)
$\emptyset + \sim^VSb^VSb = \sim^VSb$	(T _{M621})(Complement Conjunction)
$\sim^VSb^VSb = \sim^VSb$	(T _{M668})(Null Set Disjunction)
$\sim^VSb \subseteq \sim^VSb$	(A _{MX44})($\subseteq$ Equality Form)(Eqn6)
$^VSb = \sim^VSb$	(T _{M430})(Biconditional Introduction Eqns5&6)
$\sim^VSb = ^VSb$	(T _{M008})(Inverse of an Equality)

$$\sim V \sim^V V a = a \quad (T_{S298})$$

$$\sim S \sim^V S a = a \quad (T_{S117})$$

$$\sim Q \sim^V Q a = a \quad (T_{S299})$$

Example Proof of (T_{S117})

$\sim^V S a = ^V S a$	(T _{R036})(Sub-Relationship Negative Theorem)
$S \sim^V S a = S^V S a$	(T _{R026})(Sub-Relationship Deduction)
$S \sim^V S a = \underline{a}$	(T _{R012m})(Sub-relationship Closure Theorem)
$\sim S \sim^V S a = a$	(T _{M008})(Inverse of an Equality)

$$^V Q b * \underline{b} = \emptyset \quad (T_{R152})$$

Proof of (T_{R152})

$(c \subseteq \sim^V Q b) = (c * b = \emptyset)$	(T _{R079})
$(c \subseteq \sim^V Q \underline{b}) = (c * \underline{b} = \emptyset)$	Resymbolising (Eqn1)
Let $c = ^V Q b$	(Eqn2)
$(^V Q b \subseteq \sim^V Q \underline{b}) = (^V Q b * \underline{b} = \emptyset)$	(A _{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)
$^V Q b = \sim^V Q \underline{b}$	(T _{R037})(Sub-Relationship Negative Theorem)(Eqn4)
$(\sim^V Q \underline{b} \subseteq \sim^V Q \underline{b}) = (^V Q b * \underline{b} = \emptyset)$	(A _{MX19})(Substitution of Eqn4 into Eqn3)
$(\sim^V Q \underline{b} \subseteq \sim^V Q \underline{b}) = (\sim^V Q \underline{b} * \underline{b} = \emptyset)$	(T _{M735})($\subseteq$ Existential Form)
$E = (^V Q b * \underline{b} = \emptyset)$	(T _{M001})(Universality of (T _{M621}))($\sim^V Q \underline{b} \sim^V Q \underline{b} = \emptyset$)
$^V Q b * \underline{b} = \emptyset$	(A _{MX16})(Order Axiom)

$$^V Q b \diamond \underline{b} = \emptyset \quad (T_{R278})$$

Proof of (T_{R278})

$^V Q b * \underline{b} = \emptyset$	(T _{R152})
$^V Q b \diamond \underline{b} = \emptyset$	(T _{S246})

$$^V Q \underline{b} * b = \emptyset \quad (T_{R153})$$

Proof of (T_{R153})

$$\forall \mathbf{Qb} * \mathbf{b} = \emptyset$$

(T_{R152})

$$\forall \mathbf{Qb} * \mathbf{b} = \emptyset$$

Resymbolising

$$\forall \mathbf{Qb} \diamond \mathbf{b} = \emptyset$$

(T_{R279})

Proof of (T_{R279})

$$\forall \mathbf{Qb} * \mathbf{b} = \emptyset$$

(T_{R153})

$$\forall \mathbf{Qb} \diamond \mathbf{b} = \emptyset$$

(T_{S246})

$$\forall (\forall \mathbf{Sa}) * \mathbf{a} = \emptyset$$

(T_{R155})

Proof of (T_{R155})

$$(\mathbf{c} \subseteq \sim \forall \mathbf{Sa}) = (\forall \mathbf{c} * \mathbf{a} = \emptyset)$$

(T_{R084})

$$(\mathbf{c} \subseteq \sim \forall \mathbf{Sa}) = (\forall \mathbf{c} * \mathbf{a} = \emptyset)$$

Resymbolising (Eqn1)

$$\text{Let } \mathbf{c} = \forall \mathbf{Sa}$$

(Eqn2)

$$(\forall \mathbf{Sa} \subseteq \sim \forall \mathbf{Sa}) = (\forall (\forall \mathbf{Sa}) * \mathbf{a} = \emptyset)$$

($\mathbf{A}_{MX19}$) (Substitution of Eqn2 into Eqn1) (Eqn3)

$$\forall \mathbf{Sa} = \sim \forall \mathbf{Sa}$$

(T_{R036}) (Sub-Relationship Negative Theorem) (Eqn4)

$$(\sim \forall \mathbf{Sa} \subseteq \sim \forall \mathbf{Sa}) = (\forall (\forall \mathbf{Sa}) * \mathbf{a} = \emptyset)$$

($\mathbf{A}_{MX19}$) (Substitution of Eqn4 into Eqn3)

$$(\sim \forall \mathbf{Sa} \forall \mathbf{Sa} = \emptyset) = (\forall (\forall \mathbf{Sa}) * \mathbf{a} = \emptyset)$$

(T_{M735}) ($\subseteq$ Existential Form)

$$\mathbf{E} = (\forall (\forall \mathbf{Sa}) * \mathbf{a} = \emptyset)$$

(T_{M001}) (Universality of (T_{M621})) ($\sim \forall \mathbf{Sa} \forall \mathbf{Sa} = \emptyset$)

$$\forall (\forall \mathbf{Sa}) * \mathbf{a} = \emptyset$$

($\mathbf{A}_{MX16}$) (Order Axiom)

$$\forall (\forall \mathbf{Sa}) \diamond \mathbf{a} = \emptyset$$

(T_{R281})

Proof of (T_{R281})

$$\forall (\forall \mathbf{Sa}) * \mathbf{a} = \emptyset$$

(T_{R155})

$$\forall (\forall \mathbf{Sa}) \diamond \mathbf{a} = \emptyset$$

(T_{S246})

$$\forall (\forall \mathbf{Sa}) * \mathbf{a} = \emptyset$$

(T_{R156})

Proof of (T_{R156})

$$\forall (\forall \mathbf{Sa}) * \mathbf{a} = \emptyset$$

(T_{R155})

$$\forall (\forall \mathbf{Sa}) * \mathbf{a} = \emptyset$$

Resymbolising

$$\forall (\forall \mathbf{Sa}) \diamond \mathbf{a} = \emptyset$$

(T_{R282})

Proof of (T_{R282})

$$\forall (\forall \mathbf{Sa}) * \mathbf{a} = \emptyset$$

(T_{R156})

$$\forall (\forall \mathbf{Sa}) \diamond \mathbf{a} = \emptyset$$

(T_{S246})

$$(\mathbf{b} \subseteq \sim \mathbf{Qc}) = (\mathbf{c} \subseteq \sim \forall \mathbf{Qb})$$

(T_{R090})

$$(\mathbf{a} \subseteq \sim \mathbf{Sc}) = (\mathbf{c} \subseteq \sim \forall \mathbf{Sa})$$

(T_{R091})

Example Proof of (T_{R090})

$$\text{Let } \mathbf{bQc} = \emptyset$$

(Eqn1)

$$\forall \mathbf{Q}(\mathbf{bQc}) = \forall \mathbf{Q}(\mathbf{bQc})$$

($\mathbf{D}_{M00}$) (Definition of =) (Eqn2)

$$\forall \mathbf{Q}(\mathbf{bQc}) = \forall \mathbf{Q} \emptyset$$

($\mathbf{A}_{MX19}$) (Substitution of Eqn1 into Eqn2)

$$\forall \mathbf{Q}(\mathbf{bQc}) = \emptyset$$

(T_{R069}) (Null Object Theorem)

$$\forall \mathbf{Qb} \forall \mathbf{QQc} = \emptyset$$

(T_{R012c}) (Distribution of $\forall \mathbf{Q}*$ over $\cdot$) (Eqn3)

$$\mathbf{c} \subseteq \forall \mathbf{QQc}$$

(T_{R012n}) (Sub-Relationship Non-Closure Theorem)

$$\mathbf{c} \forall \mathbf{Qb} \subseteq \forall \mathbf{Qb} \forall \mathbf{QQc}$$

(T_{M029}) (Deduction) (Eqn4)

$$\mathbf{c} \forall \mathbf{Qb} \subseteq \emptyset$$

($\mathbf{A}_{MX19}$) (Substitution of Eqn3 into Eqn4)

$$\mathbf{c} \forall \mathbf{Qb} = \emptyset$$

(T_{M052}) (Subsets of the Null Set)

$$\text{Thus } (\mathbf{bQc} = \emptyset) \subseteq (\mathbf{c} \forall \mathbf{Qb} = \emptyset)$$

($\mathbf{A}_{MX02}$) (Conditional Introduction)

$$(b \subseteq \sim Qc) \subseteq (c \subseteq \sim^v Qb)$$

Let $c \subseteq \sim^v Qb$

$$c \subseteq \sim^v Qb$$

$$Qc \subseteq Q^v Qb$$

$$Qc \subseteq b$$

$$b \subseteq \sim Qc$$

Thus $(c \subseteq \sim^v Qb) \subseteq (b \subseteq \sim Qc)$

$$(b \subseteq \sim Qc) = (c \subseteq \sim^v Qb)$$

(T_{M735})($\subseteq$ Existential Form)(Eqn5)

(T_{R037})(Sub-Relationship Negative Theorem)

(T_{R027})(Sub-Relationship Deduction)

(T_{R012})(Sub-Relationship Closure Theorem)

(T_{M012})(Transposition)

(A_{MX02})(Conditional Introduction)(Eqn 6)

(T_{M430})(Biconditional Introduction Eqns5&6)

$$(b \not\subseteq \sim Qc) = (c \not\subseteq \sim^v Qb)$$

(T_{R092})

$$(a \not\subseteq \sim Sc) = (c \not\subseteq \sim^v Sa)$$

(T_{R093})

Example Proof of (T_{R092})

$$(b \subseteq \sim Qc) = (c \subseteq \sim^v Qb)$$

(T_{R090})

$$(b \not\subseteq \sim Qc) = (c \not\subseteq \sim^v Qb)$$

(T_{M008})(Inverse of an Equality)

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b) = (\sim^v Sa^1.c.^v Qb \neq \emptyset))$$

(T_{S087})

Proof of (T_{S087})

$$(a^1 \subseteq c*b) = (a^1 \subseteq S(c.^v Qb))$$

(T_{R598})(Eqn1)

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq S(c.^v Qb)) = (a^1 \not\subseteq \sim S(c.^v Qb)))$$

(T_{M090})(Inverse Unit Subset Rule)(Eqn2)

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b) = (a^1 \not\subseteq \sim S(c.^v Qb)))$$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b) = \sim(a^1 \subseteq \sim S(c.^v Qb)))$$

(T_{M259})(Definition of $\not\subseteq$)

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b) = \sim(c.^v Qb \subseteq \sim^v Sa^1))$$

(T_{R093})

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b) = \sim(\sim^v Sa^1.c.^v Qb = \emptyset))$$

(T_{M735})($\subseteq$ Existential Form)

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b) = (\sim^v Sa^1.c.^v Qb \neq \emptyset))$$

(D_{M14})(Definition of $\neq$)

$$(Qc \subseteq b) = (c \subseteq \sim^v Qb)$$

(T_{S240})

$$(Sc \subseteq a) = (c \subseteq \sim^v Sa)$$

(T_{S241})

Example Proof of (T_{S240})

$$(b \subseteq \sim Qc) = (c \subseteq \sim^v Qb)$$

(T_{R090})

$$(b \subseteq \sim Qc) = (c \subseteq \sim^v Qb)$$

(T_{R037})(Sub-Relationship Negative Theorem)

$$(Qc \subseteq b) = (c \subseteq \sim^v Qb)$$

(T_{M012})(Transposition)

$$(Qc \subseteq b) = (c \subseteq \sim^v Qb)$$

Resymbolising

$$(Qc \not\subseteq b) = (c \not\subseteq \sim^v Qb)$$

(T_{S306})

$$(Sc \not\subseteq a) = (c \not\subseteq \sim^v Sa)$$

(T_{S307})

Example Proof of (T_{S306})

$$(Qc \subseteq b) = (c \subseteq \sim^v Qb)$$

(T_{S240})

$$(Qc \not\subseteq b) = (c \not\subseteq \sim^v Qb)$$

(T_{M008})(Inverse of an Equality)

$$(Qc \subseteq \underline{b}) = (c*b = \emptyset)$$

(T_{R082})

$$(Sc \subseteq \underline{a}) = (\sim^v c*a = \emptyset)$$

(T_{R087})

Example Proof of (T_{R082})

$$(\sim^v Qb \subseteq c) = (c*b = \emptyset)$$

(T_{R079})

$$(c \subseteq \sim^v Qb) = (c*b = \emptyset)$$

(T_{M012})(Transposition)(Eqn1)

$$(b \subseteq \sim Qc) = (c \subseteq \sim^v Qb)$$

(T_{R090})(Eqn2)

$$(b \subseteq \sim Qc) = (c*b = \emptyset)$$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

$$(c \neq \emptyset) = (c*b \neq \emptyset)$$

(T_{R171})

Proof of (T_{R171})

$(Qc \subseteq b) = (c * b = \emptyset)$	(T_{R082})
$(Qc \not\subseteq b) = (c * b \neq \emptyset)$	$(T_{M008})(\text{Inverse of an Equality})(\text{Eqn1})$
Let $b = E$	(Eqn2)
$(Qc \not\subseteq \emptyset) = (c * E \neq \emptyset)$	$(A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$
$\sim(Qc \subset \emptyset) \sim(Qc = \emptyset) = (c * E \neq \emptyset)$	(T_{M198})
$\sim \emptyset \sim(Qc = \emptyset) = (c * E \neq \emptyset)$	$(T_{M053})(\text{Proper Subset of the Null Set})$
$E \sim(Qc = \emptyset) = (c * E \neq \emptyset)$	$(T_{M054})(\text{Negative of the Null Set})$
$\sim(Qc = \emptyset) = (c * E \neq \emptyset)$	$(A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn3})$
$(c = \emptyset) = (Qc = \emptyset)$	$(T_{R051})(\text{Eqn4})$
$\sim(c = \emptyset) = (c * E \neq \emptyset)$	$(A_{MX19})(\text{Substitution of Eqn4 into Eqn3})$
$(c \neq \emptyset) = (c * E \neq \emptyset)$	$(T_{M260})(\text{Definition of } \neq)$

$$(c = \emptyset) = (c * E = \emptyset) \quad (T_{S305})$$

Proof of (T_{S305})

$(c \neq \emptyset) = (c * E \neq \emptyset)$	(T_{R171})
$(c = \emptyset) = (c * E = \emptyset)$	$(T_{M008})(\text{Inverse of an Equality})$

$$(a \neq \Omega) = (\forall a \neq \Omega) \quad (T_{T820})$$

Proof of (T_{T820})

Let $a \neq \Omega$	
$\forall a \neq \Omega$	$(A_{S78})(\text{Relationship Symmetry Axiom})$
Thus $(a \neq \Omega) = (\forall a \neq \Omega)$	$(A_{MX02})(\text{Conditional Introduction})$ since this proof is reversible

Alternative Proof of (T_{T820})

$a = \Omega$	
$a = \emptyset^1, E^1, \sim \emptyset^1, \sim E^1$	$(D_{M16})(\text{Definition of } \Omega)$
$(a = \emptyset^1) + (a = E^1) + (a = \sim \emptyset^1) + (a = \sim E^1)$	$(D_{M15})(\text{Definition of a Variable Set})$
$(\forall a = \forall \emptyset^1) + (\forall a = \forall E^1) + (\forall a = \forall \sim \emptyset^1) + (\forall a = \forall \sim E^1)$	$(T_{R022})(\text{Sub-Relationship Deduction})$
$(\forall a = \emptyset^1) + (\forall a = E^1) + (\forall a = \sim \emptyset^1) + (\forall a = \sim E^1)$	$(T_{T795}), (T_{T794}), (T_{T797}), (T_{T796})$
$\forall a = \emptyset^1, E^1, \sim \emptyset^1, \sim E^1$	$(D_{M15})(\text{Definition of a Variable Set})$
$\forall a = \Omega$	$(D_{M16})(\text{Definition of } \Omega)$
Thus $(a = \Omega) = (\forall a = \Omega)$	$(A_{MX02})(\text{Conditional Introduction})$
	since this proof is reversible
$(a \neq \Omega) = (\forall a \neq \Omega)$	$(T_{M008})(\text{Inverse of an Equality})$

$$\forall a + \forall a = E \quad (T_{S293})$$

Proof of (T_{S293})

$\forall a = \sim \forall a$	(T_{R035})
$(\forall a = \sim \forall a) = E$	$(A_{MX16})(\text{Order Axiom})(\text{Eqn1})$
$(\forall a \subseteq \sim \forall a)(\sim \forall a \subseteq \forall a) = (\forall a = \sim \forall a)$	$(T_{M430})(\text{Biconditional Introduction})$
$(\forall a = \sim \forall a) \subseteq (\forall a \subseteq \sim \forall a)$	$(T_{M073b})(\text{Single Premise Syllogism})(\text{Eqn2})$
$E \subseteq (\forall a \subseteq \sim \forall a)$	$(A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$
$E = (\forall a \subseteq \sim \forall a)$	$(T_{M275})(\text{Supersets of the Universal Set})$
$\sim \forall a \subseteq \forall a$	$(A_{MX16})(\text{Order Axiom})$
$E = \forall a + \forall a$	$(T_{M064})(\subseteq \text{ Universal Form})$

$$\sim \forall a. \sim \forall a = \emptyset \quad (T_{S294})$$

Proof of (T_{S294})

$\forall a = \sim \forall a$	(T_{R035})
$(\forall a = \sim \forall a) = E$	$(A_{MX16})(\text{Order Axiom})(\text{Eqn1})$
$(\forall a \subseteq \sim \forall a)(\sim \forall a \subseteq \forall a) = (\forall a = \sim \forall a)$	$(T_{M430})(\text{Biconditional Introduction})$
$(\forall a = \sim \forall a) \subseteq (\forall a \subseteq \sim \forall a)$	$(T_{M073b})(\text{Single Premise Syllogism})(\text{Eqn2})$
$E \subseteq (\forall a \subseteq \sim \forall a)$	$(A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$

$$\begin{array}{ll}
E = (\forall a \subseteq \sim^v a) & (T_{M275})(\text{Supersets of the Universal Set}) \\
\sim^v a \subseteq \sim^v a & (A_{MX16})(\text{Order Axiom}) \\
\sim^v a. \sim^v a = \emptyset & (T_{M735})(\subseteq \text{Existential Form})
\end{array}$$

$$(a \neq \emptyset) = (\forall c \diamond a = \sim Q(\sim(\forall c) \forall Sa)) \quad (T_{R261})$$

Proof of (T_{R261})

$$\begin{array}{ll}
(b \neq \emptyset) = ((c * b) = \sim(c \diamond b)) & (A_{R10})(\text{Axiom of Complementary Relationship Sets}) \\
(a \neq \emptyset) = ((c * a) = \sim(c \diamond a)) & \text{Resymbolising} \\
(a \neq \emptyset) = (\sim(c * a) = (c \diamond a)) & (T_{M008})(\text{Inverse of an Equality}) \\
(a \neq \emptyset) = (\sim(\sim(\forall c) * a) = (\forall c \diamond a)) & \text{Resymbolising (Eqn1)} \\
\forall c * a = Q(c \forall Sa) & (T_{R020})(\text{Function Theorem}) \\
\forall c * a = Q(c \forall Sa) & \text{Resymbolising} \\
\sim(\forall c * a) = Q(c \forall Sa) & (T_{R035})(\text{Eqn2}) \\
(a \neq \emptyset) = (\forall c \diamond a = \sim Q(\sim(\forall c) \forall Sa)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
\end{array}$$

$$c \subseteq \forall Q S^v c \quad (T_{R218}) \text{ Reversibility of Matching Entities}$$

Proof of (T_{R218})

$$\begin{array}{ll}
S^v c = Qc & (T_{R016}) \\
\forall Q S^v c = \forall Q Qc & (T_{R522})(\text{Relational Deduction})(\text{Eqn1}) \\
c \subseteq \forall Q Qc & (T_{R012n})(\text{Sub-Relationship Non-Closure Theorem})(\text{Eqn2}) \\
c \subseteq \forall Q S^v c & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})
\end{array}$$

$$((a \subseteq S c) = E) \subseteq ((b = \forall c * a) \subseteq (a \subseteq c * b)) \quad (T_{S524}) \text{ Active/Passive Theorem}$$

Proof of (T_{S524})

$$\begin{array}{ll}
(b \subseteq Qc) \subseteq (b \subseteq \forall c * c * b) & (T_{R527}) \\
(a \subseteq Q^v c) \subseteq (a \subseteq c * \forall c * a) & \text{Resymbolising (Eqn1)} \\
\text{Let } b = \forall c * a & (\text{Eqn2}) \\
(a \subseteq Q^v c) \subseteq (a \subseteq c * b) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
(a \subseteq S c) \subseteq (a \subseteq c * b) & (T_{R015})(\text{Reversibility of Matching Entities}) \\
\text{Thus } ((b = \forall c * a) = E) \subseteq ((a \subseteq S c) \subseteq (a \subseteq c * b)) & (A_{MX02})(\text{Conditional Introduction}) \\
(b = \forall c * a)(a \subseteq S c) \subseteq (a \subseteq c * b) & (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\
((a \subseteq S c) = E) \subseteq ((b = \forall c * a) \subseteq (a \subseteq c * b)) & (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets}
\end{array}$$

$$(c^1 \subseteq R_0)(a \subseteq S c) \subseteq (a = c * b)(b = \forall c * a) \quad (T_{S534})$$

Proof of (T_{S534})

$$\begin{array}{ll}
((\forall c)^1 \subseteq R_0)(b \subseteq Qc) \subseteq (a = c * b)(b = \forall c * a) & (T_{S533})(\text{Eqn1}) \\
\text{Let } \forall c = c & (\text{Eqn2}) \\
\text{Let } a = b & (\text{Eqn3}) \\
(c^1 \subseteq R_0)(a \subseteq Q^v c) \subseteq (b = \forall c * a)(a = c * b) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\
(c^1 \subseteq R_0)(a \subseteq S c) \subseteq (b = \forall c * a)(a = c * b) & (T_{R015})(\text{Reversibility of Matching Entities}) \\
(c^1 \subseteq R_0)(a \subseteq S c) \subseteq (a = c * b)(b = \forall c * a) & (T_{M404})(\text{Commutative .})
\end{array}$$

$$((c^1 \subseteq R_0)(a \subseteq S c) = E) \subseteq ((b = \forall c * a) = (a = c * b)) \quad (T_{S522})$$

Proof of (T_{S522})

$$\begin{array}{ll}
(((\forall c)^1 \subseteq R_0)(b \subseteq Qc) = E) \subseteq ((b = \forall c * a) = (a = c * b)) & (T_{S528}) \\
(((\forall c)^1 \subseteq R_0)(b \subseteq Qc) = E) \subseteq ((a = c * b) = (b = \forall c * a)) & (T_{M197})(\text{Commutativity of } =)(\text{Eqn1}) \\
\text{Let } c = \forall c & (\text{Eqn2}) \\
((c^1 \subseteq R_0)(b \subseteq Q^v c) = E) \subseteq ((a = \forall c * b) = (b = c * a)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
((c^1 \subseteq R_0)(b \subseteq S c) = E) \subseteq ((a = \forall c * b) = (b = c * a)) & (T_{R015})(\text{Reversibility of Matching Entities}) \\
((c^1 \subseteq R_0)(a \subseteq S c) = E) \subseteq ((b = \forall c * a) = (a = c * b)) & \text{Resymbolising}
\end{array}$$

$$(c^1 \subseteq R_0)(a \subseteq S c) \subseteq (a = c * \forall c * a) \quad (T_{S518})$$

Proof of (T₅₅₁₈)

$$((\forall c)^1 \subseteq R_o)(a \subseteq Qc) \subseteq (a = {}^v c * c * a)$$

Let ${}^v c = c$

$$(c^1 \subseteq R_o)(a \subseteq Q^v c) \subseteq (a = c * {}^v c * a)$$

$$(c^1 \subseteq R_o)(a \subseteq Sc) \subseteq (a = c * {}^v c * a)$$

(T₅₅₂₉) Resymbolised (Eqn1)

(Eqn2)

(A_{MX19})(Substitution of Eqn2 into Eqn1)

(T_{R015})(Reversibility of Matching Entities)

$$((\forall c)^1 \subseteq R_o)(a \subseteq Qc)(b \subseteq Qc)(o^1 \subseteq R_b):(a \langle o \rangle b) = (c * a \langle o \rangle c * b) \quad (T_{R535})$$

Proof of (T_{R535})

$$(a \subseteq Qc)(b \subseteq Qc)(o^1 \subseteq R_b):(a \langle o \rangle b) \subseteq (c * a \langle o \rangle c * b) \quad (A_{R38})(2^{nd} \text{ Relational Deduction Axiom})(Eqn1)$$

Let $a = c * a$

(Eqn2)

Let $b = c * b$

(Eqn3)

Let $c = {}^v c$

(Eqn4)

$$(c * a \subseteq Q^v c)(c * b \subseteq Qc)(o^1 \subseteq R_b):(c * a \langle o \rangle c * b) \subseteq ({}^v c * c * a \langle o \rangle {}^v c * c * b)$$

(A_{MX19})(Substitution of Eqns2, 3 & 4 into Eqn1)(Eqn5)

$$((\forall c)^1 \subseteq R_o)(b \subseteq Qc):(b = {}^v c * c * b)$$

(T₅₅₂₉)(Eqn6)

$$((\forall c)^1 \subseteq R_o)(a \subseteq Qc):(a = {}^v c * c * a)$$

(T₅₅₂₉)(Eqn7)

$$((\forall c)^1 \subseteq R_o)(b \subseteq Qc)(a \subseteq Qc)(c * a \subseteq Q^v c)(c * b \subseteq Q^v c)(o^1 \subseteq R_b):(c * a \langle o \rangle c * b) \subseteq (a \langle o \rangle b)$$

(A_{MX19})(Substitution of Eqns6&7 into Eqn5)

$$((\forall c)^1 \subseteq R_o)(b \subseteq Qc)(a \subseteq Qc)(c * a \subseteq Sc)(c * b \subseteq Sc)(o^1 \subseteq R_b):(c * a \langle o \rangle c * b) = (a \langle o \rangle b)$$

(T_{R015})

$$((\forall c)^1 \subseteq R_o)(b \subseteq Qc)(a \subseteq Qc)(S(c.{}^v Qa) \subseteq Sc)(S(c.{}^v Qb) \subseteq Sc)(o^1 \subseteq R_b):(c * a \langle o \rangle c * b) = (a \langle o \rangle b)$$

(A_{R27})(Function Axiom)(Eqn8)

$$c.{}^v Qa \subseteq c$$

(T_{M083})(Conjunction Elimination)

$$S(c.{}^v Qa) \subseteq Sc$$

(T_{R196})(Relational Deduction)

$$(S(c.{}^v Qa) \subseteq Sc) = E$$

(A_{MX16})(Order Axiom)(Eqn9)

$$(S(c.{}^v Qb) \subseteq Sc) = E$$

(Resymbolising)(Eqn10)

$$((\forall c)^1 \subseteq R_o)(b \subseteq Qc)(a \subseteq Qc).E.E. (o^1 \subseteq R_b):(c * a \langle o \rangle c * b) = (a \langle o \rangle b)$$

(A_{MX19})(Substitution of Eqns 9&10 into Eqn8)

$$((\forall c)^1 \subseteq R_o)(b \subseteq Qc)(a \subseteq Qc)(o^1 \subseteq R_b):(c * a \langle o \rangle c * b) = (a \langle o \rangle b)$$

(A_{MX11})(Universal Set Conjunction Axiom)

$$(o^1 \subseteq R_b):(a \langle o \rangle b) = ({}^v S a \langle o \rangle {}^v S b)$$

(T_{R511}) Sub-Relationship Deduction

$$(o^1 \subseteq R_b):(a \langle o \rangle b) = ({}^v Q a \langle o \rangle {}^v Q b)$$

(T_{R512}) Sub-Relationship Deduction

$$(X = {}^v S, {}^v Q)(o^1 \subseteq R_b):(a \langle o \rangle b) = (X a \langle o \rangle X b)$$

(T_{R515}) Sub-Relationship Deduction

Example Proof of (T_{R511})

$$((\forall c)^1 \subseteq R_o)(a \subseteq Qc)(b \subseteq Qc)(o^1 \subseteq R_b):(a \langle o \rangle b) = (c * a \langle o \rangle c * b)$$

(T_{R535})(Eqn1)

Let $c = {}^v S$

(Eqn2)

$$(S^1 \subseteq R_o)(a \subseteq Q^v S)(b \subseteq Q^v S)(o^1 \subseteq R_b):(a \langle o \rangle b) = ({}^v S a \langle o \rangle {}^v S b)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)

$$(X = (V, S, Q))(X^1 \subseteq R_o)$$

(A_{R04})(One-to-One Sub-relationship Axiom)

$$S^1 \subseteq R_o$$

Extraction

$$(S^1 \subseteq R_o) = E$$

(A_{MX16})(Order Axiom)(Eqn4)

$$E(a \subseteq Q^v S)(b \subseteq Q^v S)(o^1 \subseteq R_b):(a \langle o \rangle b) = ({}^v S a \langle o \rangle {}^v S b)$$

(A_{MX19})(Substitution of Eqn4 into Eqn3)

$$(a \subseteq Q^v S)(b \subseteq Q^v S)(o^1 \subseteq R_b):(a \langle o \rangle b) = ({}^v S a \langle o \rangle {}^v S b)$$

(A_{MX11})(Universal Set Conjunction Axiom)(Eqn5)

$$(X = (S, Q))(Y = (V, S, Q, {}^v S, {}^v Q))(X * Y = E)$$

(A_{R34})(Universal Sub-relationship Axiom)

$$Q^v S = E$$

Extraction (Eqn6)

$$(a \subseteq E)(b \subseteq E)(o^1 \subseteq R_b):(a \langle o \rangle b) = ({}^v S a \langle o \rangle {}^v S b)$$

(A_{MX19})(Substitution of Eqn6 into Eqn5)

$$E.E. (o^1 \subseteq R_b):(a \langle o \rangle b) = ({}^v S a \langle o \rangle {}^v S b)$$

(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)

$$(o^1 \subseteq R_b):(a \langle o \rangle b) = ({}^v S a \langle o \rangle {}^v S b)$$

(A_{MX11})(Universal Set Conjunction Axiom)

The following are special cases of (T_{R515}) Sub-Relationship Deduction:

$$(a \subset b) = ({}^v S a \subset {}^v S b)$$

(T_{R023a}) Sub-Relationship Deduction

$$(a \supset b) = (\forall Sa \supset \forall Sb) \quad (T_{R023c}) \text{ Sub-Relationship Deduction}$$

$$(a \parallel b) = (\forall Sa \parallel \forall Sb) \quad (T_{R023b}) \text{ Sub-Relationship Deduction}$$

$$(a \subset b) = (\forall Qa \subset \forall Qb) \quad (T_{R024a}) \text{ Sub-Relationship Deduction}$$

$$(a \supset b) = (\forall Qa \supset \forall Qb) \quad (T_{R024c}) \text{ Sub-Relationship Deduction}$$

$$(a \parallel b) = (\forall Qa \parallel \forall Qb) \quad (T_{R024b}) \text{ Sub-Relationship Deduction}$$

Example Proof of (T_{R023a})

$$\begin{aligned} (X = \forall S, \forall Q)(o^1 \subseteq R_B): (a \langle o \rangle b) &= (Xa \langle o \rangle Xb) & (T_{R515})(\text{Sub-Relationship Deduction})(\text{Eqn1}) \\ \text{Let } X &= \forall S & (\text{Eqn2}) \\ \text{Let } o &= \{\subset\} & (\text{Eqn3}) \\ (a \subset b) &= (\forall Sa \subset \forall Sb) & (A_{MX19})(\text{Substitution of Eqns2 \& 3 into Eqn1}) \end{aligned}$$

$$(o^1 \subseteq R_B): (\forall a \langle o \rangle \forall b) = (\forall Sa \langle o \rangle \forall Sb) \quad (T_{R188}) \text{ Sub-Relationship Deduction}$$

$$(o^1 \subseteq R_B): (\forall a \langle o \rangle \forall b) = (\forall Qa \langle o \rangle \forall Qb) \quad (T_{R189}) \text{ Sub-Relationship Deduction}$$

Example Proof of (T_{R188})

$$\begin{aligned} (o^1 \subseteq R_B): (a \langle o \rangle b) &= (\forall a \langle o \rangle \forall b) & (T_{R510})(\text{Relationship Opposite of a Proposition})(\text{Eqn1}) \\ (o^1 \subseteq R_B): (a \langle o \rangle b) &= (\forall Sa \langle o \rangle \forall Sb) & (T_{R511})(\text{Sub-Relationship Deduction})(\text{Eqn2}) \\ (o^1 \subseteq R_B): (\forall a \langle o \rangle \forall b) &= (\forall Sa \langle o \rangle \forall Sb) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(o^1 \subseteq R_B): \sim(\forall a \langle o \rangle \forall b) = \sim(\forall Sa \langle o \rangle \forall Sb) \quad (T_{R191}) \text{ Sub-Relationship Deduction}$$

$$(o^1 \subseteq R_B): \sim(\forall a \langle o \rangle \forall b) = \sim(\forall Qa \langle o \rangle \forall Qb) \quad (T_{R192}) \text{ Sub-Relationship Deduction}$$

Example Proof of (T_{R191})

$$\begin{aligned} (o^1 \subseteq R_B): (\forall a \langle o \rangle \forall b) &= (\forall Sa \langle o \rangle \forall Sb) & (T_{R188})(\text{Sub-Relationship Deduction}) \\ (o^1 \subseteq R_B): \sim(\forall a \langle o \rangle \forall b) &= \sim(\forall Sa \langle o \rangle \forall Sb) & (T_{M008})(\text{Inverse of an Equality}) \end{aligned}$$

$$(o^1 \subseteq R_B): \sim(a \langle o \rangle b) = \sim(\forall Sa \langle o \rangle \forall Sb) \quad (T_{R030}) \text{ Sub-Relationship Deduction}$$

$$(o^1 \subseteq R_B): \sim(a \langle o \rangle b) = \sim(\forall Qa \langle o \rangle \forall Qb) \quad (T_{R031}) \text{ Sub-Relationship Deduction}$$

$$(X = \forall, \forall S, \forall Q)(o^1 \subseteq R_B): \sim(a \langle o \rangle b) = \sim(Xa \langle o \rangle Xb) \quad (T_{R028}) \text{ Sub-Relationship Deduction}$$

Example Proof of (T_{R028})

$$\begin{aligned} (X = \forall S, \forall Q)(o^1 \subseteq R_B): (a \langle o \rangle b) &= (Xa \langle o \rangle Xb) & (T_{R515})(\text{Sub-Relationship Deduction}) \\ (X = \forall S, \forall Q)(o^1 \subseteq R_B): \sim(a \langle o \rangle b) &= \sim(Xa \langle o \rangle Xb) & (T_{M008})(\text{Inverse of an Equality}) \end{aligned}$$

$$(o^1 \subseteq R_B): (\forall Sa \langle o \rangle \forall Sb) = (\forall Qa \langle o \rangle \forall Qb) \quad (T_{R187}) \text{ Sub-Relationship Deduction}$$

$$(o^1 \subseteq R_B): \sim(\forall Sa \langle o \rangle \forall Sb) = \sim(\forall Qa \langle o \rangle \forall Qb) \quad (T_{R190}) \text{ Sub-Relationship Deduction}$$

Proof of (T_{R187}) and (T_{R190})

$$\begin{aligned} (X = \forall S, \forall Q)(o^1 \subseteq R_B): (a \langle o \rangle b) &= (Xa \langle o \rangle Xb) & (T_{R515})(\text{Sub-Relationship Deduction}) \\ (o^1 \subseteq R_B): ((a \langle o \rangle b) &= (\forall Sa \langle o \rangle \forall Sb)) & \text{Extraction (Eqn1)} \\ (o^1 \subseteq R_B): ((a \langle o \rangle b) &= (\forall Qa \langle o \rangle \forall Qb)) & \text{Extraction (Eqn2)} \\ (o^1 \subseteq R_B): ((\forall Sa \langle o \rangle \forall Sb) &= (\forall Qa \langle o \rangle \forall Qb)) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\ (o^1 \subseteq R_B): \sim(\forall Sa \langle o \rangle \forall Sb) &= \sim(\forall Qa \langle o \rangle \forall Qb) & (T_{M008})(\text{Inverse of an Equality}) \end{aligned}$$

$$(\forall Sa \subseteq b) = (Sb \subseteq a) \quad (T_{S118})$$

$$(\forall Sa \subseteq b) = (a \subseteq \sim Sb) \quad (T_{S119})$$

$$(\forall Sa \not\subseteq b) = (a \not\subseteq \sim Sb) \quad (T_{S120})$$

Proofs of (T_{S118}), (T_{S119}) & (T_{S120})

$$\begin{aligned} \text{Let } \forall Sa &\subseteq b \\ b &\subseteq \sim \forall Sa & (T_{M012})(\text{Transposition}) \\ Sb &\subseteq S \sim \forall Sa & (T_{R026d})(\text{Sub-Relationship Deduction})(\text{Eqn1}) \\ \sim S \sim \forall Sa &= a & (T_{S117}) \end{aligned}$$

$S \sim^V Sa = a$	(T_{M008})(Inverse of an Equality)(Eqn2)
$Sb \subseteq a$	($AMX19$)(Substitution of Eqn2 into Eqn1)
Thus $(^V Sa \subseteq b) \subseteq (Sb \subseteq a)$	($AMX02$)(Conditional Introduction)(Eqn3)
Let $Sb \subseteq a$	
$^V SSb \subseteq ^V Sa$	(T_{R188})(Sub-Relationship Deduction for $\subseteq$)
$^V SSb \subseteq \sim^V Sa$	(T_{R036})(Sub-Relationship Negative Theorem)(Eqn4)
$b \subseteq ^V SSb$	(T_{R012o})(Eqn5)
$b \subseteq \sim^V Sa$	(T_{M139})(Hypothetical Syllogism Eqns4&5)
$^V Sa \subseteq b$	(T_{M012})(Transposition)
Thus $(Sb \subseteq a) \subseteq (^V Sa \subseteq b)$	($AMX02$)(Conditional Introduction)(Eqn6)
$(^V Sa \subseteq b) = (Sb \subseteq a)$	(T_{M430})(Biconditional Introduction Eqns3&6)
$(^V Sa \subseteq b) = (a \subseteq \sim Sb)$	(T_{M012})(Transposition)
$(^V Sa \not\subseteq b) = (a \not\subseteq \sim Sb)$	(T_{M008})(Inverse of an Equality)

$$(Sc \subseteq a)(Qc \subseteq b) = (c \subseteq ^V Sa^V Qb) \quad (TR500)$$

Proof of ($TR500$)

Let $Sc \subseteq a$	
$^V SSb \subseteq ^V Sa$	(T_{R023d})(Sub-relationship Deduction)(Eqn1)
$c \subseteq ^V SSb$	(T_{R012o})(Sub-relationship Non-closure Theorem)(Eqn2)
$c \subseteq ^V Sa$	(T_{M139})(Hypothetical Syllogism Eqns1 & 2)(Eqn3)
Let $Qc \subseteq b$	
$^V QQc \subseteq ^V Qb$	(T_{R024d})(Sub-relationship Deduction)(Eqn4)
$c \subseteq ^V QQc$	(T_{R012n})(Sub-relationship Non-closure Theorem)(Eqn5)
$c \subseteq ^V Qb$	(T_{M139})(Hypothetical Syllogism Eqns4 & 5)(Eqn6)
$c \subseteq ^V Sa^V Qb$	(T_{M132})(Composition Eqns3&6)
Thus $(Sc \subseteq a)(Qc \subseteq b) \subseteq (c \subseteq ^V Sa^V Qb)$	($AMX02$)(Conditional Introduction)(Eqn7)
Let $c \subseteq ^V Sa^V Qb$	
$c \subseteq ^V Sa$	(T_{M073})(Single Premise Syllogism)
$Sc \subseteq S^V Sa$	(T_{R196})(Relational Deduction)
$Sc \subseteq a$	(T_{R012m})(Sub-Relationship Closure Theorem)
Thus $(c \subseteq ^V Sa^V Qb) \subseteq (Sc \subseteq a)$	($AMX02$)(Conditional Introduction)(Eqn8)
Let $c \subseteq ^V Sa^V Qb$	
$c \subseteq ^V Qb$	(T_{M073})(Single Premise Syllogism)
$Qc \subseteq Q^V Qb$	(T_{R196})(Relational Deduction)
$Qc \subseteq b$	(T_{R012l})(Sub-relationship Closure Theorem)
Thus $(c \subseteq ^V Sa^V Qb) \subseteq (Qc \subseteq b)$	($AMX02$)(Conditional Introduction)(Eqn9)
$(c \subseteq ^V Sa^V Qb) \subseteq (Sc \subseteq a)(Qc \subseteq b)$	(T_{M132})(Composition Eqns8&9)(Eqn10)
$(Sc \subseteq a)(Qc \subseteq b) = (c \subseteq ^V Sa^V Qb)$	(T_{M430})(Biconditional Introduction Eqns7&10)

$$(Sc = a)(Qc = b) \subseteq (c \subseteq ^V Sa^V Qb) \quad (TR507)$$

Proof of ($TR507$)

$(Sc \subseteq a)(Qc \subseteq b) = (c \subseteq ^V Sa^V Qb)$	($TR500$)
$((Sc = a) + (Sc \subset a))((Qc = b) + (Qc \subset b)) = (c \subseteq ^V Sa^V Qb)$	(T_{M266})(Definition of $\subseteq$)
$(Sc = a)(Qc = b) + (Sc = a)(Qc \subset b) + (Sc \subset a)(Qc = b) + (Sc \subset a)(Qc \subset b) = (c \subseteq ^V Sa^V Qb)$	(T_{M407})(Distribution of $\cdot$ over $+$)
$(Sc = a)(Qc = b) \subseteq (c \subseteq ^V Sa^V Qb)$	(T_{M072})(Single Premise Syllogism)

$$c^1 \subseteq ^V SSb^1. ^V QQc^1 \quad (TS444)$$

Proof of ($TS444$)

$c^1 \subseteq ^V SSb^1$	(T_{R012j})(Sub-Relationship Non-Closure Theorem)(Eqn1)
$c^1 \subseteq ^V QQc^1$	(T_{R012j})(Sub-Relationship Non-Closure Theorem)(Eqn2)
$c^1. c^1 \subseteq ^V SSb^1. ^V QQc^1$	(T_{M669})(Combination of Eqns1&2)
$c^1 \subseteq ^V SSb^1. ^V QQc^1$	(T_{M667})(Conjunction Tautology)

$$(a \subseteq Sc)(Qc \subseteq b) \subseteq (a \subseteq c * b) \quad (TR505)$$

Proof of (TR505)

Let $Qc \subseteq b$	
$\forall QQc \subseteq \forall Qb$	(TR024d)(Sub-relationship Deduction)(Eqn1)
$c \subseteq \forall QQc$	(TR012n)(Sub-relationship Non-closure Theorem)(Eqn2)
$c \subseteq \forall Qb$	(TM139)(Hypothetical Syllogism Eqns1&2)
$c = c \forall Qb$	(AMX44)($\subseteq$ Equality Form)(Eqn3)
Let $a \subseteq Sc$	(Eqn4)
$a \subseteq S(c \forall Qb)$	(AMX19)(Substitution of Eqn3 into Eqn4)
$a \subseteq c * b$	(AR27)(Function Axiom)
Thus $(a \subseteq Sc)(Qc \subseteq b) \subseteq (a \subseteq c * b)$	(AMX02)(Conditional Introduction)

$$(a = Sc)(Qc = b) \subseteq (a = c * b)$$

(TR506)

Proof of (TR506)

Let $Qc \subseteq b$	
$\forall QQc \subseteq \forall Qb$	(TR024d)(Sub-relationship Deduction)(Eqn1)
$c \subseteq \forall QQc$	(TR012n)(Sub-relationship Non-closure Theorem)(Eqn2)
$c \subseteq \forall Qb$	(TM139)(Hypothetical Syllogism Eqns1&2)
$c = c \forall Qb$	(AMX44)($\subseteq$ Equality Form)(Eqn3)
Let $a = Sc$	(Eqn4)
$a = S(c \forall Qb)$	(AMX19)(Substitution of Eqn3 into Eqn4)
$a = c * b$	(AR27)(Function Axiom)
Thus $(a = Sc)(Qc \subseteq b) \subseteq (a = c * b)$	(AMX02)(Conditional Introduction)
$(a = Sc)((Qc = b) + (Qc \subset b)) \subseteq (a = c * b)$	(TM266)(Definition of $\subseteq$)
$(a = Sc)(Qc = b) + (Qc \subset b)(a = Sc) \subseteq (a = c * b)$	(TM407)(Distribution of . over +)
$(a = Sc)(Qc = b) \subseteq (a = c * b)$	(TM072)(Single Premise Syllogism)

4.7 Relational Syllogisms

A. Relational Syllogisms based on (TR196) or (TR197).

The only theorems of relationship theory used in the proofs below are (TR196) or (TR197). Apart from this the proofs use membership theory.

$$(a \subseteq c * b)(c \subseteq d) \subseteq (a \subseteq d * b)$$

(TR388) Relational Syllogism

Proof of (TR388)

$(c \subseteq d) \subseteq (c * b \subseteq d * b)$	(TR197)(Relational Deduction)
$(a \subseteq c * b)(c \subseteq d) \subseteq (a \subseteq c * b)(c * b \subseteq d * b)$	(TM029)(Deduction)
$(a \subseteq c * b)(c \subseteq d) \subseteq (a \subseteq d * b)$	(TM073)(Single Premise Syllogism)

$$(a \subseteq c * b)(b \subseteq h) \subseteq (a \subseteq c * h)$$

(TR389) Relational Syllogism

Proof of (TR389)

Let $(a \subseteq c * b).(b \subseteq h)$	
$(b \subseteq h) \subseteq (c * b \subseteq c * h)$	(TR196)(Relational Deduction)
$(a \subseteq c * b)(b \subseteq h) \subseteq (a \subseteq c * b)(c * b \subseteq c * h)$	(TM029)(Deduction)
$(a \subseteq c * b)(b \subseteq h) \subseteq (a \subseteq c * h)$	(TM139)(Hypothetical Syllogism)

$$((a \neq E)(c * b \neq E)(d * b \neq E) = E) \subseteq ((a \subseteq c * b)(d \subseteq c) \subseteq (\sim(d * b) \not\subseteq a))$$

(TR390)

Proof of (TR390)

Let $(a \subseteq c * b)$	
$(a \subseteq c * b) = E$	(AMX16)(Order Axiom)(Eqn1)
Let $(d \subseteq c)$	
$(d \subseteq c) = E$	(AMX16)(Order Axiom)(Eqn2)
$d * b \subseteq c * b$	(TR197)(Relational Deduction)

$$\begin{aligned}
& (d*b \subseteq c*b) = E & (A_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\
& ((a \neq E)(b \neq E)(c \neq E) = E) \subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b)) & (T_{M147})(\text{Second Conditional Syllogism})(\text{Eqn4}) \\
& \text{Let } b = d*b & (\text{Eqn5}) \\
& \text{Let } c = c*b & (\text{Eqn6}) \\
& ((a \neq E)(d*b \neq E)(c*b \neq E) = E) \subseteq ((a \subseteq c*b)(d*b \subseteq c*b) \subseteq (a \not\subseteq d*b)) & (A_{MX19})(\text{Substitution of Eqns5 \& 6 into Eqn4})(\text{Eqn7}) \\
& ((a \neq E)(d*b \neq E)(c*b \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq d*b)) & (A_{MX19})(\text{Substitution of Eqns1 \& 3 into Eqn7}) \\
& ((a \neq E)(d*b \neq E)(c*b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq d*b)) & (T_{M667})(\text{Conjunction Tautology}) \\
& ((a \neq E)(d*b \neq E)(c*b \neq E) = E) \subseteq (E = (a \not\subseteq d*b)) & (T_{M275})(\text{Supersets of the Universal Set}) \\
& ((a \neq E)(d*b \neq E)(c*b \neq E) \subseteq (a \not\subseteq d*b)) = E & (T_{M094})(\text{Probabilistic Rule}) \\
& (a \neq E)(d*b \neq E)(c*b \neq E) \subseteq (a \not\subseteq d*b) & (A_{MX16})(\text{Order Axiom}) \\
& \text{Thus } ((a \subseteq c*b)(d \subseteq c) = E) \subseteq ((a \neq E)(d*b \neq E)(c*b \neq E) \subseteq (a \not\subseteq d*b)) & (A_{MX02})(\text{Conditional Introduction}) \\
& (a \neq E)(d*b \neq E)(c*b \neq E)(a \subseteq c*b)(d \subseteq c) \subseteq (a \not\subseteq d*b) & (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\
& ((a \neq E)(d*b \neq E)(c*b \neq E) = E) \subseteq ((a \subseteq c*b)(d \subseteq c) \subseteq (a \not\subseteq d*b)) & (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\
& ((a \neq E)(d*b \neq E)(c*b \neq E) = E) \subseteq ((a \subseteq c*b)(d \subseteq c) \subseteq (\sim(d*b) \not\subseteq a)) & (T_{M012})(\text{Transposition})
\end{aligned}$$

$$((a \neq E)(c*b \neq E)(c*h \neq E) = E) \subseteq ((a \subseteq c*b)(h \subseteq b) \subseteq (\sim(c*h) \not\subseteq a)) \quad (T_{R391})$$

Proof of (T_{R391})

$$\begin{aligned}
& \text{Let } (a \subseteq c*b) & (A_{MX16})(\text{Order Axiom})(\text{Eqn1}) \\
& (a \subseteq c*b) = E & (A_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\
& \text{Let } (h \subseteq b) & (T_{R196})(\text{Relational Deduction}) \\
& (h \subseteq b) = E & (A_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\
& c*h \subseteq c*b & (A_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\
& (c*h \subseteq c*b) = E & (T_{M147})(\text{Second Conditional Syllogism})(\text{Eqn4}) \\
& ((a \neq E)(b \neq E)(c \neq E) = E) \subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b)) & (Eqn5) \\
& \text{Let } b = c*h & (Eqn6) \\
& \text{Let } c = c*b & (Eqn6) \\
& ((a \neq E)(c*h \neq E)(c*b \neq E) = E) \subseteq ((a \subseteq c*b)(c*h \subseteq c*b) \subseteq (a \not\subseteq c*h)) & (A_{MX19})(\text{Substitution of Eqns5 \& 6 into Eqn4})(\text{Eqn7}) \\
& ((a \neq E)(c*h \neq E)(c*b \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq c*h)) & (A_{MX19})(\text{Substitution of Eqns1 \& 3 into Eqn7}) \\
& ((a \neq E)(c*h \neq E)(c*b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq c*h)) & (T_{M667})(\text{Conjunction Tautology}) \\
& ((a \neq E)(c*h \neq E)(c*b \neq E) = E) \subseteq (E = (a \not\subseteq c*h)) & (T_{M275})(\text{Supersets of the Universal Set}) \\
& ((a \neq E)(c*h \neq E)(c*b \neq E) \subseteq (a \not\subseteq c*h)) = E & (T_{M094})(\text{Probabilistic Rule}) \\
& (a \neq E)(c*h \neq E)(c*b \neq E) \subseteq (a \not\subseteq c*h) & (A_{MX16})(\text{Order Axiom}) \\
& \text{Thus } ((a \subseteq c*b)(h \subseteq b) = E) \subseteq ((a \neq E)(c*h \neq E)(c*b \neq E) \subseteq (a \not\subseteq c*h)) & (A_{MX02})(\text{Conditional Introduction}) \\
& (a \neq E)(c*h \neq E)(c*b \neq E)(a \subseteq c*b)(h \subseteq b) \subseteq (a \not\subseteq c*h) & (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\
& ((a \neq E)(c*h \neq E)(c*b \neq E) = E) \subseteq ((a \subseteq c*b)(h \subseteq b) \subseteq (a \not\subseteq c*h)) & (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\
& ((a \neq E)(c*h \neq E)(c*b \neq E) = E) \subseteq ((a \subseteq c*b)(h \subseteq b) \subseteq (\sim(c*h) \not\subseteq a)) & (T_{M012})(\text{Transposition})
\end{aligned}$$

$$(\sim(c*b) \subseteq a)(c \subseteq d) \subseteq (\sim(d*b) \subseteq a) \quad (T_{R392})$$

Proof of (T_{R392})

$$\begin{aligned}
& \text{Let } c \subseteq d & (T_{R197})(\text{Relational Deduction}) \\
& c*b \subseteq d*b & (T_{M272})(\text{Deductive Rule}) \\
& (a \subseteq c*b) \subseteq (a \subseteq d*b) & (T_{M012})(\text{Transposition}) \\
& (\sim(c*b) \subseteq a) \subseteq (\sim(d*b) \subseteq a) & (A_{MX02})(\text{Conditional Introduction}) \\
& \text{Thus } ((c \subseteq d) = E) \subseteq ((\sim(c*b) \subseteq a) \subseteq (\sim(d*b) \subseteq a))
\end{aligned}$$

$$(\sim(c*b) \subseteq a)(c \subseteq d) \subseteq (\sim(d*b) \subseteq a)$$

(**AMX35**)(Exportation/Importation) for truth sets

$$(\sim(c*b) \subseteq a)(b \subseteq h) \subseteq (\sim(c*h) \subseteq a)$$

(**TR393**)

Proof of (**TR393**)

Let $b \subseteq h$

$$c*b \subseteq c*h$$

(**TR196**)(Relational Deduction)

$$(a \subseteq c*b) \subseteq (a \subseteq c*h)$$

(**TM272**)(Deductive Rule)

$$(\sim(c*b) \subseteq a) \subseteq (\sim(c*h) \subseteq a)$$

(**TM012**)(Transposition)

$$\text{Thus } ((b \subseteq h) = E) \subseteq ((\sim(c*b) \subseteq a) \subseteq (\sim(c*h) \subseteq a))$$

(**AMX02**)(Conditional Introduction)

$$(\sim(c*b) \subseteq a)(b \subseteq h) \subseteq (\sim(c*h) \subseteq a)$$

(**AMX35**)(Exportation/Importation) for truth sets

$$((a \neq \emptyset)(c*b \neq E)(d*b \neq E) = E) \subseteq ((\sim(c*b) \subseteq a)(d \subseteq c) \subseteq (a \not\subseteq d*b)) \quad (\textbf{TR394})$$

Proof of (**TR394**)

Let $(\sim(c*b) \subseteq a)$

$$a \subseteq c*b$$

(**TM012**)(Transposition)

$$(a \subseteq c*b) = E$$

(**AMX16**)(Order Axiom)(Eqn1)

Let $d \subseteq c$

$$(d \subseteq c) = E$$

(**AMX16**)(Order Axiom)(Eqn2)

$$d*b \subseteq c*b$$

(**TR197**)(Relational Deduction)

$$(d*b \subseteq c*b) = E$$

(**AMX16**)(Order Axiom)(Eqn3)

$$((a \neq E)(b \neq E)(c \neq E) = E) \subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b))$$

(**TM141**)(Second Conditional Syllogism)(Eqn4)

Let $a = a$

(Eqn5)

Let $c = c*b$

(Eqn6)

Let $b = d*b$

(Eqn7)

$$((a \neq E)(d*b \neq E)(c*b \neq E) = E) \subseteq ((a \subseteq c*b)(d*b \subseteq c*b) \subseteq (a \not\subseteq d*b))$$

(**AMX19**)(Substitution of Eqns5, 6 & 7 into Eqn4)(Eqn8)

$$((a \neq E)(d*b \neq E)(c*b \neq E) = E) \subseteq ((a \subseteq c*b)(d*b \subseteq c*b) \subseteq (a \not\subseteq d*b))$$

(**AMX19**)(Substitution of Eqns1 & 3 into Eqn8)

$$((a \neq E)(d*b \neq E)(c*b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq d*b))$$

(**TM667**)(Conjunction Tautology)

$$((a \neq E)(d*b \neq E)(c*b \neq E) = E) \subseteq (E = (a \not\subseteq d*b))$$

(**TM275**)(Supersets of the Universal Set)

$$((a \neq E)(d*b \neq E)(c*b \neq E) \subseteq (a \not\subseteq d*b)) = E$$

(**TM094**)(Probabilistic Rule)

$$(a \neq E)(d*b \neq E)(c*b \neq E) \subseteq (a \not\subseteq d*b)$$

(**AMX16**)(Order Axiom)

$$(a \neq \sim E)(d*b \neq E)(c*b \neq E) \subseteq (a \not\subseteq d*b)$$

(**TM008**)(Inverse of an Equality)

$$(a \neq \emptyset)(d*b \neq E)(c*b \neq E) \subseteq (a \not\subseteq d*b)$$

(**DM02**)(Definition of $\emptyset$)

$$\text{Thus } ((\sim(c*b) \subseteq a)(d \subseteq c) = E) \subseteq ((a \neq \emptyset)(d*b \neq E)(c*b \neq E) \subseteq (a \not\subseteq d*b))$$

(**AMX02**)(Conditional Introduction)

$$(\sim(c*b) \subseteq a)(d \subseteq c)(a \neq \emptyset)(d*b \neq E)(c*b \neq E) \subseteq (a \not\subseteq d*b)$$

(**AMX35**)(Exportation/Importation) for truth sets

$$((a \neq \emptyset)(d*b \neq E)(c*b \neq E) = E) \subseteq ((\sim(c*b) \subseteq a)(d \subseteq c) \subseteq (a \not\subseteq d*b))$$

(**AMX35**)(Exportation/Importation) for truth sets

$$((a \neq \emptyset)(c*b \neq E)(c*h \neq E) = E) \subseteq ((\sim(c*b) \subseteq a)(h \subseteq b) \subseteq (a \not\subseteq c*h)) \quad (\textbf{TR395})$$

Proof of (**TR395**)

Let $(\sim(c*b) \subseteq a)$

$$a \subseteq c*b$$

(**TM012**)(Transposition)

$$(a \subseteq c*b) = E$$

(**AMX16**)(Order Axiom)(Eqn1)

Let $h \subseteq b$

$$(h \subseteq b) = E$$

(**AMX16**)(Order Axiom)(Eqn2)

$$c*h \subseteq c*b$$

(**TR196**)(Relational Deduction)

$$(c*h \subseteq c*b) = E$$

(**AMX16**)(Order Axiom)(Eqn3)

$$((a \neq E)(b \neq E)(c \neq E) = E) \subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b))$$

(**TM141**)(Second Conditional Syllogism)(Eqn4)

Let $a = a$

(Eqn5)

Let $c = c*b$

(Eqn6)

Let $b = c*h$

(Eqn7)

$$\begin{aligned}
((a \neq E)(c * h \neq E)(c * b \neq E) = E) &\subseteq ((a \subseteq c * b)(c * h \subseteq c * b) \subseteq (a \not\subseteq c * h)) \\
&\quad (AMX19)(\text{Substitution of Eqns5, 6 \& 7 into Eqn4})(Eqn8) \\
((a \neq E)(c * h \neq E)(c * b \neq E) = E) &\subseteq (EE \subseteq (a \not\subseteq c * h)) \quad (AMX19)(\text{Substitution of Eqns1 \& 3 into Eqn8}) \\
((a \neq E)(c * h \neq E)(c * b \neq E) = E) &\subseteq (E \subseteq (a \not\subseteq c * h)) \quad (TM667)(\text{Conjunction Tautology}) \\
((a \neq E)(c * h \neq E)(c * b \neq E) = E) &\subseteq (E = (a \not\subseteq c * h)) \quad (TM275)(\text{Supersets of the Universal Set}) \\
((a \neq E)(c * h \neq E)(c * b \neq E) \subseteq (a \not\subseteq c * h)) &= E \quad (TM094)(\text{Probabilistic Rule}) \\
(a \neq E)(c * h \neq E)(c * b \neq E) &\subseteq (a \not\subseteq c * h) \quad (AMX16)(\text{Order Axiom}) \\
(a \neq \sim E)(c * h \neq E)(c * b \neq E) &\subseteq (a \not\subseteq c * h) \quad (TM008)(\text{Inverse of an Equality}) \\
(a \neq \emptyset)(c * h \neq E)(c * b \neq E) &\subseteq (a \not\subseteq c * h) \quad (DM02)(\text{Definition of } \emptyset) \\
\text{Thus } ((\sim(c * b) \subseteq a)(h \subseteq b) = E) &\subseteq ((a \neq \emptyset)(c * h \neq E)(c * b \neq E) \subseteq (a \not\subseteq c * h)) \\
&\quad (AMX02)(\text{Conditional Introduction}) \\
(\sim(c * b) \subseteq a)(h \subseteq b)(a \neq \emptyset)(c * h \neq E)(c * b \neq E) &\subseteq (a \not\subseteq c * h) \\
&\quad (AMX35)(\text{Exportation/Importation}) \text{ for truth sets} \\
((a \neq \emptyset)(c * h \neq E)(c * b \neq E) = E) &\subseteq ((\sim(c * b) \subseteq a)(h \subseteq b) \subseteq (a \not\subseteq c * h)) \\
&\quad (AMX35)(\text{Exportation/Importation}) \text{ for truth sets}
\end{aligned}$$

$$((a \neq E)(c * b \neq \emptyset)(d * b \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c * b))(c \subseteq d) \subseteq (d * b \not\subseteq a)) \quad (TR396)$$

Proof of (TR396)

$$\begin{aligned}
\text{Let } a &\subseteq \sim(c * b) \\
(a \subseteq \sim(c * b)) &= E \quad (AMX16)(\text{Order Axiom})(Eqn1) \\
\text{Let } c &\subseteq d \\
(c \subseteq d) &= E \quad (AMX16)(\text{Order Axiom})(Eqn2) \\
c * b &\subseteq d * b \quad (TR197)(\text{Relational Deduction}) \\
(c * b \subseteq d * b) &= E \quad (AMX16)(\text{Order Axiom})(Eqn3) \\
((a \neq E)(b \neq E)(c \neq E) = E) &\subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b)) \\
&\quad (TM141)(\text{Second Conditional Syllogism})(Eqn4) \\
\text{Let } b &= \sim(d * b) \quad (Eqn5) \\
\text{Let } c &= \sim(c * b) \quad (Eqn6) \\
((a \neq E)(\sim(d * b) \neq E)(\sim(c * b) \neq E) = E) &\subseteq ((a \subseteq \sim(c * b))(\sim(d * b) \subseteq \sim(c * b)) \subseteq (a \not\subseteq \sim(d * b))) \\
&\quad (AMX19)(\text{Substitution of Eqns5 \& 6 into Eqn4}) \\
((a \neq E)(\sim(d * b) \neq E)(\sim(c * b) \neq E) = E) &\subseteq ((a \subseteq \sim(c * b))(c * b \subseteq d * b) \subseteq (a \not\subseteq \sim(d * b))) \\
&\quad (TM012)(\text{Transposition})(Eqn7) \\
((a \neq E)(\sim(d * b) \neq E)(\sim(c * b) \neq E) = E) &\subseteq (EE \subseteq (a \not\subseteq \sim(d * b))) \\
&\quad (AMX19)(\text{Substitution of Eqns1 \& 3 into Eqn4})(Eqn7) \\
((a \neq E)(\sim(d * b) \neq E)(\sim(c * b) \neq E) = E) &\subseteq (E \subseteq (a \not\subseteq \sim(d * b))) \\
&\quad (TM667)(\text{Conjunction Tautology}) \\
((a \neq E)(\sim(d * b) \neq E)(\sim(c * b) \neq E) = E) &\subseteq (E = (a \not\subseteq \sim(d * b))) \\
&\quad (TM275)(\text{Supersets of the Universal Set}) \\
((a \neq E)(\sim(d * b) \neq E)(\sim(c * b) \neq E) \subseteq (a \not\subseteq \sim(d * b))) &= E \\
&\quad (TM094)(\text{Probabilistic Rule}) \\
(a \neq E)(\sim(d * b) \neq E)(\sim(c * b) \neq E) &\subseteq (a \not\subseteq \sim(d * b)) \quad (AMX16)(\text{Order Axiom}) \\
(a \neq E)(d * b \neq \sim E)(c * b \neq \sim E) &\subseteq (a \not\subseteq \sim(d * b)) \quad (TM008)(\text{Inverse of an Equality}) \\
(a \neq E)(d * b \neq \emptyset)(c * b \neq \emptyset) &\subseteq (a \not\subseteq \sim(d * b)) \quad (DM02)(\text{Definition of } \emptyset) \\
(a \neq E)(d * b \neq \emptyset)(c * b \neq \emptyset) &\subseteq (d * b \not\subseteq a) \quad (TM012)(\text{Transposition}) \& (TM008)(\text{Inverse of an Equality}) \\
\text{Thus } ((a \subseteq \sim(c * b))(c \subseteq d) = E) &\subseteq ((a \neq E)(d * b \neq \emptyset)(c * b \neq \emptyset) \subseteq (d * b \not\subseteq a)) \\
&\quad (AMX02)(\text{Conditional Introduction}) \\
(a \subseteq \sim(c * b))(c \subseteq d)(a \neq E)(d * b \neq \emptyset)(c * b \neq \emptyset) &\subseteq (d * b \not\subseteq a) \\
&\quad (AMX35)(\text{Exportation/Importation}) \text{ for truth sets} \\
((a \neq E)(d * b \neq \emptyset)(c * b \neq \emptyset) = E) &\subseteq ((a \subseteq \sim(c * b))(c \subseteq d) \subseteq (d * b \not\subseteq a)) \\
&\quad (AMX35)(\text{Exportation/Importation}) \text{ for truth sets}
\end{aligned}$$

$$((a \neq E)(c * b \neq \emptyset)(c * h \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c * b))(b \subseteq h) \subseteq (c * h \not\subseteq a)) \quad (TR397)$$

Proof of (TR397)

$$\begin{aligned}
\text{Let } a &\subseteq \sim(c * b) \\
(a \subseteq \sim(c * b)) &= E \quad (AMX16)(\text{Order Axiom})(Eqn1)
\end{aligned}$$

Let $b \subseteq h$
 $(b \subseteq h) = E$ ($AMX16$)(Order Axiom)(Eqn2)
 $c*b \subseteq c*h$ ($TR196$)(Relational Deduction)
 $(c*b \subseteq c*h) = E$ ($AMX16$)(Order Axiom)(Eqn3)
 $((a \neq E)(b \neq E)(c \neq E) = E) \subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b))$
($TM147$)(Second Conditional Syllogism)(Eqn4)
 Let $b = \sim(c*h)$ (Eqn5)
 Let $c = \sim(c*b)$ (Eqn6)
 $((a \neq E)(\sim(c*h) \neq E)(\sim(c*b) \neq E) = E) \subseteq ((a \subseteq \sim(c*b))(\sim(c*h) \subseteq \sim(c*b)) \subseteq (a \not\subseteq \sim(c*h)))$
($AMX19$)(Substitution of Eqns5 & 6 into Eqn4)
 $((a \neq E)(\sim(c*h) \neq E)(\sim(c*b) \neq E) = E) \subseteq ((a \subseteq \sim(c*b))(c*b \subseteq c*h) \subseteq (a \not\subseteq \sim(c*h)))$
($TM012$)(Transposition)(Eqn7)
 $((a \neq E)(\sim(c*h) \neq E)(\sim(c*b) \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq \sim(c*h)))$
($AMX19$)(Substitution of Eqns1 & 3 into Eqn7)
 $((a \neq E)(\sim(c*h) \neq E)(\sim(c*b) \neq E) = E) \subseteq (E \subseteq (a \not\subseteq \sim(c*h)))$
($TM667$)(Conjunction Tautology)
 $((a \neq E)(\sim(c*h) \neq E)(\sim(c*b) \neq E) = E) \subseteq (E = (a \not\subseteq \sim(c*h)))$
($TM275$)(Supersets of the Universal Set)
 $((a \neq E)(\sim(c*h) \neq E)(\sim(c*b) \neq E) \subseteq (a \not\subseteq \sim(c*h)) = E)$
($TM094$)(Probabilistic Rule)
 $(a \neq E)(\sim(c*h) \neq E)(\sim(c*b) \neq E) \subseteq (a \not\subseteq \sim(c*h))$ ($AMX16$)(Order Axiom)
 $(a \neq E)(c*h \neq \sim E)(c*b \neq \sim E) \subseteq (a \not\subseteq \sim(c*h))$ ($TM008$)(Inverse of an Equality)
 $(a \neq E)(c*h \neq \emptyset)(c*b \neq \emptyset) \subseteq (a \not\subseteq \sim(c*h))$ ($DM02$)(Definition of $\emptyset$)
 $(a \neq E)(c*h \neq \emptyset)(c*b \neq \emptyset) \subseteq (c*h \not\subseteq a)$ ($TM012$)(Transposition) & ($TM008$)(Inverse of an Equality)
 Thus $((a \subseteq \sim(c*b))(b \subseteq h) = E) \subseteq ((a \neq E)(c*h \neq \emptyset)(c*b \neq \emptyset) \subseteq (c*h \not\subseteq a))$
($AMX02$)(Conditional Introduction)
 $(a \subseteq \sim(c*b))(b \subseteq h)(a \neq E)(c*h \neq \emptyset)(c*b \neq \emptyset) \subseteq (c*h \not\subseteq a)$
($AMX35$)(Exportation/Importation) for truth sets
 $((a \neq E)(c*h \neq \emptyset)(c*b \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c*b))(b \subseteq h) \subseteq (c*h \not\subseteq a))$
($AMX35$)(Exportation/Importation) for truth sets

$$(a \subseteq \sim(c*b))(d \subseteq c) \subseteq (a \subseteq \sim(d*b)) \quad (TR398)$$

Proof of ($TR398$)

Let $a \subseteq \sim(c*b)$ (Eqn1)
 Let $d \subseteq c$
 $d*b \subseteq c*b$ ($TR197$)(Relational Deduction)
 $\sim(c*b) \subseteq \sim(d*b)$ ($TM012$)(Transposition)(Eqn2)
 $a \subseteq \sim(d*b)$ ($TM139$)(Hypothetical Syllogism Eqns1 & 2)
 Thus $(a \subseteq \sim(c*b))(d \subseteq c) \subseteq (a \subseteq \sim(d*b))$ ($AMX02$)(Conditional Introduction)

$$(a \subseteq \sim(c*b))(h \subseteq b) \subseteq (a \subseteq \sim(c*h)) \quad (TR399)$$

Proof of ($TR399$)

Let $a \subseteq \sim(c*b)$ (Eqn1)
 Let $h \subseteq b$
 $c*h \subseteq c*b$ ($TR196$)(Relational Deduction)
 $\sim(c*b) \subseteq \sim(c*h)$ ($TM012$)(Transposition)(Eqn2)
 $a \subseteq \sim(c*h)$ ($TM139$)(Hypothetical Syllogism Eqns1 & 2)
 Thus $(a \subseteq \sim(c*b))(h \subseteq b) \subseteq (a \subseteq \sim(c*h))$ ($AMX02$)(Conditional Introduction)

$$((a \neq \emptyset)(c*b \neq \emptyset)(d*b \neq \emptyset) = E) \subseteq ((c*b \subseteq a)(c \subseteq d) \subseteq (a \not\subseteq \sim(d*b))) \quad (TR400)$$

Proof of ($TR400$)

Let $c*b \subseteq a$
 $(c*b \subseteq a) = E$ ($AMX16$)(Order Axiom)(Eqn1)
 Let $c \subseteq d$
 $(c \subseteq d) = E$ ($AMX16$)(Order Axiom)(Eqn2)

$$\begin{aligned}
& \mathbf{c*b} \subseteq \mathbf{d*b} && (\mathcal{T}_{R197})(\text{Relational Deduction}) \\
& (\mathbf{c*b} \subseteq \mathbf{d*b}) = \mathbf{E} && (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\
& ((\mathbf{a} \neq \mathbf{E})(\mathbf{b} \neq \mathbf{E})(\mathbf{c} \neq \mathbf{E}) = \mathbf{E}) \subseteq ((\mathbf{a} \subseteq \mathbf{c})(\mathbf{b} \subseteq \mathbf{c}) \subseteq (\mathbf{a} \not\subseteq \mathbf{b})) && (\mathcal{T}_{M147})(\text{Second Conditional Syllogism})(\text{Eqn4}) \\
& \text{Let } \mathbf{a} = \underline{\mathbf{a}} && (\text{Eqn5}) \\
& \text{Let } \mathbf{b} = \sim(\mathbf{d*b}) && (\text{Eqn6}) \\
& \text{Let } \mathbf{c} = \sim(\mathbf{c*b}) && (\text{Eqn7}) \\
& ((\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{d*b}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) = \mathbf{E}) \subseteq ((\underline{\mathbf{a}} \subseteq \sim(\mathbf{c*b}))(\sim(\mathbf{d*b}) \subseteq \sim(\mathbf{c*b})) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{d*b}))) && (\mathcal{A}_{MX19})(\text{Substitution of Eqns5, 6 \& 7 into Eqn4}) \\
& ((\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{d*b}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) = \mathbf{E}) \subseteq ((\mathbf{c*b} \subseteq \mathbf{a})(\mathbf{c*b} \subseteq \mathbf{d*b}) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{d*b}))) && (\mathcal{T}_{M012})(\text{Transposition})(\text{Eqn8}) \\
& ((\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{d*b}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) = \mathbf{E}) \subseteq (\mathbf{EE} \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{d*b}))) && (\mathcal{A}_{MX19})(\text{Substitution of Eqns1 \& 3 into Eqn8}) \\
& ((\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{d*b}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) = \mathbf{E}) \subseteq (\mathbf{E} \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{d*b}))) && (\mathcal{T}_{M667})(\text{Conjunction Tautology}) \\
& ((\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{d*b}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) = \mathbf{E}) \subseteq (\mathbf{E} = (\mathbf{a} \not\subseteq \sim(\mathbf{d*b}))) && (\mathcal{T}_{M275})(\text{Supersets of the Universal Set}) \\
& ((\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{d*b}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{d*b})) = \mathbf{E}) && (\mathcal{T}_{M094})(\text{Probabilistic Rule}) \\
& (\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{d*b}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{d*b})) && (\mathcal{A}_{MX16})(\text{Order Axiom}) \\
& (\mathbf{a} \neq \sim \mathbf{E})(\mathbf{d*b} \neq \sim \mathbf{E})(\mathbf{c*b} \neq \sim \mathbf{E}) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{d*b})) && (\mathcal{T}_{M008})(\text{Inverse of an Equality}) \\
& (\mathbf{a} \neq \emptyset)(\mathbf{d*b} \neq \emptyset)(\mathbf{c*b} \neq \emptyset) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{d*b})) && (\mathcal{D}_{M02})(\text{Definition of } \emptyset) \\
& \text{Thus } ((\mathbf{c*b} \subseteq \mathbf{a})(\mathbf{c} \subseteq \mathbf{d}) = \mathbf{E}) \subseteq ((\mathbf{a} \neq \emptyset)(\mathbf{d*b} \neq \emptyset)(\mathbf{c*b} \neq \emptyset) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{d*b}))) && (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\
& (\mathbf{c*b} \subseteq \mathbf{a})(\mathbf{c} \subseteq \mathbf{d})(\mathbf{a} \neq \emptyset)(\mathbf{d*b} \neq \emptyset)(\mathbf{c*b} \neq \emptyset) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{d*b})) && (\mathcal{A}_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\
& ((\mathbf{a} \neq \emptyset)(\mathbf{d*b} \neq \emptyset)(\mathbf{c*b} \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{c*b} \subseteq \mathbf{a})(\mathbf{c} \subseteq \mathbf{d}) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{d*b}))) && (\mathcal{A}_{MX35})(\text{Exportation/Importation}) \text{ for truth sets}
\end{aligned}$$

$$((\mathbf{a} \neq \emptyset)(\mathbf{c*b} \neq \emptyset)(\mathbf{c*h} \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{c*b} \subseteq \mathbf{a})(\mathbf{b} \subseteq \mathbf{h}) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{c*h}))) \quad (\mathcal{T}_{R407})$$

Proof of ($\mathcal{T}_{R407}$)

$$\begin{aligned}
& \text{Let } \mathbf{c*b} \subseteq \mathbf{a} \\
& (\mathbf{c*b} \subseteq \mathbf{a}) = \mathbf{E} && (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn1}) \\
& \text{Let } \mathbf{b} \subseteq \mathbf{h} \\
& (\mathbf{b} \subseteq \mathbf{h}) = \mathbf{E} && (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\
& \mathbf{c*b} \subseteq \mathbf{c*h} && (\mathcal{T}_{R196})(\text{Relational Deduction}) \\
& (\mathbf{c*b} \subseteq \mathbf{c*h}) = \mathbf{E} && (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\
& ((\mathbf{a} \neq \mathbf{E})(\mathbf{b} \neq \mathbf{E})(\mathbf{c} \neq \mathbf{E}) = \mathbf{E}) \subseteq ((\mathbf{a} \subseteq \mathbf{c})(\mathbf{b} \subseteq \mathbf{c}) \subseteq (\mathbf{a} \not\subseteq \mathbf{b})) && (\mathcal{T}_{M147})(\text{Second Conditional Syllogism})(\text{Eqn4}) \\
& \text{Let } \mathbf{a} = \underline{\mathbf{a}} && (\text{Eqn5}) \\
& \text{Let } \mathbf{b} = \sim(\mathbf{c*h}) && (\text{Eqn6}) \\
& \text{Let } \mathbf{c} = \sim(\mathbf{c*b}) && (\text{Eqn7}) \\
& ((\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{c*h}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) = \mathbf{E}) \subseteq ((\underline{\mathbf{a}} \subseteq \sim(\mathbf{c*b}))(\sim(\mathbf{c*h}) \subseteq \sim(\mathbf{c*b})) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{c*h}))) && (\mathcal{A}_{MX19})(\text{Substitution of Eqns5, 6 \& 7 into Eqn4}) \\
& ((\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{d*b}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) = \mathbf{E}) \subseteq ((\mathbf{c*b} \subseteq \mathbf{a})(\mathbf{c*b} \subseteq \mathbf{d*b}) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{d*b}))) && (\mathcal{T}_{M012})(\text{Transposition})(\text{Eqn8}) \\
& ((\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{c*h}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) = \mathbf{E}) \subseteq (\mathbf{EE} \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{c*h}))) && (\mathcal{A}_{MX19})(\text{Substitution of Eqns1 \& 3 into Eqn8}) \\
& ((\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{c*h}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) = \mathbf{E}) \subseteq (\mathbf{E} \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{c*h}))) && (\mathcal{T}_{M667})(\text{Conjunction Tautology}) \\
& ((\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{c*h}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) = \mathbf{E}) \subseteq (\mathbf{E} = (\mathbf{a} \not\subseteq \sim(\mathbf{c*h}))) && (\mathcal{T}_{M275})(\text{Supersets of the Universal Set}) \\
& ((\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{c*h}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{c*h})) = \mathbf{E}) && (\mathcal{T}_{M094})(\text{Probabilistic Rule}) \\
& (\underline{\mathbf{a}} \neq \mathbf{E})(\sim(\mathbf{c*h}) \neq \mathbf{E})(\sim(\mathbf{c*b}) \neq \mathbf{E}) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{c*h})) && (\mathcal{A}_{MX16})(\text{Order Axiom}) \\
& (\mathbf{a} \neq \sim \mathbf{E})(\mathbf{c*h} \neq \sim \mathbf{E})(\mathbf{c*b} \neq \sim \mathbf{E}) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{c*h})) && (\mathcal{T}_{M008})(\text{Inverse of an Equality}) \\
& (\mathbf{a} \neq \emptyset)(\mathbf{c*h} \neq \emptyset)(\mathbf{c*b} \neq \emptyset) \subseteq (\mathbf{a} \not\subseteq \sim(\mathbf{c*h})) && (\mathcal{D}_{M02})(\text{Definition of } \emptyset)
\end{aligned}$$

Thus $((c*b \subseteq a)(b \subseteq h) = E) \subseteq ((a \neq \emptyset)(c*h \neq \emptyset)(c*b \neq \emptyset) \subseteq (a \not\subseteq \sim(c*h)))$
 $(A_{MX02})(\text{Conditional Introduction})$
 $(c*b \subseteq a)(b \subseteq h)(a \neq \emptyset)(c*h \neq \emptyset)(c*b \neq \emptyset) \subseteq (a \not\subseteq \sim(c*h))$
 $(A_{MX35})(\text{Exportation/Importation})$ for truth sets
 $((a \neq \emptyset)(c*h \neq \emptyset)(c*b \neq \emptyset) = E) \subseteq ((c*b \subseteq a)(b \subseteq h) \subseteq (a \not\subseteq \sim(c*h)))$
 $(A_{MX35})(\text{Exportation/Importation})$ for truth sets

$$(c*b \subseteq a)(d \subseteq c) \subseteq (d*b \subseteq a) \quad (T_{R402})$$

Proof of (T_{R402})

Let $c*b \subseteq a$ (Eqn1)
 Let $d \subseteq c$
 $d*b \subseteq c*b$ (T_{R197})(Relational Deduction)(Eqn2)
 $d*b \subseteq a$ (T_{M139})(Hypothetical Syllogism Eqns1&2)
 Thus $(c*b \subseteq a)(d \subseteq c) \subseteq (d*b \subseteq a)$ (A_{MX02})(Conditional Introduction)

$$(c*b \subseteq a)(h \subseteq b) \subseteq (c*h \subseteq a) \quad (T_{R403})$$

Proof of (T_{R403})

Let $c*b \subseteq a$ (Eqn1)
 Let $h \subseteq b$
 $c*h \subseteq c*b$ (T_{R196})(Relational Deduction)(Eqn2)
 $c*h \subseteq a$ (T_{M139})(Hypothetical Syllogism Eqns1&2)
 Thus $(c*b \subseteq a)(h \subseteq b) \subseteq (c*h \subseteq a)$ (A_{MX02})(Conditional Introduction)

$$((a \neq \emptyset)(c*b \neq E) = E) \subseteq ((a \not\subseteq c*b)(d \subseteq c) \subseteq (a \not\subseteq d*b)) \quad (T_{R404})$$

Proof of (T_{R404})

Let $a \not\subseteq c*b$
 $(a \not\subseteq c*b) = E$ (A_{MX16})(Order Axiom)(Eqn1)
 Let $d \subseteq c$
 $(d \subseteq c) = E$ (A_{MX16})(Order Axiom)(Eqn2)
 $d*b \subseteq c*b$ (T_{R197})(Relational Deduction)
 $(d*b \subseteq c*b) = E$ (A_{MX16})(Order Axiom)(Eqn3)
 $((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$ (T_{M140})(First Conditional Syllogism)(Eqn4)
 Let $a = d*b$ (Eqn5)
 Let $b = a$ (Eqn6)
 Let $c = c*b$ (Eqn7)
 $((a \neq \emptyset)(c*b \neq E) = E) \subseteq ((d*b \subseteq c*b)(a \not\subseteq c*b) \subseteq (a \not\subseteq d*b))$
 $(A_{MX19})(\text{Substitution of Eqns5, 6 \& 7 into Eqn4})(\text{Eqn8})$
 $((a \neq \emptyset)(c*b \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq d*b))$ (A_{MX19})(Substitution of Eqns1 \& 3 into Eqn8)
 $((a \neq \emptyset)(c*b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq d*b))$ (T_{M667})(Conjunction Tautology)
 $((a \neq \emptyset)(c*b \neq E) = E) \subseteq (E = (a \not\subseteq d*b))$ (T_{M275})(Supersets of the Universal Set)
 $((a \neq \emptyset)(c*b \neq E) \subseteq (a \not\subseteq d*b)) = E$ (T_{M094})(Probabilistic Rule)
 $(a \neq \emptyset)(c*b \neq E) \subseteq (a \not\subseteq d*b)$ (A_{MX16})(Order Axiom)
 Thus $((a \not\subseteq c*b)(d \subseteq c) = E) \subseteq ((a \neq \emptyset)(c*b \neq E) \subseteq (a \not\subseteq d*b))$
 $(A_{MX02})(\text{Conditional Introduction})$
 $(a \not\subseteq c*b)(d \subseteq c)(a \neq \emptyset)(c*b \neq E) \subseteq (a \not\subseteq d*b)$ (A_{MX35})(Exportation/Importation) for truth sets
 $((a \neq \emptyset)(c*b \neq E) = E) \subseteq ((a \not\subseteq c*b)(d \subseteq c) \subseteq (a \not\subseteq d*b))$
 $(A_{MX35})(\text{Exportation/Importation})$ for truth sets

$$((a \neq \emptyset)(c*b \neq E) = E) \subseteq ((a \not\subseteq c*b)(h \subseteq b) \subseteq (a \not\subseteq c*h)) \quad (T_{R405})$$

Proof of (T_{R405})

Let $a \not\subseteq c*b$
 $(a \not\subseteq c*b) = E$ (A_{MX16})(Order Axiom)(Eqn1)
 Let $h \subseteq b$
 $(h \subseteq b) = E$ (A_{MX16})(Order Axiom)(Eqn2)

$c * h \subseteq c * b$	(T_{R196})(Relational Deduction)
$(c * h \subseteq c * b) = E$	($AMX16$)(Order Axiom)(Eqn3)
$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$	(T_{M140})(First Conditional Syllogism)(Eqn4)
Let $a = c * h$	(Eqn5)
Let $b = a$	(Eqn6)
Let $c = c * b$	(Eqn7)
$((a \neq \emptyset)(c * b \neq E) = E) \subseteq ((c * h \subseteq c * b)(a \not\subseteq c * b) \subseteq (a \not\subseteq c * h))$	($AMX19$)(Substitution of Eqns5, 6 & 7 into Eqn4)(Eqn8)
$((a \neq \emptyset)(c * b \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq c * h))$	($AMX19$)(Substitution of Eqns1 & 3 into Eqn8)
$((a \neq \emptyset)(c * b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq c * h))$	(T_{M667})(Conjunction Tautology)
$((a \neq \emptyset)(c * b \neq E) = E) \subseteq (E = (a \not\subseteq c * h))$	(T_{M275})(Supersets of the Universal Set)
$((a \neq \emptyset)(c * b \neq E) \subseteq (a \not\subseteq c * h)) = E$	(T_{M094})(Probabilistic Rule)
$(a \neq \emptyset)(c * b \neq E) \subseteq (a \not\subseteq c * h)$	($AMX16$)(Order Axiom)
Thus $((a \not\subseteq c * b)(h \subseteq b) = E) \subseteq ((a \neq \emptyset)(c * b \neq E) \subseteq (a \not\subseteq c * h))$	($AMX02$)(Conditional Introduction)
$(a \not\subseteq c * b)(h \subseteq b)(a \neq \emptyset)(c * b \neq E) \subseteq (a \not\subseteq c * h)$	($AMX35$)(Exportation/Importation) for truth sets
$((a \neq \emptyset)(c * b \neq E) = E) \subseteq ((a \not\subseteq c * b)(h \subseteq b) \subseteq (a \not\subseteq c * h))$	($AMX35$)(Exportation/Importation) for truth sets

$$((a \neq E)(c * b \neq E) = E) \subseteq ((\sim(c * b) \not\subseteq a)(d \subseteq c) \subseteq (\sim(d * b) \not\subseteq a)) \quad (T_{R406})$$

Proof of (T_{R406})

Let $\sim(c * b) \not\subseteq a$	
$(\sim(c * b) \not\subseteq a) = E$	($AMX16$)(Order Axiom)(Eqn1)
Let $d \subseteq c$	
$(d \subseteq c) = E$	($AMX16$)(Order Axiom)(Eqn2)
$d * b \subseteq c * b$	(T_{R197})(Relational Deduction)
$(d * b \subseteq c * b) = E$	($AMX16$)(Order Axiom)(Eqn3)
$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$	(T_{M140})(First Conditional Syllogism)(Eqn4)
Let $a = d * b$	(Eqn5)
Let $b = a$	(Eqn6)
Let $c = c * b$	(Eqn7)
$((a \neq \emptyset)(c * b \neq E) = E) \subseteq ((d * b \subseteq c * b)(a \not\subseteq c * b) \subseteq (a \not\subseteq d * b))$	($AMX19$)(Substitution of Eqns5, 6 & 7 into Eqn4)(Eqn8)
$((a \neq \emptyset)(c * b \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq d * b))$	($AMX19$)(Substitution of Eqns1 & 3 into Eqn8)
$((a \neq \emptyset)(c * b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq d * b))$	(T_{M667})(Conjunction Tautology)
$((a \neq \emptyset)(c * b \neq E) = E) \subseteq (E = (a \not\subseteq d * b))$	(T_{M275})(Supersets of the Universal Set)
$((a \neq \emptyset)(c * b \neq E) \subseteq (a \not\subseteq d * b)) = E$	(T_{M094})(Probabilistic Rule)
$(a \neq \emptyset)(c * b \neq E) \subseteq (a \not\subseteq d * b)$	($AMX16$)(Order Axiom)
$(a \neq \emptyset)(c * b \neq E) \subseteq (a \not\subseteq d * b)$	(T_{M008})(Inverse of an Equality)
$(a \neq E)(c * b \neq E) \subseteq (a \not\subseteq d * b)$	(T_{M054})(Negative of Null Set)
$(a \neq E)(c * b \neq E) \subseteq (\sim(d * b) \not\subseteq a)$	(T_{M012})(Transposition)
Thus $((\sim(c * b) \not\subseteq a)(d \subseteq c) = E) \subseteq ((a \neq E)(c * b \neq E) \subseteq (\sim(d * b) \not\subseteq a))$	($AMX02$)(Conditional Introduction)
$(\sim(c * b) \not\subseteq a)(d \subseteq c)(a \neq E)(c * b \neq E) \subseteq (\sim(d * b) \not\subseteq a)$	($AMX35$)(Exportation/Importation) for truth sets
$((a \neq E)(c * b \neq E) = E) \subseteq ((\sim(c * b) \not\subseteq a)(d \subseteq c) \subseteq (\sim(d * b) \not\subseteq a))$	($AMX35$)(Exportation/Importation) for truth sets

$$((a \neq E)(c * b \neq E) = E) \subseteq ((\sim(c * b) \not\subseteq a)(h \subseteq b) \subseteq (\sim(c * h) \not\subseteq a)) \quad (T_{R407})$$

Proof of (T_{R407})

Let $\sim(c * b) \not\subseteq a$	
$(\sim(c * b) \not\subseteq a) = E$	($AMX16$)(Order Axiom)(Eqn1)
Let $h \subseteq b$	
$(h \subseteq b) = E$	($AMX16$)(Order Axiom)(Eqn2)
$c * h \subseteq c * b$	(T_{R196})(Relational Deduction)

$(c * h \subseteq c * b) = E$	($AMX16$)(Order Axiom)(Eqn3)
$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$	($TM140$)(First Conditional Syllogism)(Eqn4)
Let $a = c * h$	(Eqn5)
Let $b = a$	(Eqn6)
Let $c = c * b$	(Eqn7)
$((a \neq \emptyset)(c * b \neq E) = E) \subseteq ((c * h \subseteq c * b)(a \not\subseteq c * b) \subseteq (a \not\subseteq c * h))$	($AMX19$)(Substitution of Eqns5, 6 & 7 into Eqn4)(Eqn8)
$((a \neq \emptyset)(c * b \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq c * h))$	($AMX19$)(Substitution of Eqns1 & 3 into Eqn8)
$((a \neq \emptyset)(c * b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq c * h))$	($TM667$)(Conjunction Tautology)
$((a \neq \emptyset)(c * b \neq E) = E) \subseteq (E = (a \not\subseteq c * h))$	($TM275$)(Supersets of the Universal Set)
$((a \neq \emptyset)(c * b \neq E) \subseteq (a \not\subseteq c * h)) = E$	($TM094$)(Probabilistic Rule)
$(a \neq \emptyset)(c * b \neq E) \subseteq (a \not\subseteq c * h)$	($AMX16$)(Order Axiom)
$(a \neq \sim \emptyset)(c * b \neq E) \subseteq (a \not\subseteq c * h)$	($TM008$)(Inverse of an Equality)
$(a \neq E)(c * b \neq E) \subseteq (a \not\subseteq c * h)$	($TM054$)(Negative of Null Set)
$(a \neq E)(c * b \neq E) \subseteq (\sim(c * h) \not\subseteq a)$	($TM012$)(Transposition)
Thus $((\sim(c * b) \not\subseteq a)(h \subseteq b) = E) \subseteq ((a \neq E)(c * b \neq E) \subseteq (\sim(c * h) \not\subseteq a))$	($AMX02$)(Conditional Introduction)
$(\sim(c * b) \not\subseteq a)(h \subseteq b)(a \neq E)(c * b \neq E) \subseteq (\sim(d * b) \not\subseteq a)$	($AMX35$)(Exportation/Importation) for truth sets
$((a \neq E)(c * b \neq E) = E) \subseteq ((\sim(c * b) \not\subseteq a)(h \subseteq b) \subseteq (\sim(c * h) \not\subseteq a))$	($AMX35$)(Exportation/Importation) for truth sets

$$((a \neq \emptyset)(c * b \neq \emptyset) = E) \subseteq ((a \not\subseteq \sim(c * b))(c \subseteq d) \subseteq (a \not\subseteq \sim(d * b))) \quad (TR408)$$

Proof of ($TR408$)

Let $a \not\subseteq \sim(c * b)$	
$(a \not\subseteq \sim(c * b)) = E$	($AMX16$)(Order Axiom)(Eqn1)
Let $c \subseteq d$	
$(c \subseteq d) = E$	($AMX16$)(Order Axiom)(Eqn2)
$c * b \subseteq d * b$	($TR197$)(Relational Deduction)
$(c * b \subseteq d * b) = E$	($AMX16$)(Order Axiom)
$(\sim(d * b) \subseteq \sim(c * b)) = E$	($TM012$)(Transposition)(Eqn3)
$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$	($TM140$)(First Conditional Syllogism)(Eqn4)
Let $a = \sim(d * b)$	(Eqn5)
Let $b = a$	(Eqn6)
Let $c = \sim(c * b)$	(Eqn7)
$((a \neq \emptyset)(\sim(c * b) \neq E) = E) \subseteq ((\sim(d * b) \subseteq \sim(c * b))(a \not\subseteq \sim(c * b)) \subseteq (a \not\subseteq \sim(d * b)))$	($AMX19$)(Substitution of Eqns5, 6 & 7 into Eqn4)(Eqn8)
$((a \neq \emptyset)(\sim(c * b) \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq \sim(d * b)))$	($AMX19$)(Substitution of Eqns1 & 3 into Eqn8)
$((a \neq \emptyset)(\sim(c * b) \neq E) = E) \subseteq (E \subseteq (a \not\subseteq \sim(d * b)))$	($TM667$)(Conjunction Tautology)
$((a \neq \emptyset)(\sim(c * b) \neq E) = E) \subseteq (E = (a \not\subseteq \sim(d * b)))$	($TM275$)(Supersets of the Universal Set)
$((a \neq \emptyset)(\sim(c * b) \neq E) \subseteq (a \not\subseteq \sim(d * b))) = E$	($TM094$)(Probabilistic Rule)
$(a \neq \emptyset)(\sim(c * b) \neq E) \subseteq (a \not\subseteq \sim(d * b))$	($AMX16$)(Order Axiom)
$(a \neq \emptyset)(c * b \neq \sim E) \subseteq (a \not\subseteq \sim(d * b))$	($TM008$)(Inverse of an Equality)
$(a \neq \emptyset)(c * b \neq \emptyset) \subseteq (a \not\subseteq \sim(d * b))$	($DM02$)(Definition of $\emptyset$)
Thus $((a \not\subseteq \sim(c * b))(c \subseteq d) = E) \subseteq ((a \neq \emptyset)(c * b \neq \emptyset) \subseteq (a \not\subseteq \sim(d * b)))$	($AMX02$)(Conditional Introduction)
$(a \not\subseteq \sim(c * b))(c \subseteq d)(a \neq \emptyset)(c * b \neq \emptyset) \subseteq (a \not\subseteq \sim(d * b))$	($AMX35$)(Exportation/Importation) for truth sets
$((a \neq \emptyset)(c * b \neq \emptyset) = E) \subseteq ((a \not\subseteq \sim(c * b))(c \subseteq d) \subseteq (a \not\subseteq \sim(d * b)))$	($AMX35$)(Exportation/Importation) for truth sets

$$((a \neq \emptyset)(c * b \neq \emptyset) = E) \subseteq ((a \not\subseteq \sim(c * b))(b \subseteq h) \subseteq (a \not\subseteq \sim(c * h))) \quad (TR409)$$

Proof of ($TR409$)

Let $a \not\subseteq \sim(c * b)$	
$(a \not\subseteq \sim(c * b)) = E$	($AMX16$)(Order Axiom)(Eqn1)

Let $b \subseteq h$	
$(b \subseteq h) = E$	(A_{MX16}) (Order Axiom)(Eqn2)
$c*b \subseteq c*h$	(T_{R196}) (Relational Deduction)
$(c*b \subseteq c*h) = E$	(A_{MX16}) (Order Axiom)
$(\sim(c*h) \subseteq \sim(c*b)) = E$	(T_{M012}) (Transposition)(Eqn3)
$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$	(T_{M140}) (First Conditional Syllogism)(Eqn4)
Let $a = \sim(c*h)$	(Eqn5)
Let $b = a$	(Eqn6)
Let $c = \sim(c*b)$	(Eqn7)
$((a \neq \emptyset)(\sim(c*b) \neq E) = E) \subseteq ((\sim(c*h) \subseteq \sim(c*b))(a \not\subseteq \sim(c*b)) \subseteq (a \not\subseteq \sim(c*h)))$	(A_{MX19}) (Substitution of Eqns5, 6 & 7 into Eqn4)(Eqn8)
$((a \neq \emptyset)(\sim(c*b) \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq \sim(c*h)))$	(A_{MX19}) (Substitution of Eqns1 & 3 into Eqn8)
$((a \neq \emptyset)(\sim(c*b) \neq E) = E) \subseteq (E \subseteq (a \not\subseteq \sim(c*h)))$	(T_{M667}) (Conjunction Tautology)
$((a \neq \emptyset)(\sim(c*b) \neq E) = E) \subseteq (E = (a \not\subseteq \sim(c*h)))$	(T_{M275}) (Supersets of the Universal Set)
$((a \neq \emptyset)(\sim(c*b) \neq E) \subseteq (a \not\subseteq \sim(c*h))) = E$	(T_{M094}) (Probabilistic Rule)
$(a \neq \emptyset)(\sim(c*b) \neq E) \subseteq (a \not\subseteq \sim(c*h))$	(A_{MX16}) (Order Axiom)
$(a \neq \emptyset)(c*b \neq \sim E) \subseteq (a \not\subseteq \sim(c*h))$	(T_{M008}) (Inverse of an Equality)
$(a \neq \emptyset)(c*b \neq \emptyset) \subseteq (a \not\subseteq \sim(c*h))$	(D_{M02}) (Definition of $\emptyset$)
Thus $((a \not\subseteq \sim(c*b))(b \subseteq h) = E) \subseteq ((a \neq \emptyset)(c*b \neq \emptyset) \subseteq (a \not\subseteq \sim(c*h)))$	(A_{MX02}) (Conditional Introduction)
$(a \not\subseteq \sim(c*b))(b \subseteq h)(a \neq \emptyset)(c*b \neq \emptyset) \subseteq (a \not\subseteq \sim(c*h))$	(A_{MX35}) (Exportation/Importation) for truth sets
$((a \neq \emptyset)(c*b \neq \emptyset) = E) \subseteq ((a \not\subseteq \sim(c*b))(b \subseteq h) \subseteq (a \not\subseteq \sim(c*h)))$	(A_{MX35}) (Exportation/Importation) for truth sets

$$((a \neq E)(c*b \neq \emptyset) = E) \subseteq ((c*b \not\subseteq a)(c \subseteq d) \subseteq (d*b \not\subseteq a)) \quad (T_{R410})$$

Proof of (T_{R410})

Let $c*b \not\subseteq a$	
$(c*b \not\subseteq a) = E$	(A_{MX16}) (Order Axiom)
$(a \not\subseteq \sim(c*b)) = E$	(T_{M012}) (Transposition)(Eqn1)
Let $c \subseteq d$	
$(c \subseteq d) = E$	(A_{MX16}) (Order Axiom)(Eqn2)
$c*b \subseteq d*b$	(T_{R197}) (Relational Deduction)
$(c*b \subseteq d*b) = E$	(A_{MX16}) (Order Axiom)
$(\sim(d*b) \subseteq \sim(c*b)) = E$	(T_{M012}) (Transposition)(Eqn3)
$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$	(T_{M140}) (First Conditional Syllogism)(Eqn4)
Let $a = \sim(d*b)$	(Eqn5)
Let $b = a$	(Eqn6)
Let $c = \sim(c*b)$	(Eqn7)
$((a \neq \emptyset)(\sim(c*b) \neq E) = E) \subseteq ((\sim(d*b) \subseteq \sim(c*b))(a \not\subseteq \sim(c*b)) \subseteq (a \not\subseteq \sim(d*b)))$	(A_{MX19}) (Substitution of Eqns5, 6 & 7 into Eqn4)(Eqn8)
$((a \neq \emptyset)(\sim(c*b) \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq \sim(d*b)))$	(A_{MX19}) (Substitution of Eqns1 & 3 into Eqn8)
$((a \neq \emptyset)(\sim(c*b) \neq E) = E) \subseteq (E \subseteq (a \not\subseteq \sim(d*b)))$	(T_{M667}) (Conjunction Tautology)
$((a \neq \emptyset)(\sim(c*b) \neq E) = E) \subseteq (E = (a \not\subseteq \sim(d*b)))$	(T_{M275}) (Supersets of the Universal Set)
$((a \neq \emptyset)(\sim(c*b) \neq E) \subseteq (a \not\subseteq \sim(d*b))) = E$	(T_{M094}) (Probabilistic Rule)
$(a \neq \emptyset)(\sim(c*b) \neq E) \subseteq (a \not\subseteq \sim(d*b))$	(A_{MX16}) (Order Axiom)
$(a \neq \emptyset)(c*b \neq \sim E) \subseteq (a \not\subseteq \sim(d*b))$	(T_{M008}) (Inverse of an Equality)
$(a \neq E)(c*b \neq \emptyset) \subseteq (a \not\subseteq \sim(d*b))$	(T_{M054}) (Negative of Null Set) & (D_{M02}) (Definition of $\emptyset$)
$(a \neq E)(c*b \neq \emptyset) \subseteq (d*b \not\subseteq a)$	(T_{M012}) (Transposition)
Thus $((c*b \not\subseteq a)(c \subseteq d) = E) \subseteq ((a \neq E)(c*b \neq \emptyset) \subseteq (d*b \not\subseteq a))$	(A_{MX02}) (Conditional Introduction)
$(c*b \not\subseteq a)(c \subseteq d)(a \neq E)(c*b \neq \emptyset) \subseteq (d*b \not\subseteq a)$	(A_{MX35}) (Exportation/Importation) for truth sets
$((a \neq E)(c*b \neq \emptyset) = E) \subseteq ((c*b \not\subseteq a)(c \subseteq d) \subseteq (d*b \not\subseteq a))$	(A_{MX35}) (Exportation/Importation) for truth sets

$$((a \neq E)(c*b \neq \emptyset) = E) \subseteq ((c*b \not\subseteq a)(b \subseteq h) \subseteq (c*h \not\subseteq a)) \quad (T_{R411})$$

Proof of (T_{R411})

Let $c*b \not\subseteq a$

$$(c*b \not\subseteq a) = E$$

(A_{MX16})(Order Axiom)

$$(a \not\subseteq \sim(c*b)) = E$$

(T_{M012})(Transposition)(Eqn1)

Let $b \subseteq h$

$$(b \subseteq h) = E$$

(A_{MX16})(Order Axiom)(Eqn2)

$$c*b \subseteq c*h$$

(T_{R196})(Relational Deduction)

$$(c*b \subseteq c*h) = E$$

(A_{MX16})(Order Axiom)

$$(\sim(c*h) \subseteq \sim(c*b)) = E$$

(T_{M012})(Transposition)(Eqn3)

$$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$$

(T_{M140})(First Conditional Syllogism)(Eqn4)

Let $a = \sim(c*h)$

(Eqn5)

Let $b = a$

(Eqn6)

Let $c = \sim(c*b)$

(Eqn7)

$$((a \neq \emptyset)(\sim(c*b) \neq E) = E) \subseteq ((\sim(c*h) \subseteq \sim(c*b))(a \not\subseteq \sim(c*b)) \subseteq (a \not\subseteq \sim(c*h)))$$

(A_{MX19})(Substitution of Eqns5, 6 & 7 into Eqn4)(Eqn8)

$$((a \neq \emptyset)(\sim(c*b) \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq \sim(c*h)))$$

(A_{MX19})(Substitution of Eqns1 & 3 into Eqn8)

$$((a \neq \emptyset)(\sim(c*b) \neq E) = E) \subseteq (E \subseteq (a \not\subseteq \sim(c*h)))$$

(T_{M667})(Conjunction Tautology)

$$((a \neq \emptyset)(\sim(c*b) \neq E) = E) \subseteq (E = (a \not\subseteq \sim(c*h)))$$

(T_{M275})(Supersets of the Universal Set)

$$((a \neq \emptyset)(\sim(c*b) \neq E) \subseteq (a \not\subseteq \sim(c*h))) = E$$

(T_{M094})(Probabilistic Rule)

$$(a \neq \emptyset)(\sim(c*b) \neq E) \subseteq (a \not\subseteq \sim(c*h))$$

(A_{MX16})(Order Axiom)

$$(a \neq \sim \emptyset)(c*b \neq \sim E) \subseteq (a \not\subseteq \sim(c*h))$$

(T_{M008})(Inverse of an Equality)

$$(a \neq E)(c*b \neq \emptyset) \subseteq (a \not\subseteq \sim(c*h))$$

(T_{M054})(Negative of Null Set) & (D_{M02})(Definition of $\emptyset$)

$$(a \neq E)(c*b \neq \emptyset) \subseteq (c*h \not\subseteq a)$$

(T_{M012})(Transposition)

$$\text{Thus } ((c*b \not\subseteq a)(b \subseteq h) = E) \subseteq ((a \neq E)(c*b \neq \emptyset) \subseteq (c*h \not\subseteq a))$$

(A_{MX02})(Conditional Introduction)

$$(c*b \not\subseteq a)(b \subseteq h)(a \neq E)(c*b \neq \emptyset) \subseteq (c*h \not\subseteq a)$$

(A_{MX35})(Exportation/Importation) for truth sets

$$((a \neq E)(c*b \neq \emptyset) = E) \subseteq ((c*b \not\subseteq a)(b \subseteq h) \subseteq (c*h \not\subseteq a))$$

(A_{MX35})(Exportation/Importation) for truth sets

B. Relational Syllogisms based on (T_{R262}) or (T_{R263}).

The only theorems of relationship theory used in the proofs below are (T_{R262}) or (T_{R263}). Apart from this the proofs use membership theory.

$$(a \subseteq c \diamond b)(c \subseteq d) \subseteq (a \subseteq d \diamond b) \quad (T_{R412})$$

Proof of (T_{R412})

$$(c \subseteq d) \subseteq (c \diamond b \subseteq d \diamond b)$$

(T_{R263})

$$((c \subseteq d) \subseteq (c \diamond b \subseteq d \diamond b)) = E$$

(A_{MX16})(Order Axiom)

$$((c \subseteq d) = E) \subseteq ((c \diamond b \subseteq d \diamond b) = E)$$

(T_{M094})(Probabilistic Rule)

$$((c \subseteq d) = E) \subseteq ((a \subseteq c \diamond b) \subseteq (a \subseteq d \diamond b))$$

(T_{M272})(Deductive Rule)

$$(a \subseteq c \diamond b)(c \subseteq d) \subseteq (a \subseteq d \diamond b)$$

(A_{MX35})(Exportation/Importation) for truth sets

$$((h \neq \emptyset) = E) \subseteq ((a \subseteq c \diamond b)(h \subseteq b) \subseteq (a \subseteq c \diamond h)) \quad (T_{R413})$$

Proof of (T_{R413})

$$(h \subseteq b)(h \neq \emptyset) \subseteq (c \diamond b \subseteq c \diamond h)$$

(T_{R262})

$$((h \subseteq b)(h \neq \emptyset) \subseteq (c \diamond b \subseteq c \diamond h)) = E$$

(A_{MX16})(Order Axiom)

$$((h \subseteq b)(h \neq \emptyset) = E) \subseteq ((c \diamond b \subseteq c \diamond h) = E)$$

(T_{M094})(Probabilistic Rule)

$$((h \subseteq b)(h \neq \emptyset) = E) \subseteq ((a \subseteq c \diamond b) \subseteq (a \subseteq c \diamond h))$$

(T_{M272})(Deductive Rule)

$$((h \neq \emptyset) = E) \subseteq ((a \subseteq c \diamond b)(h \subseteq b) \subseteq (a \subseteq c \diamond h))$$

(A_{MX35})(Exportation/Importation) for truth sets

$$((a \neq E)(c \diamond b \neq E)(d \diamond b \neq E) = E) \subseteq ((a \subseteq c \diamond b)(d \subseteq c) \subseteq (\sim(d \diamond b) \not\subseteq a)) \quad (T_{R414})$$

Proof of (T_{R414})

Let $(a \subseteq c \diamond b)$
 $(a \subseteq c \diamond b) = E$ ($AMX16$)(Order Axiom)(Eqn1)
 Let $(d \subseteq c)$
 $(d \subseteq c) = E$ ($AMX16$)(Order Axiom)(Eqn2)
 $d \diamond b \subseteq c \diamond b$ (T_{R263})(Relational Deduction)
 $(d \diamond b \subseteq c \diamond b) = E$ ($AMX16$)(Order Axiom)(Eqn3)
 $((a \neq E)(b \neq E)(c \neq E) = E) \subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b))$
(T_{M147})(Second Conditional Syllogism)(Eqn4)
 Let $b = d \diamond b$ (Eqn5)
 Let $c = c \diamond b$ (Eqn6)
 $((a \neq E)(d \diamond b \neq E)(c \diamond b \neq E) = E) \subseteq ((a \subseteq c \diamond b)(d \diamond b \subseteq c \diamond b) \subseteq (a \not\subseteq d \diamond b))$
($AMX19$)(Substitution of Eqns5&6 into Eqn4)(Eqn7)
 $((a \neq E)(d \diamond b \neq E)(c \diamond b \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq d \diamond b))$ ($AMX19$)(Substitution of Eqns1&3 into Eqn7)
 $((a \neq E)(d \diamond b \neq E)(c \diamond b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq d \diamond b))$ (T_{M667})(Conjunction Tautology)
 $((a \neq E)(d \diamond b \neq E)(c \diamond b \neq E) = E) \subseteq (E = (a \not\subseteq d \diamond b))$ (T_{M275})(Supersets of the Universal Set)
 $((a \neq E)(d \diamond b \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq d \diamond b)) = E$ (T_{M094})(Probabilistic Rule)
 $(a \neq E)(d \diamond b \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq d \diamond b)$ ($AMX16$)(Order Axiom)
 Thus $((a \subseteq c \diamond b)(d \subseteq c) = E) \subseteq ((a \neq E)(d \diamond b \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq d \diamond b))$
($AMX02$)(Conditional Introduction)
 $(a \neq E)(d \diamond b \neq E)(c \diamond b \neq E)(a \subseteq c \diamond b)(d \subseteq c) \subseteq (a \not\subseteq d \diamond b)$
($AMX35$)(Exportation/Importation) for truth sets
 $((a \neq E)(d \diamond b \neq E)(c \diamond b \neq E) = E) \subseteq ((a \subseteq c \diamond b)(d \subseteq c) \subseteq (a \not\subseteq d \diamond b))$
($AMX35$)(Exportation/Importation) for truth sets
 $((a \neq E)(d \diamond b \neq E)(c \diamond b \neq E) = E) \subseteq ((a \subseteq c \diamond b)(d \subseteq c) \subseteq (\sim(d \diamond b) \not\subseteq a))$
(T_{M012})(Transposition)

$$((a \neq E)(c \diamond b \neq E)(c \diamond h \neq E) = E) \subseteq ((a \subseteq c \diamond b)(b \subseteq h) \subseteq (\sim(c \diamond h) \not\subseteq a)) \quad (T_{R415})$$

Proof of (T_{R415})

Let $(a \subseteq c \diamond b)$
 $(a \subseteq c \diamond b) = E$ ($AMX16$)(Order Axiom)(Eqn1)
 Let $(b \subseteq h)$
 $(b \subseteq h) = E$ ($AMX16$)(Order Axiom)(Eqn2)
 $c \diamond h \subseteq c \diamond b$ (T_{R262})(Relational Deduction)
 $(c \diamond h \subseteq c \diamond b) = E$ ($AMX16$)(Order Axiom)(Eqn3)
 $((a \neq E)(b \neq E)(c \neq E) = E) \subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b))$
(T_{M147})(Second Conditional Syllogism)(Eqn4)
 Let $b = c \diamond h$ (Eqn5)
 Let $c = c \diamond b$ (Eqn6)
 $((a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) = E) \subseteq ((a \subseteq c \diamond b)(c \diamond h \subseteq c \diamond b) \subseteq (a \not\subseteq c \diamond h))$
($AMX19$)(Substitution of Eqns5&6 into Eqn4)(Eqn7)
 $((a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq c \diamond h))$ ($AMX19$)(Substitution of Eqns1&3 into Eqn7)
 $((a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq c \diamond h))$ (T_{M667})(Conjunction Tautology)
 $((a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) = E) \subseteq (E = (a \not\subseteq c \diamond h))$ (T_{M275})(Supersets of the Universal Set)
 $((a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h)) = E$ (T_{M094})(Probabilistic Rule)
 $(a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h)$ ($AMX16$)(Order Axiom)
 Thus $((a \subseteq c \diamond b)(b \subseteq h) = E) \subseteq ((a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h))$
($AMX02$)(Conditional Introduction)
 $(a \neq E)(c \diamond h \neq E)(c \diamond b \neq E)(a \subseteq c \diamond b)(b \subseteq h) \subseteq (a \not\subseteq c \diamond h)$
($AMX35$)(Exportation/Importation) for truth sets
 $((a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) = E) \subseteq ((a \subseteq c \diamond b)(b \subseteq h) \subseteq (a \not\subseteq c \diamond h))$
($AMX35$)(Exportation/Importation) for truth sets
 $((a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) = E) \subseteq ((a \subseteq c \diamond b)(b \subseteq h) \subseteq (\sim(c \diamond h) \not\subseteq a))$
(T_{M012})(Transposition)

$$(\sim(c \diamond b) \subseteq a)(c \subseteq d) \subseteq (\sim(d \diamond b) \subseteq a) \quad (T_{R416})$$

Proof of (T_{R416})

$$\begin{aligned} (a \subseteq c \diamond b)(c \subseteq d) &\subseteq (a \subseteq d \diamond b) & (T_{R412}) \\ (\underline{a} \subseteq c \diamond b)(c \subseteq d) &\subseteq (\underline{a} \subseteq d \diamond b) & \text{Resymbolising} \\ (\sim(c \diamond b) \subseteq a)(c \subseteq d) &\subseteq (\sim(d \diamond b) \subseteq a) & (T_{M012})(\text{Transposition}) \end{aligned}$$

$$((h \neq \emptyset) = E) \subseteq ((\sim(c \diamond b) \subseteq a)(h \subseteq b) \subseteq (\sim(c \diamond h) \subseteq a)) \quad (T_{R417})$$

Proof of (T_{R417})

$$\begin{aligned} ((h \neq \emptyset) = E) &\subseteq ((a \subseteq c \diamond b)(h \subseteq b) \subseteq (a \subseteq c \diamond h)) & (T_{R413}) \\ ((h \neq \emptyset) = E) &\subseteq ((\underline{a} \subseteq c \diamond b)(h \subseteq b) \subseteq (\underline{a} \subseteq c \diamond h)) & \text{Resymbolising} \\ ((h \neq \emptyset) = E) &\subseteq ((\sim(c \diamond b) \subseteq a)(h \subseteq b) \subseteq (\sim(c \diamond h) \subseteq a)) & (T_{M012})(\text{Transposition}) \end{aligned}$$

$$((a \neq \emptyset)(c \diamond b \neq E)(d \diamond b \neq E) = E) \subseteq ((\sim(c \diamond b) \subseteq a)(d \subseteq c) \subseteq (a \not\subseteq d \diamond b)) \quad (T_{R418})$$

Proof of (T_{R418})

$$\begin{aligned} \text{Let } (\sim(c \diamond b) \subseteq a) & & \\ \underline{a} \subseteq c \diamond b & & (T_{M012})(\text{Transposition}) \\ (\underline{a} \subseteq c \diamond b) = E & & (A_{MX16})(\text{Order Axiom})(\text{Eqn1}) \\ \text{Let } d \subseteq c & & \\ (d \subseteq c) = E & & (A_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\ d \diamond b \subseteq c \diamond b & & (T_{R263})(\text{Relational Deduction}) \\ (d \diamond b \subseteq c \diamond b) = E & & (A_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\ ((a \neq E)(b \neq E)(c \neq E) = E) &\subseteq ((a \subseteq c)(b \subseteq c) \subseteq (\underline{a} \not\subseteq b)) & \\ & & (T_{M141})(\text{Second Conditional Syllogism})(\text{Eqn4}) \\ \text{Let } a = \underline{a} & & (\text{Eqn5}) \\ \text{Let } c = c \diamond b & & (\text{Eqn6}) \\ \text{Let } b = d \diamond b & & (\text{Eqn7}) \\ ((\underline{a} \neq E)(d \diamond b \neq E)(c \diamond b \neq E) = E) &\subseteq ((\underline{a} \subseteq c \diamond b)(d \diamond b \subseteq c \diamond b) \subseteq (\underline{a} \not\subseteq d \diamond b)) & \\ & & (A_{MX19})(\text{Substitution of Eqns5,6\&7 into Eqn4})(\text{Eqn8}) \\ ((\underline{a} \neq E)(d \diamond b \neq E)(c \diamond b \neq E) = E) &\subseteq (EE \subseteq (a \not\subseteq d \diamond b)) & (A_{MX19})(\text{Substitution of Eqns1\&3 into Eqn8}) \\ ((\underline{a} \neq E)(d \diamond b \neq E)(c \diamond b \neq E) = E) &\subseteq (E \subseteq (a \not\subseteq d \diamond b)) & (T_{M667})(\text{Conjunction Tautology}) \\ ((\underline{a} \neq E)(d \diamond b \neq E)(c \diamond b \neq E) = E) &\subseteq (E = (a \not\subseteq d \diamond b)) & (T_{M275})(\text{Supersets of the Universal Set}) \\ ((\underline{a} \neq E)(d \diamond b \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq d \diamond b)) & = E & (T_{M094})(\text{Probabilistic Rule}) \\ (\underline{a} \neq E)(d \diamond b \neq E)(c \diamond b \neq E) &\subseteq (a \not\subseteq d \diamond b) & (A_{MX16})(\text{Order Axiom}) \\ (a \neq \sim E)(d \diamond b \neq E)(c \diamond b \neq E) &\subseteq (a \not\subseteq d \diamond b) & (T_{M008})(\text{Inverse of an Equality}) \\ (a \neq \emptyset)(d \diamond b \neq E)(c \diamond b \neq E) &\subseteq (a \not\subseteq d \diamond b) & (D_{M02})(\text{Definition of } \emptyset) \\ \text{Thus } ((\sim(c \diamond b) \subseteq a)(d \subseteq c) = E) &\subseteq ((a \neq \emptyset)(d \diamond b \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq d \diamond b)) & \\ & & (A_{MX02})(\text{Conditional Introduction}) \\ (\sim(c \diamond b) \subseteq a)(d \subseteq c)(a \neq \emptyset)(d \diamond b \neq E)(c \diamond b \neq E) &\subseteq (a \not\subseteq d \diamond b) & \\ & & (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\ ((a \neq \emptyset)(d \diamond b \neq E)(c \diamond b \neq E) = E) &\subseteq ((\sim(c \diamond b) \subseteq a)(d \subseteq c) \subseteq (a \not\subseteq d \diamond b)) & \\ & & (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \end{aligned}$$

$$((a \neq \emptyset)(c \diamond b \neq E)(c \diamond h \neq E) = E) \subseteq ((\sim(c \diamond b) \subseteq a)(b \subseteq h) \subseteq (a \not\subseteq c \diamond h)) \quad (T_{R419})$$

Proof of (T_{R419})

$$\begin{aligned} \text{Let } (\sim(c \diamond b) \subseteq a) & & \\ \underline{a} \subseteq c \diamond b & & (T_{M012})(\text{Transposition}) \\ (\underline{a} \subseteq c \diamond b) = E & & (A_{MX16})(\text{Order Axiom})(\text{Eqn1}) \\ \text{Let } b \subseteq h & & \\ (b \subseteq h) = E & & (A_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\ c \diamond h \subseteq c \diamond b & & (T_{R262})(\text{Relational Deduction}) \\ (c \diamond h \subseteq c \diamond b) = E & & (A_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\ ((a \neq E)(b \neq E)(c \neq E) = E) &\subseteq ((a \subseteq c)(b \subseteq c) \subseteq (\underline{a} \not\subseteq b)) & \\ & & (T_{M141})(\text{Second Conditional Syllogism})(\text{Eqn4}) \\ \text{Let } a = \underline{a} & & (\text{Eqn5}) \\ \text{Let } c = c \diamond b & & (\text{Eqn6}) \\ \text{Let } b = c \diamond h & & (\text{Eqn7}) \end{aligned}$$

$$\begin{aligned}
& ((a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) = E) \subseteq ((a \subseteq c \diamond b)(c \diamond h \subseteq c \diamond b) \subseteq (a \not\subseteq c \diamond h)) \\
& \quad (AMX19)(\text{Substitution of Eqns 5,6\&7 into Eqn4})(Eqn8) \\
& ((a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq c \diamond h)) \quad (AMX19)(\text{Substitution of Eqns 1\&3 into Eqn8}) \\
& ((a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq c \diamond h)) \quad (TM667)(\text{Conjunction Tautology}) \\
& ((a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) = E) \subseteq (E = (a \not\subseteq c \diamond h)) \quad (TM275)(\text{Supersets of the Universal Set}) \\
& ((a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h)) = E \quad (TM094)(\text{Probabilistic Rule}) \\
& (a \neq E)(c \diamond h \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h) \quad (AMX16)(\text{Order Axiom}) \\
& (a \neq \sim E)(c \diamond h \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h) \quad (TM008)(\text{Inverse of an Equality}) \\
& (a \neq \emptyset)(c \diamond h \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h) \quad (DM02)(\text{Definition of } \emptyset) \\
& \text{Thus } ((\sim(c \diamond b) \subseteq a)(b \subseteq h) = E) \subseteq ((a \neq \emptyset)(c \diamond h \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h)) \\
& \quad (AMX02)(\text{Conditional Introduction}) \\
& (\sim(c * b) \subseteq a)(b \subseteq h)(a \neq \emptyset)(c \diamond h \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h) \\
& \quad (AMX35)(\text{Exportation/Importation}) \text{ for truth sets} \\
& ((a \neq \emptyset)(c \diamond h \neq E)(c \diamond b \neq E) = E) \subseteq ((\sim(c \diamond b) \subseteq a)(b \subseteq h) \subseteq (a \not\subseteq c \diamond h)) \\
& \quad (AMX35)(\text{Exportation/Importation}) \text{ for truth sets}
\end{aligned}$$

$$((a \neq E)(c \diamond b \neq \emptyset)(d \diamond b \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c \diamond b))(c \subseteq d) \subseteq (d \diamond b \not\subseteq a)) \quad (TR420)$$

Proof of (TR420)

$$\begin{aligned}
& \text{Let } a \subseteq \sim(c \diamond b) \\
& (a \subseteq \sim(c \diamond b)) = E \quad (AMX16)(\text{Order Axiom})(Eqn1) \\
& \text{Let } c \subseteq d \\
& (c \subseteq d) = E \quad (AMX16)(\text{Order Axiom})(Eqn2) \\
& c \diamond b \subseteq d \diamond b \quad (TR262)(\text{Relational Deduction}) \\
& (c \diamond b \subseteq d \diamond b) = E \quad (AMX16)(\text{Order Axiom})(Eqn3) \\
& ((a \neq E)(b \neq E)(c \neq E) = E) \subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b)) \\
& \quad (TM141)(\text{Second Conditional Syllogism})(Eqn4) \\
& \text{Let } b = \sim(d * b) \quad (Eqn5) \\
& \text{Let } c = \sim(c * b) \quad (Eqn6) \\
& ((a \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq ((a \subseteq \sim(c \diamond b))(\sim(d \diamond b) \subseteq \sim(c \diamond b)) \subseteq (a \not\subseteq \sim(d \diamond b))) \\
& \quad (AMX19)(\text{Substitution of Eqns 5\&6 into Eqn4}) \\
& ((a \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq ((a \subseteq \sim(c \diamond b))(c \diamond b \subseteq d \diamond b) \subseteq (a \not\subseteq \sim(d \diamond b))) \\
& \quad (TM012)(\text{Transposition})(Eqn7) \\
& ((a \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq \sim(d \diamond b))) \\
& \quad (AMX19)(\text{Substitution of Eqns 1\&3 into Eqn4})(Eqn7) \\
& ((a \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq (E \subseteq (a \not\subseteq \sim(d \diamond b))) \\
& \quad (TM667)(\text{Conjunction Tautology}) \\
& ((a \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq (E = (a \not\subseteq \sim(d \diamond b))) \\
& \quad (TM275)(\text{Supersets of the Universal Set}) \\
& ((a \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(d \diamond b))) = E \\
& \quad (TM094)(\text{Probabilistic Rule}) \\
& (a \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(d \diamond b)) \quad (AMX16)(\text{Order Axiom}) \\
& (a \neq E)(d \diamond b \neq \sim E)(c \diamond b \neq \sim E) \subseteq (a \not\subseteq \sim(d \diamond b)) \quad (TM008)(\text{Inverse of an Equality}) \\
& (a \neq E)(d \diamond b \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(d \diamond b)) \quad (DM02)(\text{Definition of } \emptyset) \\
& (a \neq E)(d \diamond b \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (d \diamond b \not\subseteq a) \quad (TM012)(\text{Transposition}) \& (TM008)(\text{Inverse of an Equality}) \\
& \text{Thus } ((a \subseteq \sim(c \diamond b))(c \subseteq d) = E) \subseteq ((a \neq E)(d \diamond b \neq \emptyset)(c * b \neq \emptyset) \subseteq (d \diamond b \not\subseteq a)) \\
& \quad (AMX02)(\text{Conditional Introduction}) \\
& (a \subseteq \sim(c \diamond b))(c \subseteq d)(a \neq E)(d \diamond b \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (d \diamond b \not\subseteq a) \\
& \quad (AMX35)(\text{Exportation/Importation}) \text{ for truth sets} \\
& ((a \neq E)(d \diamond b \neq \emptyset)(c \diamond b \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c \diamond b))(c \subseteq d) \subseteq (d \diamond b \not\subseteq a)) \\
& \quad (AMX35)(\text{Exportation/Importation}) \text{ for truth sets}
\end{aligned}$$

$$((a \neq E)(c \diamond b \neq \emptyset)(c \diamond h \neq \emptyset) \subseteq ((a \subseteq \sim(c \diamond b))(h \subseteq b) \subseteq (c \diamond h \not\subseteq a)) \quad (TR421)$$

Proof of (TR421)

$$\begin{aligned}
& \text{Let } a \subseteq \sim(c \diamond b) \\
& (a \subseteq \sim(c \diamond b)) = E \quad (AMX16)(\text{Order Axiom})(Eqn1)
\end{aligned}$$

Let $h \subseteq b$
 $(h \subseteq b) = E$ ($AMX16$)(Order Axiom)(Eqn2)
 $c \diamond b \subseteq c \diamond h$ ($TR262$)(Relational Deduction)
 $(c \diamond b \subseteq c \diamond h) = E$ ($AMX16$)(Order Axiom)(Eqn3)
 $((a \neq E)(b \neq E)(c \neq E) = E) \subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b))$
($TM147$)(Second Conditional Syllogism)(Eqn4)
 Let $b = \sim(c \diamond h)$ (Eqn5)
 Let $c = \sim(c \diamond b)$ (Eqn6)
 $((a \neq E)(\sim(c \diamond h) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq ((a \subseteq \sim(c \diamond b))(\sim(c \diamond h) \subseteq \sim(c \diamond b)) \subseteq (a \not\subseteq \sim(c \diamond h)))$
($AMX19$)(Substitution of Eqns5&6 into Eqn4)
 $((a \neq E)(\sim(c \diamond h) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq ((a \subseteq \sim(c \diamond b))(c \diamond b \subseteq c \diamond h) \subseteq (a \not\subseteq \sim(c \diamond h)))$
($TM012$)(Transposition)(Eqn7)
 $((a \neq E)(\sim(c \diamond h) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq \sim(c \diamond h)))$
($AMX19$)(Substitution of Eqns1&3 into Eqn7)
 $((a \neq E)(\sim(c \diamond h) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq (E \subseteq (a \not\subseteq \sim(c \diamond h)))$
($TM667$)(Conjunction Tautology)
 $((a \neq E)(\sim(c \diamond h) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq (E = (a \not\subseteq \sim(c \diamond h)))$
($TM275$)(Supersets of the Universal Set)
 $((a \neq E)(\sim(c \diamond h) \neq E)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(c \diamond h)) = E)$
($TM094$)(Probabilistic Rule)
 $(a \neq E)(\sim(c \diamond h) \neq E)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(c \diamond h))$ ($AMX16$)(Order Axiom)
 $(a \neq E)(c \diamond h \neq \sim E)(c \diamond b \neq \sim E) \subseteq (a \not\subseteq \sim(c \diamond h))$ ($TM008$)(Inverse of an Equality)
 $(a \neq E)(c \diamond h \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(c \diamond h))$ ($DM02$)(Definition of $\emptyset$)
 $(a \neq E)(c \diamond h \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (c \diamond h \not\subseteq a)$ ($TM012$)(Transposition) & ($TM008$)(Inverse of an Equality)
 Thus $((a \subseteq \sim(c \diamond b))(h \subseteq b) = E) \subseteq ((a \neq E)(c \diamond h \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (c \diamond h \not\subseteq a))$
($AMX02$)(Conditional Introduction)
 $(a \subseteq \sim(c \diamond b))(h \subseteq b)(a \neq E)(c \diamond h \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (c \diamond h \not\subseteq a)$
($AMX35$)(Exportation/Importation) for truth sets
 $((a \neq E)(c \diamond h \neq \emptyset)(c \diamond b \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c \diamond b))(h \subseteq b) \subseteq (c \diamond h \not\subseteq a))$
($AMX35$)(Exportation/Importation) for truth sets

$$(a \subseteq \sim(c \diamond b))(d \subseteq c) \subseteq (a \subseteq \sim(d \diamond b)) \quad (TR422)$$

Proof of ($TR422$)

Let $a \subseteq \sim(c \diamond b)$ (Eqn1)
 Let $d \subseteq c$
 $d \diamond b \subseteq c \diamond b$ ($TR263$)(Relational Deduction)
 $\sim(c \diamond b) \subseteq \sim(d \diamond b)$ ($TM012$)(Transposition)(Eqn2)
 $a \subseteq \sim(d \diamond b)$ ($TM139$)(Hypothetical Syllogism Eqns1&2)
 Thus $(a \subseteq \sim(c \diamond b))(d \subseteq c) \subseteq (a \subseteq \sim(d \diamond b))$ ($AMX02$)(Conditional Introduction)

$$(a \subseteq \sim(c \diamond b))(b \subseteq h) \subseteq (a \subseteq \sim(c \diamond h)) \quad (TR423)$$

Proof of ($TR423$)

Let $a \subseteq \sim(c \diamond b)$ (Eqn1)
 Let $b \subseteq h$
 $c \diamond h \subseteq c \diamond b$ ($TR262$)(Relational Deduction)
 $\sim(c \diamond b) \subseteq \sim(c \diamond h)$ ($TM012$)(Transposition)(Eqn2)
 $a \subseteq \sim(c \diamond h)$ ($TM139$)(Hypothetical Syllogism Eqns1&2)
 Thus $(a \subseteq \sim(c \diamond b))(b \subseteq h) \subseteq (a \subseteq \sim(c \diamond h))$ ($AMX02$)(Conditional Introduction)

$$((a \neq \emptyset)(c \diamond b \neq \emptyset)(d \diamond b \neq \emptyset) = E) \subseteq ((c \diamond b \subseteq a)(c \subseteq d) \subseteq (a \not\subseteq \sim(d \diamond b))) \quad (TR424)$$

Proof of ($TR424$)

Let $c \diamond b \subseteq a$
 $(c \diamond b \subseteq a) = E$ ($AMX16$)(Order Axiom)(Eqn1)
 Let $c \subseteq d$
 $(c \subseteq d) = E$ ($AMX16$)(Order Axiom)(Eqn2)

$c \diamond b \subseteq d \diamond b$	(T_{R263})(Relational Deduction)
$(c \diamond b \subseteq d \diamond b) = E$	($AMX16$)(Order Axiom)(Eqn3)
$((a \neq E)(b \neq E)(c \neq E) = E) \subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b))$	($TM141$)(Second Conditional Syllogism)(Eqn4)
Let $a = \underline{a}$	(Eqn5)
Let $b = \sim(d \diamond b)$	(Eqn6)
Let $c = \sim(c \diamond b)$	(Eqn7)
$((\underline{a} \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq ((\underline{a} \subseteq \sim(c \diamond b))(\sim(d \diamond b) \subseteq \sim(c \diamond b)) \subseteq (a \not\subseteq \sim(d \diamond b)))$	($AMX19$)(Substitution of Eqns5,6&7 into Eqn4)
$((\underline{a} \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq ((c \diamond b \subseteq a)(c \diamond b \subseteq d \diamond b) \subseteq (a \not\subseteq \sim(d \diamond b)))$	($TM012$)(Transposition)(Eqn8)
$((\underline{a} \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq \sim(d \diamond b)))$	($AMX19$)(Substitution of Eqns1&3 into Eqn8)
$((\underline{a} \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq (E \subseteq (a \not\subseteq \sim(d \diamond b)))$	($TM667$)(Conjunction Tautology)
$((\underline{a} \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq (E = (a \not\subseteq \sim(d \diamond b)))$	($TM275$)(Supersets of the Universal Set)
$((\underline{a} \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(d \diamond b)) = E)$	($TM094$)(Probabilistic Rule)
$(\underline{a} \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(d \diamond b))$	($AMX16$)(Order Axiom)
$(a \neq \sim E)(d \diamond b \neq \sim E)(c \diamond b \neq \sim E) \subseteq (a \not\subseteq \sim(d \diamond b))$	($TM008$)(Inverse of an Equality)
$(a \neq \emptyset)(d \diamond b \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(d \diamond b))$	($DM02$)(Definition of $\emptyset$)
Thus $((c \diamond b \subseteq a)(c \subseteq d) = E) \subseteq ((a \neq \emptyset)(d \diamond b \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(d \diamond b)))$	($AMX02$)(Conditional Introduction)
$(c \diamond b \subseteq a)(c \subseteq d)(a \neq \emptyset)(d \diamond b \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(d \diamond b))$	($AMX35$)(Exportation/Importation) for truth sets
$((a \neq \emptyset)(d \diamond b \neq \emptyset)(c \diamond b \neq \emptyset) = E) \subseteq ((c \diamond b \subseteq a)(c \subseteq d) \subseteq (a \not\subseteq \sim(d \diamond b)))$	($AMX35$)(Exportation/Importation) for truth sets

$$((a \neq \emptyset)(c \diamond b \neq \emptyset)(c \diamond h \neq \emptyset) = E) \subseteq ((c \diamond b \subseteq a)(h \subseteq b) \subseteq (a \not\subseteq \sim(c \diamond h))) \quad (T_{R425})$$

Proof of (T_{R425})

Let $c \diamond b \subseteq a$	
$(c \diamond b \subseteq a) = E$	($AMX16$)(Order Axiom)(Eqn1)
Let $h \subseteq b$	
$(h \subseteq b) = E$	($AMX16$)(Order Axiom)(Eqn2)
$c \diamond b \subseteq c \diamond h$	(T_{R262})(Relational Deduction)
$(c \diamond b \subseteq c \diamond h) = E$	($AMX16$)(Order Axiom)(Eqn3)
$((a \neq E)(b \neq E)(c \neq E) = E) \subseteq ((a \subseteq c)(b \subseteq c) \subseteq (a \not\subseteq b))$	($TM141$)(Second Conditional Syllogism)(Eqn4)
Let $a = \underline{a}$	(Eqn5)
Let $b = \sim(c \diamond h)$	(Eqn6)
Let $c = \sim(c \diamond b)$	(Eqn7)
$((\underline{a} \neq E)(\sim(c \diamond h) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq ((\underline{a} \subseteq \sim(c \diamond b))(\sim(c \diamond h) \subseteq \sim(c \diamond b)) \subseteq (a \not\subseteq \sim(c \diamond h)))$	($AMX19$)(Substitution of Eqns5,6&7 into Eqn4)
$((\underline{a} \neq E)(\sim(d \diamond b) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq ((c \diamond b \subseteq a)(c \diamond b \subseteq d \diamond b) \subseteq (a \not\subseteq \sim(d \diamond b)))$	($TM012$)(Transposition)(Eqn8)
$((\underline{a} \neq E)(\sim(c \diamond h) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq \sim(c \diamond h)))$	($AMX19$)(Substitution of Eqns1&3 into Eqn8)
$((\underline{a} \neq E)(\sim(c \diamond h) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq (E \subseteq (a \not\subseteq \sim(c \diamond h)))$	($TM667$)(Conjunction Tautology)
$((\underline{a} \neq E)(\sim(c \diamond h) \neq E)(\sim(c \diamond b) \neq E) = E) \subseteq (E = (a \not\subseteq \sim(c \diamond h)))$	($TM275$)(Supersets of the Universal Set)
$((\underline{a} \neq E)(\sim(c \diamond h) \neq E)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(c \diamond h)) = E)$	($TM094$)(Probabilistic Rule)
$(\underline{a} \neq E)(\sim(c \diamond h) \neq E)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(c \diamond h))$	($AMX16$)(Order Axiom)
$(a \neq \sim E)(c \diamond h \neq \sim E)(c \diamond b \neq \sim E) \subseteq (a \not\subseteq \sim(c \diamond h))$	($TM008$)(Inverse of an Equality)
$(a \neq \emptyset)(c \diamond h \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(c \diamond h))$	($DM02$)(Definition of $\emptyset$)

Thus $((c \diamond b \subseteq a)(h \subseteq b) = E) \subseteq ((a \neq \emptyset)(c \diamond h \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(c \diamond h)))$
 $(\mathbf{AMX02})(\text{Conditional Introduction})$
 $(c \diamond b \subseteq a)(h \subseteq b)(a \neq \emptyset)(c \diamond h \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(c \diamond h))$
 $(\mathbf{AMX35})(\text{Exportation/Importation})$ for truth sets
 $((a \neq \emptyset)(c \diamond h \neq \emptyset)(c \diamond b \neq \emptyset) = E) \subseteq ((c \diamond b \subseteq a)(h \subseteq b) \subseteq (a \not\subseteq \sim(c \diamond h)))$
 $(\mathbf{AMX35})(\text{Exportation/Importation})$ for truth sets

$$(c \diamond b \subseteq a)(d \subseteq c) \subseteq (d \diamond b \subseteq a) \quad (\mathbf{TR426})$$

Proof of ($\mathbf{TR426}$)

Let $c \diamond b \subseteq a$ (Eqn1)
 Let $d \subseteq c$
 $d \diamond b \subseteq c \diamond b$ ($\mathbf{TR263}$)(Relational Deduction) (Eqn2)
 $d \diamond b \subseteq a$ ($\mathbf{TM139}$)(Hypothetical Syllogism Eqns1 & 2)
 Thus $(c \diamond b \subseteq a)(d \subseteq c) \subseteq (d \diamond b \subseteq a)$ ($\mathbf{AMX02}$)(Conditional Introduction)

$$(c \diamond b \subseteq a)(b \subseteq h) \subseteq (c \diamond h \subseteq a) \quad (\mathbf{TR427})$$

Proof of ($\mathbf{TR427}$)

Let $c \diamond b \subseteq a$ (Eqn1)
 Let $b \subseteq h$
 $c \diamond h \subseteq c \diamond b$ ($\mathbf{TR262}$)(Relational Deduction) (Eqn2)
 $c \diamond h \subseteq a$ ($\mathbf{TM139}$)(Hypothetical Syllogism Eqns1 & 2)
 Thus $(c \diamond b \subseteq a)(b \subseteq h) \subseteq (c \diamond h \subseteq a)$ ($\mathbf{AMX02}$)(Conditional Introduction)

$$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq ((a \not\subseteq c \diamond b)(d \subseteq c) \subseteq (a \not\subseteq d \diamond b)) \quad (\mathbf{TR428})$$

Proof of ($\mathbf{TR428}$)

Let $a \not\subseteq c \diamond b$
 $(a \not\subseteq c \diamond b) = E$ ($\mathbf{AMX16}$)(Order Axiom)(Eqn1)
 Let $d \subseteq c$
 $(d \subseteq c) = E$ ($\mathbf{AMX16}$)(Order Axiom)(Eqn2)
 $d \diamond b \subseteq c \diamond b$ ($\mathbf{TR263}$)(Relational Deduction)
 $(d \diamond b \subseteq c \diamond b) = E$ ($\mathbf{AMX16}$)(Order Axiom)(Eqn3)
 $((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$ ($\mathbf{TM140}$)(First Conditional Syllogism)(Eqn4)
 Let $a = d \diamond b$ (Eqn5)
 Let $b = a$ (Eqn6)
 Let $c = c \diamond b$ (Eqn7)
 $((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq ((d \diamond b \subseteq c \diamond b)(a \not\subseteq c \diamond b) \subseteq (a \not\subseteq d \diamond b))$
 $(\mathbf{AMX19})(\text{Substitution of Eqns5,6\&7 into Eqn4})(\text{Eqn8})$
 $((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq d \diamond b))$ ($\mathbf{AMX19}$)(Substitution of Eqns1\&3 into Eqn8)
 $((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq d \diamond b))$ ($\mathbf{TM667}$)(Conjunction Tautology)
 $((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq (E = (a \not\subseteq d \diamond b))$ ($\mathbf{TM275}$)(Supersets of the Universal Set)
 $((a \neq \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq d \diamond b)) = E$ ($\mathbf{TM094}$)(Probabilistic Rule)
 $(a \neq \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq d \diamond b)$ ($\mathbf{AMX16}$)(Order Axiom)
 Thus $((a \not\subseteq c \diamond b)(d \subseteq c) = E) \subseteq ((a \neq \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq d \diamond b))$
 $(\mathbf{AMX02})(\text{Conditional Introduction})$
 $(a \not\subseteq c \diamond b)(d \subseteq c)(a \neq \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq d \diamond b)$ ($\mathbf{AMX35}$)(Exportation/Importation) for truth sets
 $((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq ((a \not\subseteq c \diamond b)(d \subseteq c) \subseteq (a \not\subseteq d \diamond b))$
 $(\mathbf{AMX35})(\text{Exportation/Importation})$ for truth sets

$$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq ((a \not\subseteq c \diamond b)(b \subseteq h) \subseteq (a \not\subseteq c \diamond h)) \quad (\mathbf{TR429})$$

Proof of ($\mathbf{TR429}$)

Let $a \not\subseteq c \diamond b$
 $(a \not\subseteq c \diamond b) = E$ ($\mathbf{AMX16}$)(Order Axiom)(Eqn1)
 Let $b \subseteq h$
 $(b \subseteq h) = E$ ($\mathbf{AMX16}$)(Order Axiom)(Eqn2)

$c \diamond h \subseteq c \diamond b$	(T_{R262})(Relational Deduction)
$(c \diamond h \subseteq c \diamond b) = E$	($AMX16$)(Order Axiom)(Eqn3)
$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$	($TM140$)(First Conditional Syllogism)(Eqn4)
Let $a = c \diamond h$	(Eqn5)
Let $b = a$	(Eqn6)
Let $c = c \diamond b$	(Eqn7)
$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq ((c \diamond h \subseteq c \diamond b)(a \not\subseteq c \diamond b) \subseteq (a \not\subseteq c \diamond h))$	($AMX19$)(Substitution of Eqns5,6&7 into Eqn4)(Eqn8)
$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq c \diamond h))$	($AMX19$)(Substitution of Eqns1&3 into Eqn8)
$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq c \diamond h))$	($TM667$)(Conjunction Tautology)
$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq (E = (a \not\subseteq c \diamond h))$	($TM275$)(Supersets of the Universal Set)
$((a \neq \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h)) = E$	($TM094$)(Probabilistic Rule)
$(a \neq \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h)$	($AMX16$)(Order Axiom)
Thus $((a \not\subseteq c \diamond b)(b \subseteq h) = E) \subseteq ((a \neq \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h))$	($AMX02$)(Conditional Introduction)
$(a \not\subseteq c \diamond b)(b \subseteq h)(a \neq \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h)$	($AMX35$)(Exportation/Importation) for truth sets
$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq ((a \not\subseteq c \diamond b)(b \subseteq h) \subseteq (a \not\subseteq c \diamond h))$	($AMX35$)(Exportation/Importation) for truth sets

$$((a \neq E)(c \diamond b \neq E) = E) \subseteq ((\sim(c \diamond b) \not\subseteq a)(d \subseteq c) \subseteq (\sim(d \diamond b) \not\subseteq a)) \quad (T_{R430})$$

Proof of (T_{R430})

Let $\sim(c \diamond b) \not\subseteq a$	
$(\sim(c \diamond b) \not\subseteq a) = E$	($AMX16$)(Order Axiom)(Eqn1)
Let $d \subseteq c$	
$(d \subseteq c) = E$	($AMX16$)(Order Axiom)(Eqn2)
$d \diamond b \subseteq c \diamond b$	(T_{R263})(Relational Deduction)
$(d \diamond b \subseteq c \diamond b) = E$	($AMX16$)(Order Axiom)(Eqn3)
$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$	($TM140$)(First Conditional Syllogism)(Eqn4)
Let $a = d \diamond b$	(Eqn5)
Let $b = a$	(Eqn6)
Let $c = c \diamond b$	(Eqn7)
$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq ((d \diamond b \subseteq c \diamond b)(a \not\subseteq c \diamond b) \subseteq (a \not\subseteq d \diamond b))$	($AMX19$)(Substitution of Eqns5,6&7 into Eqn4)(Eqn8)
$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq d \diamond b))$	($AMX19$)(Substitution of Eqns1&3 into Eqn8)
$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq d \diamond b))$	($TM667$)(Conjunction Tautology)
$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq (E = (a \not\subseteq d \diamond b))$	($TM275$)(Supersets of the Universal Set)
$((a \neq \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq d \diamond b)) = E$	($TM094$)(Probabilistic Rule)
$(a \neq \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq d \diamond b)$	($AMX16$)(Order Axiom)
$(a \neq \sim \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq d \diamond b)$	($TM008$)(Inverse of an Equality)
$(a \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq d \diamond b)$	($TM054$)(Negative of Null Set)
$(a \neq E)(c \diamond b \neq E) \subseteq (\sim(d \diamond b) \not\subseteq a)$	($TM012$)(Transposition)
Thus $((\sim(c \diamond b) \not\subseteq a)(d \subseteq c) = E) \subseteq ((a \neq E)(c \diamond b \neq E) \subseteq (\sim(d \diamond b) \not\subseteq a))$	($AMX02$)(Conditional Introduction)
$(\sim(c \diamond b) \not\subseteq a)(d \subseteq c)(a \neq E)(c \diamond b \neq E) \subseteq (\sim(d \diamond b) \not\subseteq a)$	($AMX35$)(Exportation/Importation) for truth sets
$((a \neq E)(c \diamond b \neq E) = E) \subseteq ((\sim(c \diamond b) \not\subseteq a)(d \subseteq c) \subseteq (\sim(d \diamond b) \not\subseteq a))$	($AMX35$)(Exportation/Importation) for truth sets

$$((a \neq E)(c \diamond b \neq E) = E) \subseteq ((\sim(c \diamond b) \not\subseteq a)(b \subseteq h) \subseteq (\sim(c \diamond h) \not\subseteq a)) \quad (T_{R431})$$

Proof of (T_{R431})

Let $\sim(c \diamond b) \not\subseteq a$	
$(\sim(c \diamond b) \not\subseteq a) = E$	($AMX16$)(Order Axiom)(Eqn1)
Let $b \subseteq h$	
$(b \subseteq h) = E$	($AMX16$)(Order Axiom)(Eqn2)
$c \diamond h \subseteq c \diamond b$	(T_{R262})(Relational Deduction)

$(c \diamond h \subseteq c \diamond b) = E$	($AMX16$)(Order Axiom)(Eqn3)
$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$	($TM140$)(First Conditional Syllogism)(Eqn4)
Let $a = c \diamond h$	(Eqn5)
Let $b = a$	(Eqn6)
Let $c = c \diamond b$	(Eqn7)
$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq ((c \diamond h \subseteq c \diamond b)(a \not\subseteq c \diamond b) \subseteq (a \not\subseteq c \diamond h))$	($AMX19$)(Substitution of Eqns5,6&7 into Eqn4)(Eqn8)
$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq c \diamond h))$	($AMX19$)(Substitution of Eqns1&3 into Eqn8)
$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq (E \subseteq (a \not\subseteq c \diamond h))$	($TM667$)(Conjunction Tautology)
$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq (E = (a \not\subseteq c \diamond h))$	($TM275$)(Supersets of the Universal Set)
$((a \neq \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h)) = E$	($TM094$)(Probabilistic Rule)
$(a \neq \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h)$	($AMX16$)(Order Axiom)
$(a \neq \sim \emptyset)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h)$	($TM008$)(Inverse of an Equality)
$(a \neq E)(c \diamond b \neq E) \subseteq (a \not\subseteq c \diamond h)$	($TM054$)(Negative of Null Set)
$(a \neq E)(c \diamond b \neq E) \subseteq (\sim(c \diamond h) \not\subseteq a)$	($TM012$)(Transposition)
Thus $((\sim(c \diamond b) \not\subseteq a)(b \subseteq h) = E) \subseteq ((a \neq E)(c \diamond b \neq E) \subseteq (\sim(c \diamond h) \not\subseteq a))$	($AMX02$)(Conditional Introduction)
$(\sim(c \diamond b) \not\subseteq a)(b \subseteq h)(a \neq E)(c \diamond b \neq E) \subseteq (\sim(d \diamond b) \not\subseteq a)$	($AMX35$)(Exportation/Importation) for truth sets
$((a \neq E)(c \diamond b \neq E) = E) \subseteq ((\sim(c \diamond b) \not\subseteq a)(b \subseteq h) \subseteq (\sim(c \diamond h) \not\subseteq a))$	($AMX35$)(Exportation/Importation) for truth sets

$$((a \neq \emptyset)(c \diamond b \neq \emptyset) = E) \subseteq ((a \not\subseteq \sim(c \diamond b))(c \subseteq d) \subseteq (a \not\subseteq \sim(d \diamond b))) \quad (TR432)$$

Proof of ($TR432$)

Let $a \not\subseteq \sim(c \diamond b)$	
$(a \not\subseteq \sim(c \diamond b)) = E$	($AMX16$)(Order Axiom)(Eqn1)
Let $c \subseteq d$	
$(c \subseteq d) = E$	($AMX16$)(Order Axiom)(Eqn2)
$c \diamond b \subseteq d \diamond b$	($TR263$)(Relational Deduction)
$(c \diamond b \subseteq d \diamond b) = E$	($AMX16$)(Order Axiom)
$(\sim(d \diamond b) \subseteq \sim(c \diamond b)) = E$	($TM012$)(Transposition)(Eqn3)
$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$	($TM140$)(First Conditional Syllogism)(Eqn4)
Let $a = \sim(d \diamond b)$	(Eqn5)
Let $b = a$	(Eqn6)
Let $c = \sim(c \diamond b)$	(Eqn7)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq ((\sim(d \diamond b) \subseteq \sim(c \diamond b))(a \not\subseteq \sim(c \diamond b)) \subseteq (a \not\subseteq \sim(d \diamond b)))$	($AMX19$)(Substitution of Eqns5,6&7 into Eqn4)(Eqn8)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq \sim(d \diamond b)))$	($AMX19$)(Substitution of Eqns1&3 into Eqn8)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq (E \subseteq (a \not\subseteq \sim(d \diamond b)))$	($TM667$)(Conjunction Tautology)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq (E = (a \not\subseteq \sim(d \diamond b)))$	($TM275$)(Supersets of the Universal Set)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(d \diamond b))) = E$	($TM094$)(Probabilistic Rule)
$(a \neq \emptyset)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(d \diamond b))$	($AMX16$)(Order Axiom)
$(a \neq \emptyset)(c \diamond b \neq \sim E) \subseteq (a \not\subseteq \sim(d \diamond b))$	($TM008$)(Inverse of an Equality)
$(a \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(d \diamond b))$	($DM02$)(Definition of $\emptyset$)
Thus $((a \not\subseteq \sim(c \diamond b))(c \subseteq d) = E) \subseteq ((a \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(d \diamond b)))$	($AMX02$)(Conditional Introduction)
$(a \not\subseteq \sim(c \diamond b))(c \subseteq d)(a \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(d \diamond b))$	($AMX35$)(Exportation/Importation) for truth sets
$((a \neq \emptyset)(c \diamond b \neq \emptyset) = E) \subseteq ((a \not\subseteq \sim(c \diamond b))(c \subseteq d) \subseteq (a \not\subseteq \sim(d \diamond b)))$	($AMX35$)(Exportation/Importation) for truth sets

$$((a \neq \emptyset)(c \diamond b \neq \emptyset) = E) \subseteq ((a \not\subseteq \sim(c \diamond b))(h \subseteq b) \subseteq (a \not\subseteq \sim(c \diamond h))) \quad (TR433)$$

Proof of ($TR433$)

Let $a \not\subseteq \sim(c \diamond b)$	
$(a \not\subseteq \sim(c \diamond b)) = E$	($AMX16$)(Order Axiom)(Eqn1)

Let $h \subseteq b$	
$(h \subseteq b) = E$	$(\mathbf{A}_{MX16})$ (Order Axiom)(Eqn2)
$c \diamond b \subseteq c \diamond h$	$(\mathbf{T}_{R262})$ (Relational Deduction)
$(c \diamond b \subseteq c \diamond h) = E$	$(\mathbf{A}_{MX16})$ (Order Axiom)
$(\sim(c \diamond h) \subseteq \sim(c \diamond b)) = E$	$(\mathbf{T}_{M012})$ (Transposition)(Eqn3)
$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$	$(\mathbf{T}_{M140})$ (First Conditional Syllogism)(Eqn4)
Let $a = \sim(c \diamond h)$	(Eqn5)
Let $b = a$	(Eqn6)
Let $c = \sim(c \diamond b)$	(Eqn7)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq ((\sim(c \diamond h) \subseteq \sim(c \diamond b))(a \not\subseteq \sim(c \diamond b)) \subseteq (a \not\subseteq \sim(c \diamond h)))$	$(\mathbf{A}_{MX19})$ (Substitution of Eqns5,6& 7 into Eqn4)(Eqn8)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq \sim(c \diamond h)))$	$(\mathbf{A}_{MX19})$ (Substitution of Eqns1&3 into Eqn8)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq (E \subseteq (a \not\subseteq \sim(c \diamond h)))$	$(\mathbf{T}_{M667})$ (Conjunction Tautology)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq (E = (a \not\subseteq \sim(c \diamond h)))$	$(\mathbf{T}_{M275})$ (Supersets of the Universal Set)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(c \diamond h))) = E$	$(\mathbf{T}_{M094})$ (Probabilistic Rule)
$(a \neq \emptyset)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(c \diamond h))$	$(\mathbf{A}_{MX16})$ (Order Axiom)
$(a \neq \emptyset)(c \diamond b \neq \sim E) \subseteq (a \not\subseteq \sim(c \diamond h))$	$(\mathbf{T}_{M008})$ (Inverse of an Equality)
$(a \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(c \diamond h))$	$(\mathbf{D}_{M02})$ (Definition of $\emptyset$)
Thus $((a \not\subseteq \sim(c \diamond b))(h \subseteq b) = E) \subseteq ((a \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(c \diamond h)))$	$(\mathbf{A}_{MX02})$ (Conditional Introduction)
$(a \not\subseteq \sim(c \diamond b))(h \subseteq b)(a \neq \emptyset)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(c \diamond h))$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets
$((a \neq \emptyset)(c \diamond b \neq \emptyset) = E) \subseteq ((a \not\subseteq \sim(c \diamond b))(h \subseteq b) \subseteq (a \not\subseteq \sim(c \diamond h)))$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets

$$((a \neq E)(c \diamond b \neq \emptyset) = E) \subseteq ((c \diamond b \not\subseteq a)(c \subseteq d) \subseteq (d \diamond b \not\subseteq a)) \quad (\mathbf{T}_{R434})$$

Proof of $(\mathbf{T}_{R434})$

Let $c \diamond b \not\subseteq a$	
$(c \diamond b \not\subseteq a) = E$	$(\mathbf{A}_{MX16})$ (Order Axiom)
$(a \not\subseteq \sim(c \diamond b)) = E$	$(\mathbf{T}_{M012})$ (Transposition)(Eqn1)
Let $c \subseteq d$	
$(c \subseteq d) = E$	$(\mathbf{A}_{MX16})$ (Order Axiom)(Eqn2)
$c \diamond b \subseteq d \diamond b$	$(\mathbf{T}_{R263})$ (Relational Deduction)
$(c \diamond b \subseteq d \diamond b) = E$	$(\mathbf{A}_{MX16})$ (Order Axiom)
$(\sim(d \diamond b) \subseteq \sim(c \diamond b)) = E$	$(\mathbf{T}_{M012})$ (Transposition)(Eqn3)
$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$	$(\mathbf{T}_{M140})$ (First Conditional Syllogism)(Eqn4)
Let $a = \sim(d \diamond b)$	(Eqn5)
Let $b = a$	(Eqn6)
Let $c = \sim(c \diamond b)$	(Eqn7)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq ((\sim(d \diamond b) \subseteq \sim(c \diamond b))(a \not\subseteq \sim(c \diamond b)) \subseteq (a \not\subseteq \sim(d \diamond b)))$	$(\mathbf{A}_{MX19})$ (Substitution of Eqns5,6&7 into Eqn4)(Eqn8)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq \sim(d \diamond b)))$	$(\mathbf{A}_{MX19})$ (Substitution of Eqns1&3 into Eqn8)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq (E \subseteq (a \not\subseteq \sim(d \diamond b)))$	$(\mathbf{T}_{M667})$ (Conjunction Tautology)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq (E = (a \not\subseteq \sim(d \diamond b)))$	$(\mathbf{T}_{M275})$ (Supersets of the Universal Set)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(d \diamond b))) = E$	$(\mathbf{T}_{M094})$ (Probabilistic Rule)
$(a \neq \emptyset)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(d \diamond b))$	$(\mathbf{A}_{MX16})$ (Order Axiom)
$(a \neq \emptyset)(c \diamond b \neq \sim E) \subseteq (a \not\subseteq \sim(d \diamond b))$	$(\mathbf{T}_{M008})$ (Inverse of an Equality)
$(a \neq E)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(d \diamond b))$	$(\mathbf{T}_{M054})$ (Negative of Null Set) & $(\mathbf{D}_{M02})$ (Definition of $\emptyset$)
$(a \neq E)(c \diamond b \neq \emptyset) \subseteq (d \diamond b \not\subseteq a)$	$(\mathbf{T}_{M012})$ (Transposition)
Thus $((c \diamond b \not\subseteq a)(c \subseteq d) = E) \subseteq ((a \neq E)(c \diamond b \neq \emptyset) \subseteq (d \diamond b \not\subseteq a))$	$(\mathbf{A}_{MX02})$ (Conditional Introduction)
$(c \diamond b \not\subseteq a)(c \subseteq d)(a \neq E)(c \diamond b \neq \emptyset) \subseteq (d \diamond b \not\subseteq a)$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets
$((a \neq E)(c \diamond b \neq \emptyset) = E) \subseteq ((c \diamond b \not\subseteq a)(c \subseteq d) \subseteq (d \diamond b \not\subseteq a))$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets

$$((a \neq E)(c \diamond b \neq \emptyset) = E) \subseteq ((c \diamond b \not\subseteq a)(h \subseteq b) \subseteq (c \diamond h \not\subseteq a)) \quad (\mathbf{T}_{R435})$$

Proof of (T_{R435})

Let $c \diamond b \not\subseteq a$	
$(c \diamond b \not\subseteq a) = E$	($AMX16$)(Order Axiom)
$(a \not\subseteq \sim(c \diamond b)) = E$	($TM012$)(Transposition)(Eqn1)
Let $h \subseteq b$	
$(h \subseteq b) = E$	($AMX16$)(Order Axiom)(Eqn2)
$c \diamond b \subseteq c \diamond h$	($TR262$)(Relational Deduction)
$(c \diamond b \subseteq c \diamond h) = E$	($AMX16$)(Order Axiom)
$(\sim(c \diamond h) \subseteq \sim(c \diamond b)) = E$	($TM012$)(Transposition)(Eqn3)
$((b \neq \emptyset)(c \neq E) = E) \subseteq ((a \subseteq c)(b \not\subseteq c) \subseteq (b \not\subseteq a))$	($TM140$)(First Conditional Syllogism)(Eqn4)
Let $a = \sim(c \diamond h)$	(Eqn5)
Let $b = a$	(Eqn6)
Let $c = \sim(c \diamond b)$	(Eqn7)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq ((\sim(c \diamond h) \subseteq \sim(c \diamond b))(a \not\subseteq \sim(c \diamond b)) \subseteq (a \not\subseteq \sim(c \diamond h)))$	($AMX19$)(Substitution of Eqns5,6&7 into Eqn4)(Eqn8)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq (EE \subseteq (a \not\subseteq \sim(c \diamond h)))$	($AMX19$)(Substitution of Eqns1&3 into Eqn8)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq (E \subseteq (a \not\subseteq \sim(c \diamond h)))$	($TM667$)(Conjunction Tautology)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) = E) \subseteq (E = (a \not\subseteq \sim(c \diamond h)))$	($TM275$)(Supersets of the Universal Set)
$((a \neq \emptyset)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(c \diamond h))) = E$	($TM094$)(Probabilistic Rule)
$(a \neq \emptyset)(\sim(c \diamond b) \neq E) \subseteq (a \not\subseteq \sim(c \diamond h))$	($AMX16$)(Order Axiom)
$(a \neq \sim \emptyset)(c \diamond b \neq \sim E) \subseteq (a \not\subseteq \sim(c \diamond h))$	($TM008$)(Inverse of an Equality)
$(a \neq E)(c \diamond b \neq \emptyset) \subseteq (a \not\subseteq \sim(c \diamond h))$	($TM054$)(Negative of Null Set) & ($DM02$)(Definition of $\emptyset$)
$(a \neq E)(c \diamond b \neq \emptyset) \subseteq (c \diamond h \not\subseteq a)$	($TM012$)(Transposition)
Thus $((c \diamond b \not\subseteq a)(h \subseteq b) = E) \subseteq ((a \neq E)(c \diamond b \neq \emptyset) \subseteq (c \diamond h \not\subseteq a))$	($AMX02$)(Conditional Introduction)
$(c \diamond b \not\subseteq a)(h \subseteq b)(a \neq E)(c \diamond b \neq \emptyset) \subseteq (c \diamond h \not\subseteq a)$	($AMX35$)(Exportation/Importation) for truth sets
$((a \neq E)(c \diamond b \neq \emptyset) = E) \subseteq ((c \diamond b \not\subseteq a)(h \subseteq b) \subseteq (c \diamond h \not\subseteq a))$	($AMX35$)(Exportation/Importation) for truth sets

C. Relational Syllogisms with E as a Term based on ($TR262$) or ($TR263$).

The only theorems of relationship theory used in the proofs below are the relational syllogisms in the two sub-sections above, ($TR223$) and ($TR224$). Apart from this the proofs use membership theory.

$$(a \subseteq c * b) \subseteq (a \subseteq E * b) \quad (TR436)$$

Proof of ($TR436$)

$(a \subseteq c * b)(c \subseteq d) \subseteq (a \subseteq d * b)$	($TR388$)(Relational Syllogism)(Eqn1)
Let $d = E$	(Eqn2)
$(a \subseteq c * b)(c \subseteq E) \subseteq (a \subseteq E * b)$	($AMX19$)(Substitution of Eqn2 into Eqn1)
$(a \subseteq c * b)E \subseteq (a \subseteq E * b)$	($TM007$)(Universality of ($TM744$) Universal Set Theorem)
$(a \subseteq c * b) \subseteq (a \subseteq E * b)$	($AMX11$)(Universal Set Conjunction Axiom)

$$(a \subseteq c * b) \subseteq (a \subseteq c * E) \quad (TR437)$$

Proof of ($TR437$)

$(a \subseteq c * b)(b \subseteq h) \subseteq (a \subseteq c * h)$	($TR389$)(Relational Syllogism)(Eqn1)
Let $h = E$	(Eqn2)
$(a \subseteq c * b)(b \subseteq E) \subseteq (a \subseteq c * E)$	($AMX19$)(Substitution of Eqn2 into Eqn1)
$(a \subseteq c * b)E \subseteq (a \subseteq c * E)$	($TM007$)(Universality of ($TM744$) Universal Set Theorem)
$(a \subseteq c * b) \subseteq (a \subseteq c * E)$	($AMX11$)(Universal Set Conjunction Axiom)

$$(\sim(c * b) \subseteq a) \subseteq (\sim(E * b) \subseteq a) \quad (TR438)$$

Proof of ($TR438$)

$(a \subseteq c * b) \subseteq (a \subseteq E * b)$	($TR436$)
$(\sim(c * b) \subseteq a) \subseteq (\sim(E * b) \subseteq a)$	($TM012$)(Transposition)

$$(\sim(c*b) \subseteq a) \subseteq (\sim(E*b) \subseteq a)$$

Resymbolising

$$(\sim(c*b) \subseteq a) \subseteq (\sim(c*E) \subseteq a) \quad (T_{R439})$$

Proof of (T_{R439})

$$(a \subseteq c*b) \subseteq (a \subseteq c*E)$$

(T_{R437})

$$(\sim(c*b) \subseteq a) \subseteq (\sim(c*E) \subseteq a)$$

(T_{M012})(Transposition)

$$(\sim(c*b) \subseteq a) \subseteq (\sim(c*E) \subseteq a)$$

Resymbolising

$$((a \neq E)(c*b \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c*b)) \subseteq (E*b \not\subseteq a)) \quad (T_{R440})$$

Proof of (T_{R440})

$$((a \neq E)(c*b \neq \emptyset)(d*b \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c*b))(c \subseteq d) \subseteq (d*b \not\subseteq a))$$

(T_{R396})(Eqn1)Let $d = E$

(Eqn2)

$$((a \neq E)(c*b \neq \emptyset)(E*b \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c*b))(c \subseteq E) \subseteq (E*b \not\subseteq a))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$((a \neq E)(c*b \neq \emptyset)(E*b \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c*b))E \subseteq (E*b \not\subseteq a))$$

(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)

$$((a \neq E)(c*b \neq \emptyset)(E*b \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c*b)) \subseteq (E*b \not\subseteq a))$$

(A_{MX11})(Universal Set Conjunction Axiom)(Eqn3)

$$c*b \subseteq E*b$$

(T_{R224})

$$(c*b \neq \emptyset) \subseteq (E*b \neq \emptyset)$$

(T_{M096})(Probabilistic Rule)

$$(c*b \neq \emptyset) = (c*b \neq \emptyset)(E*b \neq \emptyset)$$

(A_{MX44})($\subseteq$ Equality Form)(Eqn4)

$$((a \neq E)(c*b \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c*b)) \subseteq (E*b \not\subseteq a))$$

(A_{MX19})(Substitution of Eqn4 into Eqn3)

$$((a \neq E)(c*b \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c*b)) \subseteq (c*E \not\subseteq a)) \quad (T_{R441})$$

Proof of (T_{R441})

$$((a \neq E)(c*b \neq \emptyset)(c*h \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c*b))(b \subseteq h) \subseteq (c*h \not\subseteq a))$$

(T_{R397})(Eqn1)Let $h = E$

(Eqn2)

$$((a \neq E)(c*b \neq \emptyset)(c*E \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c*b))(b \subseteq E) \subseteq (c*E \not\subseteq a))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$((a \neq E)(c*b \neq \emptyset)(c*E \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c*b))E \subseteq (c*E \not\subseteq a))$$

(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)

$$((a \neq E)(c*b \neq \emptyset)(c*E \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c*b)) \subseteq (c*E \not\subseteq a))$$

(A_{MX11})(Universal Set Conjunction Axiom)(Eqn3)

$$c*b \subseteq c*E$$

(T_{R223})

$$(c*b \neq \emptyset) \subseteq (c*E \neq \emptyset)$$

(T_{M096})(Probabilistic Rule)

$$(c*b \neq \emptyset) = (c*b \neq \emptyset)(c*E \neq \emptyset)$$

(A_{MX44})($\subseteq$ Equality Form)(Eqn4)

$$((a \neq E)(c*b \neq \emptyset) = E) \subseteq ((a \subseteq \sim(c*b)) \subseteq (c*E \not\subseteq a)) \quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})$$

$$((a \neq \emptyset)(c*b \neq \emptyset) = E) \subseteq ((c*b \subseteq a) \subseteq (a \not\subseteq \sim(E*b))) \quad (T_{R442})$$

Proof of (T_{R442})

$$((a \neq \emptyset)(c*b \neq \emptyset)(d*b \neq \emptyset) = E) \subseteq ((c*b \subseteq a)(c \subseteq d) \subseteq (a \not\subseteq \sim(d*b)))$$

(T_{R400})(Eqn1)Let $d = E$

(Eqn2)

$$((a \neq \emptyset)(c*b \neq \emptyset)(E*b \neq \emptyset) = E) \subseteq ((c*b \subseteq a)(c \subseteq E) \subseteq (a \not\subseteq \sim(E*b)))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$((a \neq \emptyset)(c*b \neq \emptyset)(E*b \neq \emptyset) = E) \subseteq ((c*b \subseteq a)E \subseteq (a \not\subseteq \sim(E*b)))$$

(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)

$$((a \neq \emptyset)(c*b \neq \emptyset)(E*b \neq \emptyset) = E) \subseteq ((c*b \subseteq a) \subseteq (a \not\subseteq \sim(E*b)))$$

(A_{MX11})(Universal Set Conjunction Axiom)(Eqn3)

$$c*b \subseteq E*b$$

(T_{R224})

$$\begin{aligned}
(c*b \neq \emptyset) &\subseteq (E*b \neq \emptyset) && (T_{M096})(\text{Probabilistic Rule}) \\
(c*b \neq \emptyset) &= (c*b \neq \emptyset)(E*b \neq \emptyset) && (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn4}) \\
((a \neq \emptyset)(c*b \neq \emptyset) = E) &\subseteq ((c*b \subseteq a) \subseteq (a \not\subseteq \sim(E*b))) && (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})
\end{aligned}$$

$$((a \neq \emptyset)(c*b \neq \emptyset) = E) \subseteq ((c*b \subseteq a) \subseteq (a \not\subseteq \sim(c*E))) \quad (T_{R443})$$

Proof of (T_{R443})

$$\begin{aligned}
((a \neq \emptyset)(c*b \neq \emptyset)(c*h \neq \emptyset) = E) &\subseteq ((c*b \subseteq a)(b \subseteq h) \subseteq (a \not\subseteq \sim(c*h))) && (T_{R401})(\text{Eqn1}) \\
\text{Let } h &= E && (\text{Eqn2}) \\
((a \neq \emptyset)(c*b \neq \emptyset)(c*E \neq \emptyset) = E) &\subseteq ((c*b \subseteq a)(b \subseteq E) \subseteq (a \not\subseteq \sim(c*E))) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
((a \neq \emptyset)(c*b \neq \emptyset)(c*E \neq \emptyset) = E) &\subseteq ((c*b \subseteq a)E \subseteq (a \not\subseteq \sim(c*E))) && (T_{M001})(\text{Universality of (T}_{M744}\text{) Universal Set Theorem}) \\
((a \neq \emptyset)(c*b \neq \emptyset)(c*E \neq \emptyset) = E) &\subseteq ((c*b \subseteq a) \subseteq (a \not\subseteq \sim(c*E))) && (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn3}) \\
c*b &\subseteq c*E && (T_{R223}) \\
(c*b \neq \emptyset) &\subseteq (c*E \neq \emptyset) && (T_{M096})(\text{Probabilistic Rule}) \\
(c*b \neq \emptyset) &= (c*b \neq \emptyset)(c*E \neq \emptyset) && (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn4}) \\
((a \neq \emptyset)(c*b \neq \emptyset) = E) &\subseteq ((c*b \subseteq a) \subseteq (a \not\subseteq \sim(c*E))) && (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})
\end{aligned}$$

$$((a \neq \emptyset)(c*b \neq \emptyset) = E) \subseteq ((a \not\subseteq \sim(c*b)) \subseteq (a \not\subseteq \sim(E*b))) \quad (T_{R444})$$

Proof of (T_{R444})

$$\begin{aligned}
((a \neq \emptyset)(c*b \neq \emptyset) = E) &\subseteq ((a \not\subseteq \sim(c*b))(c \subseteq d) \subseteq (a \not\subseteq \sim(d*b))) && (T_{R408})(\text{Eqn1}) \\
\text{Let } d &= E && (\text{Eqn2}) \\
((a \neq \emptyset)(c*b \neq \emptyset) = E) &\subseteq ((a \not\subseteq \sim(c*b))(c \subseteq E) \subseteq (a \not\subseteq \sim(E*b))) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
((a \neq \emptyset)(c*b \neq \emptyset) = E) &\subseteq ((a \not\subseteq \sim(c*b))E \subseteq (a \not\subseteq \sim(E*b))) && (T_{M001})(\text{Universality of (T}_{M744}\text{) Universal Set Theorem}) \\
((a \neq \emptyset)(c*b \neq \emptyset) = E) &\subseteq ((a \not\subseteq \sim(c*b)) \subseteq (a \not\subseteq \sim(E*b))) && (A_{MX11})(\text{Universal Set Conjunction Axiom})
\end{aligned}$$

$$((a \neq \emptyset)(c*b \neq \emptyset) = E) \subseteq ((a \not\subseteq \sim(c*b)) \subseteq (a \not\subseteq \sim(c*E))) \quad (T_{R445})$$

Proof of (T_{R445})

$$\begin{aligned}
((a \neq \emptyset)(c*b \neq \emptyset) = E) &\subseteq ((a \not\subseteq \sim(c*b))(b \subseteq h) \subseteq (a \not\subseteq \sim(c*h))) && (T_{R409})(\text{Eqn1}) \\
\text{Let } h &= E && (\text{Eqn2}) \\
((a \neq \emptyset)(c*b \neq \emptyset) = E) &\subseteq ((a \not\subseteq \sim(c*b))(b \subseteq E) \subseteq (a \not\subseteq \sim(c*E))) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
((a \neq \emptyset)(c*b \neq \emptyset) = E) &\subseteq ((a \not\subseteq \sim(c*b))E \subseteq (a \not\subseteq \sim(c*E))) && (T_{M001})(\text{Universality of (T}_{M744}\text{) Universal Set Theorem}) \\
((a \neq \emptyset)(c*b \neq \emptyset) = E) &\subseteq ((a \not\subseteq \sim(c*b)) \subseteq (a \not\subseteq \sim(c*E))) && (A_{MX11})(\text{Universal Set Conjunction Axiom})
\end{aligned}$$

$$((a \neq E)(c*b \neq \emptyset) = E) \subseteq ((c*b \not\subseteq a) \subseteq (E*b \not\subseteq a)) \quad (T_{R446})$$

Proof of (T_{R446})

$$\begin{aligned}
((a \neq E)(c*b \neq \emptyset) = E) &\subseteq ((c*b \not\subseteq a)(c \subseteq d) \subseteq (d*b \not\subseteq a)) && (T_{R410})(\text{Eqn1}) \\
\text{Let } d &= E && (\text{Eqn2}) \\
((a \neq E)(c*b \neq \emptyset) = E) &\subseteq ((c*b \not\subseteq a)(c \subseteq E) \subseteq (E*b \not\subseteq a)) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$((a \neq E)(c * b \neq \emptyset) = E) \subseteq ((c * b \not\subseteq a)E \subseteq (E * b \not\subseteq a))$$

(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)

$$((a \neq E)(c * b \neq \emptyset) = E) \subseteq ((c * b \not\subseteq a) \subseteq (E * b \not\subseteq a))$$

($AMX11$)(Universal Set Conjunction Axiom)

$$((a \neq E)(c * b \neq \emptyset) = E) \subseteq ((c * b \not\subseteq a) \subseteq (c * E \not\subseteq a))$$

(T_{R447})

Proof of (T_{R447})

$$((a \neq E)(c * b \neq \emptyset) = E) \subseteq ((c * b \not\subseteq a)(b \subseteq h) \subseteq (c * h \not\subseteq a))$$

(T_{R411})(Eqn1)

Let $h = E$

(Eqn2)

$$((a \neq E)(c * b \neq \emptyset) = E) \subseteq ((c * b \not\subseteq a)(b \subseteq E) \subseteq (c * E \not\subseteq a))$$

($AMX19$)(Substitution of Eqn2 into Eqn1)

$$((a \neq E)(c * b \neq \emptyset) = E) \subseteq ((c * b \not\subseteq a)E \subseteq (c * E \not\subseteq a))$$

(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)

$$((a \neq E)(c * b \neq \emptyset) = E) \subseteq ((c * b \not\subseteq a) \subseteq (c * E \not\subseteq a))$$

($AMX11$)(Universal Set Conjunction Axiom)

$$((a \neq E)(c \diamond b \neq E)(c \diamond E \neq E) = E) \subseteq ((a \subseteq c \diamond b) \subseteq (\sim(c \diamond E) \not\subseteq a)) \quad (T_{R448})$$

Proof of (T_{R448})

$$((a \neq E)(c \diamond b \neq E)(c \diamond h \neq E) = E) \subseteq ((a \subseteq c \diamond b)(b \subseteq h) \subseteq (\sim(c \diamond h) \not\subseteq a))$$

(T_{R415})(Eqn1)

Let $h = E$

(Eqn2)

$$((a \neq E)(c \diamond b \neq E)(c \diamond E \neq E) = E) \subseteq ((a \subseteq c \diamond b)(b \subseteq E) \subseteq (\sim(c \diamond E) \not\subseteq a))$$

($AMX19$)(Substitution of Eqn2 into Eqn1)

$$((a \neq E)(c \diamond b \neq E)(c \diamond E \neq E) = E) \subseteq ((a \subseteq c \diamond b)E \subseteq (\sim(c \diamond E) \not\subseteq a))$$

(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)

$$((a \neq E)(c \diamond b \neq E)(c \diamond E \neq E) = E) \subseteq ((a \subseteq c \diamond b) \subseteq (\sim(c \diamond E) \not\subseteq a))$$

($AMX11$)(Universal Set Conjunction Axiom)

$$((a \neq \emptyset)(c \diamond b \neq E)(c \diamond E \neq E) = E) \subseteq ((\sim(c \diamond b) \subseteq a) \subseteq (a \not\subseteq c \diamond E)) \quad (T_{R449})$$

Proof of (T_{R449})

$$((a \neq \emptyset)(c \diamond b \neq E)(c \diamond h \neq E) = E) \subseteq ((\sim(c \diamond b) \subseteq a)(b \subseteq h) \subseteq (a \not\subseteq c \diamond h))$$

(T_{R419})(Eqn1)

Let $h = E$

(Eqn2)

$$((a \neq \emptyset)(c \diamond b \neq E)(c \diamond E \neq E) = E) \subseteq ((\sim(c \diamond b) \subseteq a)(b \subseteq E) \subseteq (a \not\subseteq c \diamond E))$$

($AMX19$)(Substitution of Eqn2 into Eqn1)

$$((a \neq \emptyset)(c \diamond b \neq E)(c \diamond E \neq E) = E) \subseteq ((\sim(c \diamond b) \subseteq a)E \subseteq (a \not\subseteq c \diamond E))$$

(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)

$$((a \neq \emptyset)(c \diamond b \neq E)(c \diamond E \neq E) = E) \subseteq ((\sim(c \diamond b) \subseteq a) \subseteq (a \not\subseteq c \diamond E))$$

($AMX11$)(Universal Set Conjunction Axiom)

$$(a \subseteq \sim(c \diamond b)) \subseteq (a \subseteq \sim(c \diamond E)) \quad (T_{R450})$$

Proof of (T_{R450})

$$(a \subseteq \sim(c \diamond b))(b \subseteq h) \subseteq (a \subseteq \sim(c \diamond h))$$

(T_{R423})(Eqn1)

Let $h = E$

(Eqn2)

$$(a \subseteq \sim(c \diamond b))(b \subseteq E) \subseteq (a \subseteq \sim(c \diamond E))$$

($AMX19$)(Substitution of Eqn2 into Eqn1)

$$(a \subseteq \sim(c \diamond b))E \subseteq (a \subseteq \sim(c \diamond E))$$

(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)

$$(a \subseteq \sim(c \diamond b)) \subseteq (a \subseteq \sim(c \diamond E))$$

($AMX11$)(Universal Set Conjunction Axiom)

$$(c \diamond b \subseteq a) \subseteq (c \diamond E \subseteq a) \quad (T_{R451})$$

Proof of (T_{R451})

$$(c \diamond b \subseteq a)(b \subseteq h) \subseteq (c \diamond h \subseteq a)$$

(T_{R427})(Eqn1)

Let $h = E$

(Eqn2)

$$(c \diamond b \subseteq a)(b \subseteq E) \subseteq (c \diamond E \subseteq a)$$

($AMX19$)(Substitution of Eqn2 into Eqn1)

$$(c \diamond b \subseteq a)E \subseteq (c \diamond E \subseteq a)$$

(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)

$$(c \diamond b \subseteq a) \subseteq (c \diamond E \subseteq a)$$

($\mathbf{A}_{MX11}$)(Universal Set Conjunction Axiom)

$$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq ((a \not\subseteq c \diamond b) \subseteq (a \not\subseteq c \diamond E)) \quad (\mathbf{T}_{R452})$$

Proof of ($\mathbf{T}_{R452}$)

$$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq ((a \not\subseteq c \diamond b)(b \subseteq h) \subseteq (a \not\subseteq c \diamond h))$$

($\mathbf{T}_{R429}$)(Eqn1)

Let $h = E$

(Eqn2)

$$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq ((a \not\subseteq c \diamond b)(b \subseteq E) \subseteq (a \not\subseteq c \diamond E))$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq ((a \not\subseteq c \diamond b)E \subseteq (a \not\subseteq c \diamond E)) \quad (\mathbf{T}_{M001})(\text{Universality of } (\mathbf{T}_{M744}) \text{ Universal Set Theorem})$$

$$((a \neq \emptyset)(c \diamond b \neq E) = E) \subseteq ((a \not\subseteq c \diamond b) \subseteq (a \not\subseteq c \diamond E)) \quad (\mathbf{A}_{MX11})(\text{Universal Set Conjunction Axiom})$$

$$((a \neq E)(c \diamond b \neq E) = E) \subseteq ((\sim(c \diamond b) \not\subseteq a) \subseteq (\sim(c \diamond E) \not\subseteq a)) \quad (\mathbf{T}_{R453})$$

Proof of ($\mathbf{T}_{R453}$)

$$((a \neq E)(c \diamond b \neq E) = E) \subseteq ((\sim(c \diamond b) \not\subseteq a)(b \subseteq h) \subseteq (\sim(c \diamond h) \not\subseteq a))$$

($\mathbf{T}_{R431}$)(Eqn1)

Let $h = E$

(Eqn2)

$$((a \neq E)(c \diamond b \neq E) = E) \subseteq ((\sim(c \diamond b) \not\subseteq a)(b \subseteq E) \subseteq (\sim(c \diamond E) \not\subseteq a))$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$((a \neq E)(c \diamond b \neq E) = E) \subseteq ((\sim(c \diamond b) \not\subseteq a)E \subseteq (\sim(c \diamond E) \not\subseteq a))$$

($\mathbf{T}_{M001}$)(Universality of ($\mathbf{T}_{M744}$) Universal Set Theorem)

$$((a \neq E)(c \diamond b \neq E) = E) \subseteq ((\sim(c \diamond b) \not\subseteq a) \subseteq (\sim(c \diamond E) \not\subseteq a))$$

($\mathbf{A}_{MX11}$)(Universal Set Conjunction Axiom)

Chapter 5 Proofs of Relationship Theorems – Part 2 – Transitive Relationships

5.1 Introduction

Many of these theorems use the symbolism $\{\triangleright\}$, $\{\triangleleft\}$, $\{=\}$, $\{\triangleright\}$, $\{\triangleleft\}$, $\{\neq\}$, $\{\neq\}$, $\{\neq\}$ and $\{\neq\}$. $\{\triangleright\}$ and $\{\triangleleft\}$ are both general transitive asymmetrical relationship sets and are interchangeable. Thus, for any such theorem $\{\triangleleft\}$ may be replaced by $\{\triangleright\}$ and vice versa providing this is done consistently throughout the equation. Theorems altered in this way retain the same reference number, however. For example, the theorem (T_{A283}) may be written $\{\triangleleft\}*\{\triangleleft\}*b^1 \subseteq \{\triangleleft\}*b^1$ or $\{\triangleright\}*\{\triangleright\}*b^1 \subseteq \{\triangleright\}*b^1$.

Unless otherwise stated, any specific transitive asymmetrical relationship set $\{<\}$, $\{>\}$, $\{<\}$ or $\{>\}$ may be substituted for $\{\triangleleft\}$ or $\{\triangleright\}$.

5.2 Proofs using Membership Theory and the Axioms and Definitions of Relationship Theory

$$(c^1 \subseteq R_T)(a^1 \subseteq Qc) \subseteq (c * c * a^1 \subseteq c * a^1) \quad (T_{R625})$$

Proof of (T_{R625})

$$\begin{aligned} (c^1 \subseteq R_T) &\subseteq (c * c * a^1 \subseteq c * a^1) & (D_{R47})(\text{Definition of a Transitive Relationship Set}) \\ (c^1 \subseteq R_T)(a^1 \subseteq Qc) &\subseteq (c * c * a^1 \subseteq c * a^1) & (T_{M089})(\text{Single Premise Syllogism}) \end{aligned}$$

$$(c * c * d^1 \not\subseteq c * d^1) \subseteq (c^1 \subseteq \sim R_T) \quad (T_{R648}) \text{ Definition of an Intransitive Relationship Set}$$

Proof of (T_{R648})

$$\begin{aligned} (c^1 \subseteq R_T) &\subseteq (c * c * d^1 \subseteq c * d^1) & (D_{R47})(\text{Definition of a Transitive Relationship Set}) \\ (c * c * d^1 \not\subseteq c * d^1) &\subseteq (c^1 \not\subseteq R_T) & (T_{M012})(\text{Transposition}) \\ (c * c * d^1 \not\subseteq c * d^1) &\subseteq (c^1 \subseteq \sim R_T) & (T_{M604}) \end{aligned}$$

$$(c * c * d^1 \not\subseteq c * d^1) \subseteq (c^1 \subseteq \sim R_U) \quad (T_{R649}) \text{ Definition of a Non Anti-transitive Relationship Set}$$

Proof of (T_{R649})

$$\begin{aligned} (c^1 \subseteq R_U) &\subseteq (c * c * d^1 \subseteq c * d^1) & (D_{R46})(\text{Definition of an Anti-transitive Relationship Set}) \\ (c * c * d^1 \not\subseteq c * d^1) &\subseteq (c^1 \not\subseteq R_U) & (T_{M012})(\text{Transposition}) \\ (c * c * d^1 \not\subseteq c * d^1) &\subseteq (c^1 \subseteq \sim R_U) & (T_{M604}) \end{aligned}$$

$$R_A \subseteq R_B + R_L \quad (T_{T718})$$

Proof of (T_{T718})

$$\begin{aligned} (o = \{=\}, \{<\}, \{>\}, \{||\}, \{<=\}, \{>=\}) &= (o^1 \subseteq R_B) & (D_{R94})(\text{Definition of a Membership Relationship Set})(\text{Eqn1}) \\ (o = \{<\}, \{>\}) &\subseteq (o = \{=\}, \{<\}, \{>\}, \{||\}, \{<=\}, \{>=\}) & (T_{M753})(\text{Eqn2}) \\ (o = \{<\}, \{>\}) &\subseteq (o^1 \subseteq R_B) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})(\text{Eqn3}) \\ (o = \{=\}, \{<\}, \{>\}, \{<=\}, \{>=\}) &= (o^1 \subseteq R_L) & (D_{R96})(\text{Definition of an Ordering Relationship Set})(\text{Eqn4}) \\ (o = \{<\}, \{>\}) &\subseteq (o = \{=\}, \{<\}, \{>\}, \{<=\}, \{>=\}) & (T_{M753})(\text{Eqn5}) \\ (o = \{<\}, \{>\}) &\subseteq (o^1 \subseteq R_L) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn5})(\text{Eqn6}) \\ (o = \{<\}, \{>\}) + (o = \{<\}, \{>\}) &\subseteq (o^1 \subseteq R_B) + (o^1 \subseteq R_L) & (A_{MX21})(\text{Combination of Eqns3\&6}) \\ (o = \{<\}, \{>\}, \{<\}, \{>\}) &\subseteq (o^1 \subseteq R_B) + (o^1 \subseteq R_L) & (T_{M754}) \\ (o = \{<\}, \{>\}, \{<\}, \{>\}) &\subseteq (o^1 \subseteq R_B + R_L) & (T_{M170})(\text{Disjunction of Necessities})(\text{Eqn7}) \\ (o = \{<\}, \{>\}, \{<\}, \{>\}) &= (o^1 \subseteq R_A) & (A_{S11})(\text{Asymmetrical Relationship Sets})(\text{Eqn8}) \\ (o^1 \subseteq R_A) &\subseteq (o^1 \subseteq R_B + R_L) & (A_{MX19})(\text{Substitution of Eqn8 into Eqn7})(\text{Eqn6}) \\ (R_A \subseteq R_B + R_L) &= E & (T_{M757})(\text{Entity Mode / Set Mode Rule}) \text{ since } (o^1 \neq \emptyset) \\ R_A &\subseteq R_B + R_L & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$\{\neq\} = \{\neq\} \cdot \{=\} \quad (T_{A059})$$

Proof of (T_{A059})

$$\begin{aligned} \{\neq\} &= \{\neq\} + \{=\} & (D_{A141})(\text{Disjunction with } \{=\}) \\ \sim\{\neq\} &= \sim(\{\neq\} + \{=\}) & (T_{M008})(\text{Inverse of an Equality}) \\ \sim\{\neq\} &= \sim\{\neq\} \cdot \sim\{=\} & (T_{M026})(\text{DeMorgan's Law}) \\ \{\neq\} &= \{\neq\} \cdot \{=\} & (D_{A142})(\text{Definition of } \sim\{=\}) \end{aligned}$$

5.3 Higher Tier Proofs

$$((c^1 \subseteq R_T) = E) \subseteq ((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) \subseteq (a^1 \subseteq c * d^1)) \quad (T_{R623})$$

Proof of (T_{R623})

$$\begin{aligned} (c^1 \subseteq R_T) &\subseteq (c * c * d^1 \subseteq c * d^1) & (D_{R47})(\text{Definition of a Transitive Relationship Set}) \\ ((c^1 \subseteq R_T) \subseteq (c * c * d^1 \subseteq c * d^1)) &= E & (A_{MX16})(\text{Order Axiom}) \\ ((c^1 \subseteq R_T) = E) &\subseteq ((c * c * d^1 \subseteq c * d^1) = E) & (T_{M094})(\text{Probabilistic Rule})(\text{Eqn1}) \\ ((c * c * d^1 \subseteq c * d^1) = E) &\subseteq ((a^1 \subseteq c * c * d^1) \subseteq (a^1 \subseteq c * d^1)) & (T_{M272})(\text{Deductive Rule})(\text{Eqn2}) \\ ((c^1 \subseteq R_T) = E) &\subseteq ((a^1 \subseteq c * c * d^1) \subseteq (a^1 \subseteq c * d^1)) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\ ((c^1 \subseteq R_T) = E) &\subseteq E \cdot ((a^1 \subseteq c * c * d^1) \subseteq (a^1 \subseteq c * d^1)) & (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn3}) \\ (a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) &\subseteq (a^1 \subseteq c * c * d^1) & (T_{R389})(\text{Relational Syllogism}) \\ E &= ((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) \subseteq (a^1 \subseteq c * c * d^1)) & (A_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\ ((c^1 \subseteq R_T) = E) &\subseteq ((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) \subseteq (a^1 \subseteq c * c * d^1)) \cdot ((a^1 \subseteq c * c * d^1) \subseteq (a^1 \subseteq c * d^1)) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})(\text{Eqn5}) \\ ((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) \subseteq (a^1 \subseteq c * c * d^1)) &\cdot ((a^1 \subseteq c * c * d^1) \subseteq (a^1 \subseteq c * d^1)) \subseteq & \\ ((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) \subseteq (a^1 \subseteq c * d^1)) & & (T_{M139})(\text{Hypothetical Syllogism})(\text{Eqn6}) \\ ((c^1 \subseteq R_T) = E) &\subseteq ((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) \subseteq (a^1 \subseteq c * d^1)) & (T_{M139})(\text{Hypothetical Syllogism Eqns5\&6}) \end{aligned}$$

$$\{=\} * b = b \quad (T_{A080})$$

Proof of (T_{A080})

$$\begin{aligned} \text{Let } (b &= b_1^1 + \dots + b_n^1) & (\text{Eqn1}) \\ \{=\} * b &= \{=\} * b_1^1 + \dots + \{=\} * b_n^1 & (T_{R001a})(\text{Distribution of } * \text{ over } +) \\ \{=\} * b &= b_1^1 + \dots + b_n^1 & (D_{A144})(\text{Definition of the Reflexive Relationship Set})(\text{Eqn2}) \\ \{=\} * b &= b & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \end{aligned}$$

$$(ab = \emptyset) = (\forall Sa. \{=\}. \forall Qb = \emptyset) \quad (T_{R892})$$

Proof of (T_{R892})

$$\begin{aligned} (a(c * b) = \emptyset) &= (\forall Sa. c. \forall Qb = \emptyset) & (T_{R228})(\text{Eqn1}) \\ \text{Let } c &= \{=\} & (\text{Eqn2}) \\ (a(\{=\} * b) = \emptyset) &= (\forall Sa. \{=\}. \forall Qb = \emptyset) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (ab = \emptyset) &= (\forall Sa. \{=\}. \forall Qb = \emptyset) & (T_{A080}) \end{aligned}$$

$$S\{=\} = E \quad (T_{A251})$$

Proof of (T_{A251})

$$\begin{aligned} \{=\} * b &= b & (T_{A080})(\text{Eqn1}) \\ \text{Let } b &= E & (\text{Eqn2}) \\ \{=\} * E &= E & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ S(\{=\}. \forall QE) &= E & (A_{R27})(\text{Function Axiom}) \\ S(\{=\}. E) &= E & (T_{R116})(\text{Sub-relationships with the Universal Set}) \\ S\{=\} &= E & (A_{MX11})(\text{Universal Set Conjunction Axiom}) \end{aligned}$$

$$Q\{=\} = E \quad (T_{A252})$$

Proof of (T_{A252})

$$\begin{array}{ll}
S\{=\} = E & (T_{A251})(\text{Eqn1}) \\
\{=\} = \forall\{=\} & (D_{A94}) \text{ Definition of } \forall\{=\}(\text{Eqn2}) \\
S^{\forall}\{=\} = E & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
Q\{=\} = E & (T_{R016})(\text{Reversibility of Matching Entities})
\end{array}$$

$$\{=\} \neq \emptyset \quad (T_{S449})$$

Proof of (T_{S449})

$$\begin{array}{ll}
S\{=\} = E & (T_{A251}) \\
S\{=\} \neq \emptyset & (T_{M058}) \\
\{=\} \neq \emptyset & (T_{R048})(\text{Triad Theorem})
\end{array}$$

$$(b \neq \emptyset) = (\{ \neq \} \diamond b = \underline{b}) \quad (T_{A085}) \text{ Definition of } \sim$$

Proof of (T_{A085})

$$\begin{array}{ll}
(b \neq \emptyset) = (c * b = c \diamond b) & (A_{R10})(\text{Axiom of Complementary Relationship Sets}) \\
(b \neq \emptyset) = (c * b = c \diamond b) & (T_{M008})(\text{Inverse of an Equality})(\text{Eqn1}) \\
\text{Let } c = \{ \neq \} & (\text{Eqn2}) \\
(b \neq \emptyset) = (\{ \neq \} * b = \{ \neq \} \diamond b) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
(b \neq \emptyset) = (\sim(\{ \neq \} * b) = \{ \neq \} \diamond b) & (D_{R04})(\text{Definition of } *) \\
(b \neq \emptyset) = (\sim b = \{ \neq \} \diamond b) & (T_{A080}) \\
(b \neq \emptyset) = (\underline{b} = \{ \neq \} \diamond b) & (D_{M03})(\text{Definition of } \sim)
\end{array}$$

$$b^1 = \{ \neq \} \diamond \sim(b^1) \quad (T_{R599})$$

Proof of (T_{R599})

$$\begin{array}{ll}
(b \neq \emptyset) = (\{ \neq \} \diamond b = \underline{b}) & (T_{A085})(\text{Definition of } \sim)(\text{Eqn1}) \\
\text{Let } \underline{b} = b^1 & (\text{Eqn2}) \\
b = \sim(b^1) & (T_{M008})(\text{Inverse of an Equality})(\text{Eqn3}) \\
(\sim(b^1) \neq \emptyset) = (\{ \neq \} \diamond \sim(b^1) = b^1) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\
(b^1 \neq \sim \emptyset) = (\{ \neq \} \diamond \sim(b^1) = b^1) & (T_{M008})(\text{Inverse of an Equality}) \\
(b^1 \neq E) = (\{ \neq \} \diamond \sim(b^1) = b^1) & (T_{M054})(\text{Negative of Null Set}) \\
E = (\{ \neq \} \diamond \sim(b^1) = b^1) & (T_{M001})(\text{Universality of } (T_{M575}) \text{ Plurality of Universal Set}) \\
\{ \neq \} \diamond \sim(b^1) = b^1 & (A_{MX16})(\text{Order Axiom})
\end{array}$$

$$(\{ \neq \} \diamond b \neq \emptyset) = (\{ \neq \} \diamond \{ \neq \} \diamond b = b) \quad (T_{A087})$$

Proof of (T_{A087})

$$\begin{array}{ll}
\sim \sim b = b & (A_{MX13})(\text{Double Complement Axiom}) \\
(b \neq \emptyset) = (\{ \neq \} \diamond b = \underline{b}) & (T_{A085})(\text{Definition of } \sim) \\
(b \neq \emptyset) = (\sim(\{ \neq \} \diamond b) = b) & (T_{M008})(\text{Inverse of an Equality}) \\
(b \neq \emptyset)(\{ \neq \} \diamond b \neq \emptyset) = (\{ \neq \} \diamond \{ \neq \} \diamond b = b) & (T_{A085})(\text{Definition of } \sim)(\text{Eqn1}) \\
(b = \emptyset) \subseteq (\{ \neq \} * b = \emptyset) & (T_{S303})(\text{Eqn2}) \\
(\{ \neq \} \diamond b) \subseteq (\{ \neq \} * b) & (T_{R241}) \\
((\{ \neq \} \diamond b) \subseteq (\{ \neq \} * b)) = E & (A_{MX16})(\text{Order Axiom}) \\
(\{ \neq \} * b = \emptyset) \subseteq (\{ \neq \} \diamond b = \emptyset) & (T_{M097})(\text{Probabilistic Rule})(\text{Eqn3}) \\
(b = \emptyset) \subseteq (\{ \neq \} \diamond b = \emptyset) & (T_{M139})(\text{Hypothetical Syllogism Eqns2\&3}) \\
(\{ \neq \} \diamond b \neq \emptyset) \subseteq (b \neq \emptyset) & (T_{M012})(\text{Transposition}) \\
(\{ \neq \} \diamond b \neq \emptyset) = (b \neq \emptyset)(\{ \neq \} \diamond b \neq \emptyset) & (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn4}) \\
(\{ \neq \} \diamond b \neq \emptyset) = (\{ \neq \} \diamond \{ \neq \} \diamond b = b) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn1})
\end{array}$$

$$(o = \{ \subset, \{ \supset, \{ \subset, \{ \supset \} \}) = (o^1 \subseteq R_T R_A (R_B + R_L)) \quad (T_{T719})$$

Proof of (T_{T719})

$$\begin{array}{ll}
(o^1 \subseteq R_A) = (o = \{ \subset, \{ \supset, \{ \subset, \{ \supset \} \}) & (A_{S11})(\text{Asymmetrical Relationship Sets})(\text{Eqn1}) \\
R_A \subseteq R_B + R_L & (T_{T718}) \\
R_A = R_A.(R_B + R_L) & (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn2})
\end{array}$$

$$\begin{aligned}
(o^1 \subseteq R_A.(R_B + R_L)) &= (o = \{\subseteq, \{\supset, \{\prec, \succ\}\}) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\
R_A &\subseteq R_T \quad (A_{S60})(\text{Transitive Asymmetrical Axiom}) \\
R_A &= R_T R_A \quad (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn4}) \\
(o^1 \subseteq R_T R_A.(R_B + R_L)) &= (o = \{\subseteq, \{\supset, \{\prec, \succ\}\}) \quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})
\end{aligned}$$

$$\{\prec\}^1 \subseteq R_T.R_A.(R_B + R_L) \quad (T_{T714}) \text{ Definition of } \prec$$

Proof of (T_{T714})

$$\begin{aligned}
(o = \{\subseteq, \{\supset, \{\prec, \succ\}\}) &= (o^1 \subseteq R_T R_A(R_B + R_L)) \quad (T_{T719})(\text{Eqn1}) \\
\{\prec\} &= \{\subseteq, \{\supset, \{\prec, \succ\}\} \quad (D_{S08})(\text{Definition of } \prec)(\text{Eqn2}) \\
(o^1 \subseteq R_T.R_A.(R_B + R_L)) &= (o = \{\prec\}) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
\{\prec\}^1 &\subseteq R_T.R_A.(R_B + R_L) \quad (A_{MX19})(\text{Substitution})
\end{aligned}$$

$$(o = \{\subseteq, \{\supset, \{\prec, \succ\}\}) = (o^1 \subseteq R_T R_A) \quad (T_{T717})$$

Proof of (T_{T717})

$$\begin{aligned}
(o = \{\subseteq, \{\supset, \{\prec, \succ\}\}) &= (o^1 \subseteq R_T R_A(R_B + R_L)) \quad (T_{T719})(\text{Eqn1}) \\
R_A &\subseteq R_B + R_L \quad (T_{T718}) \\
R_A &= R_A(R_B + R_L) \quad (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn2}) \\
(o = \{\subseteq, \{\supset, \{\prec, \succ\}\}) &= (o^1 \subseteq R_T R_A) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$\{\prec\}^1 \subseteq R_T R_A \quad (T_{T739})$$

Proof of (T_{T739})

$$\begin{aligned}
\{\prec\}^1 &\subseteq R_T.R_A.(R_B + R_L) \quad (T_{T714}) \text{ (Definition of } \prec) \text{ (Eqn1)} \\
R_A &\subseteq R_B + R_L \quad (T_{T718}) \\
R_A &= R_A(R_B + R_L) \quad (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn2}) \\
\{\prec\}^1 &\subseteq R_T R_A \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$(a^1 \subseteq \{\prec\} * c^1)(c^1 \subseteq \{\prec\} * b^1) \subseteq (a^1 \subseteq \{\prec\} * b^1) \quad (T_{S550})$$

Proof of (T_{S550})

$$\begin{aligned}
((c^1 \subseteq R_T) = E) &\subseteq ((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) \subseteq (a^1 \subseteq c * d^1)) \quad (T_{R623})(\text{Eqn1}) \\
(c^1 \subseteq R_T R_A) &\subseteq (c^1 \subseteq R_T) \quad (T_{M073})(\text{Single Premise Syllogism}) \\
((c^1 \subseteq R_T R_A) \subseteq (c^1 \subseteq R_T)) &= E \quad (A_{MX16})(\text{Order Axiom}) \\
((c^1 \subseteq R_T R_A) = E) &\subseteq ((c^1 \subseteq R_T) = E) \quad (T_{M094})(\text{Probabilistic Rule})(\text{Eqn2}) \\
((c^1 \subseteq R_T R_A) = E) &\subseteq ((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) \subseteq (a^1 \subseteq c * d^1)) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns2\&1})(\text{Eqn3}) \\
\text{Let } c &= \{\prec\} \quad (\text{Eqn4}) \\
((\{\prec\}^1 \subseteq R_T R_A) = E) &\subseteq ((a^1 \subseteq \{\prec\} * b^1)(b^1 \subseteq \{\prec\} * d^1) \subseteq (a^1 \subseteq \{\prec\} * d^1)) \quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\
(E = E) &\subseteq ((a^1 \subseteq \{\prec\} * b^1)(b^1 \subseteq \{\prec\} * d^1) \subseteq (a^1 \subseteq \{\prec\} * d^1)) \quad (T_{M001})(\text{Universality of } (T_{T739})) \\
E &\subseteq ((a^1 \subseteq \{\prec\} * b^1)(b^1 \subseteq \{\prec\} * d^1) \subseteq (a^1 \subseteq \{\prec\} * d^1)) \quad (T_{M342}) \\
E &= ((a^1 \subseteq \{\prec\} * b^1)(b^1 \subseteq \{\prec\} * d^1) \subseteq (a^1 \subseteq \{\prec\} * d^1)) \quad (T_{M275})(\text{Supersets of the Universal Set}) \\
((a^1 \subseteq \{\prec\} * b^1)(b^1 \subseteq \{\prec\} * d^1) &\subseteq (a^1 \subseteq \{\prec\} * d^1)) \quad (A_{MX16})(\text{Order Axiom}) \\
((a^1 \subseteq \{\prec\} * c^1)(c^1 \subseteq \{\prec\} * b^1) &\subseteq (a^1 \subseteq \{\prec\} * b^1)) \quad \text{Resymbolising}
\end{aligned}$$

$$((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) \not\subseteq (a^1 \subseteq c * d^1)) \subseteq ((c^1 \subseteq \sim R_T) \neq \emptyset) \quad (T_{R630})$$

Proof of (T_{R630})

$$\begin{aligned}
((c^1 \subseteq R_T) = E) &\subseteq ((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) \subseteq (a^1 \subseteq c * d^1)) \quad (T_{R623}) \\
((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) &\not\subseteq (a^1 \subseteq c * d^1)) \subseteq ((c^1 \subseteq R_T) \neq E) \quad (T_{M012})(\text{Transposition}) \\
((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) &\not\subseteq (a^1 \subseteq c * d^1)) \subseteq ((c^1 \not\subseteq R_T) \neq E) \quad (T_{M008})(\text{Inverse of an Equality}) \\
((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) &\not\subseteq (a^1 \subseteq c * d^1)) \subseteq ((c^1 \not\subseteq R_T) \neq \emptyset) \quad (D_{M02})(\text{Definition of } \emptyset)(\text{Eqn1}) \\
(c^1 \not\subseteq R_T) &\subseteq (c^1 \subseteq \sim R_T) \quad (T_{M604}) \\
((c^1 \not\subseteq R_T) \subseteq (c^1 \subseteq \sim R_T)) &= E \quad (A_{MX16})(\text{Order Axiom}) \\
((c^1 \not\subseteq R_T) \neq \emptyset) &\subseteq ((c^1 \subseteq \sim R_T) \neq \emptyset) \quad (T_{M096})(\text{Probabilistic Rule})(\text{Eqn2}) \\
((a^1 \subseteq c * b^1)(b^1 \subseteq c * d^1) &\not\subseteq (a^1 \subseteq c * d^1)) \subseteq ((c^1 \subseteq \sim R_T) \neq \emptyset) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})
\end{aligned}$$

$$(c^1 \subseteq R_T) \subseteq (c * c * a \subseteq c * a)$$

(T_{S177}) Definition of a Transitive Relationship Set

Proof of (T_{S177})

$$\begin{aligned} (c^1 \subseteq R_T) &\subseteq (c * c * a^1 \subseteq c * a^1) && (D_{R47})(\text{Definition of a Transitive Relationship Set}) \\ \text{For } r = 1 \text{ to } n \text{ let } a^1 &= a_r^1 && \\ (c^1 \subseteq R_T) &\subseteq (c * c * a_1^1 \subseteq c * a_1^1) && (\text{Eqn1}) \\ \dots &&& \dots \\ (c^1 \subseteq R_T) &\subseteq (c * c * a_n^1 \subseteq c * a_n^1) && (\text{Eqnn}) \\ (c^1 \subseteq R_T) \dots (c^1 \subseteq R_T) &\subseteq (c * c * a_1^1 \subseteq c * a_1^1) \dots (c * c * a_n^1 \subseteq c * a_n^1) && (T_{M669})(\text{Combination of Eqns1 to } n) \\ (c^1 \subseteq R_T) &\subseteq (c * c * a_1^1 \subseteq c * a_1^1) \dots (c * c * a_n^1 \subseteq c * a_n^1) && (T_{M667})(\text{Conjunction Tautology}) \\ (c^1 \subseteq R_T) &\subseteq (c * c * a_1^1 + + + c * c * a_n^1 \subseteq c * a_1^1 + + + c * a_n^1) && (T_{M131b})(\text{Conjunction of Sufficiencies}) \\ (c^1 \subseteq R_T) &\subseteq (c * (c * a_1^1 + + + c * a_n^1) \subseteq c * a_1^1 + + + c * a_n^1) && (T_{R002b})(\text{Distribution of } * \text{ over } +) \\ (c^1 \subseteq R_T) &\subseteq (c * c * (a_1^1 + + + a_n^1) \subseteq c * (a_1^1 + + + a_n^1)) && (T_{R002b})(\text{Distribution of } * \text{ over } +)(\text{Eqn2}) \\ \text{Let } a &= a_1^1 + + + a_n^1 && (\text{Eqn3}) \\ (c^1 \subseteq R_T) &\subseteq (c * c * a \subseteq c * a) && (A_{MX19})(\text{Substitution of Eqn3 into Eqn2}) \end{aligned}$$

$$((c^1 \subseteq R_T) = E) \subseteq ((a \subseteq c * b)(b \subseteq c * d) \subseteq (a \subseteq c * d)) \quad (T_{S551}) \text{ Transitive Syllogism}$$

Proof of (T_{S551})

$$\begin{aligned} \text{Let } a &\subseteq c * b && (\text{Eqn1}) \\ \text{Let } b &\subseteq c * d && \\ c * b &\subseteq c * c * d && (T_{R196})(\text{Relational Deduction})(\text{Eqn2}) \\ a &\subseteq c * c * d && (T_{M139})(\text{Hypothetical Syllogism Eqns1 \& 2})(\text{Eqn3}) \\ (c^1 \subseteq R_T) &\subseteq (c * c * d \subseteq c * d) && (T_{S177})(\text{Definition of a Transitive Relationship Set})(\text{Eqn4}) \\ (c^1 \subseteq R_T) &\subseteq (a \subseteq c * d) && (T_{M139})(\text{Hypothetical Syllogism Eqns3 \& 4}) \\ \text{Thus } ((a \subseteq c * b)(b \subseteq c * d) = E) &\subseteq ((c^1 \subseteq R_T) \subseteq (a \subseteq c * d)) && (A_{MX02})(\text{Conditional Introduction}) \\ (a \subseteq c * b)(b \subseteq c * d)(c^1 \subseteq R_T) &\subseteq (a \subseteq c * d) && (A_{MX35})(\text{Exportation/Importation for truth sets}) \\ ((c^1 \subseteq R_T) = E) &\subseteq ((a \subseteq c * b)(b \subseteq c * d) \subseteq (a \subseteq c * d)) && (A_{MX35})(\text{Exportation/Importation for truth sets}) \end{aligned}$$

$$(c^1 \subseteq R_T) \subseteq (c * c * Qc \subseteq c * Qc)$$

(T_{R644})

Proof of (T_{R644})

$$\begin{aligned} (c^1 \subseteq R_T)(a^1 \subseteq Qc) &\subseteq (c * c * a^1 \subseteq c * a^1) && (T_{R625})(\text{Eqn1}) \\ \text{Let } Qc &= a_1^1 + + + a_n^1 && (\text{Eqn2}) \\ (c^1 \subseteq R_T)(a_1^1 \subseteq Qc) &\subseteq (c * c * a_1^1 \subseteq c * a_1^1) && (\text{Resymbolising Eqn1})(\text{Eqn3}) \\ \dots &&& \dots \\ (c^1 \subseteq R_T)(a_n^1 \subseteq Qc) &\subseteq (c * c * a_n^1 \subseteq c * a_n^1) && (\text{Resymbolising Eqn1})(\text{Eqn4}) \\ (c^1 \subseteq R_T)(a_1^1 \subseteq Qc) \dots (c^1 \subseteq R_T)(a_n^1 \subseteq Qc) &\subseteq (c * c * a_1^1 \subseteq c * a_1^1) \dots (c * c * a_n^1 \subseteq c * a_n^1) && (T_{M669})(\text{Combination of Eqns3 to 4}) \\ (c^1 \subseteq R_T)(a_1^1 \subseteq Qc) \dots (a_n^1 \subseteq Qc) &\subseteq (c * c * a_1^1 \subseteq c * a_1^1) \dots (c * c * a_n^1 \subseteq c * a_n^1) && (T_{M667})(\text{Conjunction Tautology}) \\ (c^1 \subseteq R_T)(a_1^1 \dots a_n^1 \subseteq Qc \dots Qc) &\subseteq (c * c * a_1^1 \subseteq c * a_1^1) \dots (c * c * a_n^1 \subseteq c * a_n^1) && (T_{M669})(\text{Combination}) \\ (c^1 \subseteq R_T)(a_1^1 \dots a_n^1 \subseteq Qc) &\subseteq (c * c * a_1^1 \subseteq c * a_1^1) \dots (c * c * a_n^1 \subseteq c * a_n^1) && (T_{M667})(\text{Conjunction Tautology})(\text{Eqn5}) \\ (c^1 \subseteq R_T)(Qc \subseteq Qc) &\subseteq (c * c * a_1^1 \subseteq c * a_1^1) \dots (c * c * a_n^1 \subseteq c * a_n^1) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn5}) \\ (c^1 \subseteq R_T)((Qc = Qc) + (Qc \subset Qc)) &\subseteq (c * c * a_1^1 \subseteq c * a_1^1) \dots (c * c * a_n^1 \subseteq c * a_n^1) && (T_{M266})(\text{Definition of } \subseteq) \\ (c^1 \subseteq R_T)(E + (Qc \subset Qc)) &\subseteq (c * c * a_1^1 \subseteq c * a_1^1) \dots (c * c * a_n^1 \subseteq c * a_n^1) && (T_{M001})(\text{Universality of } (D_{M00})) \\ (c^1 \subseteq R_T)E &\subseteq (c * c * a_1^1 \subseteq c * a_1^1) \dots (c * c * a_n^1 \subseteq c * a_n^1) && (T_{M055})(\text{Universal Set Disjunction}) \\ (c^1 \subseteq R_T) &\subseteq (c * c * a_1^1 \subseteq c * a_1^1) \dots (c * c * a_n^1 \subseteq c * a_n^1) && (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\ (c^1 \subseteq R_T) &\subseteq (c * c * a_1^1 + + + c * c * a_n^1 \subseteq c * a_1^1 + + + c * a_n^1) && (A_{MX21})(\text{Combination Axiom}) \\ (c^1 \subseteq R_T) &\subseteq (c * (c * a_1^1 + + + c * a_n^1) \subseteq c * (a_1^1 + + + a_n^1)) && (T_{R002b})(\text{Distribution of } * \text{ over } +) \\ (c^1 \subseteq R_T) &\subseteq (c * c * (a_1^1 + + + a_n^1) \subseteq c * (a_1^1 + + + a_n^1)) && (T_{R002b})(\text{Distribution of } * \text{ over } +)(\text{Eqn6}) \\ (c^1 \subseteq R_T) &\subseteq (c * c * Qc \subseteq c * Qc) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn6}) \end{aligned}$$

$$(a^1 \subseteq Qc)(c * c * a^1 \not\subseteq c * a^1) \subseteq (c^1 \subseteq \sim R_T) \quad (T_{R632})$$

Proof of (T_{R632})

$$\begin{aligned} (c^1 \subseteq R_T)(a^1 \subseteq Qc) &\subseteq (c * c * a^1 \subseteq c * a^1) & (T_{R625}) \\ ((a^1 \subseteq Qc) = E) &\subseteq ((c^1 \subseteq R_T) \subseteq (c * c * a^1 \subseteq c * a^1)) & (A_{MX35})(\text{Exportation/Importation for truth sets}) \\ ((a^1 \subseteq Qc) = E) &\subseteq ((c * c * a^1 \not\subseteq c * a^1) \subseteq (c^1 \not\subseteq R_T)) & (T_{M012})(\text{Transposition}) \\ (a^1 \subseteq Qc)(c * c * a^1 \not\subseteq c * a^1) &\subseteq (c^1 \not\subseteq R_T) & (A_{MX35})(\text{Exportation/Importation for truth sets(Eqn1)}) \\ (c^1 \not\subseteq R_T) &\subseteq (c^1 \subseteq \sim R_T) & (T_{M604})(\text{Eqn2}) \\ (a^1 \subseteq Qc)(c * c * a^1 \not\subseteq c * a^1) &\subseteq (c^1 \subseteq \sim R_T) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$${}^V\{\triangleleft\} = \{\triangleright\} \quad (T_{T741}) \text{ Definition of } {}^V\{\triangleleft\}$$

Proof of (T_{T741})

$$\begin{aligned} {}^V\{\triangleleft\} &= \{\triangleright\} & (D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set}) \\ {}^V\{\triangleleft\} + \{=\} &= \{\triangleright\} + \{=\} & (T_{M232})(\text{Deduction}) \\ {}^V\{\triangleleft\} + {}^V\{=\} &= \{\triangleright\} + \{=\} & (T_{A485}) \text{ NB this is proven later but using only axioms and definitions} \\ {}^V(\{\triangleleft\} + \{=\}) &= \{\triangleright\} + \{=\} & (T_{S288})(\text{Distribution of } V * \text{ over } +) \\ {}^V\{\triangleleft\} &= \{\triangleright\} & (D_{A141})(\text{Disjunction with } \{=\}) \end{aligned}$$

$${}^V\{\triangleleft\} * b^1 = \{\triangleright\} * b^1 \quad (T_{A448})$$

$${}^V\{\triangleleft\} * b^1 = \{\triangleright\} * b^1 \quad (T_{A449})$$

Example Proof of (T_{A448})

$$\begin{aligned} {}^V\{\triangleleft\} &= \{\triangleright\} & (D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set}) \\ {}^V\{\triangleleft\} * b^1 &= \{\triangleright\} * b^1 & (T_{R197})(\text{Relational Deduction}) \end{aligned}$$

NB the proof of (T_{A449}) uses (T_{T741}) rather than (D_{A143})

$$\{\triangleleft\} * \{\triangleleft\} * b^1 \subseteq \{\triangleleft\} * b^1 \quad (T_{A283})$$

Proof of (T_{A283})

$$\begin{aligned} (c^1 \subseteq R_T) &\subseteq ((c * c * h) \subseteq (c * h)) & (T_{S177})(\text{Definition of a Transitive Relationship Set})(\text{Eqn1}) \\ (c^1 \subseteq R_T R_A) &\subseteq (c^1 \subseteq R_T) & (T_{M073})(\text{Single Premise Syllogism})(\text{Eqn2}) \\ (c^1 \subseteq R_T R_A) &\subseteq ((c * c * h) \subseteq (c * h)) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\ \text{Let } h &= b^1 & (\text{Eqn4}) \\ \text{Let } c &= \{\triangleleft\} & (\text{Eqn5}) \\ (\{\triangleleft\}^1 \subseteq R_T R_A) &\subseteq ((\{\triangleleft\} * \{\triangleleft\} * b^1) \subseteq (\{\triangleleft\} * b^1)) & (A_{MX19})(\text{Substitution of Eqns4\&5 into Eqn3}) \\ E &\subseteq ((\{\triangleleft\} * \{\triangleleft\} * b^1) \subseteq (\{\triangleleft\} * b^1)) & (T_{M001})(\text{Universality of (T}_{T739})) \\ E &= ((\{\triangleleft\} * \{\triangleleft\} * b^1) \subseteq (\{\triangleleft\} * b^1)) & (T_{M275})(\text{Supersets of the Universal Set}) \\ \{\triangleleft\} * \{\triangleleft\} * b^1 &\subseteq \{\triangleleft\} * b^1 & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$\{\triangleright\} * \{\triangleright\} * b^1 \subseteq \{\triangleright\} * b^1 \quad (T_{A283}) \text{ Alternative symbolism}$$

Proof of (T_{A283})

$$\begin{aligned} (c^1 \subseteq R_T) &\subseteq ((c * c * h) \subseteq (c * h)) & (T_{S177})(\text{Definition of a Transitive Relationship Set})(\text{Eqn1}) \\ (c^1 \subseteq R_T R_A) &\subseteq (c^1 \subseteq R_T) & (T_{M073})(\text{Single Premise Syllogism})(\text{Eqn2}) \\ (c^1 \subseteq R_T R_A) &\subseteq ((c * c * h) \subseteq (c * h)) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\ \text{Let } h &= b^1 & (\text{Eqn4}) \\ \text{Let } c &= \{\triangleright\} & (\text{Eqn5}) \\ (\{\triangleright\}^1 \subseteq R_T R_A) &\subseteq ((\{\triangleright\} * \{\triangleright\} * b^1) \subseteq (\{\triangleright\} * b^1)) & (A_{MX19})(\text{Substitution of Eqns4\&5 into Eqn3}) \\ E &\subseteq ((\{\triangleright\} * \{\triangleright\} * b^1) \subseteq (\{\triangleright\} * b^1)) & (T_{M001})(\text{Universality of (T}_{T739})) \\ E &= ((\{\triangleright\} * \{\triangleright\} * b^1) \subseteq (\{\triangleright\} * b^1)) & (T_{M275})(\text{Supersets of the Universal Set}) \\ \{\triangleright\} * \{\triangleright\} * b^1 &\subseteq \{\triangleright\} * b^1 & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$\{\triangleleft\} * \{\triangleleft\} * b^1 = \{\triangleleft\} * b^1 \quad (T_{A555})$$

$$\{\triangleleft\} * \{\triangleleft\} * b^1 = \{\triangleleft\} * b^1 \quad (T_{A556})$$

Example Proof of (T_{A555})

$$\begin{aligned}
\{\triangleleft\} * \{\triangleleft\} * b^1 &= \{\triangleleft\} * \{\triangleleft\} * b^1 & (D_{M00})(\text{Definition of } =) \\
\{\triangleleft\} * \{\triangleleft\} * b^1 &= (\{\triangleleft\} + \{=\}) * \{\triangleleft\} * b^1 & (T_{A059}) \\
\{\triangleleft\} * \{\triangleleft\} * b^1 &= \{\triangleleft\} * \{\triangleleft\} * b^1 + \{=\} * \{\triangleleft\} * b^1 & (T_{R002c})(\text{Distribution of } + \text{ over } *) \\
\{\triangleleft\} * \{\triangleleft\} * b^1 &= \{\triangleleft\} * \{\triangleleft\} * b^1 + \{\triangleleft\} * b^1 & (T_{A080})(\text{Eqn1}) \\
\{\triangleleft\} * \{\triangleleft\} * b^1 &\subseteq \{\triangleleft\} * b^1 & (T_{A283}) \\
\{\triangleleft\} * \{\triangleleft\} * b^1 &= (\{\triangleleft\} * \{\triangleleft\} * b^1)(\{\triangleleft\} * b^1) & (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn 2}) \\
\{\triangleleft\} * \{\triangleleft\} * b^1 &= (\{\triangleleft\} * \{\triangleleft\} * b^1)(\{\triangleleft\} * b^1) + \{\triangleleft\} * b^1 & (A_{MX19})(\text{Substitution of Eqn 2 into Eqn 1}) \\
\{\triangleleft\} * \{\triangleleft\} * b^1 &= \{\triangleleft\} * b^1 & (T_{M063})(\text{Partial Redundancy})
\end{aligned}$$

$$\{\triangleleft\} * \{\triangleleft\} * b \subseteq \{\triangleleft\} * b \quad (T_{S965})$$

Proof of (T_{S965})

$$\begin{aligned}
\text{Let } b &= b_1^1 + \dots + b_n^1 & (\text{Eqn1}) \\
\{\triangleleft\} * \{\triangleleft\} * b_r^1 &= \{\triangleleft\} * b_r^1 & (T_{A556})(\text{Transitive Function Theorem}) \\
\{\triangleleft\} * \{\triangleleft\} * b_1^1 + \dots + \{\triangleleft\} * \{\triangleleft\} * b_n^1 &= \{\triangleleft\} * b_1^1 + \dots + \{\triangleleft\} * b_n^1 & (T_{M230})(\text{Combination for } r = 1 \text{ to } n) \\
\{\triangleleft\} * (\{\triangleleft\} * b_1^1 + \dots + \{\triangleleft\} * b_n^1) &= \{\triangleleft\} * (b_1^1 + \dots + b_n^1) & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\
\{\triangleleft\} * \{\triangleleft\} * (b_1^1 + \dots + b_n^1) &= \{\triangleleft\} * (b_1^1 + \dots + b_n^1) & (T_{R002b})(\text{Distribution of } * \text{ over } +)(\text{Eqn2}) \\
\{\triangleleft\} * \{\triangleleft\} * b &= \{\triangleleft\} * b & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})
\end{aligned}$$

$$\{\triangleleft\} * \{\triangleleft\} * b \subseteq \{\triangleleft\} * b \quad (T_{S966})$$

Proof of (T_{S966})

$$\begin{aligned}
\text{Let } b &= b_1^1 + \dots + b_n^1 & (\text{Eqn1}) \\
\{\triangleleft\} * \{\triangleleft\} * b_r^1 &= \{\triangleleft\} * b_r^1 & (T_{A556})(\text{Transitive Function Theorem}) \\
\{\triangleleft\} * \{\triangleleft\} * b_1^1 + \dots + \{\triangleleft\} * \{\triangleleft\} * b_n^1 &= \{\triangleleft\} * b_1^1 + \dots + \{\triangleleft\} * b_n^1 & (T_{M230})(\text{Combination for } r = 1 \text{ to } n) \\
\{\triangleleft\} * (\{\triangleleft\} * b_1^1 + \dots + \{\triangleleft\} * b_n^1) &= \{\triangleleft\} * (b_1^1 + \dots + b_n^1) & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\
\{\triangleleft\} * \{\triangleleft\} * (b_1^1 + \dots + b_n^1) &= \{\triangleleft\} * (b_1^1 + \dots + b_n^1) & (T_{R002b})(\text{Distribution of } * \text{ over } +)(\text{Eqn2}) \\
\{\triangleleft\} * \{\triangleleft\} * b &= \{\triangleleft\} * b & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})
\end{aligned}$$

$$\{\triangleleft\} * \{\triangleleft\} * b^1 = \{\triangleleft\} * b^1 \quad (T_{A711}) \text{ Transitive Function Theorem}$$

Proof of (T_{A711})

$$\begin{aligned}
\{\triangleleft\} * b^1 + \{=\} * b^1 &= \{\triangleleft\} * b^1 + \{=\} * b^1 & (D_{M00})(\text{Definition of } =)(\text{Eqn1}) \\
\{\triangleleft\} * \{\triangleleft\} * b^1 &\subseteq \{\triangleleft\} * b^1 & (T_{A283}) \\
(\{\triangleleft\} * \{\triangleleft\} * b^1) &= (\{\triangleleft\} * \{\triangleleft\} * b^1)(\{\triangleleft\} * b^1) & (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn2}) \\
(\{\triangleleft\} * b^1) &= (\{\triangleleft\} * \{\triangleleft\} * b^1)(\{\triangleleft\} * b^1) + (\{\triangleleft\} * b^1) & (T_{M063})(\text{Partial Redundancy})(\text{Eqn3}) \\
(\{\triangleleft\} * b^1) &= (\{\triangleleft\} * \{\triangleleft\} * b^1) + (\{\triangleleft\} * b^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn3})(\text{Eqn4}) \\
\{\triangleleft\} * \{\triangleleft\} * b^1 + \{\triangleleft\} * b^1 + \{=\} * b^1 &= \{\triangleleft\} * b^1 + \{=\} * b^1 & (A_{MX19})(\text{Substitution of Eqn4 into Eqn1}) \\
\{\triangleleft\} * \{\triangleleft\} * b^1 + \{\triangleleft\} * b^1 + \{\triangleleft\} * b^1 + \{=\} * b^1 &= \{\triangleleft\} * b^1 + \{=\} * b^1 & (A_{MX01})(\text{Disjunction Tautology}) \\
\{\triangleleft\} * \{\triangleleft\} * b^1 + \{=\} * \{\triangleleft\} * b^1 + \{\triangleleft\} * \{=\} * b^1 + \{=\} * \{=\} * b^1 &= \{\triangleleft\} * b^1 + \{=\} * b^1 & (T_{A080}) \\
(\{\triangleleft\} + \{=\}) * \{\triangleleft\} * b^1 + (\{\triangleleft\} + \{=\}) * \{=\} * b^1 &= (\{\triangleleft\} + \{=\}) * b^1 & (T_{R002c})(\text{Distribution of } + \text{ over } *) \\
(\{\triangleleft\} + \{=\}) * (\{\triangleleft\} * b^1 + \{=\} * b^1) &= (\{\triangleleft\} + \{=\}) * b^1 & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\
(\{\triangleleft\} + \{=\}) * (\{\triangleleft\} + \{=\}) * b^1 &= (\{\triangleleft\} + \{=\}) * b^1 & (T_{R002c})(\text{Distribution of } + \text{ over } *) \\
\{\triangleleft\} * \{\triangleleft\} * b^1 &= \{\triangleleft\} * b^1 & (D_{A141})(\text{Disjunction with } \{=\})
\end{aligned}$$

$$\{\triangleleft\} * \{\triangleleft\} * b = \{\triangleleft\} * b \quad (T_{A817})$$

Proof of (T_{A817})

$$\begin{aligned}
\text{Let } b &= b_1^1 + \dots + b_n^1 & (\text{Eqn1}) \\
\{\triangleleft\} * \{\triangleleft\} * b_r^1 &= \{\triangleleft\} * b_r^1 & (T_{A711})(\text{Transitive Function Theorem}) \\
\{\triangleleft\} * \{\triangleleft\} * b_1^1 + \dots + \{\triangleleft\} * \{\triangleleft\} * b_n^1 &= \{\triangleleft\} * b_1^1 + \dots + \{\triangleleft\} * b_n^1 & (T_{M230})(\text{Combination for } r = 1 \text{ to } n) \\
\{\triangleleft\} * (\{\triangleleft\} * b_1^1 + \dots + \{\triangleleft\} * b_n^1) &= \{\triangleleft\} * (b_1^1 + \dots + b_n^1) & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\
\{\triangleleft\} * \{\triangleleft\} * (b_1^1 + \dots + b_n^1) &= \{\triangleleft\} * (b_1^1 + \dots + b_n^1) & (T_{R002b})(\text{Distribution of } * \text{ over } +)(\text{Eqn2}) \\
\{\triangleleft\} * \{\triangleleft\} * b &= \{\triangleleft\} * b & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})
\end{aligned}$$

Thus $(a \subseteq \{\triangleleft\} * c)(c \subseteq \{\triangleleft\} * b) \subseteq (a \subseteq \{\triangleleft\} * b)$

$(\mathbf{A}_{MX02})$ (Conditional Introduction)

$\{\triangleleft\} * a^1 \subseteq \{\triangleleft\} \diamond (\{\triangleright\} * a^1)$

$(\mathbf{T}_{S897})$

Proof of $(\mathbf{T}_{S897})$

$(a \subseteq \{\triangleleft\} * c)(c \subseteq \{\triangleleft\} * b) \subseteq (a \subseteq \{\triangleleft\} * b)$

$(\mathbf{T}_{A319})$ (Transitive Asymmetrical Syllogism)(Eqn1)

Let $a = a_r^1$

(Eqn2)

Let $b = b_s^1$

(Eqn3)

Let $c = c^1$

(Eqn4)

Let $a = a^1$

(Eqn5)

Let $\{\triangleleft\} = \{\triangleleft\}$

(Eqn6)

$(a_r^1 \subseteq \{\triangleleft\} * c^1)(c^1 \subseteq \{\triangleleft\} * b_s^1) \subseteq (a_r^1 \subseteq \{\triangleleft\} * b_s^1)$

$(\mathbf{A}_{MX19})$ (Substitution of Eqns2 to 6 into Eqn1)

$(a_r^1 \subseteq \{\triangleleft\} * c^1)(b_s^1 \subseteq \{\triangleright\} * c^1) \subseteq (a_r^1 \subseteq \{\triangleleft\} * b_s^1)$

$(\mathbf{T}_{R519})$ & $(\mathbf{D}_{A143})$ (Symmetrical Opposite of an Asymmetrical Relationship Set) (Eqn7)

Let $a = a_1^1 +++ a_n^1$

(Eqn8)

Let $b = b_1^1 +++ b_m^1$

(Eqn9)

Let $a_r^1 \subseteq a_1^1 +++ a_n^1$

(Eqn10)

Let $b_s^1 \subseteq b_1^1 +++ b_m^1$

(Eqn11)

$a_r^1 \subseteq a$

$(\mathbf{A}_{MX19})$ (Substitution of Eqn8 into Eqn10)

$(a_r^1 \subseteq a) = E$

$(\mathbf{A}_{MX16})$ (Order Axiom)(Eqn12)

$b_s^1 \subseteq b$

$(\mathbf{A}_{MX19})$ (Substitution of Eqn9 into Eqn11)

$(b_s^1 \subseteq b) = E$

$(\mathbf{A}_{MX16})$ (Order Axiom)(Eqn13)

Let $a = \{\triangleleft\} * c^1$

(Eqn14)

Let $b = \{\triangleright\} * c^1$

(Eqn15)

$(a_r^1 \subseteq a)(b_s^1 \subseteq b) \subseteq (a_r^1 \subseteq \{\triangleleft\} * b_s^1)$

$(\mathbf{A}_{MX19})$ (Substitution of Eqns10&11 into Eqn7)(Eqn16)

$EE \subseteq (a_r^1 \subseteq \{\triangleleft\} * b_s^1)$

$(\mathbf{A}_{MX19})$ (Substitution of Eqns12&13 into Eqn16)

$E \subseteq (a_r^1 \subseteq \{\triangleleft\} * b_s^1)$

$(\mathbf{T}_{M667})$ (Conjunction Tautology)

$E = (a_r^1 \subseteq \{\triangleleft\} * b_s^1)$

$(\mathbf{T}_{M275})$ (Supersets of the Universal Set)

$a_r^1 \subseteq \{\triangleleft\} * b_s^1$

$(\mathbf{A}_{MX16})$ (Order Axiom)(Eqn17)

$a_1^1 +++ a_n^1 \subseteq \{\triangleleft\} * b_s^1$

$(\mathbf{A}_{MX21})$ (Combination of Eqn17 for $r = 1$ to n)(Eqn18)

$(a_1^1 +++ a_n^1) \dots (a_1^1 +++ a_n^1) \subseteq \{\triangleleft\} * b_1^1 \dots \{\triangleleft\} * b_m^1$

$(\mathbf{T}_{M669})$ (Combination of Eqn18 for $s = 1$ to m)

$(a_1^1 +++ a_n^1) \subseteq \{\triangleleft\} * b_1^1 \dots \{\triangleleft\} * b_m^1$

$(\mathbf{T}_{M667})$ (Conjunction Tautology)

$a_1^1 +++ a_n^1 \subseteq \{\triangleleft\} \diamond (b_1^1 +++ b_m^1)$

$(\mathbf{D}_{S15})$ (Definition of the Intersection Function Operator)(Eqn19)

$a \subseteq \{\triangleleft\} \diamond b$

$(\mathbf{A}_{MX19})$ (Substitution of Eqns8&9 into Eqn19)(Eqn20)

$\{\triangleleft\} * c^1 \subseteq \{\triangleleft\} \diamond \{\triangleright\} * c^1$

$(\mathbf{A}_{MX19})$ (Substitution of Eqns14&15 into Eqn20)

$\{\triangleleft\} * a^1 \subseteq \{\triangleleft\} \diamond \{\triangleright\} * a^1$

Resymbolising

$\{\triangleleft\} * ((\{\triangleleft\} * a^1).(\{\triangleleft\} * b^1)) = (\{\triangleleft\} * \{\triangleleft\} * a^1).(\{\triangleleft\} * \{\triangleleft\} * b^1)$

$(\mathbf{T}_{S437})$

$\{\triangleleft\} * ((\{\triangleleft\} * a^1).(\{\triangleleft\} * b^1)) = (\{\triangleleft\} * \{\triangleleft\} * a^1).(\{\triangleleft\} * \{\triangleleft\} * b^1)$

$(\mathbf{T}_{S438})$

$\{\triangleleft\} * ((\{\triangleleft\} * a^1).(\{\triangleleft\} * b^1)) = (\{\triangleleft\} * \{\triangleleft\} * a^1).(\{\triangleleft\} * \{\triangleleft\} * b^1)$

$(\mathbf{T}_{S439})$

$\{\triangleleft\} * ((\{\triangleleft\} * a^1).(\{\triangleleft\} * b^1)) = (\{\triangleleft\} * \{\triangleleft\} * a^1).(\{\triangleleft\} * \{\triangleleft\} * b^1)$

$(\mathbf{T}_{S440})$

Example Proof of $(\mathbf{T}_{S437})$

Let $\{\triangleleft\} * a^1 \subseteq \{\triangleleft\} * b^1$

(Eqn1)

$\{\triangleleft\} * a^1 = (\{\triangleleft\} * a^1).(\{\triangleleft\} * b^1)$

$(\mathbf{A}_{MX44})$ ($\subseteq$ Equality Form)

$\{\triangleleft\} * \{\triangleleft\} * a^1 = \{\triangleleft\} * ((\{\triangleleft\} * a^1).(\{\triangleleft\} * b^1))$

$(\mathbf{T}_{R196})$ (Relational Deduction)(Eqn2)

$\{\triangleleft\} * a^1 \subseteq \{\triangleleft\} * b^1$

(Eqn1)

$\{\triangleleft\} * \{\triangleleft\} * a^1 \subseteq \{\triangleleft\} * \{\triangleleft\} * b^1$

$(\mathbf{T}_{R196})$ (Relational Deduction)

$\{\triangleleft\} * \{\triangleleft\} * a^1 = (\{\triangleleft\} * \{\triangleleft\} * a^1).(\{\triangleleft\} * \{\triangleleft\} * b^1)$

$(\mathbf{A}_{MX44})$ ($\subseteq$ Equality Form)(Eqn3)

$\{\triangleleft\} * ((\{\triangleleft\} * a^1).(\{\triangleleft\} * b^1)) = (\{\triangleleft\} * \{\triangleleft\} * a^1).(\{\triangleleft\} * \{\triangleleft\} * b^1)$

$(\mathbf{A}_{MX19})$ (Substitution of Eqn2 into Eqn3)

The proofs of $(\mathbf{T}_{S438})$, $(\mathbf{T}_{S439})$ and $(\mathbf{T}_{S440})$ are identical except for the substitution of $\{\triangleleft\}$ for $\{\triangleleft\}$ where appropriate.

$\{\triangleleft\} * (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1) = (\{\triangleleft\} * \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_n^1)$

$(\mathbf{T}_{S429})$

$\{\triangleleft\} * (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1) = (\{\triangleleft\} * \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_n^1)$

$(\mathbf{T}_{S430})$

$$\{\triangleleft\} * (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1) = (\{\triangleleft\} * \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_n^1) \quad (T_{S431})$$

$$\{\triangleleft\} * (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1) = (\{\triangleleft\} * \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_n^1) \quad (T_{S432})$$

Example Proof of (T_{S429})

$$\begin{aligned} \text{Let } \{\triangleleft\} * a_1^1 &\subseteq \{\triangleleft\} * a_r^1 && (\text{Eqn1}) \\ \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_1^1 &\subseteq \{\triangleleft\} * a_2^1 \dots \{\triangleleft\} * a_n^1 && (T_{M669})(\text{Combination of Eqn1 for } r = 2 \text{ to } n) \\ \{\triangleleft\} * a_1^1 &\subseteq \{\triangleleft\} * a_2^1 \dots \{\triangleleft\} * a_n^1 && (T_{M667})(\text{Conjunction Tautology}) \\ \{\triangleleft\} * a^1 &= \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1 && (A_{MX44})(\subseteq \text{Equality Form}) \\ \{\triangleleft\} * \{\triangleleft\} * a_1^1 &= \{\triangleleft\} * (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1) && (T_{R196})(\text{Relational Deduction})(\text{Eqn2}) \\ \text{Let } \{\triangleleft\} * a_1^1 &\subseteq \{\triangleleft\} * a_r^1 && (\text{Eqn1}) \\ \{\triangleleft\} * \{\triangleleft\} * a_1^1 &\subseteq \{\triangleleft\} * \{\triangleleft\} * a_r^1 && (T_{R196})(\text{Relational Deduction})(\text{Eqn3}) \\ \{\triangleleft\} * \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_1^1 &\subseteq \{\triangleleft\} * \{\triangleleft\} * a_2^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_n^1 && (T_{M669})(\text{Combination of Eqn3 for } r = 2 \text{ to } n) \\ \{\triangleleft\} * \{\triangleleft\} * a_1^1 &\subseteq \{\triangleleft\} * \{\triangleleft\} * a_2^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_n^1 && (T_{M667})(\text{Conjunction Tautology}) \\ \{\triangleleft\} * \{\triangleleft\} * a_1^1 &= \{\triangleleft\} * \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_n^1 && (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn4}) \\ \{\triangleleft\} * (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1) &= (\{\triangleleft\} * \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_n^1) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn4}) \end{aligned}$$

The proofs of (T_{S430}), (T_{S431}) and (T_{S432}) are identical except for the substitution of $\{\triangleleft\}$ for $\{\triangleleft\}$ where appropriate.

$$\{\triangleleft\} * \{\triangleleft\} \diamond a \subseteq \{\triangleleft\} \diamond a \quad (T_{S414})$$

Proof of (T_{S414})

$$\begin{aligned} \{\triangleleft\} * \{\triangleleft\} \diamond a &= \{\triangleleft\} * \{\triangleleft\} \diamond a && (D_{M00})(\text{Definition of } =)(\text{Eqn1}) \\ \text{Let } a &= a_1^1 +++ a_n^1 && (\text{Eqn2}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= \{\triangleleft\} * \{\triangleleft\} \diamond (a_1^1 +++ a_n^1) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= \{\triangleleft\} * (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1) && (D_{S15})(\text{Definition of the Intersection Function Operator}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= (\{\triangleleft\} * \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_n^1) && (T_{S429})(\text{Eqn3}) \\ \{\triangleleft\} * \{\triangleleft\} * a_r^1 &\subseteq \{\triangleleft\} * a_r^1 && (T_{A283})(\text{Eqn4}) \\ \{\triangleleft\} * \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_n^1 &\subseteq \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1 && (T_{M669})(\text{Combination of Eq4 for } r = 1 \text{ to } n)(\text{Eqn5}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &\subseteq (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1) && (A_{MX19})(\text{Substitution of Eqn3 into Eqn5}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &\subseteq \{\triangleleft\} \diamond (a_1^1 +++ a_n^1) && (D_{S15})(\text{Def. of the Intersection Function Operator})(\text{Eqn6}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &\subseteq \{\triangleleft\} \diamond a && (A_{MX19})(\text{Substitution of Eqn2 into Eqn6}) \end{aligned}$$

$$\{\triangleleft\} * \{\triangleleft\} \diamond a = \{\triangleleft\} \diamond a \quad (T_{S415})$$

Proof of (T_{S415})

$$\begin{aligned} \{\triangleleft\} * \{\triangleleft\} \diamond a &= \{\triangleleft\} * \{\triangleleft\} \diamond a && (D_{M00})(\text{Definition of } =)(\text{Eqn1}) \\ \text{Let } a &= a_1^1 +++ a_n^1 && (\text{Eqn2}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= \{\triangleleft\} * \{\triangleleft\} \diamond (a_1^1 +++ a_n^1) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= \{\triangleleft\} * (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1) && (D_{S15})(\text{Definition of the Intersection Function Operator}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= (\{\triangleleft\} * \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_n^1) && (T_{S430}) \\ \{\triangleleft\} * \{\triangleright\} \diamond a &= (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1) && (T_{A555}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= \{\triangleleft\} \diamond (a_1^1 +++ a_n^1) && (D_{S15})(\text{Def. of the Intersection Function Operator})(\text{Eqn3}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= \{\triangleleft\} \diamond a && (A_{MX19})(\text{Substitution of Eqn2 into Eqn3}) \end{aligned}$$

$$\{\triangleleft\} * \{\triangleleft\} \diamond a = \{\triangleleft\} \diamond a \quad (T_{S406})$$

Proof of (T_{S406})

$$\begin{aligned} \{\triangleleft\} * \{\triangleleft\} \diamond a &= \{\triangleleft\} * \{\triangleleft\} \diamond a && (D_{M00})(\text{Definition of } =)(\text{Eqn1}) \\ \text{Let } a &= a_1^1 +++ a_n^1 && (\text{Eqn2}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= \{\triangleleft\} * \{\triangleleft\} \diamond (a_1^1 +++ a_n^1) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= \{\triangleleft\} * (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1) && (D_{S15})(\text{Definition of the Intersection Function Operator}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= (\{\triangleleft\} * \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_n^1) && (T_{S431}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1) && (T_{A556}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= \{\triangleleft\} \diamond (a_1^1 +++ a_n^1) && (D_{S15})(\text{Def. of the Intersection Function Operator})(\text{Eqn3}) \\ \{\triangleleft\} * \{\triangleleft\} \diamond a &= \{\triangleleft\} \diamond a && (A_{MX19})(\text{Substitution of Eqn2 into Eqn3}) \end{aligned}$$

$$\{\triangleleft\} * \{\triangleleft\} \diamond a = \{\triangleleft\} \diamond a \quad (T_{S407})$$

Proof of (T_{S407})

$\{\triangleleft\} * \{\triangleleft\} \diamond a = \{\triangleleft\} * \{\triangleleft\} \diamond a$	$(D_{M00})(\text{Definition of } =)(\text{Eqn1})$
Let $a = a_1^1 + \dots + a_n^1$	(Eqn2)
$\{\triangleleft\} * \{\triangleleft\} \diamond a = \{\triangleleft\} * \{\triangleleft\} \diamond (a_1^1 + \dots + a_n^1)$	$(A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$
$\{\triangleleft\} * \{\triangleleft\} \diamond a = \{\triangleleft\} * (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1)$	$(D_{S15})(\text{Definition of the Intersection Function Operator})$
$\{\triangleleft\} * \{\triangleleft\} \diamond a = (\{\triangleleft\} * \{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * \{\triangleleft\} * a_n^1)$	(T_{S432})
$\{\triangleleft\} * \{\triangleleft\} \diamond a = (\{\triangleleft\} * a_1^1 \dots \{\triangleleft\} * a_n^1)$	(T_{A711})
$\{\triangleleft\} * \{\triangleleft\} \diamond a = \{\triangleleft\} \diamond (a_1^1 + \dots + a_n^1)$	$(D_{S15})(\text{Def. of the Intersection Function Operator})(\text{Eqn3})$
$\{\triangleleft\} * \{\triangleleft\} \diamond a = \{\triangleleft\} \diamond a$	$(A_{MX19})(\text{Substitution of Eqn2 into Eqn3})$

$$((b \neq \emptyset)(a \neq \emptyset) + (a = \emptyset)(b = \emptyset) = E) \subseteq ((a \subseteq \{\triangleleft\} \diamond b) \subseteq (\{\triangleright\} * b \subseteq \{\triangleright\} \diamond a)) \quad (T_{S558})$$

$$((b \neq \emptyset)(a \neq \emptyset) + (a = \emptyset)(b = \emptyset) = E) \subseteq ((a \subseteq \{\triangleleft\} \diamond b) \subseteq (\{\triangleright\} * b \subseteq \{\triangleright\} \diamond a)) \quad (T_{S559})$$

$$((b \neq \emptyset)(a \neq \emptyset) + (a = \emptyset)(b = \emptyset) = E) \subseteq ((a \subseteq \{\triangleleft\} \diamond b) \subseteq (\{\triangleright\} * b \subseteq \{\triangleright\} \diamond a)) \quad (T_{S560})$$

$$((b \neq \emptyset)(a \neq \emptyset) + (a = \emptyset)(b = \emptyset) = E) \subseteq ((a \subseteq \{\triangleleft\} \diamond b) \subseteq (\{\triangleright\} * b \subseteq \{\triangleright\} \diamond a)) \quad (T_{S561})$$

Proof of (T_{S558}) , (T_{S559}) , (T_{S560}) and (T_{S561})

$(b \neq \emptyset)(a \neq \emptyset) + (a = \emptyset)(b = \emptyset)$:

Let $a \subseteq \{\triangleleft\} \diamond b$

$b \subseteq \forall \{\triangleleft\} \diamond a$ $(T_{R293})(\text{Active/Passive Theorem})$

$b \subseteq \{\triangleright\} \diamond a$ $(D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set})(\text{Eqn1})$

$\{\triangleright\} * b \subseteq \{\triangleright\} * \{\triangleright\} \diamond a$ $(T_{R196})(\text{Relational Deduction})(\text{Eqn2})$

$\{\triangleright\} * \{\triangleright\} \diamond a \subseteq \{\triangleright\} \diamond a$ $(T_{S414})(\text{Eqn3})$

$\{\triangleright\} * b \subseteq \{\triangleright\} \diamond a$ $(T_{M139})(\text{Hypothetical Syllogism Eqns2\&3})(\text{Eqn4})$

Thus $(a \subseteq \{\triangleleft\} \diamond b) \subseteq (\{\triangleright\} * b \subseteq \{\triangleright\} \diamond a)$

$(A_{MX02})(\text{Conditional Introduction})$

$\{\triangleright\} * b + b \subseteq \{\triangleright\} \diamond a + \{\triangleright\} \diamond a$ $(A_{MX21})(\text{Combination of Eqns1\&4})$

$\{\triangleright\} * b + b \subseteq \{\triangleright\} \diamond a$ $(A_{MX01})(\text{Disjunction Tautology})$

$\{\triangleright\} * b + \{=\} * b \subseteq \{\triangleright\} \diamond a$ $(D_{A144})(\text{Definition of the Reflexive Relationship Set})$

$(\{\triangleright\} + \{=\}) * b \subseteq \{\triangleright\} \diamond a$ $(T_{R002c})(\text{Distribution of } + \text{ over } *)$

$\{\triangleright\} * b \subseteq \{\triangleright\} \diamond a$ $(D_{A141})(\text{Disjunction with } \{=\})(\text{Eqn5})$

Thus $(a \subseteq \{\triangleleft\} \diamond b) \subseteq (\{\triangleright\} * b \subseteq \{\triangleright\} \diamond a)$

$(A_{MX02})(\text{Conditional Introduction})$

$\{\triangleright\} = \{\triangleright\} + \{=\}$ $(D_{A141})(\text{Disjunction with } \{=\})$

$\{\triangleright\} \subseteq \{\triangleright\}$ $(T_{M072b})(\text{Single Premise Syllogism})$

$\{\triangleright\} \diamond a \subseteq \{\triangleright\} \diamond a$ $(T_{R263})(\text{Relational Deduction})(\text{Eqn6})$

$\{\triangleright\} * b \subseteq \{\triangleright\} \diamond a$ $(T_{M139})(\text{Hypothetical Syllogism Eqns5\&6})$

Thus $(a \subseteq \{\triangleleft\} \diamond b) \subseteq (\{\triangleright\} * b \subseteq \{\triangleright\} \diamond a)$

$(A_{MX02})(\text{Conditional Introduction})$

$\{\triangleright\} * b \subseteq \{\triangleright\} \diamond a$ $(T_{M139})(\text{Hypothetical Syllogism Eqns4 \&6})$

Thus $(a \subseteq \{\triangleleft\} \diamond b) \subseteq (\{\triangleright\} * b \subseteq \{\triangleright\} \diamond a)$

$(A_{MX02})(\text{Conditional Introduction})$

$$((b^1 \neq \emptyset)(a^1 \neq \emptyset) + (a^1 = \emptyset)(b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\triangleleft\} * b^1) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * a^1)) \quad (T_{A312})$$

$$((b^1 \neq \emptyset)(a^1 \neq \emptyset) + (a^1 = \emptyset)(b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\triangleleft\} * b^1) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * a^1)) \quad (T_{S408})$$

$$((b^1 \neq \emptyset)(a^1 \neq \emptyset) + (a^1 = \emptyset)(b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\triangleleft\} * b^1) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * a^1)) \quad (T_{A313})$$

$$((b^1 \neq \emptyset)(a^1 \neq \emptyset) + (a^1 = \emptyset)(b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\triangleleft\} * b^1) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * a^1)) \quad (T_{A314})$$

Example Proof of (T_{A312})

$((b \neq \emptyset)(a \neq \emptyset) + (a = \emptyset)(b = \emptyset) = E) \subseteq ((a \subseteq \{\triangleleft\} \diamond b) \subseteq (\{\triangleright\} * b \subseteq \{\triangleright\} \diamond a))$

(T_{S558})

Let $a = a^1$

(Eqn2)

Let $b = b^1$

(Eqn3)

$((b^1 \neq \emptyset)(a^1 \neq \emptyset) + (a^1 = \emptyset)(b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\triangleleft\} \diamond b^1) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} \diamond a^1))$

$(A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})$

$((b^1 \neq \emptyset)(a^1 \neq \emptyset) + (a^1 = \emptyset)(b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\triangleleft\} * b^1) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * a^1))$

$(T_{R232})(\text{Interchangeability of } * \text{ and } \diamond \text{ for Entities})$

$$(\{\triangleright\} * b^1 \neq \emptyset) \subseteq (\{\triangleleft\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1)$$

(T_{A404})Proof of (T_{A404})

$$((b^1 \neq \emptyset)(a^1 \neq \emptyset) + (a^1 = \emptyset)(b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\triangleleft\} * b^1) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * a^1))$$

(T_{A312})

$$((b^1 \neq \emptyset)(a^1 \neq \emptyset) + (a^1 = \emptyset)(b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\triangleleft\} * b^1) + (a^1 = \{\triangleleft\} * b^1) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * a^1))$$

(T_{M266})(Definition of $\subseteq$)(Eqn1)

$$\text{Let } a^1 = \{\triangleleft\} * b^1$$

(Eqn2)

$$(a^1 = \{\triangleleft\} * b^1) = E$$

(A_{MX16})(Order Axiom)(Eqn3)

$$((b^1 \neq \emptyset)(\{\triangleleft\} * b^1 \neq \emptyset) + (\{\triangleleft\} * b^1 = \emptyset)(b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\triangleleft\} * b^1) + E \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1))$$

(A_{MX19})(Substitution of Eqns2&3 into Eqn1)

$$((b^1 \neq \emptyset)(\{\triangleleft\} * b^1 \neq \emptyset) + (\{\triangleleft\} * b^1 = \emptyset)(b^1 = \emptyset) = E) \subseteq (E \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1))$$

(T_{M055})(Universal Set Disjunction)

$$((b^1 \neq \emptyset)(\{\triangleleft\} * b^1 \neq \emptyset) + (\{\triangleleft\} * b^1 = \emptyset)(b^1 = \emptyset) = E) \subseteq (E = (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1))$$

(T_{M275})(Supersets of the Universal Set)(Eqn4)

$$(b^1 = \emptyset) \subseteq (\{\triangleleft\} * b^1 = \emptyset)$$

(T_{S303})

$$(b^1 = \emptyset) = (b^1 = \emptyset)(\{\triangleleft\} * b^1 = \emptyset)$$

(A_{MX44})($\subseteq$ Equality Form)(Eqn5)

$$(\{\triangleleft\} * b^1 \neq \emptyset) \subseteq (b^1 \neq \emptyset)$$

(T_{R170})

$$(\{\triangleleft\} * b^1 \neq \emptyset) = (b^1 \neq \emptyset)(\{\triangleleft\} * b^1 \neq \emptyset)$$

(A_{MX44})($\subseteq$ Equality Form)(Eqn6)

$$((\{\triangleleft\} * b^1 \neq \emptyset) + (b^1 = \emptyset) = E) \subseteq (E = (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1))$$

(A_{MX19})(Substitution of Eqns5&6 into Eqn4)

$$((\{\triangleleft\} * b^1 \neq \emptyset) + (b^1 = \emptyset) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)) = E$$

(T_{M094})(Probabilistic Rule)

$$(\{\triangleleft\} * b^1 \neq \emptyset) + (b^1 = \emptyset) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(A_{MX16})(Order Axiom)(Eqn7)

$$(b^1 \subseteq Q\{\triangleleft\}) \subseteq (b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(T_{R526})(Eqn8)

$$(\{\triangleleft\} * b^1 \neq \emptyset) \subseteq (b^1 \subseteq Q\{\triangleleft\})$$

(T_{S410b})(Eqn9)

$$(\{\triangleleft\} * b^1 \neq \emptyset) \subseteq (b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(T_{M139})(Hypothetical Syllogism)

Eqns9&8)(Eqn10)

$$((\{\triangleleft\} * b^1 \neq \emptyset) + (b^1 = \emptyset)).(\{\triangleleft\} * b^1 \neq \emptyset) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1).(b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(T_{M669})(Combination of Eqns7&10)

$$(\{\triangleleft\} * b^1 \neq \emptyset)(\{\triangleleft\} * b^1 \neq \emptyset) + (b^1 = \emptyset).(\{\triangleleft\} * b^1 \neq \emptyset) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1).(b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(T_{M407})(Distribution of $\cdot$ over $+$)

$$(\{\triangleleft\} * b^1 \neq \emptyset) + (b^1 = \emptyset).(\{\triangleleft\} * b^1 \neq \emptyset) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1).(b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(T_{M667})(Conjunction Tautology)(Eqn13)

$$(\{\triangleleft\} * b^1 \neq \emptyset) + (b^1 = \emptyset).(b^1 \neq \emptyset).(\{\triangleleft\} * b^1 \neq \emptyset) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1).(b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(A_{MX19})(Substitution of Eqn6 into Eqn13)

$$(\{\triangleleft\} * b^1 \neq \emptyset) + \emptyset.(\{\triangleleft\} * b^1 \neq \emptyset) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1).(b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(T_{M621})(Complement Conjunction)

$$(\{\triangleleft\} * b^1 \neq \emptyset) + \emptyset \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1).(b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(T_{M050})(Null Set Conjunction)

$$(\{\triangleleft\} * b^1 \neq \emptyset) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1).(b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(T_{M668})(Null Set Disjunction)

$$(\{\triangleleft\} * b^1 \neq \emptyset) \subseteq (\{\triangleright\} * b^1 + b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(T_{M131b})(Conjunction of Sufficiencies)

$$(\{\triangleleft\} * b^1 \neq \emptyset) \subseteq (\{\triangleright\} * b^1 + \{=\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(D_{A144})(Def. of the Reflexive Relationship Set)

$$(\{\triangleleft\} * b^1 \neq \emptyset) \subseteq ((\{\triangleright\} + \{=\}) * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(T_{R002c})(Distribution of $+$ over $*$)

$$(\{\triangleleft\} * b^1 \neq \emptyset) \subseteq (\{\triangleright\} * b^1 \subseteq \{\triangleright\} * \{\triangleleft\} * b^1)$$

(D_{A141})(Disjunction with $\{=\}$)

$$(\{\triangleright\} * b^1 \neq \emptyset) \subseteq (\{\triangleleft\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1)$$

Resymbolising

$$(\{\triangleright\} * b^1 \neq \emptyset) \subseteq (b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1)$$

(T_{A127})Proof of (T_{A127})

$$(\{\triangleright\} * b^1 \neq \emptyset) \subseteq (\{\triangleleft\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1)$$

(T_{A404})

$$(\{\triangleright\} * b^1 \neq \emptyset) \subseteq ((\{\triangleleft\} + \{=\}) * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1)$$

(D_{A141})(Disjunction with $\{=\}$)

$$(\{\triangleright\} * b^1 \neq \emptyset) \subseteq (\{\triangleleft\} * b^1 + \{=\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1)$$

(T_{R002c})(Distribution of $+$ over $*$)

$$(\{\triangleright\} * b^1 \neq \emptyset) \subseteq (\{=\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1)$$

(T_{M072})(Single Premise Syllogism)

$$(\{\triangleright\} * b^1 \neq \emptyset) \subseteq (b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1)$$

(D_{A144})(Definition of the Reflexive Relationship Set)

$$(\{\triangleright\} * b^1 \neq \emptyset) \subseteq (\{\triangleleft\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1)$$

(T_{A123})Proof of (T_{A123})

$$(\{\triangleright\} * b^1 \neq \emptyset) \subseteq (b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1) \quad (T_{A165})$$

Proof of (T_{A165})

$$\begin{aligned} (\{\triangleright\} * b^1 \neq \emptyset) &\subseteq (b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1) && (T_{A143}) \\ (\{\triangleright\} * b^1 \neq \emptyset) &\subseteq (b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1 + \{=\} * \{\triangleright\} * b^1) && (T_{M082})(\text{Disjunction Introduction}) \\ (\{\triangleright\} * b^1 \neq \emptyset) &\subseteq (b^1 \subseteq (\{\triangleleft\} + \{=\}) * \{\triangleright\} * b^1) && (T_{R002c})(\text{Distribution of } + \text{ over } *) \\ (\{\triangleright\} * b^1 \neq \emptyset) &\subseteq (b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1) && (D_{A141})(\text{Disjunction with } \{=\}) \end{aligned}$$

$$(\{\triangleright\} * b^1 \neq \emptyset) \subseteq (\{\triangleleft\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1) \quad (T_{A163})$$

Proof of (T_{A163})

$$\begin{aligned} (\{\triangleright\} * b^1 \neq \emptyset) &\subseteq (\{\triangleleft\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1) && (T_{A130}) \\ (\{\triangleright\} * b^1 \neq \emptyset) &\subseteq (\{\triangleleft\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1 + \{=\} * \{\triangleright\} * b^1) && (T_{M082})(\text{Disjunction Introduction}) \\ (\{\triangleright\} * b^1 \neq \emptyset) &\subseteq (\{\triangleleft\} * b^1 \subseteq (\{\triangleleft\} + \{=\}) * \{\triangleright\} * b^1) && (T_{R002c})(\text{Distribution of } + \text{ over } *) \\ (\{\triangleright\} * b^1 \neq \emptyset) &\subseteq (\{\triangleleft\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1) && (D_{A141})(\text{Disjunction with } \{=\}) \end{aligned}$$

$$(\{\triangleright\} * b^1 \neq \emptyset) \subseteq (\{\triangleleft\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1) \quad (T_{A161})$$

Proof of (T_{A161})

$$\begin{aligned} (\{\triangleright\} * b^1 \neq \emptyset) &\subseteq (\{\triangleleft\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1) && (T_{A128}) \\ (\{\triangleright\} * b^1 \neq \emptyset) &\subseteq (\{\triangleleft\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1 + \{=\} * \{\triangleright\} * b^1) && (T_{M082})(\text{Disjunction Introduction}) \\ (\{\triangleright\} * b^1 \neq \emptyset) &\subseteq (\{\triangleleft\} * b^1 \subseteq (\{\triangleleft\} + \{=\}) * \{\triangleright\} * b^1) && (T_{R002c})(\text{Distribution of } + \text{ over } *) \\ (\{\triangleright\} * b^1 \neq \emptyset) &\subseteq (\{\triangleleft\} * b^1 \subseteq \{\triangleleft\} * \{\triangleright\} * b^1) && (D_{A141})(\text{Disjunction with } \{=\}) \end{aligned}$$

$$\{\triangleright\} \diamond a \subseteq \{\triangleleft\} * a \quad (T_{S380})$$

Proof of (T_{S380})

$$\begin{aligned} \{\triangleleft\}^1 &\subseteq R_{\tau} R_A && (T_{739}) \\ \{\triangleleft\}^1 &\subseteq R_A && (T_{M073})(\text{Single Premise Syllogism}) \\ (\{\triangleleft\}^1 \subseteq R_A) &= E && (A_{MX16})(\text{Order Axiom})(\text{Eqn1}) \\ (c^1 \subseteq R_A) &= (\forall c \subseteq c) && (D_{A145})(\text{Definition of an Asymmetrical Relationship Set}) \\ (\{\triangleleft\}^1 \subseteq R_A) &= (\forall \{\triangleleft\} \subseteq \sim \{\triangleleft\}) && \text{Resymbolising} \\ (\{\triangleleft\}^1 \subseteq R_A) &= (\{\triangleright\} \subseteq \sim \{\triangleleft\}) && (D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set})(\text{Eqn2}) \\ E &= (\{\triangleright\} \subseteq \sim \{\triangleleft\}) && (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\ \{\triangleright\} &\subseteq \sim \{\triangleleft\} && (A_{MX16})(\text{Order Axiom}) \\ \{\triangleright\} * a_{r^1} &\subseteq \sim \{\triangleleft\} * a_{r^1} && (T_{R197})(\text{Relational Deduction}) \\ (a_{r^1} \neq \emptyset) &= (\{\triangleright\} * a_{r^1} \subseteq \{\triangleleft\} * a_{r^1}) && (T_{R245})(\text{Negative Rule for Single Element Object})(\text{Eqn3}) \\ \text{Let } a_{r^1} &= \emptyset && (\text{Eqn4}) \\ \text{Let } \{\triangleright\} * a_{r^1} &\not\subseteq \{\triangleleft\} * a_{r^1} && (\text{Eqn5}) \\ \{\triangleright\} * \emptyset &\not\subseteq \{\triangleleft\} * \emptyset && (A_{MX19})(\text{Substitution of Eqn4 into Eqn5}) \\ \{\triangleright\} * \emptyset &\not\subseteq \sim(\{\triangleleft\} * \emptyset) && (D_{R04})(\text{Definition of } *) \\ \emptyset &\not\subseteq \sim \emptyset && (T_{S181})(\text{Null Object Theorem}) \\ \emptyset &\not\subseteq E && (T_{M054})(\text{Negative of Null Set}) \\ (\emptyset \not\subseteq E) &= E && (A_{MX16})(\text{Order Axiom}) \\ (\emptyset \subseteq E) &= \sim E && (T_{M012})(\text{Transposition}) \\ (\emptyset \subseteq E) &= \emptyset && (D_{M02})(\text{Definition of } \emptyset) \\ E &= \emptyset && (T_{M001})(\text{Universality of } (T_{M670}) \text{ Null Set Theorem}) \\ \emptyset &&& (T_{M347}) \\ \text{Thus } (a_{r^1} = \emptyset) &(\{\triangleright\} * a_{r^1} \not\subseteq \{\triangleleft\} * a_{r^1}) = \emptyset && (A_{MX02})(\text{Conditional Introduction}) \\ (a_{r^1} = \emptyset) &\subseteq (\{\triangleright\} * a_{r^1} \subseteq \{\triangleleft\} * a_{r^1}) && (T_{M735})(\subseteq \text{Existential Form})(\text{Eqn6}) \\ (a_{r^1} = \emptyset) + (a_{r^1} \neq \emptyset) &= (\{\triangleright\} * a_{r^1} \subseteq \{\triangleleft\} * a_{r^1}) + (\{\triangleright\} * a_{r^1} \subseteq \{\triangleleft\} * a_{r^1}) && (A_{MX21})(\text{Combination of Eqns3\&6}) \\ (a_{r^1} = \emptyset) + (a_{r^1} \neq \emptyset) &= (\{\triangleright\} * a_{r^1} \subseteq \{\triangleleft\} * a_{r^1}) && (A_{MX01})(\text{Disjunction Tautology}) \\ E &= (\{\triangleright\} * a_{r^1} \subseteq \{\triangleleft\} * a_{r^1}) && (T_{M671})(\text{Tertium non Datur}) \\ \{\triangleright\} * a_{r^1} &\subseteq \{\triangleleft\} * a_{r^1} && (A_{MX16})(\text{Order Axiom})(\text{Eqn7}) \\ \{\triangleright\} * a_1 \dots \{\triangleright\} * a_n &\subseteq \{\triangleleft\} * a_1 \dots \{\triangleleft\} * a_n && (T_{M669})(\text{Combination of Eqn7 For } r = 1 \text{ to } n) \end{aligned}$$

$$\{\triangleright\} * a_1^1 \dots \{\triangleright\} * a_n^1 \subseteq \sim(\{\triangleleft\} * a_1^1 +++ \{\triangleleft\} * a_n^1)$$

$$\{\triangleright\} \diamond (a_1^1 +++ a_n^1) \subseteq \sim(\{\triangleleft\} * a_1^1 +++ \{\triangleleft\} * a_n^1)$$

$$\{\triangleright\} \diamond (a_1^1 +++ a_n^1) \subseteq \sim\{\triangleleft\} * (a_1^1 +++ a_n^1)$$

$$\{\triangleright\} \diamond (a_1^1 +++ a_n^1) \subseteq \{\triangleleft\} \underline{*} (a_1^1 +++ a_n^1)$$

$$\text{Let } \mathbf{a} = a_1^1 +++ a_n^1$$

$$\{\triangleright\} \diamond \mathbf{a} \subseteq \{\triangleleft\} \underline{*} \mathbf{a}$$

(T_{M188})(DeMorgan's Law for Series)

(D_{S15})(Definition of the Intersection Function Operator)

(A_{S64})(Distribution of $*$ over $+$)

(D_{R04})(Definition of $\underline{*}$)(Eqn8)

(Eqn9)

(A_{MX19})(Substitution of Eqn9 into Eqn8)

Chapter 6 Proofs of Relationship Theorems – Part 3 – Binary Relations and Operators

6.1 Introduction

This chapter sets out the proofs of theorems associated with the theory of binary relations, such as $\subseteq$ and $\supset$, and binary operators, such as $+$ and $\cdot$ (dot).

The proofs are not however set out in these categories but rather in the order in which they are proven from axioms, definitions, and previous theorems. As a rule, therefore a theorem is not used in a proof before it has itself been proven. This helps to avoid circularity.

6.2 Proofs using the Axioms and Definitions of Relationship Theory Only

$$\{=\}^1 \subseteq R_B \quad (T_{695})$$

$$\{<\}^1 \subseteq R_B \quad (T_{696})$$

$$\{>\}^1 \subseteq R_B \quad (T_{697})$$

$$\{||\}^1 \subseteq R_B \quad (T_{698})$$

$$\{\subseteq\}^1 \subseteq R_B \quad (T_{699})$$

$$\{\supset\}^1 \subseteq R_B \quad (T_{700})$$

Example Proof of (T_{695})

$$(o = \{=\}, \{<\}, \{>\}, \{||\}, \{\subseteq\}, \{\supset\}) = (o^1 \subseteq R_B)$$

(D_{R94}) (Definition of a Membership Relationship Set)
Extraction

$$\{=\}^1 \subseteq R_S \quad (T_{603}) \text{ Third Peano Axiom}$$

Proof of (T_{603})

$$(o = \{=\}, \{||\}): (o^1 \subseteq R_S)$$

(A_{509}) (Symmetrical Relationship Sets)
Extraction

$$\forall \{=\} = \{=\} \quad (T_{485})$$

$$\forall \{||\} = \{||\} \quad (T_{488})$$

$$\forall \{<\} = \{<\} \quad (T_{491})$$

$$\forall \{>\} = \{>\} \quad (T_{494})$$

Example Proof of (T_{485})

$$(o = \{=\}, \{||\}): (o^1 \subseteq R_S)$$

(A_{509}) (Symmetrical Relationship Sets)
Extraction

$$\{=\}^1 \subseteq R_S$$

(D_{A94}) (Definition of a Symmetrical Relationship Set)

$$(a^1 \subseteq \{<\} * b^1) = (b^1 \subseteq \{>\} * a^1) \quad (T_{775})$$

Proof of (T_{775})

$$(a^1 \subseteq \{<\} * b^1) = (b^1 \subseteq \forall \{<\} * a^1)$$

(T_{519}) (Active/Passive Theorem)

$$(a^1 \subseteq \{<\} * b^1) = (b^1 \subseteq \{>\} * a^1)$$

(D_{A143}) (Symmetrical Opposite of an Asymmetrical Relationship Set)

6.3 Proofs using The Axioms and Definitions of Membership and Relationship Theory

$$\{=\} * \emptyset^1 = \emptyset^1 \quad (T_{7873})$$

Proof of (T₇₈₇₃)

$$\begin{aligned} \{=\} * \emptyset^1 &= \{=\} * \emptyset^1 & (D_{M00})(\text{Definition of } =) \\ \{=\} * \emptyset^1 &= \emptyset^1 & (D_{A144})(\text{Definition of the Reflexive Relationship Set}) \end{aligned}$$

$$\{=\} * E^1 = E^1 \quad (T_{7874})$$

Proof of (T₇₈₇₄)

$$\begin{aligned} \{=\} * E^1 &= \{=\} * E^1 & (D_{M00})(\text{Definition of } =) \\ \{=\} * E^1 &= E^1 & (D_{A144})(\text{Definition of the Reflexive Relationship Set}) \end{aligned}$$

$$\{\subseteq\}^1 \subseteq R_T \quad (T_{R600})$$

$$\{\supseteq\}^1 \subseteq R_T \quad (T_{R602})$$

Example Proof of (T_{R600})

$$\begin{aligned} \{\triangleleft\} &= \{\subseteq, \supseteq, \triangleleft, \triangleright\} & (D_{S08})(\text{Definition of } \triangleleft) \\ \{\triangleleft\} &= \{\subseteq\} & \text{Extraction (Eqn1)} \\ (\{\triangleleft\} + \{=\})^1 &\subseteq R_T & (A_{R22})(\text{Transitive Relationships Disjunction Axiom})(\text{Eqn2}) \\ (\{\subseteq\} + \{=\})^1 &\subseteq R_T & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\ \{\subseteq\}^1 &\subseteq R_T & (D_{A141})(\text{Disjunction with } \{=\}) \end{aligned}$$

6.4 Proofs using Membership Theory and the Axioms and Definitions of Relationship Theory

$$(b^1 \neq \emptyset) = (\{ \neq \} * b^1 = \sim(b^1)) \quad (T_{A084})$$

Proof of (T_{A084})

$$\begin{aligned} (b^1 \neq \emptyset) &= ((c * b^1) = \sim(c * b^1)) & (T_{R245})(\text{Negative Rule for Single Element Object})(\text{Eqn1}) \\ \text{Let } c &= \{=\} & (\text{Eqn2}) \\ (b^1 \neq \emptyset) &= (\{=\} * b^1 = \sim(\{ \neq \} * b^1)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (b^1 \neq \emptyset) &= (\{ \neq \} * b^1 = \sim(\{=\} * b^1)) & (T_{M008})(\text{Inverse of an Equality}) \\ (b^1 \neq \emptyset) &= (\{ \neq \} * b^1 = \sim(b^1)) & (D_{A144})(\text{Definition of the Reflexive Relationship Set}) \end{aligned}$$

$$(a = a^1) + (a = \emptyset) = (a. \{ \neq \} * a = \emptyset) \quad (T_{S661})$$

Proof of (T_{S661})

$$\begin{aligned} (a \neq a^1)(a \neq \emptyset) &= (a. \{ \neq \} * a \neq \emptyset) & (A_{R37})(\text{Axiom of Plural Sets}) \\ \sim((a \neq a^1) + (a \neq \emptyset)) &= (a. \{ \neq \} * a = \emptyset) & (T_{M008})(\text{Inverse of an Equality}) \\ (a = a^1) + (a = \emptyset) &= (a. \{ \neq \} * a = \emptyset) & (T_{M026})(\text{DeMorgan's Law}) \end{aligned}$$

$$(\forall Sb^1 \forall Qb^1) \subseteq \{=\} \quad (T_{A299}) \text{ Definition of a Reflexive Relationship}$$

Proof of (T_{A299})

$$\begin{aligned} o &= [(\forall Sb_s^1 \forall Qb_r^1)(b_s \langle o \rangle b_r) \text{+++}]^E_{r=1} \text{+++}]^E_{s=1} & (D_{A02})(\text{Definition of a Relationship Set})(\text{Eqn1}) \\ \text{Let } o &= \{=\} & (\text{Eqn2}) \\ \{=\} &= [(\forall Sb_s^1 \forall Qb_r^1)(b_s \langle \{=\} \rangle b_r) \text{+++}]^E_{r=1} \text{+++}]^E_{s=1} & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \{=\} &= [(\forall Sb_s^1 \forall Qb_r^1)(b_s = b_r) \text{+++}]^E_{r=1} \text{+++}]^E_{s=1} & (D_{A229})(\text{Definition of } \langle \rangle) \\ \{=\} &= [(\forall Sb_r^1 \forall Qb_r^1) \text{+++}]^E_{r=1} \text{+++}]^E_{s=1} & (A_{MX19})(\text{Substitution}) \\ \{=\} &= [(\forall Sb_r^1 \forall Qb_r^1) \text{+++}]^E_{r=1} & (A_{MX01})(\text{Disjunction Tautology}) \\ (\forall Sb^1 \forall Qb^1) &\subseteq \{=\} & (T_{M072})(\text{Single Premise Syllogism}) \\ (\forall Sb^1 \forall Qb^1) &\subseteq \{=\} & \text{Resymbolising} \end{aligned}$$

$$\{\subset\} \subseteq \{\neq\} \quad (T_{A103})$$

$$\{\supset\} \subseteq \{\neq\} \quad (T_{A104})$$

$$\{\|\} \subseteq \{\neq\} \quad (T_{A105})$$

$$\{\supset\} \subseteq \{\sim\subset\} \quad (T_{A106})$$

$$\{\|\} \subseteq \{\sim\subset\} \quad (T_{A107})$$

$$\{\|\} \subseteq \{\sim\supset\} \quad (T_{A108})$$

Example Proof of (T_{A103})

$$\begin{aligned} (a \subset b)(a = b) &= \emptyset & (T_{M037})(\text{Conjunction of Binary Relations}) \\ (a \subset b) &\subseteq \sim(a = b) & (T_{M735})(\subseteq \text{Existential Form}) \\ {}^v\text{Sa}^{\text{IV}}\text{Qb}^1(a \subset b) &\subseteq {}^v\text{Sa}^{\text{IV}}\text{Qb}^1(a \neq b) & (T_{M029})(\text{Deduction}) \\ {}^v\text{Sa}_r^{\text{IV}}\text{Qb}_s^1(a_r \subset b_s) &\subseteq {}^v\text{Sa}_r^{\text{IV}}\text{Qb}_s^1(a_r \neq b_s) & \text{Resymbolising(Eqns1)} \\ [[{}^v\text{Sa}_r^{\text{IV}}\text{Qb}_s^1(a_r \subset b_s)+++]]_{r=1}^E +++]_{s=1}^E &\subseteq [[{}^v\text{Sa}_r^{\text{IV}}\text{Qb}_s^1(a_r \neq b_s)+++]]_{r=1}^E +++]_{s=1}^E & (T_{M669})(\text{Combination of Eqn1 for } r=1 \text{ to } E \text{ and } s=1 \text{ to } E)(\text{Eqn2}) \\ \{\subset\} &= [[({}^v\text{Sa}_s^{\text{IV}}\text{Qb}_r^1)(a_s \subset b_r)+++]]_{r=1}^E +++]_{s=1}^E & (D_{A02}) (o = \{\subset\})(\text{Eqn3}) \\ \{\neq\} &= [[({}^v\text{Sa}_s^{\text{IV}}\text{Qb}_r^1)(a_s \neq b_r)+++]]_{r=1}^E +++]_{s=1}^E & (D_{A02}) (o = \{\neq\})(\text{Eqn4}) \\ \{\subset\} &\subseteq \{\neq\} & (A_{MX19})(\text{Substitution of Eqns 3\&4 into Eqn2}) \end{aligned}$$

$$(\{\supset\} * b^1 = \emptyset) \subseteq (\{\|\} * b^1 = \emptyset) \quad (T_{A692})$$

Proof of (T_{A692})

$$\begin{aligned} (\{\supset\} * b^1 = \emptyset) &\subseteq (\{\supset\} * b^1 = \emptyset) + (\{\subset\} * b^1 = \emptyset) & (T_{M082})(\text{Disjunction Introduction}) \\ (\{\supset\} * b^1 = \emptyset) &\subseteq (\{\|\} * b^1 = \emptyset) & (A_{A72})(\text{Intersection Axiom}) \end{aligned}$$

$$(\{\subset\} * b^1 = \emptyset) \subseteq (\{\|\} * b^1 = \emptyset) \quad (T_{A693})$$

Proof of (T_{A693})

$$\begin{aligned} (\{\subset\} * b^1 = \emptyset) &\subseteq (\{\supset\} * b^1 = \emptyset) + (\{\subset\} * b^1 = \emptyset) & (T_{M082})(\text{Disjunction Introduction}) \\ (\{\subset\} * b^1 = \emptyset) &\subseteq (\{\|\} * b^1 = \emptyset) & (A_{A72})(\text{Intersection Axiom}) \end{aligned}$$

$$(\{\subset\} * b = \emptyset) \subseteq (\{\|\} * b = \emptyset) \quad (T_{882})$$

$$(\{\supset\} * b = \emptyset) \subseteq (\{\|\} * b = \emptyset) \quad (T_{883})$$

Example Proof of (T_{882})

$$\begin{aligned} (\{\subset\} * b^1 = \emptyset) &\subseteq (\{\|\} * b^1 = \emptyset) & (T_{A693}) \\ (\{\|\} * b^1 \subseteq \{\subset\} * b^1) &= E & (T_{M097})(\text{Probabilistic Rule}) \\ \{\|\} * b^1 &\subseteq \{\subset\} * b^1 & (A_{MX16})(\text{Order Axiom})(\text{Eqn1}) \\ \{\|\} * b_1^1 +++ \{\|\} * b_n^1 &\subseteq \{\subset\} * b_1^1 +++ \{\subset\} * b_n^1 & (A_{MX21})(\text{Combination of Eqn1 for } b^1 = b_1^1 \text{ to } b_n^1) \\ \{\|\} * (b_1^1 +++ b_n^1) &\subseteq \{\subset\} * (b_1^1 +++ b_n^1) & (A_{S64})(\text{Distribution of } * \text{ over } +)(\text{Eqn1}) \\ \text{Let } b &= b_1^1 +++ b_n^1 & (\text{Eqn2}) \\ \{\|\} * b &\subseteq \{\subset\} * b & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (\{\|\} * b \subseteq \{\subset\} * b) &= E & (A_{MX16})(\text{Order Axiom}) \\ (\{\subset\} * b = \emptyset) &\subseteq (\{\|\} * b = \emptyset) & (T_{M097})(\text{Probabilistic Rule}) \end{aligned}$$

$$(\{\supset\} * b^1 \neq \emptyset).(\{\subset\} * b^1 \neq \emptyset) = (\{\|\} * b^1 \neq \emptyset) \quad (T_{A689})$$

Proof of (T_{A689})

$$\begin{aligned} (\{\supset\} * b^1 = \emptyset) + (\{\subset\} * b^1 = \emptyset) &= (\{\|\} * b^1 = \emptyset) & (A_{A72})(\text{Intersection Axiom}) \\ \sim((\{\supset\} * b^1 = \emptyset) + (\{\subset\} * b^1 = \emptyset)) &= \sim(\{\|\} * b^1 = \emptyset) & (T_{M008})(\text{Inverse of an Equality}) \\ \sim(\{\supset\} * b^1 = \emptyset). \sim(\{\subset\} * b^1 = \emptyset) &= \sim(\{\|\} * b^1 = \emptyset) & (T_{M026})(\text{DeMorgan's Law}) \\ (\{\supset\} * b^1 \neq \emptyset).(\{\subset\} * b^1 \neq \emptyset) &= (\{\|\} * b^1 \neq \emptyset) & (T_{M260})(\text{Definition of } \neq) \end{aligned}$$

$$(a \neq \emptyset) \subseteq (a^1 \subseteq \{\supset\} * \{\supset\} \diamond a) \quad (TS220)$$

Proof of (TS220)

$$\begin{aligned} (a \neq \emptyset) &\subseteq (a^1 = (\{\supset\} \diamond a).(\{\supset\} * \{\supset\} \diamond a)) && (DR98)(\text{General Definition of Aggregation})(\text{Eqn1}) \\ (a^1 = (\{\supset\} \diamond a).(\{\supset\} * \{\supset\} \diamond a)) &\subseteq (a^1 \subseteq \{\supset\} * \{\supset\} \diamond a) && (TM073b)(\text{Single Premise Syllogism})(\text{Eqn2}) \\ (a \neq \emptyset) &\subseteq (a^1 \subseteq \{\supset\} * \{\supset\} \diamond a) && (TM139)(\text{Hypothetical Syllogism Eqns 1\&2}) \end{aligned}$$

$$(a \neq \emptyset) \subseteq (a^1 \subseteq \{\supset\} \diamond a) \quad (TS215)$$

Proof of (TS215)

$$\begin{aligned} (a \neq \emptyset) &\subseteq (a^1 = (\{\supset\} \diamond a).(\{\supset\} * \{\supset\} \diamond a)) && (DR98)(\text{General Definition of Aggregation})(\text{Eqn1}) \\ (a^1 = (\{\supset\} \diamond a).(\{\supset\} * \{\supset\} \diamond a)) &\subseteq (a^1 \subseteq \{\supset\} \diamond a) && (TM073b)(\text{Single Premise Syllogism})(\text{Eqn2}) \\ (a \neq \emptyset) &\subseteq (a^1 \subseteq \{\supset\} \diamond a) && (TM139)(\text{Hypothetical Syllogism Eqns 1\&2}) \end{aligned}$$

$$\{\|\}^1 \subseteq R_S.R_B \quad (T720) \text{ Intersecting Relationships Theorem}$$

Proof of (T720)

$$\begin{aligned} (o = \{=\}, \{\subset\}, \{\supset\}, \{\|\}, \{\subseteq\}, \{\supseteq\}) &= (o^1 \subseteq R_B) && (DR94)(\text{Definition of a Membership Relationship Set}) \\ \{\|\}^1 &\subseteq R_B && \text{Extraction (Eqn1)} \\ (o^1 \subseteq R_S): (o = \{=\}, \{\|\}) &&& (AS09)(\text{Symmetrical Relationship Sets}) \\ \{\|\}^1 &\subseteq R_S && \text{Extraction (Eqn2)} \\ \{\|\}^1 \{\|\}^1 &\subseteq R_S R_B && (TM669)(\text{Combination of Eqns1\&2}) \\ \{\|\}^1 &\subseteq R_S R_B && (TM667)(\text{Conjunction Tautology}) \end{aligned}$$

$$(o = \{=\}, \{\subset\}, \{\supset\}, \{\|\}, \{\subseteq\}, \{\supseteq\}, \{\<\}, \{\>\}, \{\leq\}, \{\geq\}) = (o^1 \subseteq R_B + R_L) \quad (T711) \text{ Definition of a Binary Relationship Set}$$

Proof of (T711)

$$\begin{aligned} (o = \{=\}, \{\subset\}, \{\supset\}, \{\|\}, \{\subseteq\}, \{\supseteq\}) &= (o^1 \subseteq R_B) && (DR94)(\text{Definition of a Membership Relationship Set})(\text{Eqn1}) \\ (o = \{=\}, \{\<\}, \{\>\}, \{\leq\}, \{\geq\}) &= (o^1 \subseteq R_L) && (DR96)(\text{Definition of an Ordering Relationship Set})(\text{Eqn2}) \\ (o = \{=\}, \{\subset\}, \{\supset\}, \{\|\}, \{\subseteq\}, \{\supseteq\}) + (o = \{=\}, \{\<\}, \{\>\}, \{\leq\}, \{\geq\}) &= (o^1 \subseteq R_B) + (o^1 \subseteq R_L) && (AMX21)(\text{Combination of Eqns1\&2}) \\ (o = \{=\}, \{\subset\}, \{\supset\}, \{\|\}, \{\subseteq\}, \{\supseteq\}, \{\<\}, \{\>\}, \{\leq\}, \{\geq\}) &= (o^1 \subseteq R_B) + (o^1 \subseteq R_L) && (TM753) \\ (o = \{=\}, \{\subset\}, \{\supset\}, \{\|\}, \{\subseteq\}, \{\supseteq\}, \{\<\}, \{\>\}, \{\leq\}, \{\geq\}) &= (o^1 \subseteq R_B + R_L) && (TM170)(\text{Disjunction of Necessities}) \end{aligned}$$

6.5 Higher Tier Proofs

$$(o^1 \subseteq R_B) = ((\vee o)^1 \subseteq R_B) \quad (T779)$$

Proof of (T779)

$$\begin{aligned} \text{Let } o^1 &\subseteq R_B && (\text{Eqn1}) \\ o &= \{=\}, \{\subset\}, \{\supset\}, \{\|\}, \{\subseteq\}, \{\supseteq\} && (DR94)(\text{Definition of a Membership Relationship Set})(\text{Eqn2}) \\ \vee o &= \vee \{=\}, \vee \{\subset\}, \vee \{\supset\}, \vee \{\|\}, \vee \{\subseteq\}, \vee \{\supseteq\} && (TR022)(\text{Relationship Opposite of a Proposition}) \\ \vee o &= \vee \{=\}, \{\subset\}, \{\supset\}, \vee \{\|\}, \vee \{\subseteq\}, \vee \{\supseteq\} && (DA143)(\text{Symmetrical Opposite of an Asymmetrical Relationship Set}) \\ \vee o &= \vee \{=\}, \{\subset\}, \{\supset\}, \vee \{\|\}, \{\subseteq\}, \{\supseteq\} && (T741)(\text{Definition of } \vee \{\leq\}) \\ \vee o &= \{=\}, \{\subset\}, \{\supset\}, \{\|\}, \{\subseteq\}, \{\supseteq\} && (TA485) \& (TA488)(\text{Eqn3}) \\ \vee o &= o && (AMX19)(\text{Substitution of Eqn3 into Eqn2})(\text{Eqn4}) \\ (\vee o)^1 &\subseteq R_B && (AMX19)(\text{Substitution of Eqn4 into Eqn1}) \\ \text{Thus } (o^1 \subseteq R_B) &= ((\vee o)^1 \subseteq R_B) && (AMX02)(\text{Conditional Introduction}) \text{ since this proof is reversible.} \end{aligned}$$

$$(b^1 \neq \emptyset)(c^1 \neq \emptyset)(c^1 * b^1 = b^1) \subseteq (c^1 \subseteq \{=\}) \quad (TA246) \text{ Definition of a Reflexive Relationship}$$

Proof of (TA246)

$$\begin{aligned} \text{Let } c^1 * b^1 &= b^1 && (\text{Eqn1}) \\ S(c^1 \vee Qb^1) &= b^1 && (AR27)(\text{Function Axiom})(\text{Eqn2}) \end{aligned}$$

Let $\mathbf{b}^1 \neq \emptyset$	(Eqn3)
Let $\mathbf{c}^1 \neq \emptyset$	(Eqn4)
$\mathbf{S}(\mathbf{c}^1 \vee \mathbf{Qb}^1) \neq \emptyset$	($\mathbf{A}_{MX19}$)(Substitution of Eqn2 into Eqn3)
$\mathbf{c}^1 \vee \mathbf{Qb}^1 \neq \emptyset$	($\mathbf{T}_{R052}$)(Triad Theorem)
$\sim(\mathbf{c}^1 \vee \mathbf{Qb}^1 = \emptyset)$	($\mathbf{T}_{M260}$)(Definition of $\neq$)
$\sim(\mathbf{c}^1 \subseteq \sim \vee \mathbf{Qb}^1)$	($\mathbf{T}_{M735}$)($\subseteq$ Existential Form)
$\mathbf{c}^1 \not\subseteq \sim \vee \mathbf{Qb}^1$	($\mathbf{T}_{M259}$)(Definition of $\not\subseteq$)
$\mathbf{c}^1 \subseteq \vee \mathbf{Qb}^1$	($\mathbf{T}_{M604}$)
$\mathbf{c}^1 = \mathbf{c}^1 \vee \mathbf{Qb}^1$	($\mathbf{A}_{MX44}$)($\subseteq$ Equality Form)(Eqn5)
$\{\neq\} * \mathbf{b}^1 = \sim \mathbf{b}^1$	($\mathbf{T}_{A084}$) since $\mathbf{b}^1 \neq \emptyset$ (Eqn6)
$\mathbf{b}^1 . \sim \mathbf{b}^1 = \emptyset$	($\mathbf{T}_{M621}$)(Complement Conjunction) (Eqn7)
$(\mathbf{c}^1 * \mathbf{b}^1)(\{\neq\} * \mathbf{b}^1) = \emptyset$	($\mathbf{A}_{MX19}$)(Substitution of Eqns1&6 into Eqn7)
$(\mathbf{c}^1 \{\neq\}) * \mathbf{b}^1 = \emptyset$	($\mathbf{T}_{R239}$)(Distribution of $.$ over $*$ for Entities)
$\mathbf{S}(\mathbf{c}^1 \{\neq\} \vee \mathbf{Qb}^1) = \emptyset$	($\mathbf{A}_{R27}$)(Function Axiom)
$\mathbf{c}^1 \{\neq\} \vee \mathbf{Qb}^1 = \emptyset$	($\mathbf{T}_{R052}$)(Triad Theorem)(Eqn8)
$\mathbf{c}^1 \{\neq\} = \emptyset$	($\mathbf{A}_{MX19}$)(Substitution of Eqn5 into Eqn8)
$\mathbf{c}^1 \subseteq \sim \{\neq\}$	($\mathbf{T}_{M735}$)($\subseteq$ Existential Form)
$\mathbf{c}^1 \subseteq \{=\}$	($\mathbf{D}_{A142}$)(Definition of $\sim \{\}$)
Thus $(\mathbf{b} \neq \mathbf{b}^1)(\mathbf{b} \neq \emptyset)(\mathbf{c}^1 * \mathbf{b}^1 = \mathbf{b}^1) \subseteq (\mathbf{c}^1 \subseteq \{=\})$	($\mathbf{A}_{MX02}$)(Conditional Introduction)

$$(\mathbf{b} \neq \mathbf{b}^1)(\mathbf{b} \neq \emptyset) \subseteq (\{\neq\} * \mathbf{b} = \mathbf{E}) \quad (\mathbf{T}_{A086})$$

Proof of ($\mathbf{T}_{A086}$)

Let $\mathbf{b} \neq \mathbf{b}^1$	(Eqn1)
Let $(\mathbf{b} = \mathbf{b}_1^1 + \dots + \mathbf{b}_n^1)$	($\mathbf{T}_{M072b}$)(Single Premise Syllogism)(Eqn2)
$\mathbf{b}_1^1 \subseteq \mathbf{b}$	
Let $(\mathbf{b}_1^1 \neq \emptyset) \dots (\mathbf{b}_n^1 \neq \emptyset)$	
$\mathbf{b} \neq \emptyset$	($\mathbf{T}_{M096}$)(Probabilistic Rule)
$(\mathbf{b}_1^1 \neq \emptyset) \dots (\mathbf{b}_n^1 \neq \emptyset) = \mathbf{E}$	($\mathbf{A}_{MX16}$)(Order Axiom)(Eqn3)
Let $\mathbf{b}_1^1 \neq \mathbf{b}_n^1$	i.e. $\mathbf{b}_1^1$ and $\mathbf{b}_n^1$ are different individual entities
$(\mathbf{b}_1^1 \neq \mathbf{b}_n^1) = \mathbf{E}$	($\mathbf{A}_{MX16}$)(Order Axiom)(Eqn4)
$(\mathbf{b}_1^1 \not\subseteq \mathbf{b}_n^1) = (\mathbf{b}_1^1 \sim \mathbf{b}_n^1) . (\mathbf{b}_1^1 \neq \mathbf{b}_n^1)$	($\mathbf{T}_{M198}$)(Eqn5)
$(\mathbf{b}_1^1 \subset \mathbf{b}_n^1) = (\mathbf{b}_n^1 \neq \emptyset)(\mathbf{b}_1^1 = \emptyset)$	($\mathbf{A}_{MX40}$)(Proper Subset of Unit Set Axiom)
$(\mathbf{b}_1^1 \sim \mathbf{b}_n^1) = \sim((\mathbf{b}_n^1 \neq \emptyset)(\mathbf{b}_1^1 = \emptyset))$	($\mathbf{T}_{M008}$)(Inverse of an Equality)
$(\mathbf{b}_1^1 \sim \mathbf{b}_n^1) = (\mathbf{b}_n^1 = \emptyset) + (\mathbf{b}_1^1 \neq \emptyset)$	($\mathbf{T}_{M025}$)(DeMorgan's Law)
$(\mathbf{b}_1^1 \sim \mathbf{b}_n^1) = (\mathbf{b}_n^1 = \emptyset) + \mathbf{E}$	since $\mathbf{b}_1^1 \neq \emptyset$
$(\mathbf{b}_1^1 \sim \mathbf{b}_n^1) = \mathbf{E}$	($\mathbf{T}_{M055}$)(Universal Set Disjunction)(Eqn6)
$(\mathbf{b}_1^1 \not\subseteq \mathbf{b}_n^1) = \mathbf{E} . (\mathbf{b}_1^1 \neq \mathbf{b}_n^1)$	($\mathbf{A}_{MX19}$)(Substitution of Eqn6 into Eqn5)
$(\mathbf{b}_1^1 \not\subseteq \mathbf{b}_n^1) = (\mathbf{b}_1^1 \neq \mathbf{b}_n^1)$	($\mathbf{A}_{MX11}$)(Universal Set Conjunction Axiom)(Eqn7)
$(\mathbf{b}_1^1 \not\subseteq \mathbf{b}_n^1) = \mathbf{E}$	($\mathbf{A}_{MX19}$)(Substitution of Eqn4 into Eqn7)
$\mathbf{b}_1^1 \not\subseteq \mathbf{b}_n^1$	($\mathbf{A}_{MX16}$)(Order Axiom)
$\mathbf{b}_1^1 \subseteq \sim \mathbf{b}_n^1$	($\mathbf{T}_{M604}$)
$\mathbf{b}_1^1 \mathbf{b}_n^1 = \emptyset$	($\mathbf{T}_{M735}$)($\subseteq$ Existential Form)(Eqn8)
$\mathbf{b}_1^1 \dots \mathbf{b}_n^1 = \mathbf{b}_1^1 \dots \mathbf{b}_n^1 \mathbf{b}_1^1 \mathbf{b}_n^1$	($\mathbf{T}_{M667}$)(Conjunction Tautology)(Eqn9)
$\mathbf{b}_1^1 \dots \mathbf{b}_n^1 = \mathbf{b}_1^1 \dots \mathbf{b}_n^1 \emptyset$	($\mathbf{A}_{MX19}$)(Substitution of Eqn8 into Eqn9)
$\mathbf{b}_1^1 \dots \mathbf{b}_n^1 = \emptyset$	($\mathbf{T}_{M050}$)(Conjunction with the Null Set)(Eqn10)
$\{\neq\} * \mathbf{b} = \{\neq\} * \mathbf{b}_1^1 + \dots + \{\neq\} * \mathbf{b}_n^1$	($\mathbf{T}_{R001a}$)(Distribution of $*$ over $+$)
$(\mathbf{b}_1^1 \neq \emptyset) \dots (\mathbf{b}_n^1 \neq \emptyset) \subseteq (\{\neq\} * \mathbf{b} = \sim(\mathbf{b}_1^1) + \dots + \sim(\mathbf{b}_n^1))$	($\mathbf{T}_{A084}$)(Eqn11)
$\mathbf{E} \subseteq (\{\neq\} * \mathbf{b} = \sim(\mathbf{b}_1^1) + \dots + \sim(\mathbf{b}_n^1))$	($\mathbf{A}_{MX19}$)(Substitution of Eqn3 into Eqn11)
$\mathbf{E} = (\{\neq\} * \mathbf{b} = \sim(\mathbf{b}_1^1) + \dots + \sim(\mathbf{b}_n^1))$	($\mathbf{T}_{M275}$)(Supersets of the Universal Set)
$\{\neq\} * \mathbf{b} = \sim(\mathbf{b}_1^1) + \dots + \sim(\mathbf{b}_n^1)$	($\mathbf{A}_{MX16}$)(Order Axiom)
$\{\neq\} * \mathbf{b} = \sim(\mathbf{b}_1^1 \dots \mathbf{b}_n^1)$	($\mathbf{T}_{M187}$)(DeMorgan's Law for Series)(Eqn12)
$\{\neq\} * \mathbf{b} = \sim \emptyset$	($\mathbf{A}_{MX19}$)(Substitution of Eqn10 into Eqn12)
$\{\neq\} * \mathbf{b} = \mathbf{E}$	($\mathbf{T}_{M054}$)(Negative of the Null Set)
Thus $(\mathbf{b} \neq \mathbf{b}^1)(\mathbf{b} \neq \emptyset) \subseteq (\{\neq\} * \mathbf{b} = \mathbf{E})$	($\mathbf{A}_{MX02}$)(Conditional Introduction)

$$\{\subset\} \diamond E = \emptyset \quad (T_{S192})$$

Proof of (T_{S192})

$$\begin{aligned} \{\subset\} &\subseteq \{\neq\} & (T_{A103}) \\ \{\subset\} \diamond E &\subseteq \{\neq\} \diamond E & (T_{R263})(\text{Relational Deduction})(\text{Eqn1}) \\ (E \neq \emptyset) &= (\{\neq\} \diamond E = \sim E) & (T_{A085})(\text{Definition of } \sim) \\ E &= (\{\neq\} \diamond E = \sim E) & (T_{M344}) \\ \{\neq\} \diamond E &= \sim E & (A_{MX16})(\text{Order Axiom}) \\ \{\neq\} \diamond E &= \emptyset & (D_{M02})(\text{Definition of } \emptyset)(\text{Eqn2}) \\ \{\subset\} \diamond E &\subseteq \emptyset & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \{\subset\} \diamond E &= \emptyset & (T_{M052})(\text{Subsets of the Null Set}) \end{aligned}$$

$$\{\neq\} * \emptyset^1 = \sim \emptyset^1 \quad (T_{S451})$$

Proof of (T_{S451})

$$\begin{aligned} \{\neq\} * \emptyset^1 &= \{\neq\} * \emptyset^1 & (D_{M00})(\text{Definition of } =) \\ \{\neq\} * \emptyset^1 &= \sim \emptyset^1 & (T_{A084}) \text{ since } (T_{M780})(\text{Existence of } \emptyset^1) \text{ deems that } \emptyset^1 \neq \emptyset \end{aligned}$$

$$\{\neq\} * E^1 = \sim E^1 \quad (T_{S452})$$

Proof of (T_{S452})

$$\begin{aligned} \{\neq\} * E^1 &= \{\neq\} * E^1 & (D_{M00})(\text{Definition of } =) \\ \{\neq\} * E^1 &= \sim E^1 & (T_{A084}) \text{ since } (T_{M773})(\text{Existence of } E^1) \text{ deems that } E^1 \neq \emptyset \end{aligned}$$

$$\{\neq\} * E = E \quad (T_{S453})$$

Proof of (T_{S453})

$$\begin{aligned} (b \neq b^1)(b \neq \emptyset) &\subseteq (\{\neq\} * b = E) & (T_{A086})(\text{Eqn1}) \\ \text{Let } b &= E & (\text{Eqn2}) \\ b^1 &= E^1 & (T_{M488})(\text{Eqn3}) \\ (E \neq E^1)(E \neq \emptyset) &\subseteq (\{\neq\} * E = E) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\ E(E \neq \emptyset) &\subseteq (\{\neq\} * E^1 = \sim E^1) & (T_{M001})(\text{Universality of } (T_{M492})) \\ (E \neq \emptyset) &\subseteq (\{\neq\} * E^1 = \sim E^1) & (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\ E &\subseteq (\{\neq\} * E^1 = \sim E^1) & (T_{M001})(\text{Universality of } (T_{M352})) \\ E &= (\{\neq\} * E^1 = \sim E^1) & (T_{M275})(\text{Supersets of the Universal Set}) \\ \{\neq\} * E^1 &= \sim E^1 & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$S\{\neq\} = E \quad (T_{A253})$$

Proof of (T_{A253})

$$\begin{aligned} \{\neq\} * E &= E & (T_{S453}) \\ S(\{\neq\}.^V Q E) &= E & (A_{R27})(\text{Function Axiom}) \\ S(\{\neq\}.E) &= E & (T_{R116})(\text{Sub-relationships with the Universal Set}) \\ S\{\neq\} &= E & (A_{MX11})(\text{Universal Set Conjunction Axiom}) \end{aligned}$$

$$Q\{\neq\} = E \quad (T_{A254})$$

Proof of (T_{A254})

$$\begin{aligned} S\{\neq\} &= E & (T_{A253})(\text{Eqn1}) \\ \{\neq\} &= ^V \{\neq\} & (D_{A94}) \text{ Definition of } ^V \{\neq\} (\text{Eqn2}) \\ S^V \{\neq\} &= E & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ Q\{\neq\} &= E & (T_{R016})(\text{Reversibility of Matching Entities}) \end{aligned}$$

$$\{=\} * b^1 \subseteq \{\subset\} * b^1 \quad (T_{S212})$$

$$\{=\} * b^1 \subseteq \{\supset\} * b^1 \quad (T_{S213})$$

$$\{=\} * b^1 \subseteq \{\|\} * b^1 \quad (T_{S562})$$

$$\{\subset\} * b^1 \subseteq \{\supset\} * b^1 \quad (T_{S563})$$

$$\{\subset\} * b^1 \subseteq \{\|\} * b^1 \quad (T_{S564})$$

$$\{\supset\} * b^1 \subseteq \{\|\} * b^1 \quad (T_{S565})$$

Example Proof of (T_{S212})

$$\begin{aligned} \{\subset\} &\subseteq \{\neq\} & (T_{A103}) \\ \{\subset\} * b^1 &\subseteq \{\neq\} * b^1 & (T_{R197})(\text{Eqn1}) \\ (b^1 \neq \emptyset) &= ((\{=\} * b^1) = \sim(\{=\} * b^1)) & (T_{R245})(\text{Negative Rule for Single Element Object})(\text{Eqn2}) \\ (b^1 \neq \emptyset) &\subseteq (\{\subset\} * b^1 \subseteq \{=\} * b^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (b^1 \neq \emptyset) &\subseteq (\{=\} * b^1 \subseteq \{\subset\} * b^1) & (T_{M012})(\text{Transposition})(\text{Eqn3}) \\ \text{Let } b^1 &= \emptyset & (\text{Eqn4}) \\ \text{Let } \{=\} * b^1 &\not\subseteq \{\subset\} * b^1 & (\text{Eqn5}) \\ \{=\} * \emptyset &\not\subseteq \{\subset\} * \emptyset & (A_{MX19})(\text{Substitution of Eqn4 into Eqn5}) \\ \emptyset &\not\subseteq \sim \emptyset & (T_{S181})(\text{Null Object Theorem}) \\ \emptyset &\not\subseteq E & (T_{M054})(\text{Negative of Null Set}) \\ \sim(\emptyset \subseteq E) & & (T_{M259})(\text{Definition of } \not\subseteq) \\ \sim E & & (T_{M001})(\text{Universality of } (T_{M670}) \text{ Null Set Theorem}) \\ \emptyset & & (D_{M02})(\text{Definition of } \emptyset) \\ \text{Thus } (b^1 = \emptyset) &(\{=\} * b^1 \not\subseteq \{\subset\} * b^1) = \emptyset & (A_{MX02})(\text{Conditional Introduction}) \\ (b^1 = \emptyset) &\subseteq (\{=\} * b^1 \not\subseteq \{\subset\} * b^1) & (T_{M735})(\subseteq \text{Existential Form})(\text{Eqn6}) \\ (b^1 \neq \emptyset) + (b^1 = \emptyset) &\subseteq (\{=\} * b^1 \subseteq \{\subset\} * b^1) + (\{=\} * b^1 \subseteq \{\subset\} * b^1) & (A_{MX21})(\text{Combination of Eqns3\&6}) \\ (b^1 \neq \emptyset) + (b^1 = \emptyset) &\subseteq (\{=\} * b^1 \subseteq \{\subset\} * b^1) & (A_{MX01})(\text{Disjunction Tautology}) \\ E &\subseteq (\{=\} * b^1 \subseteq \{\subset\} * b^1) & (T_{M671})(\text{Tertium non Datur}) \\ E &= (\{=\} * b^1 \subseteq \{\subset\} * b^1) & (T_{M275})(\text{Supersets of the Universal Set}) \\ \{=\} * b^1 &\subseteq \{\subset\} * b^1 & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

The other theorems can be obtained in the same way by substituting the relevant membership relationship sets.

$$a^1 = \{\subset\} a^1 . (\{\subset\} * \{\subset\} a^1) \quad (T_{T271})$$

Proof of (T_{T271})

$$\begin{aligned} \{\subset\} a^1 . (\{\subset\} * \{\subset\} a^1) &= \{\subset\} a^1 . (\{\subset\} * \{\subset\} a^1) & (D_{M00})(\text{Definition of } =) \\ \{\subset\} a^1 . (\{\subset\} * \{\subset\} a^1) &= \{\subset\} a^1 . \sim(\{\subset\} * \{\subset\} a^1) & (D_{M03})(\text{Definition of } \sim) \\ \{\subset\} a^1 . (\{\subset\} * \{\subset\} a^1) &= \{\subset\} a^1 . \sim(\{\subset\} a^1) & (T_{A556}) \\ \{\subset\} a^1 . (\{\subset\} * \{\subset\} a^1) &= \{\subset\} a^1 . \{\subset\} * a^1 & (D_{M03})(\text{Definition of } \sim) \\ \{\subset\} a^1 . (\{\subset\} * \{\subset\} a^1) &= (\{\subset\} + \{=\}) a^1 . \{\subset\} * a^1 & (D_{A141})(\text{Disjunction with } \{=\}) \\ \{\subset\} a^1 . (\{\subset\} * \{\subset\} a^1) &= (\{\subset\} * a^1 + \{=\} * a^1) . \{\subset\} * a^1 & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\ \{\subset\} a^1 . (\{\subset\} * \{\subset\} a^1) &= \{\subset\} * a^1 . \{\subset\} * a^1 + \{=\} * a^1 . \{\subset\} * a^1 & (T_{M407})(\text{Distribution of } . \text{ over } +) \\ \{\subset\} a^1 . (\{\subset\} * \{\subset\} a^1) &= \emptyset + \{=\} * a^1 . \{\subset\} * a^1 & (T_{M621})(\text{Complement Conjunction}) \\ \{\subset\} a^1 . (\{\subset\} * \{\subset\} a^1) &= \{=\} * a^1 . \{\subset\} * a^1 & (T_{M668})(\text{Null Set Disjunction})(\text{Eqn1}) \\ \{=\} * a^1 &\subseteq \{\subset\} * a^1 & (T_{S212}) \\ \{=\} * a^1 &= \{=\} * a^1 . \{\subset\} * a^1 & (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn2}) \\ \{\subset\} a^1 . (\{\subset\} * \{\subset\} a^1) &= \{=\} * a^1 & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \{\subset\} a^1 . (\{\subset\} * \{\subset\} a^1) &= a^1 & (D_{A144})(\text{Definition of the Reflexive Relationship Set}) \end{aligned}$$

$$(\forall Sa. \{ \neq \}. \forall Qa \neq \emptyset) \subseteq (a \neq a^1)(a \neq \emptyset) \quad (T_{S640})$$

Proof of (T_{S640})

$$\begin{aligned} (a. \{ \neq \} * a \neq \emptyset) &= (a \neq a^1)(a \neq \emptyset) & (A_{R37})(\text{Axiom of Plural Sets}) \\ (a. S(\{ \neq \}. \forall Qa) \neq \emptyset) &= (a \neq a^1)(a \neq \emptyset) & (A_{R27})(\text{Function Axiom}) \\ (S \forall Sa. S(\{ \neq \}. \forall Qa) \neq \emptyset) &= (a \neq a^1)(a \neq \emptyset) & (T_{R012m})(\text{Sub-Relationship Closure Theorem})(\text{Eqn1}) \\ S(\forall Sa. \{ \neq \}. \forall Qa) &\subseteq S \forall Sa. S(\{ \neq \}. \forall Qa) & (T_{R012d})(\text{Distribution of } S * \text{ over } .) \\ (S(\forall Sa. \{ \neq \}. \forall Qa) &\subseteq S \forall Sa. S(\{ \neq \}. \forall Qa)) = E & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$\begin{aligned}
(\mathbf{S}(\forall \mathbf{Sa}.\{\neq\}.\forall \mathbf{Qa}) \neq \emptyset) &\subseteq (\forall \mathbf{Sa}.\mathbf{S}(\{\neq\}.\forall \mathbf{Qa}) \neq \emptyset) & (\mathbf{T}_{M096})(\text{Probabilistic Rule}) \\
(\forall \mathbf{Sa}.\{\neq\}.\forall \mathbf{Qa} \neq \emptyset) &\subseteq (\forall \mathbf{Sa}.\mathbf{S}(\{\neq\}.\forall \mathbf{Qa}) \neq \emptyset) & (\mathbf{T}_{R052})(\text{Eqn2}) \\
(\forall \mathbf{Sa}.\{\neq\}.\forall \mathbf{Qa} \neq \emptyset) &\subseteq (\mathbf{a} \neq \mathbf{a}^1)(\mathbf{a} \neq \emptyset) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn1 into Eqn2})
\end{aligned}$$

$$(\mathbf{b}^1 \neq \emptyset) \subseteq (\{\|\} * \mathbf{b}^1 = (\{\supset\} * \mathbf{b}) + \{\supset\} \diamond \mathbf{b} + (\{\supset\} * \mathbf{b}) + \{\supset\} \diamond \mathbf{b}) \quad (\mathbf{T}_{A097})$$

Proof of ($\mathbf{T}_{A097}$)

$$\begin{aligned}
\{\|\} * \mathbf{b}^1 &= (\{\supset\} * \mathbf{b}).(\{\supset\} \diamond \mathbf{b}).(\{\supset\} * \mathbf{b}).(\{\supset\} \diamond \mathbf{b}) & (\mathbf{D}_{A08})(\text{Definition of Intersection}) \\
\sim(\{\|\} * \mathbf{b}^1) &= \sim((\{\supset\} * \mathbf{b}).(\{\supset\} \diamond \mathbf{b}).(\{\supset\} * \mathbf{b}).(\{\supset\} \diamond \mathbf{b})) & (\mathbf{T}_{M008})(\text{Inverse of an Equality}) \\
(\mathbf{b}^1 \neq \emptyset) &\subseteq (\{\|\} * \mathbf{b}^1 = \sim((\{\supset\} * \mathbf{b}).(\{\supset\} \diamond \mathbf{b}).(\{\supset\} * \mathbf{b}).(\{\supset\} \diamond \mathbf{b}))) & (\mathbf{T}_{R245})(\text{Negative Rule for Single Element Object}) \\
(\mathbf{b}^1 \neq \emptyset) &\subseteq (\{\|\} * \mathbf{b}^1 = (\{\supset\} * \mathbf{b}) + \{\supset\} \diamond \mathbf{b} + (\{\supset\} * \mathbf{b}) + \{\supset\} \diamond \mathbf{b}) & (\mathbf{T}_{M025})(\text{DeMorgan's Law})
\end{aligned}$$

$$(\mathbf{a} \neq \emptyset) \subseteq (\{\supset\} * \mathbf{a} \neq \emptyset) \quad (\mathbf{T}_{S619})$$

Proof of ($\mathbf{T}_{S619}$)

$$\begin{aligned}
\mathbf{a} &\subseteq \mathbf{a} + \{\supset\} * \mathbf{a} & (\mathbf{T}_{M082})(\text{Disjunction Introduction}) \\
\mathbf{a} &\subseteq \{\neq\} * \mathbf{a} + \{\supset\} * \mathbf{a} & (\mathbf{T}_{A080}) \\
\mathbf{a} &\subseteq (\{\neq\} + \{\supset\}) * \mathbf{a} & (\mathbf{T}_{R002c})(\text{Distribution of } + \text{ over } *) \\
\mathbf{a} &\subseteq \{\supset\} * \mathbf{a} & (\mathbf{D}_{A141})(\text{Disjunction with } \{\neq\}) \\
(\mathbf{a} \subseteq \{\supset\} * \mathbf{a}) &= \mathbf{E} & (\mathbf{A}_{MX16})(\text{Order Axiom}) \\
(\mathbf{a} \neq \emptyset) &\subseteq (\{\supset\} * \mathbf{a} \neq \emptyset) & (\mathbf{T}_{M096})(\text{Probabilistic Rule})
\end{aligned}$$

$$\{\subset\}\{\neq\} = \emptyset \quad (\mathbf{T}_{A109})$$

$$\{\supset\}\{\neq\} = \emptyset \quad (\mathbf{T}_{A110})$$

$$\{\|\}\{\neq\} = \emptyset \quad (\mathbf{T}_{A111})$$

$$\{\supset\}\{\subset\} = \emptyset \quad (\mathbf{T}_{A112})$$

$$\{\|\}\{\subset\} = \emptyset \quad (\mathbf{T}_{A113})$$

$$\{\|\}\{\supset\} = \emptyset \quad (\mathbf{T}_{A114})$$

Example Proof of ($\mathbf{T}_{A109}$)

$$\begin{aligned}
\{\subset\} &\subseteq \{\neq\} & (\mathbf{T}_{A103}) \\
\{\subset\}.\sim\{\neq\} &= \emptyset & (\mathbf{T}_{M735})(\subseteq \text{ Existential Form}) \\
\{\subset\}.\{\neq\} &= \emptyset & (\mathbf{D}_{A142})(\text{Definition of } \sim\{\neq\})
\end{aligned}$$

$$\{\subset\} = \{\subset\}\{\neq\} \quad (\mathbf{T}_{A227})$$

$$\{\supset\} = \{\supset\}\{\neq\} \quad (\mathbf{T}_{A228})$$

Example Proof of ($\mathbf{T}_{A227}$)

$$\begin{aligned}
\{\subset\} + \{\neq\} &= \{\subset\} & (\mathbf{D}_{A141})(\text{Disjunction with } \{\neq\}) \\
(\{\subset\} + \{\neq\})\{\neq\} &= \{\subset\}\{\neq\} & (\mathbf{T}_{M029})(\text{Deduction}) \\
\{\subset\}\{\neq\} + \{\neq\}\{\neq\} &= \{\subset\}\{\neq\} & (\mathbf{T}_{M407})(\text{Distribution of } . \text{ over } +) \\
\{\subset\}\{\neq\} + \emptyset &= \{\subset\}\{\neq\} & (\mathbf{T}_{M621})(\text{Complement Conjunction}) \\
\{\subset\}\{\neq\} &= \{\subset\}\{\neq\} & (\mathbf{T}_{M668})(\text{Null Set Disjunction})(\text{Eqn 1}) \\
\{\subset\} &\subseteq \{\neq\} & (\mathbf{T}_{A103}) \\
\{\subset\} &= \{\subset\}\{\neq\} & (\mathbf{A}_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn 2}) \\
\{\subset\} &= \{\subset\}\{\neq\} & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$\{\subset\}\{\supset\} = \{\neq\} \quad (\mathbf{T}_{A226})$$

Proof of ($\mathbf{T}_{A226}$)

$$\begin{aligned}
\{\subset\}\{\supset\} &= \{\subset\}\{\supset\} & (\mathbf{D}_{M00})(\text{Definition of } =) \\
\{\subset\}\{\supset\} &= (\{\subset\} + \{\neq\})(\{\subset\} + \{\neq\}) & (\mathbf{D}_{A141})(\text{Disjunction with } \{\neq\}) \\
\{\subset\}\{\supset\} &= \{\subset\}\{\supset\} + \{\neq\}\{\supset\} + \{\subset\}\{\neq\} + \{\neq\}\{\neq\} & (\mathbf{T}_{M407})(\text{Distribution of } . \text{ over } +) \\
\{\subset\}\{\supset\} &= \emptyset + \{\neq\}\{\supset\} + \{\subset\}\{\neq\} + \{\neq\}\{\neq\} & (\mathbf{T}_{A112})
\end{aligned}$$

$$E = \{\subset\} + \{=\} + \{\supset\} + \{\|\} + \{\sim\subset\}\{\neq\}\{\sim\supset\}\{\|\}$$

(**AMX19**)(Substitution of Eqns2to7 into Eqn1)

$$\{=\}^1 \subseteq R_T \quad (T_{R594}) \text{ Fourth Peano Axiom}$$

Example Proof of (T_{R594})

$$\begin{aligned} (\{=\} \subseteq \{\subset\} + \{=\})((\{\subset\} + \{=\})^1 \subseteq R_T) &\subseteq (\{=\}^1 \subseteq R_T) & (T_{T534})(\text{Eqn1}) \\ \{=\} \subseteq \{\subset\} + \{=\} & & (T_{M082})(\text{Disjunction Introduction}) \\ (\{=\} \subseteq \{\subset\} + \{=\}) = E & & (AMX16)(\text{Order Axiom})(\text{Eqn2}) \\ (\{\subset\} + \{=\})^1 \subseteq R_T & & (AR22)(\text{Transitive Relationships Disjunction Axiom}) \\ ((\{\subset\} + \{=\})^1 \subseteq R_T) = E & & (AMX16)(\text{Order Axiom})(\text{Eqn3}) \\ E.E \subseteq (\{=\}^1 \subseteq R_T) & & (AMX19)(\text{Substitution of Eqns2\&3 into Eqn1}) \\ E \subseteq (\{=\}^1 \subseteq R_T) & & (AMX17)(\text{Universal Set Conjunction Axiom}) \\ E = (\{=\}^1 \subseteq R_T) & & (T_{M275})(\text{Supersets of the Universal Set}) \\ \{=\}^1 \subseteq R_T & & (AMX16)(\text{Order Axiom}) \end{aligned}$$

$$\{=\}^1 \subseteq R_T R_S.(R_B + R_L) \quad (T_{T740}) \text{ Reflexive Relationships Theorem}$$

Proof of (T_{T740})

$$\begin{aligned} (o = \{=\}, \{\subset\}, \{\supset\}, \{\|\}, \{\sim\subset\}, \{\neq\}) &= (o^1 \subseteq R_B) & (DR94)(\text{Definition of a Membership Relationship Set}) \\ \{=\}^1 \subseteq R_B & & \text{Extraction (Eqn1)} \\ (o = \{=\}, \{\subset\}, \{\supset\}, \{\leq\}, \{\geq\}) &= (o^1 \subseteq R_L) & (DR96)(\text{Definition of an Ordering Relationship Set}) \\ \{=\}^1 \subseteq R_L & & \text{Extraction (Eqn2)} \\ \{=\}^1 + \{=\}^1 \subseteq R_B + R_L & & (AMX21)(\text{Combination of Eqns1\&2}) \\ \{=\}^1 \subseteq R_B + R_L & & (AMX01)(\text{Disjunction Tautology})(\text{Eqn3}) \\ \{=\}^1 \subseteq R_S & & (TR603)(\text{Third Peano Axiom})(\text{Eqn4}) \\ \{=\}^1 \subseteq R_T & & (T_{R594})(\text{Fourth Peano Axiom})(\text{Eqn5}) \\ \{=\}^1 \{=\}^1 \{=\}^1 \subseteq R_T R_S.(R_B + R_L) & & (TM669)(\text{Combination of Eqns3,4\&5}) \\ \{=\}^1 \subseteq R_T R_S.(R_B + R_L) & & (TM667)(\text{Conjunction Tautology}) \end{aligned}$$

$$\{=\}^1 \subseteq R_O \quad (TA007)$$

$$\{=\}^1 \subseteq R_P \quad (T_{T421})$$

Proof of (T_{T421}) & ($TA007$)

$$\begin{aligned} ((c^1 \subseteq R_O) = E) &= ((b^1 \subseteq Qc)(b^1 \neq \emptyset) \subseteq (c * b^1 = (c * b^1)^1)) & (DR30)(\text{Definition of a One-to-One Relationship Set})(\text{Eqn1}) \\ \text{Let } c = \{=\} & & (\text{Eqn2}) \\ ((\{=\}^1 \subseteq R_O) = E) &= ((b^1 \subseteq Q\{=\})(b^1 \neq \emptyset) \subseteq (\{=\}b^1 = (\{=\}b^1)^1)) & (AMX19)(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\ (\{=\}b^1 \neq \emptyset) &\subseteq (b^1 \subseteq Q\{=\}) & (TS410b) \\ (\{=\}b^1 \neq \emptyset)(b^1 \neq \emptyset) &\subseteq (b^1 \subseteq Q\{=\})(b^1 \neq \emptyset) & (TM029)(\text{Deduction}) \\ (\{=\}b^1 \neq \emptyset)(\{=\}b^1 \neq \emptyset) &\subseteq (b^1 \subseteq Q\{=\})(b^1 \neq \emptyset) & (DA144)(\text{Definition of the Reflexive Relationship Set}) \\ (\{=\}b^1 \neq \emptyset) &\subseteq (b^1 \subseteq Q\{=\})(b^1 \neq \emptyset) & (TM667)(\text{Conjunction Tautology})(\text{Eqn4}) \\ ((\{=\}^1 \subseteq R_O) = E) &= ((\{=\}b^1 \neq \emptyset) \subseteq (\{=\}b^1 = (\{=\}b^1)^1)) & (TM139)(\text{Hypothetical Syllogism Eqns4\&3}) \\ ((\{=\}^1 \subseteq R_O) = E) = E & & (TM001)(\text{Universality of } (DM17) \text{ Definition of } a^{11}) \\ (\{=\}^1 \subseteq R_O) = E & & (AMX16)(\text{Order Axiom}) \\ \{=\}^1 \subseteq R_O & & (AMX16)(\text{Order Axiom}) \\ (\forall \{=\})^1 \subseteq R_O & & (TA485) \\ \{=\}^1 \subseteq R_P & & (DR31)(\text{Definition of a Pairing Relationship Set}) \end{aligned}$$

$$(a \subseteq \{=\} * b)(b \subseteq \{=\} * c) \subseteq (a \subseteq \{=\} * c) \quad (TS351)$$

Proof of ($TS351$)

$$\begin{aligned} ((c^1 \subseteq R_T) = E) &\subseteq ((a \subseteq c * b)(b \subseteq c * d) \subseteq (a \subseteq c * d)) & (TS551)(\text{Transitive Syllogism})(\text{Eqn1}) \\ \text{Let } c = \{=\} & & (\text{Eqn2}) \\ ((\{=\}^1 \subseteq R_T) = E) &\subseteq ((a \subseteq \{=\} * c)(c \subseteq \{=\} * b) \subseteq (a \subseteq \{=\} * b)) & (AMX19)(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\ \{=\}^1 \subseteq R_T & & (T_{R594})(\text{Fourth Peano Axiom}) \end{aligned}$$

$(\{=\}^1 \subseteq R_T) = E$	$(A_{MX16})(\text{Order Axiom})(\text{Eqn4})$
$(E = E) \subseteq ((a \subseteq \{=\} * c)(c \subseteq \{=\} * b) \subseteq (a \subseteq \{=\} * b))$	$(A_{MX19})(\text{Substitution of Eqn4 into Eqn3})$
$E \subseteq ((a \subseteq \{=\} * c)(c \subseteq \{=\} * b) \subseteq (a \subseteq \{=\} * b))$	(T_{M342})
$E = ((a \subseteq \{=\} * c)(c \subseteq \{=\} * b) \subseteq (a \subseteq \{=\} * b))$	$(T_{M275})(\text{Supersets of the Universal Set})$
$(a \subseteq \{=\} * c)(c \subseteq \{=\} * b) \subseteq (a \subseteq \{=\} * b)$	$(A_{MX16})(\text{Order Axiom})$

$$(c \subseteq \{=\}) \subseteq (Sc = Qc)$$

(T_{A250}) Definition of a Reflexive Set

Proof of (T_{A250})

$(c \subseteq \{=\})(\{=\}^1 \subseteq R_s) \subseteq (c^1 \subseteq R_s)$	$(T_{T534})(\text{Eqn1})$
$\{=\}^1 \subseteq R_s$	$(T_{R603})(\text{Third Peano Axiom})$
$(\{=\}^1 \subseteq R_s) = E$	$(A_{MX16})(\text{Order Axiom})(\text{Eqn2})$
$(c \subseteq \{=\})E \subseteq (c^1 \subseteq R_s)$	$(A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$
$(c \subseteq \{=\}) \subseteq (c^1 \subseteq R_s)$	$(A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn3})$
$(c^1 \subseteq R_s) = (c = {}^v c)$	$(D_{A94})(\text{Definition of a Symmetrical Relationship Set})(\text{Eqn4})$
$(c \subseteq \{=\}) \subseteq (c = {}^v c)$	$(A_{MX19})(\text{Substitution of Eqn4 into Eqn3})(\text{Eqn5})$
$(c = {}^v c) \subseteq (Sc = S^v c)$	(T_{R026})
$(c = {}^v c) \subseteq (Sc = Qc)$	$(T_{R016})(\text{Reversibility of Matching Entities})(\text{Eqn6})$
$(c \subseteq \{=\}) \subseteq (Sc = Qc)$	$(T_{M139})(\text{Hypothetical Syllogism Eqns5\&6})$

$$(c \subseteq \{=\}) \subseteq (Qc = c * Qc)$$

(T_{A301}) Definition of a Reflexive Set

Proof of (T_{A301})

$(c \subseteq \{=\}) \subseteq (Sc = Qc)$	$(T_{A250})(\text{Definition of a Reflexive Set})(\text{Eqn1})$
$Sc = c * Qc$	$(T_{R020a})(\text{Function Theorem})(\text{Eqn2})$
$(c \subseteq \{=\}) \subseteq (Qc = c * Qc)$	$(A_{MX19})(\text{Substitution of Eqn 2 into Eqn 1})$

$$\{\subset\}^1 \subseteq R_T R_A$$

(T_{T715})

$$\{\supset\}^1 \subseteq R_T R_A$$

(T_{R601})

Example Proof of (T_{T715})

$\{\triangleleft\} = \{\subset, \supset, \triangleleft, \triangleright\}$	$(D_{S08})(\text{Definition of } \triangleleft)$
$\{\triangleleft\} = \{\subset\}$	Extraction (Eqn1)
$\{\triangleleft\}^1 \subseteq R_T R_A$	$(T_{T739})(\text{Eqn2})$
$\{\subset\}^1 \subseteq R_T R_A$	$(A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$

$$\{\subset\}^1 \subseteq R_T$$

(T_{A241})

$$\{\supset\}^1 \subseteq R_T$$

(T_{A247})

Example Proof of (T_{A241})

$\{\subset\}^1 \subseteq R_T R_A$	(T_{T715})
$\{\subset\}^1 \subseteq R_T$	$(T_{M073})(\text{Single Premise Syllogism})$

$$(c \subseteq \{=\}) = ({}^v c \subseteq \{=\})$$

(T_{A175})

Proof of (T_{A175})

Let $c \subseteq \{=\}$	
${}^v c \subseteq {}^v \{=\}$	$(T_{R022d})(\text{Relationship Opposite of a Proposition})$
${}^v c \subseteq \{=\}$	$(D_{A94})(\text{Definition of } {}^v \{=\})$
Thus $(c \subseteq \{=\}) = ({}^v c \subseteq \{=\})$	$(A_{MX02})(\text{Conditional Introduction})$ Since this proof is reversible

$$(c \subseteq \{=\}) = ({}^v c \subseteq \{=\})$$

(T_{A176})

Proof of (T_{A176})

Let $c \subseteq \{=\}$	
${}^v c \subseteq {}^v \{=\}$	$(T_{R022d})(\text{Relationship Opposite of a Proposition})$
${}^v c \subseteq \{=\}$	(D_{A94}) Definition of ${}^v \{=\}$

Thus $(c \subseteq \{ \neq \}) = (\forall c \subseteq \{ \neq \})$ ($\mathbf{A}_{MX02}$)(Conditional Introduction) Since this proof is reversible.

$$\{=\} \diamond b^1 = b^1 \quad (\mathbf{T}_{A079})$$

Proof of ($\mathbf{T}_{A079}$)

$$\{=\} * b^1 = b^1 \quad (\mathbf{D}_{A144}) \text{(Definition of the Reflexive Relationship Set)}$$

$$\{=\} \diamond b^1 = b^1 \quad (\mathbf{T}_{R232}) \text{(Interchangeability of } * \text{ and } \diamond \text{ for Entities)}$$

$$((b \neq b^1) \subseteq (\{=\} \diamond b = \emptyset)) \quad (\mathbf{T}_{A081}) \text{ Reflexive Intersection Function Theorem}$$

Proof of ($\mathbf{T}_{A081}$)

$$(((\forall c)^1 \subseteq R_0) = E) \subseteq ((b \neq b^1) = (c \diamond b = \emptyset))$$

$$(\mathbf{A}_{R14}) \text{(One-to-One Relationship Intersection Function Axiom)} (\text{Eqn1})$$

$$\text{Let } c = \{=\}$$

$$(\text{Eqn2})$$

$$(((\forall \{=\})^1 \subseteq R_0) = E) \subseteq ((b \neq b^1) = (\{=\} \diamond b = \emptyset))$$

$$(\mathbf{A}_{MX19}) \text{(Substitution of Eqn1 into Eqn2)}$$

$$(((\{=\}^1 \subseteq R_0) = E) \subseteq ((b \neq b^1) = (\{=\} \diamond b = \emptyset)))$$

$$(\mathbf{T}_{A485}) (\text{Eqn3})$$

$$\{=\}^1 \subseteq R_0$$

$$(\mathbf{T}_{A007})$$

$$(\{=\}^1 \subseteq R_0) = E$$

$$(\mathbf{A}_{MX16}) \text{(Order Axiom)} (\text{Eqn4})$$

$$(E = E) \subseteq ((b \neq b^1) = (\{=\} \diamond b = \emptyset))$$

$$(\mathbf{A}_{MX19}) \text{(Substitution of Eqn4 into Eqn3)}$$

$$E \subseteq ((b \neq b^1) = (\{=\} \diamond b = \emptyset))$$

$$(\mathbf{T}_{M342})$$

$$E = ((b \neq b^1) = (\{=\} \diamond b = \emptyset))$$

$$(\mathbf{T}_{M275}) \text{(Supersets of the Universal Set)}$$

$$(b \neq b^1) = (\{=\} \diamond b = \emptyset)$$

$$(\mathbf{A}_{MX16}) \text{(Order Axiom)}$$

$$E^1 = (\{\supset\} \diamond E).(\{\supset\} * \{\supset\} \diamond E) \quad (\mathbf{T}_{S462})$$

Proof of ($\mathbf{T}_{A462}$)

$$(a \neq \emptyset) \subseteq (a^1 = (\{\supset\} \diamond a).(\{\supset\} * \{\supset\} \diamond a))$$

$$(\mathbf{D}_{R98}) \text{(General Definition of Aggregation)} (\text{Eqn1})$$

$$\text{Let } a = E$$

$$(\text{Eqn2})$$

$$a^1 = E^1$$

$$(\mathbf{T}_{M488}) (\text{Eqn3})$$

$$(E \neq \emptyset) \subseteq (E^1 = (\{\supset\} \diamond E).(\{\supset\} * \{\supset\} \diamond E))$$

$$(\mathbf{A}_{MX19}) \text{(Substitution of Eqns2\&3 into Eqn1)} (\text{Eqn3})$$

$$E \subseteq (E^1 = (\{\supset\} \diamond E).(\{\supset\} * \{\supset\} \diamond E))$$

$$(\mathbf{T}_{M344})$$

$$E = (E^1 = (\{\supset\} \diamond E).(\{\supset\} * \{\supset\} \diamond E))$$

$$(\mathbf{T}_{M275}) \text{(Supersets of the Universal Set)}$$

$$E^1 = (\{\supset\} \diamond E).(\{\supset\} * \{\supset\} \diamond E)$$

$$(\mathbf{A}_{MX16}) \text{(Order Axiom)}$$

$$a \subseteq \{\supset\} * a^1 \quad (\mathbf{T}_{S581}) \text{(General Definition of Disaggregation)}$$

$$a \subseteq \{\supset\} * a^1 \quad (\mathbf{T}_{S581A}) \text{(General Definition of Reverse Abstraction)}$$

Example Proof of ($\mathbf{T}_{S581}$)

$$(a \neq \emptyset) \subseteq (a^1 = (\{\supset\} \diamond a).(\{\supset\} * \{\supset\} \diamond a))$$

$$(\mathbf{D}_{R98}) \text{(General Definition of Aggregation)} (\text{Eqn1})$$

$$(a^1 = (\{\supset\} \diamond a).(\{\supset\} * \{\supset\} \diamond a)) \subseteq (a^1 \subseteq \{\supset\} \diamond a)$$

$$(\mathbf{T}_{M073b}) \text{(Single Premise Syllogism)} (\text{Eqn2})$$

$$(a \neq \emptyset) \subseteq (a^1 \subseteq \{\supset\} \diamond a)$$

$$(\mathbf{T}_{M139}) \text{(Hypothetical Syllogism Eqns1\&2)}$$

$$((a \neq \emptyset) \subseteq (a^1 \subseteq \{\supset\} \diamond a)) = E$$

$$(\mathbf{A}_{MX16}) \text{(Order Axiom)}$$

$$((a \neq \emptyset) = E) \subseteq ((a^1 \subseteq \{\supset\} \diamond a) = E)$$

$$(\mathbf{T}_{M094}) \text{(Probabilistic Rule)} (\text{Eqn3})$$

$$((b \neq \emptyset)(a^1 \neq \emptyset) = E) + ((a^1 = \emptyset)(b = \emptyset) = E) = ((a^1 \subseteq c \diamond b) = (b \subseteq \forall c * a^1))$$

$$(\mathbf{T}_{S028}) \text{(Active/Passive Theorem)}$$

$$((b \neq \emptyset)(a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c \diamond b) = (b \subseteq \forall c * a^1))$$

$$(\mathbf{T}_{M072b}) \text{(Single Premise Syllogism)} (\text{Eqn4})$$

$$\text{Let } b = a$$

$$(\text{Eqn5})$$

$$\text{Let } c = \{\supset\}$$

$$(\text{Eqn6})$$

$$\forall c = \forall \{\supset\}$$

$$(\mathbf{T}_{R022}) \text{(Relationship Opposite of a Proposition)}$$

$$\forall c = \{\supset\}$$

$$(\mathbf{T}_{T741}) \text{(Definition of } \forall \{\supset\}) (\text{Eqn7})$$

$$((a \neq \emptyset)(a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq \{\supset\} \diamond a) = (a \subseteq \{\supset\} * a^1))$$

$$(\mathbf{A}_{MX19}) \text{(Substitution of Eqns5,6\&7 into Eqn4)} (\text{Eqn8})$$

$$(a \neq \emptyset) \subseteq (a^1 \neq \emptyset)$$

$$(\mathbf{T}_{M503})$$

$$(a \neq \emptyset) = (a \neq \emptyset)(a^1 \neq \emptyset)$$

$$(\mathbf{A}_{MX44}) \text{(} \subseteq \text{ Equality Form)} (\text{Eqn9})$$

$$((a \neq \emptyset) = E) \subseteq ((a^1 \subseteq \{\supset\} \diamond a) = (a \subseteq \{\supset\} * a^1))$$

$$(\mathbf{A}_{MX19}) \text{(Substitution of Eqn9 into Eqn8)} (\text{Eqn10})$$

$((a \neq \emptyset) = E) \subseteq (E = (a \subseteq \{\subseteq\} * a^1))$	($\mathbf{A}_{MX19}$)(Substitution of Eqn3 into Eqn10)
$((a \neq \emptyset) \subseteq (a \subseteq \{\subseteq\} * a^1)) = E$	($\mathbf{T}_{M094}$)(Probabilistic Rule)
$(a \neq \emptyset) \subseteq (a \subseteq \{\subseteq\} * a^1)$	($\mathbf{A}_{MX16}$)(Order Axiom)(Eqn11)
Let $a = \emptyset$	(Eqn12)
Let $a \not\subseteq \{\subseteq\} * a^1$	(Eqn13)
$\emptyset \not\subseteq \{\subseteq\} * a^1$	($\mathbf{A}_{MX19}$)(Substitution of Eqn12 into Eqn13)
$(\emptyset \not\subseteq \{\subseteq\} * a^1) = E$	($\mathbf{A}_{MX16}$)(Order Axiom)
$\sim E = E$	($\mathbf{T}_{M001}$)(Universality of ($\mathbf{T}_{M670}$) Null Set Theorem)
$\emptyset = E$	($\mathbf{D}_{M02}$)(Definition of $\emptyset$)
$\emptyset$	($\mathbf{T}_{M347}$)
Thus $(a = \emptyset)(a \not\subseteq \{\subseteq\} * a^1) = \emptyset$	($\mathbf{A}_{MX02}$)(Conditional Introduction)
$(a = \emptyset) \subseteq (a \subseteq \{\subseteq\} * a^1)$	($\mathbf{T}_{M735}$)($\subseteq$ Existential Form)(Eqn14)
$(a \neq \emptyset) + (a = \emptyset) \subseteq (a \subseteq \{\subseteq\} * a^1) + (a \subseteq \{\subseteq\} * a^1)$	($\mathbf{A}_{MX21}$)(Combination of Eqns11&14)
$(a \neq \emptyset) + (a = \emptyset) \subseteq (a \subseteq \{\subseteq\} * a^1)$	($\mathbf{A}_{MX01}$)(Disjunction Tautology)
$E \subseteq (a \subseteq \{\subseteq\} * a^1)$	($\mathbf{T}_{M671}$)(Tertium non Datur)
$E = (a \subseteq \{\subseteq\} * a^1)$	($\mathbf{T}_{M275}$)(Supersets of the Universal Set)
$a \subseteq \{\subseteq\} * a^1$	($\mathbf{A}_{MX16}$)(Order Axiom)

$$\{\subseteq\} * a \subseteq \{\subseteq\} * a^1 \quad (\mathbf{T}_{S843})$$

Proof of ($\mathbf{T}_{S843}$)

$a \subseteq \{\subseteq\} * a^1$	($\mathbf{T}_{S581}$)(General Definition of Disaggregation)
$\{\subseteq\} * a \subseteq \{\subseteq\} * \{\subseteq\} * a^1$	($\mathbf{T}_{R196}$)(Relational Deduction)
$\{\subseteq\} * a \subseteq \{\subseteq\} * a^1$	($\mathbf{T}_{A556}$)

$$(a^1 \neq \emptyset) \subseteq (E^1 \subseteq \{\supseteq\} * a^1) \quad (\mathbf{T}_{T848})$$

Proof of ($\mathbf{T}_{T848}$)

$E^1 = (\{\supseteq\} \diamond E) \cdot (\{\supseteq\} \cdot \{\supseteq\} \diamond E)$	($\mathbf{T}_{S462}$)
$E^1 \subseteq \{\supseteq\} \diamond E$	($\mathbf{T}_{M073b}$)(Single Premise Syllogism)(Eqn1)
$a^1 \subseteq E$	($\mathbf{T}_{M744}$)(Universal Set Theorem)
$(a^1 \neq \emptyset) \subseteq (\{\supseteq\} \diamond E \subseteq \{\supseteq\} \diamond a^1)$	($\mathbf{T}_{R262}$)(Relational Deduction)(Eqn2)
$(a^1 \neq \emptyset) \subseteq (E^1 \subseteq \{\supseteq\} \diamond a^1)$	($\mathbf{T}_{M139}$)(Hypothetical Syllogism Eqns1&2)
$(a^1 \neq \emptyset) \subseteq (E^1 \subseteq \{\supseteq\} * a^1)$	($\mathbf{T}_{R232}$)(Interchangeability of $*$ and $\diamond$ for Entities)

$$\forall S\emptyset^1 \subseteq \{\subseteq\} \quad (\mathbf{T}_{T857})$$

$$\forall SE^1 \subseteq \{\supseteq\} \quad (\mathbf{T}_{T858})$$

$$\forall Q\emptyset^1 \subseteq \{\supseteq\} \quad (\mathbf{T}_{T859})$$

$$\forall QE^1 \subseteq \{\subseteq\} \quad (\mathbf{T}_{T860})$$

Example Proof of ($\mathbf{T}_{T857}$)

$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a \langle o \rangle b) = (\forall Sa^1.o.\forall Qb^1 \neq \emptyset))$	($\mathbf{D}_{A00}$)(Definition of an Equation)(Eqn1)
Let $o = \{\not\subseteq\}$	(Eqn2)
Let $a = \emptyset$	(Eqn3)
Let $b = a_n$	(Eqn4)
Let $a_n^1 \neq \emptyset$	
$(a_n^1 \neq \emptyset) = E$	($\mathbf{A}_{MX16}$)(Order Axiom)(Eqn5)
$((\emptyset^1 \neq \emptyset)(a_n^1 \neq \emptyset) = E) \subseteq ((\emptyset \not\subseteq a_n) = (\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_n^1 \neq \emptyset))$	($\mathbf{A}_{MX19}$)(Substitution of Eqns2,3&4 into Eqn1)
$(E(a_n^1 \neq \emptyset) = E) \subseteq ((\emptyset \not\subseteq a_n) = (\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_n^1 \neq \emptyset))$	($\mathbf{T}_{M001}$)(Universality of ($\mathbf{T}_{M780}$) Existence of $\emptyset^1$)(Eqn6)
$(EE = E) \subseteq ((\emptyset \not\subseteq a_n) = (\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_n^1 \neq \emptyset))$	($\mathbf{A}_{MX19}$)(Substitution of Eqn5 into Eqn6)
$(E = E) \subseteq ((\emptyset \not\subseteq a_n) = (\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_n^1 \neq \emptyset))$	($\mathbf{T}_{M667}$)(Conjunction Tautology)
$E \subseteq ((\emptyset \not\subseteq a_n) = (\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_n^1 \neq \emptyset))$	($\mathbf{T}_{M342}$)

$E = ((\emptyset \not\subseteq a_n) = (\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_n^1 \neq \emptyset))$	(T_{M275})(Supersets of the Universal Set)
$(\emptyset \not\subseteq a_n) = (\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_n^1 \neq \emptyset)$	($AMX16$)(Order Axiom)
$(\emptyset \subseteq a_n) = (\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_n^1 = \emptyset)$	(T_{M008})(Inverse of an Equality)
$E = (\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_n^1 = \emptyset)$	(T_{M001})(Universality of (T_{M670}) Null Set Theorem)
$\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_n^1 = \emptyset$	($AMX16$)(Order Axiom)
$(\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_1^1 = \emptyset) \dots (\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_E^1 = \emptyset)$	Resymbolising for $a_n = a_1$ to $a_n = a_E$
$\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_1^1 +++ \forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_E^1 = \emptyset$	(T_{M071})(Probabilistic Conjunction)
$\forall S\emptyset^1.\{\not\subseteq\}.\forall Qa_1^1 +++ \forall Qa_E^1 = \emptyset$	(T_{M407})(Distribution of . over +)
$\forall S\emptyset^1.\{\not\subseteq\}.\forall Q(a_1^1 +++ a_E^1) = \emptyset$	($AS64$)(Distribution of * over +)
$\forall S\emptyset^1.\{\not\subseteq\}.\forall QE = \emptyset$	($DM11$)(Definition of Universal Set using Entities)
$\forall S\emptyset^1.\{\not\subseteq\}.E = \emptyset$	($TR116$)
$\forall S\emptyset^1.\{\not\subseteq\} = \emptyset$	($AMX11$)(Universal Set Conjunction Axiom)
$\forall S\emptyset^1 \subseteq \{\not\subseteq\}$	(T_{M735})($\subseteq$ Existential Form)

$$\forall S\emptyset^1 = \{\not\subseteq\}.\forall S\emptyset^1 \quad (T_{T861})$$

$$\forall SE^1 = \{\supset\}.\forall SE^1 \quad (T_{T862})$$

$$\forall Q\emptyset^1 = \{\supset\}.\forall Q\emptyset^1 \quad (T_{T863})$$

$$\forall QE^1 = \{\not\subseteq\}.\forall QE^1 \quad (T_{T864})$$

Example Proof of (T_{T863})

$$\begin{aligned} \forall Q\emptyset^1 &\subseteq \{\supset\} & (T_{T859}) \\ \forall Q\emptyset^1 &= \forall Q\emptyset^1.\{\supset\} & (AMX44)(\subseteq \text{ Equality Form}) \end{aligned}$$

$$\{\supset\} * E^1 = \emptyset \quad (T_{T875})$$

Proof of (T_{T875})

$$\begin{aligned} \{\supset\} * E^1 &= \{\supset\} * E^1 & (DM00)(\text{Definition of } =) \\ \{\supset\} * E^1 &= S(\{\supset\}.\forall QE^1) & (AR27)(\text{Function Axiom}) \\ \{\supset\} * E^1 &= S(\{\supset\}.\{\not\subseteq\}.\forall QE^1) & (T_{T864}) \\ \{\supset\} * E^1 &= S(\{\supset\}.\{\not\subseteq\} + \{=\}).\forall QE^1 & (DA141)(\text{Disjunction with } \{=\}) \\ \{\supset\} * E^1 &= S((\{\supset\}.\{\not\subseteq\} + \{\supset\}.\{=\}).\forall QE^1) & (T_{M407})(\text{Distribution of . over +}) \\ \{\supset\} * E^1 &= S(\emptyset + \emptyset).\forall QE^1 & (TA112) \& (TA110) \\ \{\supset\} * E^1 &= S(\emptyset).\forall QE^1 & (TM668)(\text{Null Set Disjunction}) \\ \{\supset\} * E^1 &= S\emptyset & (TM050)(\text{Null Set Conjunction}) \\ \{\supset\} * E^1 &= \emptyset & (TR066)(\text{Null Object Theorem}) \end{aligned}$$

$$\{\supset\} * E^1 = E^1 \quad (TA215)$$

Proof of ($TA215$)

$$\begin{aligned} \{\supset\} * E^1 & \\ \{\supset\} * E^1 + \{=\} * E^1 & \quad (TR002c)(\text{Distribution of + over *}) \\ \emptyset + \{=\} * E^1 & \quad (T_{T875}) \\ \emptyset + E^1 & \quad (DA144)(\text{Definition of the Reflexive Relationship Set}) \\ E^1 & \quad (TM668)(\text{Null Set Disjunction}) \end{aligned}$$

$$\{\supset\} \diamond E^1 = E^1 \quad (TA216)$$

Proof of ($TA216$)

$$\begin{aligned} \{\supset\} \diamond E^1 & \\ \{\supset\} * E^1 & \quad (TR232)(\text{Interchangeability of } \diamond \text{ and } * \text{ for entities}) \\ E^1 & \quad (TA215) \end{aligned}$$

$$\{\parallel\} * \emptyset^1 = \emptyset \quad (TA276)$$

Proof of ($TA276$)

$\{\parallel\} * \emptyset^1 = \{\parallel\} * \emptyset^1$	(D_{M00})(Definition of $=$)
$\{\parallel\} * \emptyset^1 = S(\{\parallel\}.^vQ\emptyset^1)$	(A_{R27})(Function Axiom)
$\{\parallel\} * \emptyset^1 = S(\{\parallel\}. \{\supset\}.^vQ\emptyset^1)$	(T_{863})
$\{\parallel\} * \emptyset^1 = S(\{\parallel\}. (\{\supset\} + \{=\}) .^vQ\emptyset^1)$	(D_{A141})(Disjunction with $\{=\}$)
$\{\parallel\} * \emptyset^1 = S((\{\parallel\}. \{\supset\} + \{\parallel\}\{=\}) .^vQ\emptyset^1)$	(T_{M407})(Distribution of $.$ over $+$)
$\{\parallel\} * \emptyset^1 = S((\emptyset + \emptyset).^vQ\emptyset^1)$	(T_{A114}) & (T_{A111})
$\{\parallel\} * \emptyset^1 = S(\emptyset.^vQ\emptyset^1)$	(T_{M668})(Null Set Disjunction)
$\{\parallel\} * \emptyset^1 = S\emptyset$	(T_{M050})(Null Set Conjunction)
$\{\parallel\} * \emptyset^1 = \emptyset$	(T_{R066})(Null Object Theorem)

$$\{\parallel\} * \emptyset^1 = E \quad (T_{A278})$$

Proof of (T_{A278})

$\{\parallel\} * \emptyset^1 = \{\parallel\} * \emptyset^1$	(D_{M00})(Definition of $=$)
$\{\parallel\} * \emptyset^1 = S(\{\parallel\}.^vQ\emptyset^1)$	(A_{R27})(Function Axiom)
$\{\parallel\} * \emptyset^1 = S(\{\parallel\}. \{\supset\}.^vQ\emptyset^1)$	(T_{863})
$\{\parallel\} * \emptyset^1 = S(\{\parallel\}. (\{\supset\} + \{=\}) .^vQ\emptyset^1)$	(D_{A141})(Disjunction with $\{=\}$)
$\{\parallel\} * \emptyset^1 = S((\{\parallel\}. \{\supset\} + \{\parallel\}. \{=\}) .^vQ\emptyset^1)$	(T_{M407})(Distribution of $.$ over $+$)(Eqn1)
$\{\parallel\} \subseteq \{\sim\}$	(T_{A108})
$\{\supset\} \subseteq \{\parallel\}$	(T_{M012})(Transposition)
$\{\supset\} = \{\parallel\}. \{\supset\}$	(A_{MX44})($\subseteq$ Equality Form)(Eqn2)
$\{\parallel\} \subseteq \{\neq\}$	(T_{A105})
$\{=\} \subseteq \{\parallel\}$	(T_{M012})(Transposition)
$\{=\} = \{\parallel\}. \{=\}$	(A_{MX44})($\subseteq$ Equality Form)(Eqn3)
$\{\parallel\} * \emptyset^1 = S((\{\supset\} + \{=\}) .^vQ\emptyset^1)$	(A_{MX19})(Substitution of Eqns2&3 into Eqn1)
$\{\parallel\} * \emptyset^1 = S(\{\supset\}.^vQ\emptyset^1)$	(D_{A141})(Disjunction with $\{=\}$)
$\{\parallel\} * \emptyset^1 = S^vQ\emptyset^1$	(T_{863})
$\{\parallel\} * \emptyset^1 = E$	(T_{R055}) since (T_{M80})(Existence of $\emptyset^1$) deems that $\emptyset^1 \neq \emptyset$

$$\{\subseteq\} * E = E \quad (T_{A221})$$

Proof of (T_{A221})

$\{\subseteq\} * E$	
$(\{\subseteq\} + \{=\}) * E$	(D_{A141})(Disjunction with $\{=\}$)
$\{\subseteq\} * E + \{=\} * E$	(T_{R002c})(Distribution of $+$ over $*$)
$\{\subseteq\} * E + E$	(T_{A080})
E	(T_{M055})(Universal Set Disjunction)

$$S\{\subseteq\} = E \quad (T_{A268})$$

Proof of (T_{A268})

$\{\subseteq\} * E = E$	(T_{A221})
$S(\{\subseteq\}.^vQE) = E$	(A_{R27})(Function Axiom)
$S(\{\subseteq\}.E) = E$	(T_{R116})(Sub-relationships with the Universal Set)
$S\{\subseteq\} = E$	(A_{MX11})(Universal Set Conjunction Axiom)

$$Q\{\supset\} = E \quad (T_{A269})$$

Proof of (T_{A269})

$S\{\subseteq\} = E$	(T_{A268})(Eqn1)
$\{\subseteq\} = ^v\{\supset\}$	(T_{7741})(Definition of $^v\{\supset\}$)(Eqn2)
$S^v\{\supset\} = E$	(A_{MX19})(Substitution of Eqn2 into Eqn1)
$Q\{\supset\} = E$	(T_{R016})(Reversibility of Matching Entities)

$$\{\neq\} \diamond E = \emptyset \quad (T_{A273})$$

Proof of (T_{A273})

$\{\subseteq\} * E = E$	(T_{A221})
$\{\subset\} \diamond E = E$	(T_{A199})
$\{\subset\} \diamond E = \sim E$	(T_{M008})(Inverse of an Equality)
$\{\subset\} \diamond E = \emptyset$	(D_{M02})(Definition of $\emptyset$)

$$\{\sim\} * E^1 = E \quad (T_{A255})$$

Proof of (T_{A255})

$\{\sim\} * E^1 = \{\sim\} * E^1$	(D_{M00})(Definition of =)
$\{\sim\} * E^1 = S(\{\sim\}. \forall QE^1)$	(A_{R27})(Function Axiom)
$\{\sim\} * E^1 = S(\{\sim\}. \{\subseteq\}. \forall QE^1)$	(T_{T864})
$\{\sim\} * E^1 = S(\{\sim\}. (\{\subset\} + \{=\}). \forall QE^1)$	(D_{A141})(Disjunction with $\{=\}$)
$\{\sim\} * E^1 = S(((\{\sim\}. \{\subset\} + \{\sim\}. \{=\}). \forall QE^1)$	(T_{M407})(Distribution of . over +)(Eqn1)
$\{\subset\} \subseteq \{\sim\}$	(T_{A106})
$\{\subset\} \subseteq \{\sim\}$	(T_{M012})(Transposition)
$\{\subset\} = \{\sim\}. \{\subset\}$	(A_{MX44})($\subseteq$ Equality Form)(Eqn2)
$\{\subset\} \subseteq \{\neq\}$	(T_{A104})
$\{=\} \subseteq \{\sim\}$	(T_{M012})(Transposition)
$\{=\} = \{\sim\}. \{=\}$	(A_{MX44})($\subseteq$ Equality Form)(Eqn3)
$\{\sim\} * E^1 = S((\{\subset\} + \{=\}). \forall QE^1)$	(A_{MX19})(Substitution of Eqns2&3 into Eqn1)
$\{\sim\} * E^1 = S(\{\subseteq\}. \forall QE^1)$	(D_{A141})(Disjunction with $\{=\}$)
$\{\sim\} * E^1 = S^{\forall QE^1}$	(T_{T864})
$\{\sim\} * E^1 = E$	(T_{A084}) since (T_{M780})(Existence of $\emptyset^1$) states that $\emptyset^1 \neq \emptyset$

$$\{\subset\} * \emptyset^1 = \emptyset \quad (T_{S576}) \text{ Null Set Entity Theorem}$$

Proof of (T_{S576})

$\{\subset\} * \emptyset^1 = \{\subset\} * \emptyset^1$	(D_{M00})(Definition of =)
$\{\subset\} * \emptyset^1 = S(\{\subset\}. \forall Q \emptyset^1)$	(A_{R27})(Function Axiom)
$\{\subset\} * \emptyset^1 = S(\{\subset\}. \{\supset\}. \forall Q \emptyset^1)$	(T_{T863})
$\{\subset\} * \emptyset^1 = S(\{\subset\}. (\{\supset\} + \{=\}). \forall Q \emptyset^1)$	(D_{A141})(Disjunction with $\{=\}$)
$\{\subset\} * \emptyset^1 = S(((\{\subset\}. \{\supset\} + \{\subset\}. \{=\}). \forall Q \emptyset^1)$	(T_{M407})(Distribution of . over +)
$\{\subset\} * \emptyset^1 = S((\emptyset + \emptyset). \forall Q \emptyset^1)$	(T_{A112}) & (T_{A109})
$\{\subset\} * \emptyset^1 = S(\emptyset. \forall Q \emptyset^1)$	(A_{MX01})(Disjunction Tautology)
$\{\subset\} * \emptyset^1 = S \emptyset$	(T_{M050})(Null Set Conjunction)
$\{\subset\} * \emptyset^1 = \emptyset$	(T_{R066})(Null Object Theorem)

$$\{\|\} * E^1 = \emptyset \quad (T_{A275})$$

Proof of (T_{A275})

$\{\ \} * E^1 = \{\ \} * E^1$	(D_{M00})(Definition of =)
$\{\ \} * E^1 = S(\{\ \}. \forall QE^1)$	(A_{R27})(Function Axiom)
$\{\ \} * E^1 = S(\{\ \}. \{\subseteq\}. \forall QE^1)$	(T_{T864})
$\{\ \} * E^1 = S(\{\ \}. (\{\subset\} + \{=\}). \forall QE^1)$	(D_{A141})(Disjunction with $\{=\}$)
$\{\ \} * E^1 = S(((\{\ \}. \{\subset\} + \{\ \}. \{=\}). \forall QE^1)$	(T_{M407})(Distribution of . over +)
$\{\ \} * E^1 = S((\emptyset + \emptyset). \forall QE^1)$	(T_{A113}) & (T_{A111})
$\{\ \} * E^1 = S(\emptyset. \forall QE^1)$	(T_{M668})(Null Set Disjunction)
$\{\ \} * E^1 = S \emptyset$	(T_{M050})(Null Set Conjunction)
$\{\ \} * E^1 = \emptyset$	(T_{R066})(Null Object Theorem)

$$\{\|\} * E^1 = E \quad (T_{A277})$$

Proof of (T_{A277})

$\{\ \} * E^1 = \{\ \} * E^1$	(D_{M00})(Definition of =)
$\{\ \} * E^1 = S(\{\ \}. \forall QE^1)$	(A_{R27})(Function Axiom)
$\{\ \} * E^1 = S(\{\ \}. \{\subseteq\}. \forall QE^1)$	(T_{T864})
$\{\ \} * E^1 = S(\{\ \}. (\{\subset\} + \{=\}). \forall QE^1)$	(D_{A141})(Disjunction with $\{=\}$)

$$\begin{aligned}
\{/\} * E^1 &= S((\{/\}. \{ \subseteq \} + \{/\}. \{ = \}). \vee Q E^1) & (T_{M407})(\text{Distribution of } . \text{ over } +)(\text{Eqn1}) \\
\{/\} &\subseteq \{ \sim \} & (T_{A107}) \\
\{ \subseteq \} &\subseteq \{ /\} & (T_{M012})(\text{Transposition}) \\
\{ \subseteq \} &= \{ /\}. \{ \subseteq \} & (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn2}) \\
\{/\} &\subseteq \{ \neq \} & (T_{A105}) \\
\{ = \} &\subseteq \{ /\} & (T_{M012})(\text{Transposition}) \\
\{ = \} &= \{ /\}. \{ = \} & (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn3}) \\
\{/\} * E^1 &= S((\{ \subseteq \} + \{ = \}). \vee Q E^1) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\
\{/\} * E^1 &= S(\{ \subseteq \}. \vee Q E^1) & (D_{A141})(\text{Disjunction with } \{ = \}) \\
\{/\} * E^1 &= S^{\vee} Q E^1 & (T_{864}) \\
\{/\} * E^1 &= E & (T_{A084}) \text{ since } (T_{M780})(\text{Existence of } \emptyset^1) \text{ states that } \emptyset^1 \neq \emptyset
\end{aligned}$$

$$\{ \sim \} * \emptyset^1 = \emptyset^1 \quad (T_{A258})$$

Proof of (T_{A258})

$$\begin{aligned}
\{ \sim \} * \emptyset^1 &= \{ \sim \} * \emptyset^1 & (D_{M00})(\text{Definition of } =) \\
\{ \sim \} * \emptyset^1 &= S(\{ \sim \}. \vee Q \emptyset^1) & (A_{R27})(\text{Function Axiom}) \\
\{ \sim \} * \emptyset^1 &= S(\{ \sim \}. \{ \supset \}. \vee Q \emptyset^1) & (T_{863}) \\
\{ \sim \} * \emptyset^1 &= S(\{ \sim \}. \{ \supset \} + \{ = \}). \vee Q \emptyset^1 & (D_{A141})(\text{Disjunction with } \{ = \}) \\
\{ \sim \} * \emptyset^1 &= S((\{ \sim \}. \{ \supset \} + \{ \sim \} \{ = \}). \vee Q \emptyset^1) & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
\{ \sim \} * \emptyset^1 &= S((\emptyset + \{ \sim \} \{ = \}). \vee Q \emptyset^1) & (T_{M627})(\text{Complement Conjunction}) \\
\{ \sim \} * \emptyset^1 &= S(\{ \sim \} \{ = \}. \vee Q \emptyset^1) & (T_{M668})(\text{Null Set Disjunction})(\text{Eqn1}) \\
\{ \supset \} &\subseteq \{ \neq \} & (T_{A104}) \\
\{ = \} &\subseteq \{ \sim \} & (T_{M012})(\text{Transposition}) \\
\{ = \} &= \{ \sim \}. \{ = \} & (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn2}) \\
\{ \sim \} * \emptyset^1 &= S(\{ = \}. \vee Q \emptyset^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
\{ \sim \} * \emptyset^1 &= \{ = \} * \emptyset^1 & (A_{R27})(\text{Function Axiom}) \\
\{ \sim \} * \emptyset^1 &= \emptyset^1 & (T_{873})
\end{aligned}$$

$$\{ \sim \} * \emptyset^1 = E \quad (T_{A257})$$

Proof of (T_{A257})

$$\begin{aligned}
\{ \sim \} * \emptyset^1 &= \{ \sim \} * \emptyset^1 & (D_{M00})(\text{Definition of } =) \\
\{ \sim \} * \emptyset^1 &= S(\{ \sim \}. \vee Q \emptyset^1) & (A_{R27})(\text{Function Axiom}) \\
\{ \sim \} * \emptyset^1 &= S(\{ \sim \}. \{ \supset \}. \vee Q \emptyset^1) & (T_{863}) \\
\{ \sim \} * \emptyset^1 &= S(\{ \sim \}. \{ \supset \} + \{ = \}). \vee Q \emptyset^1 & (D_{A141})(\text{Disjunction with } \{ = \}) \\
\{ \sim \} * \emptyset^1 &= S((\{ \sim \}. \{ \supset \} + \{ \sim \}. \{ = \}). \vee Q \emptyset^1) & (T_{M407})(\text{Distribution of } . \text{ over } +)(\text{Eqn1}) \\
\{ \supset \} &\subseteq \{ \sim \} & (T_{A106}) \\
\{ \supset \} &= \{ \sim \}. \{ \supset \} & (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn2}) \\
\{ \subseteq \} &\subseteq \{ \neq \} & (T_{A103}) \\
\{ = \} &\subseteq \{ \sim \} & (T_{M012})(\text{Transposition}) \\
\{ = \} &= \{ \sim \}. \{ = \} & (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn3}) \\
\{ \sim \} * \emptyset^1 &= S((\{ \supset \} + \{ = \}). \vee Q \emptyset^1) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\
\{ \sim \} * \emptyset^1 &= S(\{ \supset \}. \vee Q \emptyset^1) & (D_{A141})(\text{Disjunction with } \{ = \}) \\
\{ \sim \} * \emptyset^1 &= S^{\vee} Q \emptyset^1 & (T_{863}) \\
\{ \sim \} * \emptyset^1 &= E & (T_{R055}) \text{ since } (T_{M780})(\text{Existence of } \emptyset^1) \text{ states that } \emptyset^1 \neq \emptyset
\end{aligned}$$

$$\{ \subseteq \} * \emptyset^1 = \emptyset^1 \quad (T_{A151})$$

Proof of (T_{A151})

$$\begin{aligned}
\{ \subseteq \} * \emptyset^1 &= \{ \subseteq \} * \emptyset^1 & (D_{M00})(\text{Definition of } =) \\
\{ \subseteq \} * \emptyset^1 &= (\{ \subseteq \} + \{ = \}) * \emptyset^1 & (D_{A141})(\text{Disjunction with } \{ = \}) \\
\{ \subseteq \} * \emptyset^1 &= \{ \subseteq \} * \emptyset^1 + \{ = \} * \emptyset^1 & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\
\{ \subseteq \} * \emptyset^1 &= \emptyset + \{ = \} * \emptyset^1 & (T_{S576})(\text{Null Set Entity Theorem}) \\
\{ \subseteq \} * \emptyset^1 &= \emptyset + \emptyset^1 & (D_{A144})(\text{Definition of the Reflexive Relationship Set}) \\
\{ \subseteq \} * \emptyset^1 &= \emptyset^1 & (T_{M668})(\text{Null Set Disjunction})
\end{aligned}$$

$$\{\subseteq\} \diamond \emptyset^1 = \emptyset^1 \quad (\mathcal{T}_{A152})$$

Proof of ($\mathcal{T}_{A152}$)

$$\{\subseteq\} * \emptyset^1 = \emptyset^1 \quad (\mathcal{T}_{A151})$$

$$\{\subseteq\} \diamond \emptyset^1 = \emptyset^1 \quad (\mathcal{T}_{R232})$$

$$\mathbf{b} \subseteq \{\subseteq\} \diamond \{\supseteq\} \diamond \mathbf{b} \quad (\mathcal{T}_{S540})$$

Proof of ($\mathcal{T}_{S540}$)

$$(\mathbf{b} \subseteq \mathbf{Qc}) \subseteq (\mathbf{b} \subseteq {}^v\mathbf{c} \diamond \mathbf{c} \diamond \mathbf{b}) \quad (\mathcal{A}_{R21})(\text{Intersection Function Axiom})(\text{Eqn1})$$

$$\text{Let } \mathbf{c} = \{\supseteq\} \quad (\text{Eqn2})$$

$$(\mathbf{b} \subseteq \mathbf{Q}\{\supseteq\}) \subseteq (\mathbf{b} \subseteq {}^v\{\supseteq\} \diamond \{\supseteq\} \diamond \mathbf{b}) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(\mathbf{b} \subseteq \mathbf{Q}\{\supseteq\}) \subseteq (\mathbf{b} \subseteq \{\subseteq\} \diamond \{\supseteq\} \diamond \mathbf{b}) \quad (\mathcal{T}_{T741})(\text{Definition of } {}^v\{\supseteq\})$$

$$(\mathbf{b} \subseteq \mathbf{E}) \subseteq (\mathbf{b} \subseteq \{\subseteq\} \diamond \{\supseteq\} \diamond \mathbf{b}) \quad (\mathcal{T}_{A269})$$

$$\mathbf{E} \subseteq (\mathbf{b} \subseteq \{\subseteq\} \diamond \{\supseteq\} \diamond \mathbf{b}) \quad (\mathcal{T}_{M001})(\text{Universality of } (\mathcal{T}_{M744}) \text{ Universal Set Theorem})$$

$$\mathbf{E} = (\mathbf{b} \subseteq \{\subseteq\} \diamond \{\supseteq\} \diamond \mathbf{b}) \quad (\mathcal{T}_{M275})(\text{Supersets of the Universal Set})$$

$$\mathbf{b} \subseteq \{\subseteq\} \diamond \{\supseteq\} \diamond \mathbf{b} \quad (\mathcal{A}_{MX16})(\text{Order Axiom})$$

6.6 Membership Equations in Relationship Form

$$((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\mathbf{a}^1 \subseteq \mathbf{o} * \mathbf{b}^1)) \quad (\mathcal{T}_{R653}) \text{ Definition of a Relationship}$$

Proof of ($\mathcal{T}_{R653}$)

$$((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = ({}^v\mathbf{S}\mathbf{a}^1.\mathbf{o}.\mathbf{Q}\mathbf{b}^1 \neq \emptyset)) \quad (\mathcal{D}_{A00})(\text{Definition of an Equation})$$

$$((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = ({}^v\mathbf{S}\mathbf{a}^1 \not\subseteq \sim(\mathbf{o}.\mathbf{Q}\mathbf{b}^1))) \quad (\mathcal{T}_{M735})(\subseteq \text{ Existential Form})$$

$$((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\mathbf{a}^1 \not\subseteq \sim\mathbf{S}(\mathbf{o}.\mathbf{Q}\mathbf{b}^1))) \quad (\mathcal{T}_{S120})$$

$$((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\mathbf{a}^1 \subseteq \mathbf{S}(\mathbf{o}.\mathbf{Q}\mathbf{b}^1))) \quad (\mathcal{T}_{M090})(\text{Inverse Unit Subset Rule})$$

$$((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\mathbf{a}^1 \subseteq \mathbf{o} * \mathbf{b}^1)) \quad (\mathcal{A}_{R27})(\text{Function Axiom})$$

$$((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a}^1 \langle \mathbf{o} \rangle \mathbf{b}^1) = (\mathbf{a}^1 \subseteq \mathbf{o} * \mathbf{b}^1)) \quad (\mathcal{T}_{T736}) \text{ Definition of a Relationship}$$

Proof of ($\mathcal{T}_{T736}$)

$$((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\mathbf{a}^1 \subseteq \mathbf{o} * \mathbf{b}^1)) \quad (\mathcal{T}_{R653})(\text{Definition of a Relationship})(\text{Eqn1})$$

$$\text{Let } \mathbf{a} = \mathbf{a}^1 \quad (\text{Eqn2})$$

$$\text{Let } \mathbf{b} = \mathbf{b}^1 \quad (\text{Eqn3})$$

$$((\mathbf{a}^{11} \neq \emptyset)(\mathbf{b}^{11} \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a}^1 \langle \mathbf{o} \rangle \mathbf{b}^1) = (\mathbf{a}^{11} \subseteq \mathbf{o} * \mathbf{b}^{11})) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})$$

$$((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a}^1 \langle \mathbf{o} \rangle \mathbf{b}^1) = (\mathbf{a}^1 \subseteq \mathbf{o} * \mathbf{b}^1)) \quad (\mathcal{D}_{M17})(\text{Definition of } \mathbf{a}^{11})$$

$$((\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \subseteq \mathbf{b}) = (\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1)) \quad (\mathcal{T}_{S412})$$

$$((\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \parallel \mathbf{b}) = (\mathbf{a}^1 \subseteq \{\parallel\} * \mathbf{b}^1)) \quad (\mathcal{T}_{S567})$$

$$((\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \subset \mathbf{b}) = (\mathbf{a}^1 \subseteq \{\subset\} * \mathbf{b}^1)) \quad (\mathcal{T}_{S416})$$

Example Proof of ($\mathcal{T}_{S412}$)

$$((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \langle \mathbf{o} \rangle \mathbf{b}) = (\mathbf{a}^1 \subseteq \mathbf{o} * \mathbf{b}^1)) \quad (\mathcal{T}_{R653})(\text{Definition of a Relationship})$$

$$((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \langle \{\subseteq\} \rangle \mathbf{b}) = (\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1)) \quad \text{Resymbolising}$$

$$((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \subseteq \mathbf{b}) = (\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1)) \quad (\mathcal{D}_{A229})(\text{Definition of } \langle \rangle)$$

$$(\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset)(\mathbf{a} \subseteq \mathbf{b}) = (\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset)(\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1) \quad (\mathcal{T}_{M444b})(\text{Exportation from a Biconditional})(\text{Eqn1})$$

$$\text{Let } \mathbf{a}^1 = \emptyset \quad (\text{Eqn2})$$

$$\mathbf{a} = \emptyset$$

$$\text{Let } (\mathbf{a} \subseteq \mathbf{b}) \neq (\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1) \quad (\mathcal{A}_{MX50})(\text{Components of a Null Entity})(\text{Eqn3})$$

$$(\emptyset \subseteq \mathbf{b}) \neq (\emptyset \subseteq \{\subseteq\} * \mathbf{b}^1) \quad (\text{Eqn4})$$

$$(\emptyset \subseteq \mathbf{b}) \neq (\emptyset \subseteq \{\subseteq\} * \mathbf{b}^1) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqns2\&3 into Eqn4})$$

$$\mathbf{E} \neq \mathbf{E} \quad (\mathcal{T}_{M001})(\text{Universality of } (\mathcal{T}_{M670}) \text{ Null Set Theorem})$$

$$\emptyset \quad (\mathcal{T}_{M345})$$

$$\text{Thus } ((\mathbf{a}^1 = \emptyset) = \mathbf{E})((\mathbf{a} \subseteq \mathbf{b}) \neq (\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1)) = \emptyset \quad (\mathcal{A}_{MX02})(\text{Conditional Introduction})$$

$((a^1 = \emptyset) = E) \subseteq ((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))$	$(T_{M735})(\subseteq \text{ Existential Form})(\text{Eqn5})$
$(a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq (a^1 = \emptyset)$	$(T_{M083})(\text{Conjunction Elimination})$
$((a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq (a^1 = \emptyset)) = E$	$(A_{MX16})(\text{Order Axiom})$
$((a^1 = \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 = \emptyset) = E)$	$(T_{M094})(\text{Probabilistic Rule})(\text{Eqn6})$
$((a^1 = \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))$	$(T_{M139})(\text{Hypothetical Syllogism Eqns5\&6})$
$(a^1 = \emptyset)(b^1 \neq \emptyset)(a \subseteq b) = (a^1 = \emptyset)(b^1 \neq \emptyset)(a^1 \subseteq \{\subseteq\} * b^1)$	$(T_{M444b})(\text{Exportation from a Biconditional})(\text{Eqn7})$
$(a^1 \neq \emptyset)(b^1 \neq \emptyset)(a \subseteq b) + (a^1 = \emptyset)(b^1 \neq \emptyset)(a \subseteq b) =$	
$(a^1 \neq \emptyset)(b^1 \neq \emptyset)(a^1 \subseteq \{\subseteq\} * b^1) + (a^1 = \emptyset)(b^1 \neq \emptyset)(a^1 \subseteq \{\subseteq\} * b^1)$	$(A_{MX21})(\text{Combination of Eqns1\&7})$
$((a^1 \neq \emptyset) + (a^1 = \emptyset)).(b^1 \neq \emptyset)(a \subseteq b) = ((a^1 \neq \emptyset) + (a^1 = \emptyset)).(b^1 \neq \emptyset)(a^1 \subseteq \{\subseteq\} * b^1)$	$(T_{M407})(\text{Distribution of } . \text{ over } +)$
$E.(b^1 \neq \emptyset)(a \subseteq b) = E.(b^1 \neq \emptyset)(a^1 \subseteq \{\subseteq\} * b^1)$	$(T_{M671})(\text{Tertium non Datur})$
$(b^1 \neq \emptyset)(a \subseteq b) = (b^1 \neq \emptyset)(a^1 \subseteq \{\subseteq\} * b^1)$	$(A_{MX11})(\text{Universal Set Conjunction Axiom})$
$((b^1 \neq \emptyset) = E) \subseteq ((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))$	$(T_{M444b})(\text{Exportation from a Biconditional})$

$$((a^1 \neq \emptyset) = E) \subseteq ((a \supset b) = (a^1 \subseteq \{\supset\} * b^1)) \quad (T_{S413})$$

Proof of (T_{S413})

$((b^1 \neq \emptyset) = E) \subseteq ((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))$	(T_{S412})
$((a^1 \neq \emptyset) = E) \subseteq ((b \subseteq a) = (b^1 \subseteq \{\subseteq\} * a^1))$	Resymbolising
$((a^1 \neq \emptyset) = E) \subseteq ((a \supset b) = (b^1 \subseteq \{\subseteq\} * a^1))$	$(T_{M196})(\subseteq \text{ Reverse of } \supset)$
$((a^1 \neq \emptyset) = E) \subseteq ((a \supset b) = (a^1 \subseteq \forall \{\subseteq\} * b^1))$	$(T_{R519})(\text{Active/Passive Theorem})$
$((a^1 \neq \emptyset) = E) \subseteq ((a \supset b) = (a^1 \subseteq \{\supset\} * b^1))$	$(T_{T741})(\text{Definition of } \forall \{\subseteq\})$

$$((a^1 \neq \emptyset) = E) \subseteq ((a \supset b) = (a^1 \subseteq \{\supset\} * b^1)) \quad (T_{S417})$$

Proof of (T_{S417})

$((b^1 \neq \emptyset) = E) \subseteq ((a \subset b) = (a^1 \subseteq \{\subset\} * b^1))$	(T_{S416})
$((a^1 \neq \emptyset) = E) \subseteq ((b \subset a) = (b^1 \subseteq \{\subset\} * a^1))$	Resymbolising
$((a^1 \neq \emptyset) = E) \subseteq ((a \supset b) = (b^1 \subseteq \{\subset\} * a^1))$	$(T_{M195})(\subset \text{ Reverse of } \supset)$
$((a^1 \neq \emptyset) = E) \subseteq ((a \supset b) = (a^1 \subseteq \forall \{\subset\} * b^1))$	$(T_{R519})(\text{Active/Passive Theorem})$
$((a^1 \neq \emptyset) = E) \subseteq ((a \supset b) = (a^1 \subseteq \{\supset\} * b^1))$	$(D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set})$

$$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a = b) = (a^1 \subseteq \{=\} * b^1)) \quad (T_{S308})$$

Proof of (T_{S308})

$((b^1 \neq \emptyset) = E) \subseteq ((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))$	$(T_{S412})(\text{Eqn1})$
$((a^1 \neq \emptyset) = E) \subseteq ((a \supset b) = (a^1 \subseteq \{\supset\} * b^1))$	$(T_{S413})(\text{Eqn2})$
$((a^1 \neq \emptyset) = E)((b^1 \neq \emptyset) = E) \subseteq ((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))((a \supset b) = (a^1 \subseteq \{\supset\} * b^1))$	$(T_{M669})(\text{Combination of Eqns1\&2})$
$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))((a \supset b) = (a^1 \subseteq \{\supset\} * b^1))$	$(T_{M108})(\text{Conjunction Introduction})(\text{Eqn3})$
$((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))((a \supset b) = (a^1 \subseteq \{\supset\} * b^1)) \subseteq ((a \subseteq b)(a \supset b) = (a^1 \subseteq \{\subseteq\} * b^1)(a^1 \subseteq \{\supset\} * b^1))$	$(T_{M669})(\text{Combination})$
$((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))((a \supset b) = (a^1 \subseteq \{\supset\} * b^1)) \subseteq ((a \subseteq b)(b \subseteq a) = (a^1 \subseteq \{\subseteq\} * b^1)(a^1 \subseteq \{\supset\} * b^1))$	$(T_{M196})(\subseteq \text{ reverse of } \supset)$
$((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))((a \supset b) = (a^1 \subseteq \{\supset\} * b^1)) \subseteq ((a = b) = (a^1 \subseteq \{\subseteq\} * b^1)(a^1 \subseteq \{\supset\} * b^1))$	$(T_{M430})(\text{Biconditional Introduction})$
$((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))((a \supset b) = (a^1 \subseteq \{\supset\} * b^1)) \subseteq ((a = b) = (a^1 \subseteq (\{\subseteq\} * b^1)(\{\supset\} * b^1)))$	$(T_{M132})(\text{Composition})$
$((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))((a \supset b) = (a^1 \subseteq \{\supset\} * b^1)) \subseteq ((a = b) = (a^1 \subseteq (\{\subseteq\} \{\supset\}) * b^1))$	$(T_{R239})(\text{Distribution of } . \text{ over } * \text{ for Entities})$
$((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))((a \supset b) = (a^1 \subseteq \{\supset\} * b^1)) \subseteq ((a = b) = (a^1 \subseteq \{=\} * b^1))$	$(T_{A226})(\text{Eqn4})$
$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a = b) = (a^1 \subseteq \{=\} * b^1))$	$(T_{M139})(\text{Hypothetical Syllogism Eqns3\&4})$

$$((b \neq \emptyset) = E) \subseteq ((b \subseteq \{\supset\}a) \subseteq (a \not\subseteq \sim\{\subseteq\}b)) \quad (T_{R553})$$

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq \{\subseteq\}b) \subseteq (b \not\subseteq \sim\{\supset\}a)) \quad (T_{R956})$$

Example Proof of (T_{R956})

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq c * b) \subseteq (b \not\subseteq {}^V c * a)) \quad (T_{7385})(\text{Eqn1})$$

$$\text{Let } c = \{\subseteq\} \quad (\text{Eqn2})$$

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq \{\subseteq\} * b) \subseteq (b \not\subseteq {}^V \{\subseteq\} * a)) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq \{\subseteq\} * b) \subseteq (b \not\subseteq \{\supset\} * a)) \quad (T_{7747})(\text{Definition of } {}^V \{\subseteq\})$$

$$(a \neq \emptyset) \subseteq (\{\supset\} \diamond a + \{=\} * a^1 = \{\supset\} \diamond a) \quad (T_{S391})$$

Proof of (T_{S391})

$$(a \neq \emptyset) \subseteq (a^1 = (\{\supset\} \diamond a).(\{\supset\} * \{\supset\} \diamond a)) \quad (D_{R98})(\text{General Definition of Aggregation})$$

$$(a \neq \emptyset) \subseteq (a^1 = (\{\supset\} \diamond a). \sim(\{\supset\} * \{\supset\} \diamond a)) \quad (D_{R04})(\text{Definition of } *)$$

$$(a \neq \emptyset) \subseteq (a^1 = (\{\supset\} \diamond a). \sim(\{\supset\} \diamond a)) \quad (T_{S406})(\text{Eqn3})$$

$$(a^1 = (\{\supset\} \diamond a). \sim(\{\supset\} \diamond a)) \subseteq (a^1 + \{\supset\} \diamond a = (\{\supset\} \diamond a). \sim(\{\supset\} \diamond a) + (\{\supset\} \diamond a)) \quad (T_{M030})(\text{Deduction})(\text{Eqn4})$$

$$(a \neq \emptyset) \subseteq (a^1 + \{\supset\} \diamond a = (\{\supset\} \diamond a). \sim(\{\supset\} \diamond a) + (\{\supset\} \diamond a)) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns3\&4})$$

$$(a \neq \emptyset) \subseteq (a^1 + \{\supset\} \diamond a = (\{\supset\} \diamond a). \sim(\{\supset\} \diamond a) + E(\{\supset\} \diamond a)) \quad (A_{MX11})(\text{Universal Set Conjunction Axiom})$$

$$(a \neq \emptyset) \subseteq (a^1 + \{\supset\} \diamond a = (\{\supset\} \diamond a). \sim(\{\supset\} \diamond a) + (\{\supset\} \diamond a + \{\supset\} \diamond a)(\{\supset\} \diamond a)) \quad (T_{M671})(\text{Tertium non Datur})$$

$$(a \neq \emptyset) \subseteq (a^1 + \{\supset\} \diamond a = (\{\supset\} \diamond a). \sim(\{\supset\} \diamond a) + (\{\supset\} \diamond a)(\{\supset\} \diamond a) + (\{\supset\} \diamond a)(\{\supset\} \diamond a)) \quad (T_{M407})(\text{Distribution of } . \text{ over } +)$$

$$(a \neq \emptyset) \subseteq (a^1 + \{\supset\} \diamond a = (\{\supset\} \diamond a). (\sim(\{\supset\} \diamond a) + (\{\supset\} \diamond a)) + (\{\supset\} \diamond a)(\{\supset\} \diamond a)) \quad (T_{M407})(\text{Distribution of } . \text{ over } +)$$

$$(a \neq \emptyset) \subseteq (a^1 + \{\supset\} \diamond a = (\{\supset\} \diamond a). E + (\{\supset\} \diamond a)(\{\supset\} \diamond a)) \quad (T_{M671})(\text{Tertium non Datur})$$

$$(a \neq \emptyset) \subseteq (a^1 + \{\supset\} \diamond a = (\{\supset\} \diamond a) + (\{\supset\} \diamond a)(\{\supset\} \diamond a)) \quad (A_{MX11})(\text{Universal Set Conjunction Axiom})$$

$$(a \neq \emptyset) \subseteq (a^1 + \{\supset\} \diamond a = (\{\supset\} \diamond a) + (\{\supset\} * a)(\{\sim\} * a)) \quad (A_{R10})(\text{Axiom of Complementary Relationship Sets})$$

$$(a \neq \emptyset) \subseteq (a^1 + \{\supset\} \diamond a = (\{\supset\} \diamond a) + (\{\sim\} * a)(\{\sim\} * a)) \quad (T_{A059})(\text{Eqn5})$$

$$\{\sim\} * a \subseteq \{\sim\} * a. \{\neq\} * a \quad (T_{R220})(\text{Conditional Distribution of } . \text{ over } *)$$

$$\{\sim\} * a = \{\sim\} * a. \{\neq\} * a. \{\neq\} * a \quad (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn6})$$

$$(a \neq \emptyset) \subseteq (a^1 + \{\supset\} \diamond a = (\{\supset\} \diamond a) + (\{\sim\} * a). (\{\neq\} * a). (\{\neq\} * a)) \quad (A_{MX19})(\text{Substitution of Eqn6 into Eqn5})$$

$$(a \neq \emptyset) \subseteq (a^1 + \{\supset\} \diamond a = (\{\supset\} \diamond a) + (\{\sim\} * a). (\{\neq\} * a). \emptyset) \quad (T_{M621})(\text{Complement Conjunction})$$

$$(a \neq \emptyset) \subseteq (a^1 + \{\supset\} \diamond a = \{\supset\} \diamond a + \emptyset) \quad (T_{M050})(\text{Null Set Conjunction})$$

$$(a \neq \emptyset) \subseteq (a^1 + \{\supset\} \diamond a = \{\supset\} \diamond a) \quad (T_{M668})(\text{Null Set Disjunction})(\text{Eqn7})$$

$$(a \neq \emptyset) \subseteq (\{=\} * a^1 + \{\supset\} \diamond a = \{\supset\} \diamond a) \quad (D_{A144})(\text{Definition of the Reflexive Relationship Set})$$

$$(a \neq \emptyset) \subseteq (\{\supset\} * a^1 = \{\supset\} \diamond a) \quad (T_{S402})$$

Proof of (T_{S402})

$$(a \neq \emptyset) \subseteq (\{\supset\} \diamond a = \{\supset\} * a^1) \quad (A_{S92})(\text{Inclusion Axiom})$$

$$(a \neq \emptyset) \subseteq (\{=\} * a^1 + \{\supset\} \diamond a = \{=\} * a^1 + \{\supset\} * a^1) \quad (T_{M232})(\text{Deduction})$$

$$(a \neq \emptyset) \subseteq (\{\supset\} \diamond a = \{=\} * a^1 + \{\supset\} * a^1) \quad (T_{S391})$$

$$(a \neq \emptyset) \subseteq (\{\supset\} \diamond a = (\{=\} + \{\supset\}) * a^1) \quad (T_{R002c})(\text{Distribution of } + \text{ over } *)$$

$$(a \neq \emptyset) \subseteq (\{\supset\} \diamond a = \{\supset\} * a^1) \quad (D_{A141})(\text{Disjunction with } \{=\})$$

$$(a \neq \emptyset) \subseteq (\{\supset\} * a^1 \subseteq \{\supset\} * a) \quad (T_{S844})$$

Proof of (T_{S844})

$$(a \neq \emptyset) \subseteq (\{\supset\} * a^1 = \{\supset\} \diamond a) \quad (A_{S92})(\text{Inclusion Axiom})(\text{Eqn1})$$

$$\begin{aligned} \{\supset\} \diamond a \subseteq \{\supset\} * a & \quad (T_{R241})(\text{Eqn2}) \\ (a \neq \emptyset) \subseteq (\{\supset\} * a^1 \subseteq \{\supset\} * a) & \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$(a \neq \emptyset) \subseteq (\{\supset\} * a^1 \subseteq \{\supset\} * a) \quad (T_{S381})$$

Proof of (T_{S381})

$$\begin{aligned} (a \neq \emptyset) \subseteq (\{\supset\} \diamond a = \{\supset\} * a^1) & \quad (A_{S92})(\text{Inclusion Axiom})(\text{Eqn1}) \\ \{\supset\} \diamond a \subseteq \{\supset\} * a & \quad (T_{S380})(\text{Eqn2}) \\ (a \neq \emptyset) \subseteq (\{\supset\} * a^1 \subseteq \{\supset\} * a) & \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \end{aligned}$$

$$(a \neq \emptyset)(a^1 \neq \emptyset) \subseteq (\{\supset\} * a^1 = \{\supset\} \diamond a) \quad (T_{A095})$$

Proof of (T_{A095})

$$\begin{aligned} (a \neq \emptyset) \subseteq (\{\supset\} * a^1 = \{\supset\} \diamond a) & \quad (T_{S402}) \\ (a \neq \emptyset) \subseteq (\{\supset\} * a^1 = \{\supset\} \diamond a) & \quad (T_{M008})(\text{Inverse of an Equality}) \\ (a \neq \emptyset)(a^1 \neq \emptyset) \subseteq (\{\supset\} * a^1 = \{\supset\} \diamond a) & \quad (T_{R245})(\text{Negative Rule for Single Element Object})(\text{Eqn1}) \end{aligned}$$

$$\{\|\} * b^1 = \{\|\} * (b)^1 \quad (T_{A190})$$

Proof of (T_{A190})

$$\begin{aligned} ((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a \parallel b) = (a^1 \subseteq \{\|\} * b^1)) & \quad (T_{R653})(\text{Definition of a Relationship})(\text{Eqn1}) \\ ((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a \parallel \underline{b}) = (a^1 \subseteq \{\|\} * \underline{b}^1)) & \quad (T_{R653})(\text{Definition of a Relationship})(\text{Eqn2}) \\ a \parallel b = a \parallel \underline{b} & \quad (T_{M194})(\text{Co-Intersection})(\text{Eqn3}) \\ ((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a \parallel b) = (a^1 \subseteq \{\|\} * \underline{b}^1)) & \quad (A_{MX19})(\text{Substitution of Eqn3 into Eqn2})(\text{Eqn4}) \\ ((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq \{\|\} * b^1 = (a^1 \subseteq \{\|\} * \underline{b}^1)) & \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn4})(\text{Eqn5}) \\ \text{Let } a^1 = \emptyset & \quad (\text{Eqn6}) \\ \text{Let } (a^1 \subseteq \{\|\} * b^1) \neq (a^1 \subseteq \{\|\} * \underline{b}^1) & \quad (\text{Eqn7}) \\ (\emptyset \subseteq \{\|\} * b^1) \neq (\emptyset \subseteq \{\|\} * \underline{b}^1) & \quad (A_{MX19})(\text{Substitution of Eqn6 into Eqn7}) \\ E \neq E & \quad (T_{M001})(\text{Universality of (T}_{M670}\text{) Null Set Theorem}) \\ \emptyset & \quad (T_{M345}) \end{aligned}$$

$$\text{Thus } ((a^1 = \emptyset) = E)((a^1 \subseteq \{\|\} * b^1) \neq (a^1 \subseteq \{\|\} * \underline{b}^1)) = \emptyset$$

$$\begin{aligned} ((a^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) & \quad (A_{MX02})(\text{Conditional Introduction}) \\ ((a^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) & \quad (T_{M735})(\subseteq \text{Existential Form})(\text{Eqn8}) \\ \text{Let } b^1 = \emptyset & \quad (\text{Eqn9}) \\ b = \emptyset & \quad (A_{MX50})(\text{Components of a Null Entity}) \\ \underline{b} = \sim \emptyset & \quad (T_{M008})(\text{Inverse of an Equality}) \\ \underline{b} = E & \quad (T_{M054})(\text{Negative of Null Set}) \\ (b)^1 = E^1 & \quad (T_{M488})(\text{Eqn10}) \\ \text{Let } (a^1 \subseteq \{\|\} * b^1) \neq (a^1 \subseteq \{\|\} * \underline{b}^1) & \quad (\text{Eqn11}) \\ (a^1 \subseteq \{\|\} * \emptyset) \neq (a^1 \subseteq \{\|\} * E^1) & \quad (A_{MX19})(\text{Substitution of Eqns9\&10 into Eqn11}) \\ (a^1 \subseteq \emptyset) \neq (a^1 \subseteq \{\|\} * E^1) & \quad (T_{S181})(\text{Null Object Theorem}) \\ (a^1 \subseteq \emptyset) \neq (a^1 \subseteq \emptyset) & \quad (T_{A275}) \\ ((a^1 \subseteq \emptyset) \neq (a^1 \subseteq \emptyset)) = E & \quad (A_{MX16})(\text{Order Axiom}) \\ ((a^1 \subseteq \emptyset) = (a^1 \subseteq \emptyset)) = \sim E & \quad (T_{M008})(\text{Inverse of an Equality}) \\ ((a^1 \subseteq \emptyset) = (a^1 \subseteq \emptyset)) = \emptyset & \quad (T_{M054})(\text{Negative of Null Set}) \\ E = \emptyset & \quad (T_{M001})(\text{Universality of (D}_{M00}\text{) Definition of =}) \\ \emptyset & \quad (T_{M347}) \end{aligned}$$

$$\text{Thus } ((b^1 = \emptyset) = E)((a^1 \subseteq \{\|\} * b^1) \neq (a^1 \subseteq \{\|\} * \underline{b}^1)) = \emptyset$$

$$\begin{aligned} ((b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) & \quad (A_{MX02})(\text{Conditional Introduction}) \\ ((b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) & \quad (T_{M735})(\subseteq \text{Existential Form})(\text{Eqn12}) \\ ((a^1 = \emptyset) = E) + ((b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) + ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) & \quad (A_{MX21})(\text{Combination of Eqns8\&12}) \\ ((a^1 = \emptyset) = E) + ((b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) & \quad (A_{MX01})(\text{Disjunction Tautology}) \\ ((a^1 = \emptyset) + (b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) & \end{aligned}$$

$$\begin{aligned}
& ((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) + ((a^1 = \emptyset) + (b^1 = \emptyset) = E) \subseteq \\
& ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) + ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) \\
& \quad (T_{M160})(\text{Probabilistic Disjunction})(\text{Eqn13}) \\
& \quad (A_{MX21})(\text{Combination of Eqns5\&13}) \\
& ((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) + ((a^1 = \emptyset) + (b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) \\
& \quad (A_{MX01})(\text{Disjunction Tautology}) \\
& ((a^1 \neq \emptyset)(b^1 \neq \emptyset) + (a^1 = \emptyset) + (b^1 = \emptyset) = E) \subseteq ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) \\
& \quad (T_{M160})(\text{Probabilistic Disjunction}) \\
& ((a^1 \neq \emptyset)(b^1 \neq \emptyset) + \sim((a^1 \neq \emptyset)(b^1 \neq \emptyset)) = E) \subseteq ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) \\
& \quad (T_{M025})(\text{DeMorgan's Law}) \\
& (E = E) \subseteq ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) \\
& \quad (T_{M671})(\text{Tertium non Datur}) \\
& E \subseteq ((a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1)) \\
& \quad (T_{M342}) \\
& E = (a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1) \\
& \quad (T_{M275})(\text{Supersets of the Universal Set}) \\
& (a^1 \subseteq \{\|\} * b^1) = (a^1 \subseteq \{\|\} * \underline{b}^1) \\
& \quad (A_{MX16})(\text{Order Axiom})
\end{aligned}$$

$$\{\supset\} \diamond E = \emptyset \quad (T_{S456})$$

Proof of (T_{S456})

$$\begin{aligned}
& E^1 \subset E \quad (T_{M481})(\text{Domain of the Universal Set Entity}) \\
& E^1 \neq E \quad (T_{M492}) \\
& (E^1 \neq E) = E \quad (A_{MX16})(\text{Order Axiom}) \\
& (E^1 = E) = \sim E \quad (T_{M008})(\text{Inverse of an Equality}) \\
& (E^1 = E) = \emptyset \quad (D_{M02})(\text{Definition of } \emptyset)(\text{Eqn1}) \\
& E^1 \subset E \quad (T_{M481})(\text{Domain of the Universal Set Entity}) \\
& (E^1 \subset E) + \emptyset \quad (T_{M668})(\text{Null Set Disjunction})(\text{Eqn2}) \\
& (E^1 \subset E) + (E^1 = E) \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\
& E^1 \subseteq E \quad (T_{M266})(\text{Definition of } \subseteq) \\
& (E^1 \subseteq E) = E \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\
& (E^1 \neq \emptyset)(E^1 \subseteq E) \subseteq (\{\supset\} \diamond E \subseteq \{\supset\} \diamond E^1) \quad (T_{R262})(\text{Relational Deduction})(\text{Eqn4}) \\
& (E^1 \neq \emptyset)E \subseteq (\{\supset\} \diamond E \subseteq \{\supset\} \diamond E^1) \quad (A_{MX19})(\text{Substitution of Eqn3 into Eqn4}) \\
& (E^1 \neq \emptyset) \subseteq (\{\supset\} \diamond E \subseteq \{\supset\} \diamond E^1) \quad (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn5}) \\
& \text{Let } E^1 = \emptyset \quad (\text{Eqn6}) \\
& \text{Let } \{\supset\} \diamond E \not\subseteq \{\supset\} \diamond E^1 \quad (\text{Eqn7}) \\
& \{\supset\} \diamond E \not\subseteq \{\supset\} \diamond \emptyset \quad (A_{MX19})(\text{Substitution of Eqn6 into Eqn7}) \\
& \{\supset\} \diamond E \not\subseteq \emptyset \quad (T_{R174})(\text{Null Object Theorem}) \\
& \sim E \quad (T_{M001})(\text{Universality of } (T_{M744}) \text{ Universal Set Theorem}) \\
& \emptyset \quad (D_{M02})(\text{Definition of } \emptyset) \\
& \text{Thus } (E^1 = \emptyset)(\{\supset\} \diamond E \not\subseteq \{\supset\} \diamond E^1) = \emptyset \quad (A_{MX02})(\text{Conditional Introduction}) \\
& (E^1 = \emptyset) \subseteq (\{\supset\} \diamond E \subseteq \{\supset\} \diamond E^1) \quad (T_{M735})(\subseteq \text{ Existential Form})(\text{Eqn8}) \\
& (E^1 = \emptyset) + (E^1 \neq \emptyset) \subseteq (\{\supset\} \diamond E \subseteq \{\supset\} \diamond E^1) + (\{\supset\} \diamond E \subseteq \{\supset\} \diamond E^1) \\
& \quad (A_{MX21})(\text{Combination of Eqns5\&8}) \\
& (E^1 = \emptyset) + (E^1 \neq \emptyset) \subseteq (\{\supset\} \diamond E \subseteq \{\supset\} \diamond E^1) \quad (A_{MX01})(\text{Disjunction Tautology}) \\
& E \subseteq (\{\supset\} \diamond E \subseteq \{\supset\} \diamond E^1) \quad (T_{M671})(\text{Tertium non Datur}) \\
& E = (\{\supset\} \diamond E \subseteq \{\supset\} \diamond E^1) \quad (T_{M275})(\text{Supersets of the Universal Set}) \\
& \{\supset\} \diamond E \subseteq \{\supset\} \diamond E^1 \quad (A_{MX16})(\text{Order Axiom}) \\
& \{\supset\} \diamond E \subseteq \emptyset \quad (T_{R75}) \\
& \{\supset\} \diamond E = \emptyset \quad (T_{M052})(\text{Subsets of the Null Set})
\end{aligned}$$

$$\{\subset\} * E = \sim E^1 \quad (T_{S455})$$

Proof of (T_{S455})

$$\begin{aligned}
& a \subseteq \{\subset\} * a^1 \quad (T_{S581})(\text{General Definition of Disaggregation})(\text{Eqn1}) \\
& \text{Let } a = E \quad (\text{Eqn2}) \\
& a^1 = E^1 \quad (T_{M479})(\text{Eqn3}) \\
& E \subseteq \{\subset\} * E^1 \quad (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\
& E \subseteq (\{\subset\} + \{=\}) * E^1 \quad (D_{A141})(\text{Disjunction with } \{=\}) \\
& E \subseteq \{\subset\} * E^1 + \{=\} * E^1 \quad (T_{R002c})(\text{Distribution of } + \text{ over } *) \\
& E \subseteq \{\subset\} * E^1 + E^1 \quad (D_{A144})(\text{Definition of the Reflexive Relationship Set})
\end{aligned}$$

$E \cdot \sim E^1 \subseteq \{ \subset \} * E^1$	(T_{M468})
$\sim E^1 \subseteq \{ \subset \} * E^1$	($AMX11$)(Universal Set Conjunction Axiom)
$\{ \subset \} * E^1 \subseteq E^1$	(T_{M012})(Transposition)
$(\{ \subset \} * E^1 \subset E^1) + (\{ \subset \} * E^1 = E^1)$	(T_{M266})(Definition of $\subseteq$)
$(\{ \subset \} * E^1 = \emptyset)(E^1 \neq \emptyset) + (\{ \subset \} * E^1 = E^1)$	($AMX40$)(Proper Subset of Unit Set Axiom)(Eqn4)
$\{ = \} * E^1 \subseteq \{ \subset \} * E^1$	(T_{S212})
$E^1 \subseteq \{ \subset \} * E^1$	($DA144$)(Definition of the Reflexive Relationship Set)
$(E^1 \subseteq \{ \subset \} * E^1) = E$	($AMX16$)(Order Axiom)
$(E^1 \neq \emptyset) \subseteq (\{ \subset \} * E^1 \neq \emptyset)$	(T_{M096})(Probabilistic Rule)
$(E^1 \neq \emptyset) = (E^1 \neq \emptyset)(\{ \subset \} * E^1 \neq \emptyset)$	($AMX44$)($\subseteq$ Equality Form)(Eqn5)
$(\{ \subset \} * E^1 = \emptyset)(\{ \subset \} * E^1 \neq \emptyset)(E^1 \neq \emptyset) + (\{ \subset \} * E^1 = E^1)$	($AMX19$)(Substitution of Eqn5 into Eqn4)
$\emptyset(E^1 \neq \emptyset) + (\{ \subset \} * E^1 = E^1)$	(T_{M667})(Conjunction Tautology)
$\emptyset + (\{ \subset \} * E^1 = E^1)$	(T_{M050})(Null Set Conjunction)
$\{ \subset \} * E^1 = E^1$	(T_{M668})(Null Set Disjunction)
$\{ \subset \} * E = \sim E^1$	(T_{M012})(Transposition)

$$\{ \subset \} * E = E^1 \quad (T_{S196})$$

Proof of (T_{S196})

$\{ \subset \} * E = \sim E^1$	(T_{S455})
$\{ \subset \} * E = E^1$	(T_{M008})(Inverse of an Equality)

$$S\{ \subset \} = \sim E^1 \quad (T_{A259})$$

Proof of (T_{A259})

$\{ \subset \} * E = \sim E^1$	(T_{S455})
$S(\{ \subset \} \cdot \forall QE) = \sim E^1$	($AR27$)(Function Axiom)
$S(\{ \subset \} \cdot E) = \sim E^1$	($TR116$)(Sub-relationships with the Universal Set)
$S\{ \subset \} = \sim E^1$	($AMX11$)(Universal Set Conjunction Axiom)

$$Q\{ \supset \} = \sim E^1 \quad (T_{A260})$$

Proof of (T_{A260})

$S\{ \subset \} = \sim E^1$	(T_{A259})(Eqn1)
$\{ \subset \} = \forall \{ \supset \}$	($DA143$)(Symmetrical Opposite of an Asymmetrical Relationship Set)(Eqn2)
$S^V \{ \supset \} = \sim E^1$	($AMX19$)(Substitution of Eqn2 into Eqn1)
$Q\{ \supset \} = \sim E^1$	($TR016$)(Reversibility of Matching Entities)

$$\{ \supset \} \diamond E = E^1 \quad (T_{A643})$$

Proof of (T_{A643})

$(a \neq \emptyset) \subseteq (\{ \supset \} * a^1 = \{ \supset \} \diamond a)$	(T_{S402})(Eqn1)
Let $a = E$	(Eqn2)
$a^1 = E^1$	(T_{M479})(Eqn3)
$(E \neq \emptyset) \subseteq (\{ \supset \} * E^1 = \{ \supset \} \diamond E)$	($AMX19$)(Substitution of Eqn2&3 into Eqn1)
$E \subseteq (\{ \supset \} * E^1 = \{ \supset \} \diamond E)$	(T_{M344})
$E = (\{ \supset \} * E^1 = \{ \supset \} \diamond E)$	(T_{M275})(Supersets of the Universal Set)
$\{ \supset \} * E^1 = \{ \supset \} \diamond E$	($AMX16$)(Order Axiom)
$(\{ \supset \} + \{ = \}) * E^1 = \{ \supset \} \diamond E$	($DA141$)(Disjunction with $\{ = \}$)
$\{ \supset \} * E^1 + \{ = \} * E^1 = \{ \supset \} \diamond E$	($TR002c$)(Distribution of $+$ over $*$)
$\emptyset + \{ = \} * E^1 = \{ \supset \} \diamond E$	(T_{T875})
$\emptyset + E^1 = \{ \supset \} \diamond E$	($DA13$)(Definition of the Reflexive Function)
$E^1 = \{ \supset \} \diamond E$	(T_{M668})(Null Set Disjunction)

$$\{ \supset \} * E = \sim E^1 \quad (T_{A272})$$

Proof of (T_{A272})

$\{ \supset \} \diamond E = E^1$	(T_{A643})

$$\begin{aligned} \{\supset\} * E &= E^1 & (T_{A200}) \\ \{\supset\} * E &= \sim E^1 & (T_{M008}) \text{(Inverse of an Equality)} \end{aligned}$$

$$\{\supset\} \diamond E = \{\subset\} * E \quad (T_{S199})$$

Proof of (T_{S199})

$$\begin{aligned} \{\supset\} \diamond E &= E^1 & (T_{A643}) \text{(Eqn1)} \\ \{\subset\} * E &= E^1 & (T_{S196}) \text{(Eqn2)} \\ \{\supset\} \diamond E &= \{\subset\} * E & (A_{MX19}) \text{(Substitution of Eqn2 into Eqn1)} \end{aligned}$$

$$\{\subset\} * E^1 = E \quad (T_{A644})$$

Proof of (T_{A644})

$$\begin{aligned} E^1 &= \{\supset\} \diamond E & (T_{A643}) \\ \forall \{\supset\} \diamond E^1 &= E & (T_{S028}) \text{(Active/Passive Theorem)} \\ \{\subset\} \diamond E^1 &= E & (T_{T741}) \text{(Definition of } \forall \{\supset\}) \\ \{\subset\} * E^1 &= E & (T_{R232}) \text{(Interchangeability of } * \text{ and } \diamond \text{ for Entities)} \end{aligned}$$

$$\{\subset\} \diamond E^1 = E \quad (T_{A218})$$

Proof of (T_{A218})

$$\begin{aligned} \{\subset\} \diamond E^1 &= \{\subset\} \diamond E^1 & (D_{M00}) \text{(Definition of } =) \\ \{\subset\} \diamond E^1 &= \{\subset\} * E^1 & (T_{R232}) \text{(Interchangeability of } \diamond \text{ and } * \text{ for entities)} \\ \{\subset\} \diamond E^1 &= E & (T_{A644}) \end{aligned}$$

$$\{\subset\} * E^1 = \sim E^1 \quad (T_{A231})$$

Proof of (T_{A231})

$$\begin{aligned} \{\subset\} * E^1 &= E & (T_{A644}) \\ (\{\subset\} + \{=\}) * E^1 &= E & (D_{A141}) \text{(Disjunction with } \{=\}) \\ \{\subset\} * E^1 + \{=\} * E^1 &= E & (T_{R002c}) \text{(Distribution of } + \text{ over } *) \\ (\{\subset\} * E^1 + \{=\} * E^1) \cdot \{=\} * E^1 &= E \cdot \{=\} * E^1 & (T_{M233}) \text{(Deduction)} \\ \{\subset\} * E^1 \cdot \{=\} * E^1 + \{=\} * E^1 \cdot \{=\} * E^1 &= E \cdot \{=\} * E^1 & (T_{M407}) \text{(Distribution of } \cdot \text{ over } +) \\ \{\subset\} * E^1 \cdot \{=\} * E^1 + \emptyset &= E \cdot \{=\} * E^1 & (T_{M627}) \text{(Complement Conjunction)} \\ \{\subset\} * E^1 \cdot \{=\} * E^1 &= E \cdot \{=\} * E^1 & (T_{M668}) \text{(Null Set Disjunction)(Eqn1)} \\ \{=\} * E^1 \subseteq \{\subset\} * E^1 & & (T_{S212}) \\ \{\subset\} * E^1 \subseteq \{=\} * E^1 & & (T_{M012}) \text{(Transposition)} \\ \{\subset\} * E^1 &= \{\subset\} * E^1 \cdot \{=\} * E^1 & (A_{MX44}) \text{(} \subseteq \text{ Equality Form)(Eqn2)} \\ \{\subset\} * E^1 &= E \cdot \{=\} * E^1 & (A_{MX19}) \text{(Substitution of Eqn2 into Eqn1)} \\ \{\subset\} * E^1 &= E \cdot \sim E^1 & (D_{A144}) \text{(Definition of the Reflexive Relationship Set)} \\ \{\subset\} * E^1 &= \sim E^1 & (A_{MX11}) \text{(Universal Set Conjunction Axiom)} \end{aligned}$$

$$\{\supset\} \diamond \sim E^1 = E^1 \quad (T_{S195})$$

Proof of (T_{S195})

$$\begin{aligned} \{\subset\} * E^1 &= \sim E^1 & (T_{A231}) \\ \{\subset\} \diamond E^1 &= \sim E^1 & (T_{R232}) \text{(Interchangeability of } * \text{ and } \diamond \text{ for Entities)} \\ (\{\subset\} \diamond E^1 = \sim E^1) &= E & (A_{MX16}) \text{(Order Axiom)(Eqn1)} \\ ((b \neq \emptyset)(a \neq \emptyset) = E) + ((a = \emptyset)(b = \emptyset) = E) &\subseteq ((a \subseteq c \diamond b) = (b \subseteq \forall c \diamond a)) & (T_{R293}) \text{(Active/Passive Theorem)(Eqn2)} \\ \text{Let } a &= \sim E^1 & \text{(Eqn3)} \\ \text{Let } b &= E^1 & \text{(Eqn4)} \\ ((E^1 \neq \emptyset)(\sim E^1 \neq \emptyset) = E) + ((\sim E^1 = \emptyset)(E^1 = \emptyset) = E) &\subseteq ((\sim E^1 \subseteq \{\subset\} \diamond E^1) = (E^1 \subseteq \forall \{\subset\} \diamond \sim E^1)) & (A_{MX19}) \text{(Substitution of Eqns3\&4 into Eqn2)} \\ ((E^1 \neq \emptyset)(E^1 \neq \sim \emptyset) = E) + ((\sim E^1 = \emptyset)(E^1 = \emptyset) = E) &\subseteq ((\sim E^1 \subseteq \{\subset\} \diamond E^1) = (E^1 \subseteq \forall \{\subset\} \diamond \sim E^1)) & (T_{M008}) \text{ \& } (T_{M008}) \text{(Inverse of an Equality)} \\ ((E^1 \neq \emptyset)(E^1 \neq E) = E) + ((\sim E^1 = \emptyset)(E^1 = \emptyset) = E) &\subseteq ((\sim E^1 \subseteq \{\subset\} \diamond E^1) = (E^1 \subseteq \forall \{\subset\} \diamond \sim E^1)) \end{aligned}$$

$(E(E^1 \neq E) = E) + ((\sim E^1 = \emptyset)(E^1 = \emptyset) = E) \subseteq ((\sim E^1 \subseteq \{c\} \diamond E^1) = (E^1 \subseteq \vee \{c\} \diamond \sim E^1))$	(T_{M054})(Negative of Null Set)
$(EE = E) + ((\sim E^1 = \emptyset)(E^1 = \emptyset) = E) \subseteq ((\sim E^1 \subseteq \{c\} \diamond E^1) = (E^1 \subseteq \vee \{c\} \diamond \sim E^1))$	(T_{M001})(Universality of (T_{M773} Existence of E^1))
$(E = E) + ((\sim E^1 = \emptyset)(E^1 = \emptyset) = E) \subseteq ((\sim E^1 \subseteq \{c\} \diamond E^1) = (E^1 \subseteq \vee \{c\} \diamond \sim E^1))$	(T_{M001})(Universality of (T_{M492}))
$E + ((\sim E^1 = \emptyset)(E^1 = \emptyset) = E) \subseteq ((\sim E^1 \subseteq \{c\} \diamond E^1) = (E^1 \subseteq \vee \{c\} \diamond \sim E^1))$	(T_{M667})(Conjunction Tautology)
$E \subseteq ((\sim E^1 \subseteq \{c\} \diamond E^1) = (E^1 \subseteq \vee \{c\} \diamond \sim E^1))$	(T_{M342})
$E = ((\sim E^1 \subseteq \{c\} \diamond E^1) = (E^1 \subseteq \vee \{c\} \diamond \sim E^1))$	(T_{M055})(Universal Set Disjunction)
$(\sim E^1 \subseteq \{c\} \diamond E^1) = (E^1 \subseteq \vee \{c\} \diamond \sim E^1)$	(T_{M275})(Supersets of the Universal Set)
$(\sim E^1 \subseteq \{c\} \diamond E^1) = (E^1 \subseteq \vee \{c\} \diamond \sim E^1)$	(A_{MX16})(Order Axiom)
$(\sim E^1 \subseteq \{c\} \diamond E^1) + (\sim E^1 = \{c\} \diamond E^1) = (E^1 \subseteq \vee \{c\} \diamond \sim E^1)$	(T_{M266})(Definition of $\subseteq$)
$(\sim E^1 = \{c\} \diamond E^1) \subseteq (E^1 \subseteq \vee \{c\} \diamond \sim E^1)$	(T_{M072b})(Single Premise Syllogism)(Eqn5)
$E \subseteq (E^1 \subseteq \vee \{c\} \diamond \sim E^1)$	(A_{MX19})(Substitution of Eqn1 into Eqn5)
$E = (E^1 \subseteq \vee \{c\} \diamond \sim E^1)$	(T_{M275})(Supersets of the Universal Set)
$E^1 \subseteq \vee \{c\} \diamond \sim E^1$	(A_{MX16})(Order Axiom)
$E^1 \subseteq \{c\} \diamond \sim E^1$	(D_{A143})(Symmetrical Opposite of an Asymmetrical Relationship Set)(Eqn6)
$\{c\} \subseteq \{ \neq \}$	(T_{A104})
$\{c\} \diamond \sim E^1 \subseteq \{ \neq \} \diamond \sim E^1$	(T_{R263})(Relational Deduction)(Eqn7)
$(\sim E^1 \neq \emptyset) = (\{ \neq \} \diamond \sim E^1 = \sim \sim E^1)$	(T_{A085})(Definition of $\sim$)
$(E^1 \neq \sim \emptyset) = (\{ \neq \} \diamond \sim E^1 = \sim \sim E^1)$	(T_{M008})(Inverse of an Equality)
$(E^1 \neq E) = (\{ \neq \} \diamond \sim E^1 = \sim \sim E^1)$	(T_{M054})(Negative of Null Set)
$E = (\{ \neq \} \diamond \sim E^1 = \sim \sim E^1)$	(T_{M001})(Universality of (T_{M492}))
$\{ \neq \} \diamond \sim E^1 = \sim \sim E^1$	(A_{MX16})(Order Axiom)
$\{ \neq \} \diamond \sim E^1 = E^1$	(A_{MX13})(Double Complement Axiom)(Eqn8)
$\{c\} \diamond \sim E^1 \subseteq E^1$	(A_{MX19})(Substitution of Eqn8 into Eqn7)(Eqn9)
$E^1 = \{c\} \diamond \sim E^1$	(T_{M430})(Biconditional Introduction Eqns6&9)

$$\{\sim c\} * E^1 = E^1 \quad (T_{A256})$$

Proof of (T_{A256})

$\{\sim c\} * E^1 = \{\sim c\} * E^1$	(D_{M00})(Definition of $=$)
$\{\sim c\} * E^1 = S(\{\sim c\}. \vee QE^1)$	(A_{R27})(Function Axiom)
$\{\sim c\} * E^1 = S(\{\sim c\}. \{ \subseteq \}. \vee QE^1)$	(T_{R64})
$\{\sim c\} * E^1 = S(\{\sim c\}. (\{c\} + \{=\}). \vee QE^1)$	(D_{A141})(Disjunction with $\{=\}$)
$\{\sim c\} * E^1 = S((\{\sim c\}. \{c\} + \{\sim c\}\{=\}). \vee QE^1)$	(T_{M407})(Distribution of $.$ over $+$)
$\{\sim c\} * E^1 = S((\emptyset + \{\sim c\}\{=\}). \vee QE^1)$	(T_{M621})(Complement Conjunction)
$\{\sim c\} * E^1 = S(\{\sim c\}\{=\}. \vee QE^1)$	(T_{M668})(Null Set Disjunction)(Eqn1)
$\{c\} \subseteq \{ \neq \}$	(T_{A103})
$\{=\} \subseteq \{\sim c\}$	(T_{M012})(Transposition)
$\{=\} = \{\sim c\}. \{=\}$	(A_{MX44})($\subseteq$ Equality Form)(Eqn2)
$\{\sim c\} * E^1 = S(\{=\}. \vee QE^1)$	(A_{MX19})(Substitution of Eqn2 into Eqn1)
$\{\sim c\} * E^1 = \{=\} * E^1$	(A_{R27})(Function Axiom)
$\{\sim c\} * E^1 = E^1$	(T_{R74})

$$(a \neq \emptyset) \subseteq (\{ \supset \} * a^1 \subseteq \{ \supset \} * a) \quad (T_{A533})$$

Proof of (T_{A533})

$(a \neq \emptyset) \subseteq (\{ \supset \} * a^1 = \{ \supset \} \diamond a)$	(T_{S402})(Eqn1)
$\{ \supset \} \diamond a \subseteq \{ \supset \} * a$	(T_{R241})(Eqn2)
$(a \neq \emptyset) \subseteq (\{ \supset \} * a^1 \subseteq \{ \supset \} * a)$	(A_{MX19})(Substitution of Eqn1 into Eqn2)

Further theorems of binary operators are proven in a later chapter. They are proven there because they rely on a theorem of aspect symmetry, (T_{R79}), for their proof.

Chapter 7 Proofs of Relationship Theorems – Part 4 – Binary Operator Sets

7.1 Proofs for the $\{+\}$, $\{.\}$, $\{*\}$ and $\{\diamond\}$ Functions

A. Proofs using the Axioms and Definitions of Relationship Theory Only

$$\{+\}^1 \subseteq R_c \quad (T_{S568}) \text{ Commutative } +$$

$$\{.\}^1 \subseteq R_c \quad (T_{S569}) \text{ Commutative } .$$

$$\{+\}^1 \subseteq R_v \quad (T_{S570}) \text{ Associative } +$$

$$\{.\}^1 \subseteq R_v \quad (T_{S571}) \text{ Associative } .$$

Example Proof of (T_{S568})

$$\begin{aligned} (o = \{.\}, \{+\}) &\subseteq (o^1 \subseteq R_c R_v) & (A_{R12}) (\text{Commutative \& Associative } . \& +) \\ \{+\}^1 &\subseteq R_c & \text{Extraction} \end{aligned}$$

$$\forall \{*\} = \{*\} \quad (T_{T724})$$

$$\forall \{\diamond\} = \{\diamond\} \quad (T_{T725})$$

Example Proof of (T_{T724})

$$\begin{aligned} (o = \{*\}, \{*\}, \{\diamond\}, \{\diamond\}) &\subseteq (o^1 \subseteq \sim R_c R_s) & (A_{S61}) (\text{Function Operator Axiom}) \\ \{*\}^1 &\subseteq R_s & \text{Extraction} \\ \forall \{*\} = \{*\} & & (D_{A94}) (\text{Definition of a Symmetrical Relationship Set}) \end{aligned}$$

B. Proofs using The Axioms and Definitions of Membership and Relationship Theory

$$(a * b)^1 = (\{*\} * b^1) * a^1 \quad (T_{R592}) \text{ Definition of } \{*\}$$

$$(a \diamond b)^1 = (\{\diamond\} * b^1) * a^1 \quad (T_{R593}) \text{ Definition of } \{\diamond\}$$

Example Proof of (T_{R592})

$$\begin{aligned} (a \langle o \rangle b)^1 &= (o * b^1) * a^1 & (D_{A11}) (\text{Definition of a Binary Operator}) (\text{Eqn1}) \\ \text{Let } o &= \{*\} & (\text{Eqn2}) \\ (a \langle \{*\} \rangle b)^1 &= (\{*\} * b^1) * a^1 & (A_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\ (a * b)^1 &= (\{*\} * b^1) * a^1 & (D_{A229}) (\text{Definition of } \langle \rangle) \end{aligned}$$

C. Proofs using Membership Theory and the Axioms and Definitions of Relationship Theory

$$(o^1 \subseteq R_c) \subseteq ((a \langle o \rangle b)^1 = (o * a^1) * b^1) \quad (T_{T490})$$

Proof of (T_{R490})

$$\begin{aligned} (a \langle o \rangle b)^1 &= (o * b^1) * a^1 & (D_{A11}) (\text{Definition of a Binary Operator}) (\text{Eqn1}) \\ (o^1 \subseteq R_c) &= (a \langle o \rangle b = b \langle o \rangle a) & (D_{R40}) (\text{Definition of a Commutative Operator}) \\ (o^1 \subseteq R_c) &= ((a \langle o \rangle b)^1 = (b \langle o \rangle a)^1) & (T_{M479}) (\text{Eqn2}) \\ (o^1 \subseteq R_c) &= ((o * b^1) * a^1 = (b \langle o \rangle a)^1) & (A_{MX19}) (\text{Substitution of Eqn1 into Eqn2}) \\ (o^1 \subseteq R_c) &= ((o * a^1) * b^1 = (a \langle o \rangle b)^1) & \text{Resymbolising} \end{aligned}$$

D. Higher Tier Proofs

$$(a + b)^1 = (\{+\} * a^1) * b^1 \quad (T_{R590}) \text{ Definition of } \{+\}$$

$$(a.b)^1 = (\{.\} * a^1) * b^1 \quad (T_{R591}) \text{ Definition of } \{.\}$$

Example Proof of (T_{R591})

$$\begin{aligned} (o^1 \subseteq R_c) &\subseteq ((a \langle o \rangle b)^1 = (o * a^1) * b^1) & (T_{490})(\text{Eqn1}) \\ \text{Let } o &= \{.\} & (\text{Eqn2}) \\ o^1 &= \{.\}^1 & (T_{M479})(\text{Eqn3}) \\ (\{.\}^1 \subseteq R_c) &\subseteq ((a \langle \{.\} \rangle b)^1 = (\{.\} * a^1) * b^1) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\ (\{.\}^1 \subseteq R_c) &\subseteq ((a.b)^1 = (\{.\} * a^1) * b^1) & (D_{A229})(\text{Definition of } \langle \rangle) \\ E &\subseteq ((a.b)^1 = (\{.\} * a^1) * b^1) & (T_{M001b})(\text{Universality of } (T_{S569}) \text{ Commutative.}) \\ E &= ((a.b)^1 = (\{.\} * a^1) * b^1) & (T_{M275})(\text{Supersets of the Universal Set}) \\ (a.b)^1 &= (\{.\} * a^1) * b^1 & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$\{\subseteq\} = \{.\} * E \quad (T_{A177})$$

Proof of (T_{A177})

$$\begin{aligned} \text{Let } E &= [b_r + + +]_{r=1}^E \text{ where for } r = 1 \text{ to } E, b_r \neq \emptyset & (\text{Eqn1}) \\ \text{and } E &= [b_s + + +]_{s=1}^E \text{ where for } s = 1 \text{ to } E, b_s \neq \emptyset & (\text{Eqn2}) \\ \text{and } E &= [b_t + + +]_{t=1}^E \text{ where for } t = 1 \text{ to } E, b_t \neq \emptyset & (\text{Eqn3}) \\ E &= [b_t^1 + + +]_{t=1}^E & (T_{M228})(\text{Eqn4}) \\ \{\subseteq\} &= [[(\vee S b_s^1 \vee Q b_r^1)(b_s \subseteq b_r) + + +]_{r=1}^E + + +]_{s=1}^E & (D_{A02})(o = \{\subseteq\}) \\ \{\subseteq\} &= [[[(\vee S b_s^1 \vee Q b_r^1)(b_s = b_r.b_t) + + +]_{t=1}^E + + +]_{r=1}^E + + +]_{s=1}^E & (T_{M227})(\text{Eqn5}) \\ \text{Let } b_s &= b_r.b_t & \\ b_s^1 &= (b_t.b_r)^1 & (T_{M479})(\text{Eqn6}) \\ (b_t.b_r)^1 &= (\{.\} * b_t^1) * b_r^1 & (T_{R591})(\text{Definition of } \{.\})(\text{Eqn7}) \\ b_s^1 &= (\{.\} * b_t^1) * b_r^1 & (A_{MX19})(\text{Substitution of Eqn7 into Eqn6})(\text{Eqn8}) \\ \text{Thus } (b_s = b_r.b_t) &= (b_s^1 = (\{.\} * b_t^1) * b_r^1) & (A_{MX02})(\text{Conditional Introduction})(\text{Eqn9}) \\ \{\subseteq\} &= [[[(\vee S b_s^1 \vee Q b_r^1)(b_s^1 = (\{.\} * b_t^1) * b_r^1) + + +]_{t=1}^E + + +]_{r=1}^E + + +]_{s=1}^E & \\ & & (A_{MX19})(\text{Substitution of Eqn9 into Eqn5}) \\ \{\subseteq\} &= [[(\vee S b_s^1 \vee Q b_r^1).[(b_s^1 = (\{.\} * b_t^1) * b_r^1) + + +]_{t=1}^E + + +]_{r=1}^E + + +]_{s=1}^E & \\ & & (T_{M407})(\text{Distribution of } . \text{ over } +) \\ \{\subseteq\} &= [[(\vee S b_s^1 \vee Q b_r^1).[(b_s^1 + + +]_{t=1}^E = [(\{.\} * b_t^1) * b_r^1 + + +]_{t=1}^E) + + +]_{r=1}^E + + +]_{s=1}^E & \\ & & (T_{M230})(\text{Combination for } t = 1 \text{ to } E) \\ \{\subseteq\} &= [[(\vee S b_s^1 \vee Q b_r^1) (b_s^1 = [(\{.\} * b_t^1) * b_r^1 + + +]_{t=1}^E) + + +]_{r=1}^E + + +]_{s=1}^E & \\ & & (A_{MX01})(\text{Disjunction Tautology}) \\ \{\subseteq\} &= [[(\vee S b_s^1 \vee Q b_r^1) (b_s^1 = [(\{.\} * b_t^1) + + +]_{t=1}^E * b_r^1) + + +]_{r=1}^E + + +]_{s=1}^E & \\ & & (T_{R002c})(\text{Distribution of } + \text{ over } *) \\ \{\subseteq\} &= [[(\vee S b_s^1 \vee Q b_r^1) (b_s^1 = (\{.\} * [b_t^1 + + +]_{t=1}^E) * b_r^1) + + +]_{r=1}^E + + +]_{s=1}^E & \\ & & (T_{R002b})(\text{Distribution of } * \text{ over } +)(\text{Eqn10}) \\ \{\subseteq\} &= [[(\vee S b_s^1 \vee Q b_r^1) (b_s^1 = (\{.\} * E) * b_r^1) + + +]_{r=1}^E + + +]_{s=1}^E & \\ & & (A_{MX19})(\text{Substitution of Eqn4 into Eqn10}) \\ \{\subseteq\} &= [[(\vee S b_s^1 \vee Q b_r^1) (b_s \langle \{.\} * E \rangle b_r) + + +]_{r=1}^E + + +]_{s=1}^E & \\ & & (T_{R653})(\text{Definition of a Relationship})(\text{Eqn11}) \\ (\{.\} * E) &= [[(\vee S b_s^1 \vee Q b_r^1) (b_s \langle \{.\} * E \rangle b_r) + + +]_{r=1}^E + + +]_{s=1}^E & \\ & & (D_{A02})(o = \{.\} * E)(\text{Eqn12}) \\ \{\subseteq\} &= \{.\} * E & (A_{MX19})(\text{Substitution of Eqn11 into Eqn12}) \end{aligned}$$

$$\{\supseteq\} = \{+\} * E \quad (T_{A178})$$

Proof of (T_{A178})

$$\begin{aligned} \text{Let } E &= [b_r + + +]_{r=1}^E & (\text{Eqn1}) \text{ where for } r = 1 \text{ to } E, b_r \neq \emptyset \\ \text{and } E &= [b_s + + +]_{s=1}^E & (\text{Eqn2}) \text{ where for } s = 1 \text{ to } E, b_s \neq \emptyset \\ \text{and } E &= [b_t + + +]_{t=1}^E & (\text{Eqn3}) \text{ where for } t = 1 \text{ to } E, b_t \neq \emptyset \\ E &= [b_t^1 + + +]_{t=1}^E & (T_{M228})(\text{Eqn4}) \\ \{\supseteq\} &= [[(\vee S b_s^1 \vee Q b_r^1)(b_s \subseteq b_r) + + +]_{r=1}^E + + +]_{s=1}^E & (D_{A02})(o = \{\supseteq\}) \\ \{\supseteq\} &= [[[(\vee S b_s^1 \vee Q b_r^1)(b_s = b_r.b_t) + + +]_{t=1}^E + + +]_{r=1}^E + + +]_{s=1}^E & (T_{M226})(\text{Eqn5}) \\ \text{Let } b_s &= b_r.b_t & \\ b_s^1 &= (b_t.b_r)^1 & (T_{M479})(\text{Eqn6}) \\ (b_t.b_r)^1 &= (\{+\} * b_t^1) * b_r^1 & (T_{R590})(\text{Definition of } \{+\})(\text{Eqn7}) \end{aligned}$$

$$\begin{aligned}
b_s^1 &= (\{+\} * b_t^1) * b_r^1 & (A_{MX19}) & \text{(Substitution of Eqn7 into Eqn6) (Eqn8)} \\
\text{Thus } (b_s = b_r + b_t) &= (b_s^1 \subseteq (\{+\} * b_t^1) * b_r^1) & (A_{MX02}) & \text{(Conditional Introduction)(Eqn9)} \\
\{\supseteq\} &= [[[(\forall S b_s^1 \forall Q b_r^1) (b_s^1 = (\{+\} * b_t^1) * b_r^1) \text{+++}]_{t=1}^{E_{t=1}} \text{+++}]_{r=1}^{E_{r=1}} \text{+++}]_{s=1}^{E_{s=1}} & (A_{MX19}) & \text{(Substitution of Eqn9 into Eqn5)} \\
\{\supseteq\} &= [[(\forall S b_s^1 \forall Q b_r^1) [(b_s^1 = (\{+\} * b_t^1) * b_r^1) \text{+++}]_{t=1}^{E_{t=1}} \text{+++}]_{r=1}^{E_{r=1}} \text{+++}]_{s=1}^{E_{s=1}} & (T_{M407}) & \text{(Distribution of . over +)} \\
\{\supseteq\} &= [[(\forall S b_s^1 \forall Q b_r^1) [(b_s^1 \text{+++}]_{t=1}^{E_{t=1}} = [(\{+\} * b_t^1) * b_r^1 \text{+++}]_{t=1}^{E_{t=1}}) \text{+++}]_{r=1}^{E_{r=1}} \text{+++}]_{s=1}^{E_{s=1}} & (A_{MX21}) & \text{(Combination for } t = 1 \text{ to } E) \\
\{\supseteq\} &= [[(\forall S b_s^1 \forall Q b_r^1) (b_s^1 = [(\{+\} * b_t^1) * b_r^1 \text{+++}]_{t=1}^{E_{t=1}}) \text{+++}]_{r=1}^{E_{r=1}} \text{+++}]_{s=1}^{E_{s=1}} & (A_{MX01}) & \text{(Disjunction Tautology)} \\
\{\supseteq\} &= [[(\forall S b_s^1 \forall Q b_r^1) (b_s^1 = [(\{+\} * b_t^1) \text{+++}]_{t=1}^{E_{t=1}} * b_r^1) \text{+++}]_{r=1}^{E_{r=1}} \text{+++}]_{s=1}^{E_{s=1}} & (T_{R002c}) & \text{(Distribution of + over *)} \\
\{\supseteq\} &= [[(\forall S b_s^1 \forall Q b_r^1) (b_s^1 = (\{+\} * [b_t^1 \text{+++}]_{t=1}^{E_{t=1}}) * b_r^1) \text{+++}]_{r=1}^{E_{r=1}} \text{+++}]_{s=1}^{E_{s=1}} & (T_{R002b}) & \text{(Distribution of * over +)(Eqn10)} \\
\{\supseteq\} &= [[(\forall S b_s^1 \forall Q b_r^1) (b_s^1 = (\{+\} * E) * b_r^1) \text{+++}]_{r=1}^{E_{r=1}} \text{+++}]_{s=1}^{E_{s=1}} & (A_{MX19}) & \text{(Substitution of Eqn4 into Eqn10)} \\
\{\supseteq\} &= [[(\forall S b_s^1 \forall Q b_r^1) (b_s \langle \{+\} * E \rangle b_r) \text{+++}]_{r=1}^{E_{r=1}} \text{+++}]_{s=1}^{E_{s=1}} & (T_{R653}) & \text{(Definition of a Relationship)(Eqn11)} \\
(\{+\} * E) &= [[(\forall S b_s^1 \forall Q b_r^1) (b_s \langle \{+\} * E \rangle b_r) \text{+++}]_{r=1}^{E_{r=1}} \text{+++}]_{s=1}^{E_{s=1}} & (D_{A02}) & \text{(} o = \{+\} * E \text{) (Eqn12)} \\
\{\supseteq\} &= \{+\} * E & (A_{MX19}) & \text{(Substitution of Eqn11 into Eqn12)}
\end{aligned}$$

$$\{.\} * E = \forall (\{+\} * E) \quad (T_{A179})$$

Proof of (T_{A179})

$$\begin{aligned}
\{\subseteq\} &= \{.\} * E & (T_{A177}) & \text{(Eqn1)} \\
\{\supseteq\} &= \{+\} * E & (T_{A178}) & \text{(Eqn2)} \\
\forall \{\subseteq\} &= \{\supseteq\} & (T_{T741}) & \text{(Definition of } \forall \{\subseteq\} \text{)(Eqn3)} \\
\forall (\{.\} * E) &= (\{+\} * E) & (A_{MX19}) & \text{(Substitution of Eqns1&2 into Eqn3)} \\
\forall \forall (\{.\} * E) &= \forall (\{+\} * E) & (T_{R022}) & \text{(Relationship Opposite of a Proposition)} \\
\{.\} * E &= \forall (\{+\} * E) & (T_{R012k}) & \text{(Sub-Relationship Closure Theorem)}
\end{aligned}$$

$$\{\supseteq\} = S\{+\} \quad (T_{A180})$$

Proof of (T_{A180})

$$\begin{aligned}
\{\supseteq\} &= \{+\} * E & (T_{A178}) & \\
\{\supseteq\} &= S(\{+\} \cdot \forall QE) & (A_{R27}) & \text{(Function Axiom)} \\
\{\supseteq\} &= S(\{+\} \cdot E) & (T_{R116}) & \text{(Sub-relationships with Universal Set)} \\
\{\supseteq\} &= S\{+\} & (A_{MX11}) & \text{(Universal Set Conjunction Axiom)}
\end{aligned}$$

$$\{\subseteq\} = S\{.\} \quad (T_{A181})$$

Proof of (T_{A181})

$$\begin{aligned}
\{\subseteq\} &= \{.\} * E & (T_{A177}) & \\
\{\subseteq\} &= S(\{.\} \cdot \forall QE) & (A_{R27}) & \text{(Function Axiom)} \\
\{\subseteq\} &= S(\{.\} \cdot E) & (T_{R116}) & \text{(Sub-relationships with Universal Set)} \\
\{\subseteq\} &= S\{.\} & (A_{MX11}) & \text{(Universal Set Conjunction Axiom)}
\end{aligned}$$

$$\{.\} \subseteq \forall S\{\subseteq\} \quad (T_{T826})$$

Proof of (T_{T826})

$$\begin{aligned}
\{\subseteq\} &= S\{.\} & (T_{A181}) & \\
\forall S\{\subseteq\} &= \forall SS\{.\} & (T_{R023d}) & \text{(Sub-Relationship Deduction) (Eqn1)} \\
\{.\} \subseteq \forall SS\{.\} & & (T_{R012o}) & \text{(Sub-Relationship Non-Closure Theorem)(Eqn2)} \\
\{.\} \subseteq \forall S\{\subseteq\} & & (A_{MX19}) & \text{(Substitution of Eqn1 into Eqn2)}
\end{aligned}$$

$$\{+\} \subseteq \forall S\{\supseteq\} \quad (T_{T827})$$

Proof of (T₈₂₇)

$$\begin{aligned} \{\supseteq\} &= \mathbf{S}\{+\} & (T_{A180}) \\ {}^v\mathbf{S}\{\supseteq\} &= {}^v\mathbf{SS}\{+\} & (T_{R023d})(\text{Sub-Relationship Deduction}) (\text{Eqn1}) \\ \{+\} &\subseteq {}^v\mathbf{SS}\{.\} & (T_{R012o})(\text{Sub-Relationship Non-Closure Theorem})(\text{Eqn2}) \\ \{+\} &\subseteq {}^v\mathbf{S}\{\supseteq\} & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \end{aligned}$$

$${}^v\mathbf{S}\emptyset^1 \subseteq \mathbf{S}\{.\} \quad (T_{865})$$

$${}^v\mathbf{SE}^1 \subseteq \mathbf{S}\{+\} \quad (T_{866})$$

Example Proof of (T₈₆₅)

$$\begin{aligned} {}^v\mathbf{S}\emptyset^1 &\subseteq \{\subseteq\} & (T_{857}) \\ {}^v\mathbf{S}\emptyset^1 &\subseteq \mathbf{S}\{.\} & (T_{A181}) \end{aligned}$$

$$\emptyset^1 \subseteq \mathbf{SS}\{.\} \quad (T_{867})$$

$$\mathbf{E}^1 \subseteq \mathbf{SS}\{+\} \quad (T_{868})$$

Example Proof of (T₈₆₇)

$$\begin{aligned} {}^v\mathbf{S}\emptyset^1 &\subseteq \mathbf{S}\{.\} & (T_{865}) \\ \mathbf{S}^v\mathbf{S}\emptyset^1 &\subseteq \mathbf{SS}\{.\} & (T_{R026d}) \\ \emptyset^1 &\subseteq \mathbf{SS}\{.\} & (T_{R012h})(\text{Sub-relationship Closure Theorem}) \end{aligned}$$

$${}^v(\mathbf{S}\{+\}) = \mathbf{S}\{.\} \quad (T_{A182})$$

Proof of (T_{A182})

$$\begin{aligned} \{\supseteq\} &= \mathbf{S}\{+\} & (T_{A180})(\text{Eqn1}) \\ \{\subseteq\} &= \mathbf{S}\{.\} & (T_{A181})(\text{Eqn2}) \\ {}^v\{\subseteq\} &= \{\supseteq\} & (T_{7741})(\text{Definition of } {}^v\{\subseteq\})(\text{Eqn3}) \\ {}^v(\mathbf{S}\{.\}) &= \mathbf{S}\{+\} & (A_{MX19})(\text{Substitution of Eqns1\&2 into Eqn3}) \\ {}^{vv}(\mathbf{S}\{.\}) &= {}^v(\mathbf{S}\{+\}) & (T_{R022})(\text{Relationship Opposite of a Proposition}) \\ \mathbf{S}\{.\} &= {}^v(\mathbf{S}\{+\}) & (T_{R012k})(\text{Sub-Relationship Closure Theorem}) \end{aligned}$$

$${}^v\{+\} = \{.\} \quad (T_{5583}) \text{ Definition of } {}^v\{+\}$$

Proof of (T₅₅₈₃)

$$\begin{aligned} {}^v\{.\} &= \{+\} & (D_{R51})(\text{Definition of } {}^v\{.\}) \\ {}^{vv}\{.\} &= {}^v\{+\} & (T_{R522})(\text{Relational Deduction}) \\ \{.\} &= {}^v\{+\} & (T_{R012k})(\text{Sub-Relationship Closure Theorem}) \end{aligned}$$

$$\{.\} * \mathbf{a} \subseteq \{\subseteq\} \quad (T_{5585})$$

$$\{+\} * \mathbf{a} \subseteq \{\supseteq\} \quad (T_{5586})$$

Example Proof of (T₅₅₈₅)

$$\begin{aligned} \{.\} * \mathbf{a} &= \mathbf{S}(\{.\} \cdot {}^v\mathbf{Qa}) & (A_{R27})(\text{Function Axiom})(\text{Eqn1}) \\ \mathbf{S}(\{.\} \cdot {}^v\mathbf{Qa}) &\subseteq \mathbf{S}\{.\} \cdot \mathbf{S}^v\mathbf{Qa} & (T_{R012d})(\text{Distribution of } \mathbf{S} * \text{ over } \cdot)(\text{Eqn2}) \\ \{.\} * \mathbf{a} &\subseteq \mathbf{S}\{.\} \cdot \mathbf{S}^v\mathbf{Qa} & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\ \{.\} * \mathbf{a} &\subseteq \{\subseteq\} \cdot \mathbf{S}^v\mathbf{Qa} & (T_{A181}) \\ (\mathbf{a} \neq \emptyset) &\subseteq (\{.\} * \mathbf{a} \subseteq \{\subseteq\} \cdot \mathbf{E}) & (T_{R060}) \\ (\mathbf{a} \neq \emptyset) &\subseteq (\{.\} * \mathbf{a} \subseteq \{\subseteq\}) & (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn3}) \\ \text{Let } \mathbf{a} &= \emptyset & (\text{Eqn4}) \\ \text{Let } \{.\} * \mathbf{a} &\not\subseteq \{\subseteq\} & (\text{Eqn5}) \\ \{.\} * \emptyset &\not\subseteq \{\subseteq\} & (A_{MX19})(\text{Substitution of Eqn4 into Eqn5}) \\ \emptyset &\not\subseteq \{\subseteq\} & (T_{S181})(\text{Null Object Theorem}) \\ (\emptyset \not\subseteq \{\subseteq\}) &= \mathbf{E} & (A_{MX16})(\text{Order Axiom}) \\ (\emptyset \subseteq \{\subseteq\}) &= \sim \mathbf{E} & (T_{M008})(\text{Inverse of an Equality}) \\ (\emptyset \subseteq \{\subseteq\}) &= \emptyset & (D_{M02})(\text{Definition of } \emptyset) \\ \mathbf{E} &= \emptyset & (T_{M001})(\text{Universality of } (T_{M670}) \text{ Null Set Theorem}) \end{aligned}$$

$\emptyset$	(T_{M347})
Thus $(a = \emptyset)(\{.\} * a \subseteq \{\subseteq\}) = \emptyset$	($AMX02$)(Conditional Introduction)
$(a = \emptyset) \subseteq (\{.\} * a \subseteq \{\subseteq\})$	(T_{M735})($\subseteq$ Existential Form)(Eqn6)
$(a = \emptyset) + (a \neq \emptyset) \subseteq (\{.\} * a \subseteq \{\subseteq\}) + (\{.\} * a \subseteq \{\subseteq\})$	($AMX21$)(Combination Axiom)
$(a = \emptyset) + (a \neq \emptyset) \subseteq (\{.\} * a \subseteq \{\subseteq\})$	($AMX01$)(Disjunction Tautology)
$E \subseteq (\{.\} * a \subseteq \{\subseteq\})$	(T_{M671})(Tertium non Datur)
$E = (\{.\} * a \subseteq \{\subseteq\})$	(T_{M275})(Supersets of the Universal Set)
$\{.\} * a \subseteq \{\subseteq\}$	($AMX16$)(Order Axiom)
The proof of (T_{S586}) is the same except that (T_{A180}) is used rather than (T_{A181}).	

$$Q\{.\} = \{\supseteq\} \quad (T_{S593})$$

$$Q\{+\} = \{\subseteq\} \quad (T_{S594})$$

Example Proof of (T_{S593})

$\forall\{.\} = \{+\}$	($DR51$)(Definition of $\forall\{.\}$)
$S^{\forall}\{.\} = S\{+\}$	(T_{R522})(Relational Deduction)
$S^{\forall}\{.\} = \{\supseteq\}$	(T_{A180})
$Q\{.\} = \{\supseteq\}$	(T_{R016})(Reversibility of Matching Entities)

The proof of (T_{S594}) is the same except that (T_{A181}) is used rather than (T_{A180}) and (T_{R015}) is used rather than (T_{R016}).

$$Q\{.\} = S\{+\} \quad (T_{S591})$$

$$Q\{+\} = S\{.\} \quad (T_{S592})$$

Example Proof of (T_{S591})

$Q\{.\} = \{\supseteq\}$	(T_{S593})(Eqn1)
$S\{+\} = \{\supseteq\}$	(T_{A180})(Eqn2)
$Q\{.\} = S\{+\}$	($AMX19$)(Substitution of Eqn2 into Eqn1)

The proof of (T_{S592}) is the same except that (T_{S594}) is used rather than (T_{S593}) and (T_{A181}) is used rather than (T_{A180}).

$$(a^1 \neq \emptyset)(b^1 \neq \emptyset) \subseteq (a^1.b^1)^1 = (\{.\} * a^1) * b^1 \quad (T_{S468})$$

Proof of (T_{S468})

$(a.b)^1 = (\{.\} * a^1) * b^1$	(T_{R591})(Definition of $\{.\}$)
Let $a = a^1$	(Eqn2)
Let $b = b^1$	(Eqn3)
$(a^1.b^1)^1 = (\{.\} * a^{11}) * b^{11}$	($AMX19$)(Substitution of Eqns2&3 into Eqn1)
$(a^1 \neq \emptyset)(b^1 \neq \emptyset) \subseteq (a^1.b^1)^1 = (\{.\} * a^1) * b^1$	($DM17$)(Definition of a^{11})

$$E = ((\{.\} * a^1) * b^1 = (a^1.b^1)^1) + ((\{.\} * a^1) * b^1 = \emptyset) \quad (T_{T881})$$

Proof of (T_{T881})

$(a^1 \neq \emptyset)(b^1 \neq \emptyset) \subseteq (a^1.b^1)^1 = (\{.\} * a^1) * b^1$	(T_{S468})(Eqn1)
Let $a^1 = \emptyset$	(Eqn2)
Let $(\{.\} * a^1) * b^1 \neq \emptyset$	(Eqn3)
$(\{.\} * \emptyset) * b^1 \neq \emptyset$	($AMX19$)(Substitution of Eqn2 into Eqn3)
$\emptyset * b^1 \neq \emptyset$	(T_{S181})(Null Object Theorem)
$\emptyset \neq \emptyset$	(T_{R072})(Null Function Theorem)
$\emptyset$	(T_{M346})
Thus $(a^1 = \emptyset)((\{.\} * a^1) * b^1 \neq \emptyset) = \emptyset$	($AMX02$)(Conditional Introduction)
$(a^1 = \emptyset) \subseteq ((\{.\} * a^1) * b^1 = \emptyset)$	(T_{M735})($\subseteq$ Existential Form)(Eqn4)
Let $b^1 = \emptyset$	(Eqn5)
Let $(\{.\} * a^1) * b^1 \neq \emptyset$	(Eqn6)
$(\{.\} * a^1) * \emptyset \neq \emptyset$	($AMX19$)(Substitution of Eqn5 into Eqn6)
$\emptyset \neq \emptyset$	(T_{S181})(Null Object Theorem)
$\emptyset$	(T_{M346})

$$\begin{aligned}
&\text{Thus } (b^1 = \emptyset)((\{.\} * a^1) * b^1 \neq \emptyset) = \emptyset & (AMX02)(\text{Conditional Introduction}) \\
&(b^1 = \emptyset) \subseteq ((\{.\} * a^1) * b^1 = \emptyset) & (TM735)(\subseteq \text{Existential Form})(Eqn7) \\
&(a^1 \neq \emptyset)(b^1 \neq \emptyset) + (a^1 = \emptyset) + (b^1 = \emptyset) \subseteq ((a^1.b^1)^1 = (\{.\} * a^1) * b^1) + ((\{.\} * a^1) * b^1 = \emptyset) + ((\{.\} * a^1) * b^1 \neq \emptyset) \\
& & (AMX21)(\text{Combination of Eqns1,4\&7}) \\
&(a^1 \neq \emptyset)(b^1 \neq \emptyset) + (a^1 = \emptyset) + (b^1 = \emptyset) \subseteq ((a^1.b^1)^1 = (\{.\} * a^1) * b^1) + ((\{.\} * a^1) * b^1 = \emptyset) \\
& & (AMX01)(\text{Disjunction Tautology}) \\
&(a^1 \neq \emptyset)(b^1 \neq \emptyset) + \sim((a^1 \neq \emptyset)(b^1 \neq \emptyset)) \subseteq ((a^1.b^1)^1 = (\{.\} * a^1) * b^1) + ((\{.\} * a^1) * b^1 = \emptyset) \\
&E \subseteq ((a^1.b^1)^1 = (\{.\} * a^1) * b^1) + ((\{.\} * a^1) * b^1 = \emptyset) & (TM026)(\text{DeMorgan's Law}) \\
&E = ((a^1.b^1)^1 = (\{.\} * a^1) * b^1) + ((\{.\} * a^1) * b^1 = \emptyset) & (TM275)(\text{Supersets of the Universal Set}) \\
&E = ((\{.\} * a^1) * b^1 = (a^1.b^1)^1) + ((\{.\} * a^1) * b^1 = \emptyset) & (TM197)(\text{Commutativity of } =)
\end{aligned}$$

$$a.b = (\{.\} * a) * b \quad (TS127)$$

Proof of (TS127)

$$\begin{aligned}
&\text{Let } a = a_1^1 +++ a_a^1 & (\text{Eqn1}) \text{ where } r = 1 \text{ to } a, a_r^1 \neq \emptyset \\
&\text{Let } b = b_1^1 +++ b_b^1 & (\text{Eqn2}) \text{ where } r = 1 \text{ to } b, b_r^1 \neq \emptyset \\
&a.b & (\text{Eqn3}) \\
&(a_1^1 +++ a_a^1).(b_1^1 +++ b_b^1) & (AMX19)(\text{Substitution of Eqns1\&2 into Eqn3}) \\
&a_1^1.(b_1^1 +++ b_b^1) +++ a_a^1.(b_1^1 +++ b_b^1) & (TM407)(\text{Distribution of } . \text{ over } +) \\
&(a_1^1.b_1^1 +++ a_1^1.b_b^1) +++ (a_a^1.b_1^1 +++ a_a^1.b_b^1) & (TM407)(\text{Distribution of } . \text{ over } +) \\
&((a_1^1.b_1^1)^1 +++ (a_1^1.b_b^1)^1) +++ ((a_a^1.b_1^1)^1 +++ (a_a^1.b_b^1)^1) & (TM555) \\
&\text{Thus } a.b = ((a_1^1.b_1^1)^1 +++ (a_1^1.b_b^1)^1) +++ ((a_a^1.b_1^1)^1 +++ (a_a^1.b_b^1)^1) & (AMX02)(\text{Conditional Introduction})(Eqn4)^* \\
& & *N.B. some of the terms on the right-hand side of this equation, i.e., where $a_x^1 \neq b_y^1$, will be null. See (TM774). \\
&(\{.\} * a) * b & (\text{Eqn5}) \\
&(\{.\} * (a_1^1 +++ a_a^1)) * (b_1^1 +++ b_b^1) & (AMX19)(\text{Substitution of Eqns1\&2 into Eqn5}) \\
&(\{.\} * a_1^1 +++ \{.\} * a_a^1) * (b_1^1 +++ b_b^1) & (AS64)(\text{Distribution of } * \text{ over } +) \\
&(\{.\} * a_1^1) * (b_1^1 +++ b_b^1) +++ (\{.\} * a_a^1) * (b_1^1 +++ b_b^1) & (TR002c)(\text{Distribution of } + \text{ over } *) \\
&((\{.\} * a_1^1) * b_1^1 +++ (\{.\} * a_1^1) * b_b^1) +++ ((\{.\} * a_a^1) * b_1^1 +++ (\{.\} * a_a^1) * b_b^1) & (AS64)(\text{Distribution of } * \text{ over } +) \\
&((a_1^1.b_1^1)^1 +++ (a_1^1.b_b^1)^1) +++ ((a_a^1.b_1^1)^1 +++ (a_a^1.b_b^1)^1) & (TS468) \\
&\text{Thus } (\{.\} * a) * b = ((a_1^1.b_1^1)^1 +++ (a_1^1.b_b^1)^1) +++ ((a_a^1.b_1^1)^1 +++ (a_a^1.b_b^1)^1) & (AMX02)(\text{Conditional Introduction})(Eqn6)** \\
& & **N.B. some of the terms on the right-hand side of this equation, i.e., where $a_x^1 \neq b_y^1$, will also be null. See (TR881). However, these are the same terms as Eqn4 and so the right hand sides of Eqns4 and 6 are identical. \\
&a.b = (\{.\} * a) * b & (AMX19)(\text{Substitution of Eqn6 into Eqn4})
\end{aligned}$$

7.2 Proofs for the @ Function

A. Proofs using The Axioms and Definitions of Membership and Relationship Theory

$$(@b^1)a^1 = \{\subseteq\}a^1.\{\subseteq\}b^1.\{\subseteq\}(\{.\} * (\{\subseteq\}a^1.\{\subseteq\}b^1)) \quad (TR983)$$

Proof of (TR983)

$$\begin{aligned}
&(@a^1)b^1 = \{\subseteq\}a^1.\{\subseteq\}b^1.\{\subseteq\}(\{.\} * (\{\subseteq\}a^1.\{\subseteq\}b^1)) & (DR05)(\text{Definition of } @) \\
&(@b^1)a^1 = \{\subseteq\}a^1.\{\subseteq\}b^1.\{\subseteq\}(\{.\} * (\{\subseteq\}a^1.\{\subseteq\}b^1)) & \text{Resymbolising}
\end{aligned}$$

$$(@b^1)a^1 = (a @ b)^1 \quad (T772)$$

Proof of (T772)

$$\begin{aligned}
&(a @ b)^1 = (o * b^1) * a^1 & (DA11)(\text{Definition of a Binary Operator})(Eqn1) \\
&\text{Let } o = @ & (\text{Eqn2}) \\
&(a @ b)^1 = (@ * b^1) * a^1 & (AMX19)(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$(a = a_1^1 +++ a_n^1) \subseteq ((@a)b^1 = (@a_1^1)b^1 +++ (@a_n^1)b^1) \quad (T694)$$

Proof of (T₆₉₄)

$$\begin{aligned}
 \text{Let } a &= a_1^1 \text{+++} a_n^1 & (\text{Eqn1}) \\
 (@a^1)b^1 &= (@a^1)b^1 & (D_{M00})(\text{Definition of } =)(\text{Eqn2}) \\
 (@a^1)b^1 \text{+++} (@a_n^1)b^1 &= (@a^1)b^1 \text{+++} (@a_n^1)b^1 & (T_{M237})(\text{Combination of Eqn2 for } r = 1 \text{ to } n) \\
 (@a^1 \text{+++} @a_n^1)b^1 &= (@a^1)b^1 \text{+++} (@a_n^1)b^1 & (A_{S65})(\text{Distribution of } + \text{ over } *) \\
 (@(a^1 \text{+++} a_n^1))b^1 &= (@a^1)b^1 \text{+++} (@a_n^1)b^1 & (A_{S64})(\text{Distribution of } * \text{ over } +)(\text{Eqn3}) \\
 (@a)b^1 &= (@a^1)b^1 \text{+++} (@a_n^1)b^1 & (A_{MX19})(\text{Substitution of Eqn1 into Eqn3})
 \end{aligned}$$

$$(b = b_1^1 \text{+++} b_n^1) \subseteq ((@a^1)b = (@a^1)b_1^1 \text{+++} (@a^1)b_n^1) \quad (T_{693})$$

Proof of (T₆₉₃)

$$\begin{aligned}
 \text{Let } b &= b_1^1 \text{+++} b_n^1 & (\text{Eqn1}) \\
 (@a^1)b_r^1 &= (@a^1)b_r^1 & (D_{M00})(\text{Definition of } =)(\text{Eqn2}) \\
 (@a^1)b_1^1 \text{+++} (@a^1)b_n^1 &= (@a^1)b_1^1 \text{+++} (@a^1)b_n^1 & (T_{M237})(\text{Combination of Eqn2 for } r = 1 \text{ to } n) \\
 (@a^1)(b_1^1 \text{+++} b_n^1) &= (@a^1)b_1^1 \text{+++} (@a^1)b_n^1 & (A_{S64})(\text{Distribution of } * \text{ over } +)(\text{Eqn3}) \\
 (@a^1)b &= (@a^1)b_1^1 \text{+++} (@a^1)b_n^1 & (A_{MX19})(\text{Substitution of Eqn1 into Eqn3})
 \end{aligned}$$

B. Proofs using Membership Theory and the Axioms and Definitions of Relationship Theory

$$(@a^1)b^1 \subseteq \{\subseteq\}a^1 \quad (T_{369})$$

$$(@a^1)b^1 \subseteq \{\subseteq\}b^1 \quad (T_{370})$$

Example Proof of (T₃₆₉)

$$\begin{aligned}
 ((@a^1)b^1 &= \{\subseteq\}a^1.\{\subseteq\}b^1.\{\subseteq\}*(\{\subseteq\}a^1.\{\subseteq\}b^1)) & (D_{R05})(\text{Definition of } @) \\
 (@a^1)b^1 &\subseteq \{\subseteq\}a^1 & (T_{M073b})(\text{Single Premise Syllogism})
 \end{aligned}$$

$$(\{\subseteq\}a^1.\{\subseteq\}b^1 = \emptyset) \subseteq ((@a^1)b^1 = \emptyset) \quad (T_{267})$$

Proof of (T₂₆₇)

$$\begin{aligned}
 (@a^1)b^1 &= \{\subseteq\}a^1.\{\subseteq\}b^1.\{\subseteq\}*(\{\subseteq\}a^1.\{\subseteq\}b^1) & (D_{R05})(\text{Definition of } @) \\
 (@a^1)b^1 &\subseteq \{\subseteq\}a^1.\{\subseteq\}b^1 & (T_{M073b})(\text{Single Premise Syllogism}) \\
 ((@a^1)b^1 &\subseteq \{\subseteq\}a^1.\{\subseteq\}b^1) = E & (A_{MX16})(\text{Order Axiom}) \\
 (\{\subseteq\}a^1.\{\subseteq\}b^1 = \emptyset) &\subseteq ((@a^1)b^1 = \emptyset) & (T_{M097})(\text{Probabilistic Rule})
 \end{aligned}$$

C. Higher Tier Proofs

$$(@a^1)b^1 = (@b^1)a^1 \quad (T_{276})$$

Proof of (T₂₇₆)

$$\begin{aligned}
 (@a^1)b^1 &= \{\subseteq\}a^1.\{\subseteq\}b^1.\{\subseteq\}*(\{\subseteq\}a^1.\{\subseteq\}b^1) & (D_{R05})(\text{Definition of } @)(\text{Eqn1}) \\
 (@b^1)a^1 &= \{\subseteq\}a^1.\{\subseteq\}b^1.\{\subseteq\}*(\{\subseteq\}a^1.\{\subseteq\}b^1) & (T_{R983})(\text{Eqn2}) \\
 (@a^1)b^1 &= (@b^1)a^1 & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})
 \end{aligned}$$

$$a^1 = (@a^1)a^1 \quad (T_{R984})$$

Proof of (T_{R984})

$$\begin{aligned}
 (@a^1)b^1 &= \{\subseteq\}a^1.\{\subseteq\}b^1.\{\subseteq\}*(\{\subseteq\}a^1.\{\subseteq\}b^1) & (D_{R05})(\text{Definition of } @)(\text{Eqn1}) \\
 \text{Let } b^1 &= a^1 & (\text{Eqn2}) \\
 (@a^1)a^1 &= \{\subseteq\}a^1.\{\subseteq\}a^1.\{\subseteq\}*(\{\subseteq\}a^1.\{\subseteq\}a^1) & (A_{MX19})(\text{Substitution of Eqns2 into Eqn1}) \\
 (@a^1)a^1 &= \{\subseteq\}a^1.\{\subseteq\}*(\{\subseteq\}a^1) & (T_{M667})(\text{Conjunction Tautology}) \\
 (@a^1)a^1 &= a^1 & (T_{271})
 \end{aligned}$$

$$((@a^1)b^1 \neq \emptyset) \subseteq ((@a^1)b^1 = ((@a^1)b^1)^1) \quad (T_{368})$$

$$((@a^1)b^1 \neq \emptyset) \subseteq ((@Sa^1)b^1 = ((@Sa^1)b^1)^1) \quad (T_{361})$$

$$((@a^1)b^1 \neq \emptyset) \subseteq ((@Qa^1)b^1 = ((@Qa^1)b^1)^1) \quad (T_{7362})$$

Example Proof of (T₇₃₆₈)

$$\begin{aligned} (@a^1)^1 &\subseteq R_0 & (A_{518})(1^{st} \text{ @ Function Axiom}) \\ ((@a^1)^1 \subseteq R_0) &= E & (A_{MX16})(\text{Order Axiom})(Eqn1) \\ ((c^1 \subseteq R_0) = E) &= ((b^1 \subseteq Qc)(b^1 \neq \emptyset) \subseteq (c * b^1 = (c * b^1)^1)) & (D_{R30})(\text{Definition of a One-to-One Relationship Set})(Eqn2) \\ \text{Let } c &= @a^1 & (Eqn3) \\ (((@a^1)^1 \subseteq R_0) = E) &= ((b^1 \subseteq Q(@a^1))(b^1 \neq \emptyset) \subseteq ((@a^1)b^1 = ((@a^1)b^1)^1)) & (A_{MX19})(\text{Substitution of Eqn3 into Eqn2})(Eqn4) \\ (E = E) &= ((b^1 \subseteq Q(@a^1))(b^1 \neq \emptyset) \subseteq ((@a^1)b^1 = ((@a^1)b^1)^1)) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn4}) \\ E &= ((b^1 \subseteq Q(@a^1))(b^1 \neq \emptyset) \subseteq ((@a^1)b^1 = ((@a^1)b^1)^1)) & (T_{M342}) \\ (b^1 \subseteq Q(@a^1))(b^1 \neq \emptyset) &\subseteq ((@a^1)b^1 = ((@a^1)b^1)^1) & (A_{MX16})(\text{Order Axiom})(Eqn5) \\ ((@a^1)b^1 \neq \emptyset) &\subseteq (b^1 \subseteq Q(@a^1)) & (T_{S410b}) \\ ((@a^1)b^1 \neq \emptyset)(b^1 \neq \emptyset) &\subseteq (b^1 \subseteq Q(@a^1))(b^1 \neq \emptyset) & (T_{M029})(\text{Deduction})(Eqn6) \\ ((@a^1)b^1 \neq \emptyset) &\subseteq (b^1 \neq \emptyset) & (T_{R171b}) \\ ((@a^1)b^1 \neq \emptyset) &= ((@a^1)b^1 \neq \emptyset)(b^1 \neq \emptyset) & (A_{MX44})(\subseteq \text{Equality Form})(Eqn7) \\ ((@a^1)b^1 \neq \emptyset) &\subseteq (b^1 \subseteq Q(@a^1))(b^1 \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqn7 into Eqn6})(Eqn8) \\ ((@a^1)b^1 \neq \emptyset) &\subseteq ((@a^1)b^1 = ((@a^1)b^1)^1) & (T_{M139})(\text{Hypothetical Syllogism Eqns8\&5}) \end{aligned}$$

$$@^1 \subseteq R_c \quad (T_{7721})$$

Proof of (T₇₇₂₁)

$$\begin{aligned} (@a^1)b^1 &= \{\subseteq\}a^1.\{\subseteq\}b^1.(\{\subseteq\} * (\{\subseteq\}a^1.\{\subseteq\}b^1)) & (D_{R05})(\text{Definition of @})(Eqn1) \\ (@b^1)a^1 &= \{\subseteq\}a^1.\{\subseteq\}b^1.(\{\subseteq\} * (\{\subseteq\}a^1.\{\subseteq\}b^1)) & (T_{R983})(Eqn2) \\ (@a^1)b^1 &= (@b^1)a^1 & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\ (b \langle @ \rangle a^1) &= (a \langle @ \rangle b)^1 & (T_{7772}) \\ (b \langle @ \rangle a) &= (a \langle @ \rangle b) & (T_{M479}) \\ @^1 &\subseteq R_c & (D_{R40})(\text{Definition of a Commutative Operator}) \end{aligned}$$

$$(@a^1)E^1 = a^1 \quad (T_{R985})$$

Proof of (T_{R985})

$$\begin{aligned} (@a^1)b^1 &= \{\subseteq\}a^1.\{\subseteq\}b^1.(\{\subseteq\} * (\{\subseteq\}a^1.\{\subseteq\}b^1)) & (D_{R05})(\text{Definition of @})(Eqn1) \\ \text{Let } b^1 &= E^1 & (Eqn2) \\ (@a^1)E^1 &= \{\subseteq\}a^1.\{\subseteq\}E^1.(\{\subseteq\} * (\{\subseteq\}a^1.\{\subseteq\}E^1)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (@a^1)E^1 &= \{\subseteq\}a^1.E.(\{\subseteq\} * (\{\subseteq\}a^1.E)) & (T_{A644}) \\ (@a^1)E^1 &= \{\subseteq\}a^1(\{\subseteq\} * \{\subseteq\}a^1) & (A_{MX17})(\text{Universal Set Conjunction Axiom}) \\ (@a^1)E^1 &= a^1 & (T_{7271}) \end{aligned}$$

$$E^1 = (@E^1)E^1 \quad (T_{R986})$$

Proof of (T_{R986})

$$\begin{aligned} a^1 &= (@a^1)E^1 & (T_{R985})(Eqn1) \\ \text{Let } a^1 &= E^1 & (Eqn2) \\ E^1 &= (@E^1)E^1 & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$\{\subseteq\}a^1 = S@a^1 \quad (T_{7294})$$

Proof of (T₇₂₉₄)

$$\begin{aligned} (@a^1) * E^1 &= a^1 & (T_{R985})(Eqn1) \\ (@a^1) * \{\subseteq\} * E^1 &= \{\subseteq\} * (@a^1) * E^1 & (A_{508})(2^{nd} \text{ @ Function Axiom}) \\ (@a^1) * E &= \{\subseteq\} (@a^1) * E^1 & (T_{A644})(Eqn2) \\ (@a^1) * E &= \{\subseteq\} a^1 & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\ S(@a^1) &= \{\subseteq\} a^1 & (T_{R210}) \end{aligned}$$

$$\{\subseteq\}a^1 = (@a^1)E \quad (T_{R988})$$

Proof of (TR988)

$$\begin{aligned} \{\subseteq\}a^1 &= S@a^1 & (T_{294}) \\ \{\subseteq\}a^1 &= (@a^1)E & (T_{210}) \end{aligned}$$

$$(@E^1)a^1 = a^1 \quad (T_{277})$$

Proof of (T277)

$$\begin{aligned} (@a^1)E^1 &= a^1 & (T_{985}) \\ (@E^1)a^1 &= a^1 & (T_{276}) \end{aligned}$$

$$(@E)a^1 = \{\subseteq\}a^1 \quad (T_{987})$$

Proof of (TR987)

$$\begin{aligned} (@a^1)E &= \{\subseteq\}a^1 & (T_{988})(Eqn1) \\ \text{Let } a_1^1 +++ a_n^1 &= E & (Eqn2) \\ (@a_1^1 +++ a_n^1) &= @E & (T_{196})(\text{Relational Deduction}) \\ @a_1^1 +++ @a_n^1 &= @E & (T_{002b})(\text{Distribution of } * \text{ over } +) \\ (@a_1^1 +++ @a_n^1)a^1 &= (@E)a^1 & (T_{197})(\text{Relational Deduction}) \\ (@a_1^1)a^1 +++ (@a_n^1)a^1 &= (@E)a^1 & (T_{002c})(\text{Distribution of } + \text{ over } *) \\ (@a^1)a_1^1 +++ (@a^1)a_n^1 &= (@E)a^1 & (T_{276}) \\ (@a^1)(a_1^1 +++ a_n^1) &= (@E)a^1 & (T_{002b})(\text{Distribution of } * \text{ over } +)(Eqn3) \\ (@a^1)E &= (@E)a^1 & (A_{MX19})(\text{Substitution of Eqn2 into Eqn3})(Eqn4) \\ (@E)a^1 &= \{\subseteq\}a^1 & (A_{MX19})(\text{Substitution of Eqn1 into Eqn4}) \end{aligned}$$

$$(a = a_1^1 +++ a_n^1)(b = b_1^1 +++ b_m^1) \subseteq ((@a)b = [(@a_r^1)b_s^1 +++]_{r=1}^n]_{s=1}^m = 1) \quad (T_{690})$$

Proof of (T690)

$$\begin{aligned} \text{Let } b &= b_1^1 +++ b_m^1 & (Eqn1) \\ (a = a_1^1 +++ a_n^1) &\subseteq ((@a)b_s^1 = (@a_1^1)b_s^1 +++ (@a_n^1)b_s^1) & (T_{694})(Eqn2) \\ (a = a_1^1 +++ a_n^1) &+++ (a = a_1^1 +++ a_n^1) \subseteq \\ ((@a)b_1^1 &= (@a_1^1)b_1^1 +++ (@a_n^1)b_1^1) +++ ((@a)b_m^1 = (@a_1^1)b_m^1 +++ (@a_n^1)b_m^1) \\ & & (A_{MX21})(\text{Combination of Eqn2 for } s = 1 \text{ to } n) \\ (a = a_1^1 +++ a_n^1) &\subseteq ((@a)b_1^1 = (@a_1^1)b_1^1 +++ (@a_n^1)b_1^1) +++ ((@a)b_m^1 = (@a_1^1)b_m^1 +++ (@a_n^1)b_m^1) \\ & & (A_{MX01})(\text{Disjunction Tautology}) \\ (a = a_1^1 +++ a_n^1) &\subseteq ((@a)b_1^1 +++ (@a)b_m^1 = ((@a_1^1)b_1^1 +++ (@a_n^1)b_1^1) +++ ((@a_1^1)b_m^1 +++ (@a_n^1)b_m^1)) \\ & & (T_{M231})(\text{Combination}) \\ (a = a_1^1 +++ a_n^1) &\subseteq ((@a)(b_1^1 +++ b_m^1) = ((@a_1^1)b_1^1 +++ (@a_n^1)b_1^1) +++ ((@a_1^1)b_m^1 +++ (@a_n^1)b_m^1)) \\ & & (A_{S64})(\text{Distribution of } * \text{ over } +)(Eqn3) \\ (a = a_1^1 +++ a_n^1) &\subseteq ((@a)b = ((@a_1^1)b_1^1 +++ (@a_n^1)b_1^1) +++ ((@a_1^1)b_m^1 +++ (@a_n^1)b_m^1)) \\ & & (A_{MX19})(\text{Substitution of Eqn1 into Eqn3}) \\ (a = a_1^1 +++ a_n^1) &\subseteq ((@a)b = [(@a_r^1)b_s^1 +++]_{r=1}^n]_{s=1}^m = 1) & (D_{M09})(\text{Simplified Disjunction of a Series}) \\ \text{Thus } (a = a_1^1 +++ a_n^1)(b = b_1^1 +++ b_m^1) &\subseteq ((@a)b = [(@a_r^1)b_s^1 +++]_{r=1}^n]_{s=1}^m = 1) \\ & & (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(@a)E^1 = a \quad (T_{689})$$

Proof of (T689)

$$\begin{aligned} \text{Let } a &= a_1^1 +++ a_n^1 & (Eqn1) \\ (@a_r^1)E^1 &= a_r^1 & (T_{985})(Eqn2) \\ (@a_1^1)E^1 &+++ (@a_n^1)E^1 = a_1^1 +++ a_n^1 & (T_{M231})(\text{Combination of Eqn2 for } r = 1 \text{ to } n) \\ ((@a_1^1) &+++ (@a_n^1))E^1 = a_1^1 +++ a_n^1 & (A_{S65})(\text{Distribution of } + \text{ over } *) \\ (@(a_1^1 &+++ a_n^1))E^1 = a_1^1 +++ a_n^1 & (A_{S64})(\text{Distribution of } * \text{ over } +)(Eqn3) \\ (@a)E^1 &= a & (A_{MX19})(\text{Substitution of Eqn1 into Eqn3}) \end{aligned}$$

$$(@a)\{\subseteq\}b^1 = \{\subseteq\}(@a)b^1 \quad (T_{676})$$

Proof of (T676)

$$\text{Let } a = a_1^1 +++ a_n^1 \quad (Eqn1)$$

$$\begin{aligned}
& (@a_r^1)\{\subseteq\}b^1 = \{\subseteq\}(@a_r^1)b^1 & (A_{S08})(2^{nd} \text{ @ Function Axiom})(Eqn2) \\
& (@a_1^1)\{\subseteq\}b^1 +++ (@a_n^1)\{\subseteq\}b^1 = \{\subseteq\}(@a_1^1)b^1 +++ \{\subseteq\}(@a_n^1)b^1 & (T_{M231})(\text{Combination of Eqn2 for } r = 1 \text{ to } n) \\
& (@a_1^1)\{\subseteq\}b^1 +++ (@a_n^1)\{\subseteq\}b^1 = \{\subseteq\}((@a_1^1)b^1 +++ (@a_n^1)b^1) & (A_{S64})(\text{Distribution of } * \text{ over } +) \\
& ((@a_1^1) +++ (@a_n^1))\{\subseteq\}b^1 = \{\subseteq\}((@a_1^1) +++ (@a_n^1))b^1 & (A_{S65})(\text{Distribution of } + \text{ over } *) \\
& (@(a_1^1 +++ a_n^1))\{\subseteq\}b^1 = \{\subseteq\}(@(a_1^1 +++ a_n^1))b^1 & (A_{S64})(\text{Distribution of } * \text{ over } +)(Eqn3) \\
& (@a)\{\subseteq\}b^1 = \{\subseteq\}(@a)b^1 & (A_{MX19})(\text{Substitution of Eqn1 into Eqn3})
\end{aligned}$$

$$(@a)\{\subseteq\}b = \{\subseteq\}(@a)b \quad (T_{T636})$$

Proof of (T_{T636})

$$\begin{aligned}
& \text{Let } b = b_1^1 +++ b_n^1 & (Eqn1) \\
& (@a)\{\subseteq\}b_r^1 = \{\subseteq\}(@a)b_r^1 & (T_{T676})(Eqn2) \\
& (@a)\{\subseteq\}b_1^1 +++ (@a)\{\subseteq\}b_n^1 = \{\subseteq\}(@a)b_1^1 +++ \{\subseteq\}(@a)b_n^1 & (T_{M231})(\text{Combination of Eqn2 for } r = 1 \text{ to } n) \\
& (@a)\{\subseteq\}(b_1^1 +++ b_n^1) = \{\subseteq\}(@a)(b_1^1 +++ b_n^1) & (A_{S64})(\text{Distribution of } * \text{ over } +) \\
& (@a)\{\subseteq\}b = \{\subseteq\}(@a)b & (A_{MX19})(\text{Substitution of Eqn1 into Eqn3})
\end{aligned}$$

$$(@a)E = \{\subseteq\}a \quad (T_{T688})$$

Proof of (T_{T688})

$$\begin{aligned}
& (@a)E^1 = a & (T_{T689}) \\
& \{\subseteq\}(@a)E^1 = \{\subseteq\}a & (T_{R196})(\text{Relational Deduction}) \\
& (@a)\{\subseteq\}E^1 = \{\subseteq\}a & (T_{T676}) \\
& (@a)E = \{\subseteq\}a & (T_{A644})
\end{aligned}$$

$$(@a)b = (@a)(@b)E^1 \quad (T_{T643})$$

Proof of (T_{T643})

$$\begin{aligned}
& (@a)b = (@a)b & (D_{M00})(\text{Definition of } =)(Eqn1) \\
& (@b)E^1 = b & (T_{T689})(Eqn2) \\
& (@a)b = (@a)(@b)E^1 & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$(@a)b = (@b)a \quad (T_{T642})$$

Proof of (T_{T642})

$$\begin{aligned}
& \text{Let } a = a_1^1 +++ a_n^1 \\
& \text{Let } b = b_1^1 +++ b_m^1 \\
& (@a_r^1)b_s^1 = (@b_s^1)a_r^1 & (T_{T276})(Eqn1) \\
& [((@a_r^1)b_s^1 +++)_{r=1}^n]_{s=1}^m = [((@b_s^1)a_r^1 +++)_{s=1}^m]_{r=1}^n & (A_{MX21})(\text{Combination of Eqn1 for } s = 1 \text{ to } n \text{ and } r = 1 \text{ to } m) \\
& (@a)b = (@b)a & (T_{T690})
\end{aligned}$$

$$(@a)(@b)E^1 = (@b)(@a)E^1 \quad (T_{T638})$$

$$(@a)(@b)E = (@b)(@a)E \quad (T_{T640})$$

Proof of (T_{T638}) & (T_{T640})

$$\begin{aligned}
& (@a)b = (@b)a & (T_{T642})(Eqn1) \\
& (@a)(@b)E^1 = (@b)(@a)E^1 & (T_{T643}) \\
& \{\subseteq\}(@a)(@b)E^1 = \{\subseteq\}(@b)(@a)E^1 & (T_{R522})(\text{Relational Deduction Theorem}) \\
& (@a)\{\subseteq\}(@b)E^1 = (@b)\{\subseteq\}(@a)E^1 & (T_{T636}) \\
& (@a)(@b)\{\subseteq\}E^1 = (@b)(@a)\{\subseteq\}E^1 & (T_{T676}) \\
& (@a)(@b)E = (@b)(@a)E & (T_{A644})
\end{aligned}$$

$$\{\subseteq\} * (@a)(@b)E^1 = (@a)(@b)E \quad (T_{T587})$$

Proof of (T_{T587})

$$(@a)b = (@b)a \quad (T_{T642})(Eqn1)$$

$$\begin{aligned}
(@a)(@b)E^1 &= (@b)(@a)E^1 & (T_{643}) \\
\{\subseteq\}(@a)(@b)E^1 &= \{\subseteq\}(@b)(@a)E^1 & (T_{522})(\text{Relational Deduction Theorem}) \\
\{\subseteq\}(@a)(@b)E^1 &= (@b)\{\subseteq\}(@a)E^1 & (T_{636}) \\
\{\subseteq\}(@a)(@b)E^1 &= (@b)(@a)\{\subseteq\}E^1 & (T_{676}) \\
\{\subseteq\}(@a)(@b)E^1 &= (@b)(@a)E & (T_{644})
\end{aligned}$$

$$\{\subseteq\} * (@a)(@b)E^1 = \{\subseteq\}a.\{\subseteq\}b \quad (T_{595})$$

Proof of (T₅₉₅)

$$\begin{aligned}
(@a)(@b)E &= (@a)E.(@b)E & (A_{525})(3^{\text{rd}} @ \text{Function Axiom}) \\
\{\subseteq\} * (@a)(@b)E^1 &= (@a)E.(@b)E & (T_{587}) \\
\{\subseteq\} * (@a)(@b)E^1 &= \{\subseteq\}a.\{\subseteq\}b & (T_{688})
\end{aligned}$$

$$\{\subseteq\} * (@a)(@b)E^1 = \{\subseteq\} * (@a)E^1.\{\subseteq\} * (@b)E^1 \quad (T_{579})$$

Proof of (T₅₇₉)

$$\begin{aligned}
\{\subseteq\} * (@a)(@b)E^1 &= \{\subseteq\}a.\{\subseteq\}b & (T_{595}) \\
\{\subseteq\} * (@a)(@b)E^1 &= \{\subseteq\} * (@a)E^1.\{\subseteq\} * (@b)E^1 & (T_{689})
\end{aligned}$$

$$\{\subseteq\} * (@a)(@b)E^1 = \{\subseteq\} * a + \{\subseteq\} * b \quad (T_{580})$$

Proof of (T₅₈₀)

$$\begin{aligned}
\{\subseteq\} * (@a)(@b)E^1 &= \{\subseteq\}a.\{\subseteq\}b & (T_{595}) \\
\{\subseteq\} * (@a)(@b)E^1 &= \sim(\{\subseteq\}a.\{\subseteq\}b) & (T_{008})(\text{Inverse of an Equality}) \\
\{\subseteq\} * (@a)(@b)E^1 &= \{\subseteq\} * a + \{\subseteq\} * b & (T_{025})(\text{DeMorgan's Law})
\end{aligned}$$

$$\{\subseteq\} * (@a)(@b)E^1 = \{\subseteq\} * (@a)E^1 + \{\subseteq\} * (@b)E^1 \quad (T_{563})$$

Proof of (T₅₆₃)

$$\begin{aligned}
\{\subseteq\} * (@a)(@b)E^1 &= \{\subseteq\} * (@a)E^1.\{\subseteq\} * (@b)E^1 & (T_{579}) \\
\{\subseteq\} * (@a)(@b)E^1 &= \sim(\{\subseteq\} * (@a)E^1.\{\subseteq\} * (@b)E^1) & (T_{008})(\text{Inverse of an Equality}) \\
\{\subseteq\} * (@a)(@b)E^1 &= \{\subseteq\} * (@a)E^1 + \{\subseteq\} * (@b)E^1 & (T_{025})(\text{DeMorgan's Law})
\end{aligned}$$

$$(a = a_1^1 +++ a_n^1) \subseteq ((@a)b^1 = \{\subseteq\}a_1^1.\{\subseteq\}b^1.(\{\subseteq\} * (\{\subseteq\}a_1^1.\{\subseteq\}b^1)) +++ \{\subseteq\}a_n^1.\{\subseteq\}b^1.(\{\subseteq\} * (\{\subseteq\}a_n^1.\{\subseteq\}b^1))) \quad (T_{610})$$

$$(b = b_1^1 +++ b_m^1) \subseteq ((@a^1)b = \{\subseteq\}a^1.\{\subseteq\}b_1^1.(\{\subseteq\} * (\{\subseteq\}a^1.\{\subseteq\}b_1^1)) +++ \{\subseteq\}a^1.\{\subseteq\}b_m^1.(\{\subseteq\} * (\{\subseteq\}a^1.\{\subseteq\}b_m^1))) \quad (T_{627})$$

Example Proof of (T₆₁₀)

$$\begin{aligned}
(a = a_1^1 +++ a_n^1) &\subseteq ((@a)b^1 = (@a^1)b^1 +++ (@a_n^1)b^1) & (T_{694}) \\
(a = a_1^1 +++ a_n^1) &\subseteq ((@a)b^1 = \{\subseteq\}a_1^1.\{\subseteq\}b^1.(\{\subseteq\} * (\{\subseteq\}a_1^1.\{\subseteq\}b^1)) +++ \{\subseteq\}a_n^1.\{\subseteq\}b^1.(\{\subseteq\} * (\{\subseteq\}a_n^1.\{\subseteq\}b^1))) & (D_{005})(\text{Definition of } @)
\end{aligned}$$

$$\begin{aligned}
(a = a_1^1 +++ a_n^1).(b = b_1^1 +++ b_m^1) &\subseteq ((@a)b = ((\{\subseteq\}a_1^1.\{\subseteq\}b_1^1.(\{\subseteq\} * (\{\subseteq\}a_1^1.\{\subseteq\}b_1^1)) +++ \\
&\{\subseteq\}a_1^1.\{\subseteq\}b_m^1.(\{\subseteq\} * (\{\subseteq\}a_1^1.\{\subseteq\}b_m^1))) +++ (\{\subseteq\}a_n^1.\{\subseteq\}b_1^1.(\{\subseteq\} * (\{\subseteq\}a_n^1.\{\subseteq\}b_1^1)) +++ \\
&\{\subseteq\}a_n^1.\{\subseteq\}b_m^1.(\{\subseteq\} * (\{\subseteq\}a_n^1.\{\subseteq\}b_m^1)))) & (T_{652})
\end{aligned}$$

Proof of (T₆₅₂)

$$\begin{aligned}
(a = a_1^1 +++ a_n^1)(b = b_1^1 +++ b_m^1) &\subseteq ((@a)b = [((@a_r^1)b_s^1 +++]_{r=1}^n]_{s=1}^m) & (T_{690}) \\
(a = a_1^1 +++ a_n^1)(b = b_1^1 +++ b_m^1) &\subseteq ((@a)b = [(\{\subseteq\}a_r^1.\{\subseteq\}b_s^1.(\{\subseteq\} * (\{\subseteq\}a_r^1.\{\subseteq\}b_s^1)) +++]_{r=1}^n]_{s=1}^m) & (D_{005})(\text{Definition of } @) \\
(a = a_1^1 +++ a_n^1).(b = b_1^1 +++ b_m^1) &\subseteq ((@a)b = ((\{\subseteq\}a_1^1.\{\subseteq\}b_1^1.(\{\subseteq\} * (\{\subseteq\}a_1^1.\{\subseteq\}b_1^1)) +++ \\
&\{\subseteq\}a_1^1.\{\subseteq\}b_m^1.(\{\subseteq\} * (\{\subseteq\}a_1^1.\{\subseteq\}b_m^1))) +++ (\{\subseteq\}a_n^1.\{\subseteq\}b_1^1.(\{\subseteq\} * (\{\subseteq\}a_n^1.\{\subseteq\}b_1^1)) +++ \\
&\{\subseteq\}a_n^1.\{\subseteq\}b_m^1.(\{\subseteq\} * (\{\subseteq\}a_n^1.\{\subseteq\}b_m^1)))) & (D_{008})(\text{Simplified Disjunction of a Series})
\end{aligned}$$

$@(a + b) = @a + @b$ (T_{T530})

Proof of (T_{T530})

$@(a + b) = @a + @b$ (D_{M00}) (Definition of =)

$@(a + b) = @a + @b$ (T_{R002b}) (Distribution of * over +)

Chapter 8 Proofs of Relationship Theorems – Part 5 – Continuum Sets, Discrete Sets and Discrete Entities

8.1 Introduction

The theorems of continuum sets, discrete sets and discrete entities are proven below.

The theorems for continuum sets are of particular importance because all axioms, definitions, and theorems are universal and are, therefore, continuum sets. These former theorems can therefore be applied to the latter. Thus, for example, axioms, definitions and theorems may replace the terms **a** and **b** in the continuum theorems below.

The relationship between continuum sets, discrete sets, and discrete entities are shown in the following theorems:

$$S_C \subseteq \sim S_D \quad (T_{S627})$$

$$S_D \subseteq \sim S_E \quad (T_{S628})$$

$$S_E \subseteq \sim S_C \quad (T_{S629})$$

$$S_C \subseteq \sim S_D \quad (T_{S627_A})$$

$$S_D \subseteq \sim S_E \quad (T_{S628_A})$$

$$S_E \subseteq \sim S_C \quad (T_{S629_A})$$

Example Proof of (T_{S627})

$$\begin{aligned} (X, Y = S_C, S_D, S_E, S_C, S_D, S_E): (X \neq Y) &= (XY = \emptyset) & (A_{R03}) & \text{(Axiom of Continuum and Discrete Sets)} \\ S_C \cdot S_D &= \emptyset & & \text{Extraction} \\ S_C &\subseteq \sim S_D & (T_{M735}) & (\subseteq \text{Existential Form}) \end{aligned}$$

8.2 Theorems of Discrete Entities

$$(a^1 = \emptyset^1) \subseteq (a^1 \subseteq S_E) \quad (T_{S224}) \text{ The Null Set Entity is Discrete}$$

$$(a^1 = E^1) \subseteq (a^1 \subseteq S_E) \quad (T_{S224_A})$$

Example Proof of (T_{S224})

$$\begin{aligned} (a^1 \subseteq S_E) &= (a^1 = a) & (D_{R03}) & \text{(Definition of a Discrete Entity)(Eqn1)} \\ \text{Let } a^1 &= \emptyset^1 & & \text{(Eqn2)} \\ (a = \emptyset) + (a = \emptyset^1) & & (T_{M497}) & \text{(Eqns3\&4)} \\ (a^1 \subseteq S_E) &= (\emptyset^1 = \emptyset) & (A_{MX19}) & \text{(Substitution of Eqns2\&3 into Eqn1)(Eqn5)} \\ (a^1 \subseteq S_E) &= (\emptyset^1 = \emptyset^1) & (A_{MX19}) & \text{(Substitution of Eqns2\&4 into Eqn1)(Eqn6)} \\ (a^1 \subseteq S_E) + (a^1 \subseteq S_E) &= (\emptyset^1 = \emptyset) + (\emptyset^1 = \emptyset^1) & (A_{MX21}) & \text{(Combination of Eqns5\&6)} \\ (a^1 \subseteq S_E) &= (\emptyset^1 = \emptyset) + (\emptyset^1 = \emptyset^1) & (A_{MX01}) & \text{(Disjunction Tautology)} \\ (a^1 \subseteq S_E) &= (\emptyset^1 = \emptyset) + E & (T_{M001}) & \text{(Universality of (D}_{M00}) \text{ Definition of } =) \\ (a^1 \subseteq S_E) &= E & (T_{M055}) & \text{(Universal Set Disjunction)} \\ a^1 &\subseteq S_E & (A_{MX16}) & \text{(Order Axiom)} \\ \text{Thus } (a^1 = \emptyset^1) &\subseteq (a^1 \subseteq S_E) & (A_{MX02}) & \text{(Conditional Introduction)} \end{aligned}$$

$$\emptyset^1 \subseteq S_E \quad (T_{S355}) \text{ The Null Set Entity is Discrete}$$

$$E^1 \subseteq S_E \quad (T_{S355_A})$$

Example Proof of (T_{S355})

$$(a^1 = \emptyset^1) \subseteq (a^1 \subseteq S_E) \quad (T_{S224})$$

$$\emptyset^1 \subseteq S_E$$

(A_{MX19})(Substitution)

8.3 Theorems of Discrete Sets

$$((a^1 \subseteq S_D) = E) \subseteq ((b^1 \subseteq S_C)(a \subseteq b) \neq \emptyset) \quad (T_{S606})$$

$$((a^1 \subseteq S_D) = E) \subseteq ((b^1 \subseteq S_C)(a \subseteq b) \neq \emptyset) \quad (T_{S606A})$$

$$((a^1 \subseteq S_E) = E) \subseteq ((b^1 \subseteq S_C)(a \subseteq b) \neq \emptyset) \quad (T_{S624})$$

$$((a^1 \subseteq S_E) = E) \subseteq ((b^1 \subseteq S_C)(a \subseteq b) \neq \emptyset) \quad (T_{S624A})$$

Example Proof of (T_{S606})

$$((a^1 \subseteq \sim S_C) = E) \subseteq ((b^1 \subseteq S_C)(a \subseteq b) \neq \emptyset) \quad (A_{R17})(\text{Non Continuum Sets Axiom})(\text{Eqn1})$$

$$S_C \subseteq \sim S_D \quad (T_{S627})$$

$$S_D \subseteq \sim S_C \quad (T_{M012})(\text{Transposition})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \subseteq \sim S_C) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})$$

$$((a^1 \subseteq S_D) \subseteq (a^1 \subseteq \sim S_C)) = E \quad (A_{MX16})(\text{Order Axiom})$$

$$((a^1 \subseteq S_D) = E) \subseteq ((a^1 \subseteq \sim S_C) = E) \quad (T_{M094})(\text{Probabilistic Rule})(\text{Eqn2})$$

$$((a^1 \subseteq S_D) = E) \subseteq ((b^1 \subseteq S_C)(a \subseteq b) \neq \emptyset) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$$

$$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\supset\} * \{\subset\} * a^1) \quad (T_{R684})$$

$$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\subset\} * \{\supset\} * a^1) \quad (T_{R684A})$$

Example Proof of (T_{R684})

$$(a^1 \subseteq S_D) = (a = (\{\subset\} * a^1).(\{\supset\} * \{\subset\} * a^1)) \quad (D_{R93})(\text{Definition of Disaggregation for Discrete Sets})(\text{Eqn1})$$

$$(a = (\{\subset\} * a^1).(\{\supset\} * \{\subset\} * a^1)) \subseteq (a \subseteq \{\supset\} * \{\subset\} * a^1) \quad (T_{M073b})(\text{Single Premise Syllogism})(\text{Eqn2})$$

$$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\supset\} * \{\subset\} * a^1) \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \not\subseteq a) \quad (T_{S614})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \not\subseteq a) \quad (T_{S614A})$$

Example Proof of (T_{S614})

$$(a^1 \subseteq S_D)(b^1 \subseteq S_D) \subseteq (a^1 \not\subseteq b) \quad (D_{R97})(\text{Definition of a Discrete Set})(\text{Eqn1})$$

$$\text{Let } b = a \quad (\text{Eqn2})$$

$$(a^1 \subseteq S_D)(a^1 \subseteq S_D) \subseteq (a^1 \not\subseteq a) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \not\subseteq a) \quad (T_{M667})(\text{Conjunction Tautology})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \sim a) \quad (T_{R717})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \sim a) \quad (T_{R717A})$$

Example Proof of (T_{R717})

$$(a^1 \subseteq S_D) \subseteq (a^1 \not\subseteq a) \quad (T_{S614})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \sim a)(a^1 \neq a) \quad (T_{M198})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \sim a) \quad (T_{M073})(\text{Single Premise Syllogism})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \neq a) \quad (T_{S625})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \neq a) \quad (T_{S625A})$$

Example Proof of (T_{S625})

$$(a^1 \subseteq S_D) \subseteq (a^1 \not\subseteq a) \quad (T_{S614})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \sim a)(a^1 \neq a) \quad (T_{M198})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \neq a) \quad (T_{M073})(\text{Single Premise Syllogism})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \neq \emptyset) \quad (T_{S626})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \neq \emptyset) \quad (T_{S626A})$$

Example Proof of (T_{S626})

$$\begin{aligned} (a \not\subseteq b) &\subseteq (a \neq \emptyset)(b \neq E) && (T_{M251})(\text{Conditions for } \not\subseteq)(\text{Eqn1}) \\ \text{Let } b &= a && (\text{Eqn2}) \\ (a^1 \not\subseteq a) &\subseteq (a^1 \neq \emptyset)(a \neq E) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (a^1 \not\subseteq a) &\subseteq (a^1 \neq \emptyset) && (T_{M073})(\text{Single Premise Syllogism})(\text{Eqn3}) \\ (a^1 \subseteq S_D) &\subseteq (a^1 \not\subseteq a) && (T_{S614})(\text{Eqn4}) \\ (a^1 \subseteq S_D) &\subseteq (a^1 \neq \emptyset) && (T_{M139})(\text{Hypothetical Syllogism Eqns4\&3}) \end{aligned}$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \subseteq \{\supset\} * a) \quad (T_{S234})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \subseteq \{\subset\} * a) \quad (T_{S234A})$$

Example Proof of (T_{S234})

$$\begin{aligned} (a \neq \emptyset) &\subseteq (a^1 = (\{\supset\} \diamond a).(\{\supset\} * \{\supset\} \diamond a)) && (D_{R98})(\text{General Definition of Aggregation})(\text{Eqn1}) \\ (a^1 = (\{\supset\} \diamond a).(\{\supset\} * \{\supset\} \diamond a)) &\subseteq (a^1 \subseteq (\{\supset\} \diamond a)) && (T_{M083})(\text{Conjunction Elimination})(\text{Eqn2}) \\ (a \neq \emptyset) &\subseteq (a^1 \subseteq (\{\supset\} \diamond a)) && (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\ \{\supset\} \diamond a &\subseteq \{\supset\} * a && (T_{R241}) \\ ((a^1 \neq \emptyset) = E) &\subseteq ((a^1 \subseteq (\{\supset\} \diamond a)) \subseteq (a^1 \subseteq (\{\supset\} * a))) && (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn4}) \\ (a \neq \emptyset) &\subseteq (a^1 \subseteq (\{\supset\} * a)) && (T_{M139})(\text{Hypothetical Syllogism Eqns3 \& 4}) \\ (a \neq \emptyset) &\subseteq (a^1 \subseteq (\{\supset\} + \{=\}) * a) && (D_{A141})(\text{Disjunction with } \{=\}) \\ (a \neq \emptyset) &\subseteq (a^1 \subseteq \{\supset\} * a + \{=\} * a) && (T_{R002c})(\text{Distribution of } + \text{ over } *) \\ (a \neq \emptyset) &\subseteq (a^1 \subseteq \{\supset\} * a) + (a^1 \subseteq \{=\} * a) && (T_{M526}) \\ (a \neq \emptyset) &\subseteq (a^1 \subseteq \{\supset\} * a) + (a^1 \subseteq a) && (T_{A080}) \\ (a \neq \emptyset)(a^1 \not\subseteq a) &\subseteq (a^1 \subseteq \{\supset\} * a) && (T_{M468}) \\ (a \neq \emptyset)(a^1 \neq a)(a^1 \sim \subset a) &\subseteq (a^1 \subseteq \{\supset\} * a) && (T_{M198})(\text{Eqn5}) \\ (a^1 \subseteq S_D) &\subseteq (a^1 \sim \subset a) && (T_{R717}) \\ (a^1 \subseteq S_D)(a \neq a^1)(a \neq \emptyset) &\subseteq (a \neq \emptyset)(a \neq a^1)(a^1 \sim \subset a) && (T_{M029})(\text{Deduction})(\text{Eqn6}) \\ (a^1 \subseteq S_D)(a \neq \emptyset)(a^1 \neq a) &\subseteq (a^1 \subseteq \{\supset\} * a) && (T_{M139})(\text{Hypothetical Syllogism Eqns5\&6})(\text{Eqn7}) \\ (a^1 \subseteq S_D) &\subseteq (a^1 \neq a) && (T_{S625})(\text{Eqn8}) \\ (a^1 \subseteq S_D) &\subseteq (a^1 \neq \emptyset) && (T_{S626})(\text{Eqn9}) \\ (a^1 \subseteq S_D)(a^1 \subseteq S_D) &\subseteq (a^1 \neq a)(a^1 \neq \emptyset) && (T_{M669})(\text{Combination of Eqns8\&9}) \\ (a^1 \subseteq S_D) &\subseteq (a^1 \neq a)(a^1 \neq \emptyset) && (T_{M667})(\text{Conjunction Tautology}) \\ (a^1 \subseteq S_D) &= (a^1 \subseteq S_D)(a^1 \neq a)(a^1 \neq \emptyset) && (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn10}) \\ (a^1 \subseteq S_D) &\subseteq (a^1 \subseteq \{\supset\} * a) && (A_{MX19})(\text{Substitution of Eqn10 into Eqn7}) \end{aligned}$$

$$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\subset\} * a^1) \quad (T_{S623})$$

$$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\subset\} * a^1) \quad (T_{S623A})$$

Example Proof of (T_{S623})

$$\begin{aligned} (a^1 \subseteq S_D) &= (a = (\{\subset\} * a^1).(\{\supset\} * \{\subset\} * a^1)) && (D_{R93})(\text{Definition of Disaggregation for Discrete Sets}) \\ (a^1 \subseteq S_D) &\subseteq (a \subseteq \{\subset\} * a^1) && (T_{M073b})(\text{Single Premise Syllogism}) \end{aligned}$$

$$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\supset\} * a) \quad (T_{R719})$$

$$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\subset\} * a) \quad (T_{R719A})$$

Example Proof of (T_{R719})

$$\begin{aligned} (a^1 \subseteq S_D) &\subseteq (a \subseteq \{\subset\} * a^1) && (T_{S623}) \\ (a^1 \subseteq S_D) &\subseteq (\{\supset\} * a \subseteq \{\supset\} * \{\subset\} * a^1) && (T_{R196})(\text{Relational Deduction}) \\ (a^1 \subseteq S_D) &\subseteq (\{\supset\} * \{\subset\} * a^1 \subseteq \{\supset\} * a) && (T_{M012})(\text{Transposition}) \\ ((a^1 \subseteq S_D) = E) &\subseteq ((a \subseteq \{\supset\} * \{\subset\} * a^1) \subseteq (a \subseteq \{\supset\} * a)) && (T_{M272})(\text{Deductive Rule})(\text{Eqn1}) \\ (a^1 \subseteq S_D) &\subseteq (a \subseteq \{\supset\} * \{\subset\} * a^1) && (T_{R684}) \\ ((a^1 \subseteq S_D) \subseteq (a \subseteq \{\supset\} * \{\subset\} * a^1)) &= E && (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$\begin{aligned}
((a^1 \subseteq S_D) = E) &\subseteq ((a \subseteq \{\supset\} * \{\subset\} * a^1) = E) & (T_{M094})(\text{Probabilistic Rule})(\text{Eqn2}) \\
((a^1 \subseteq S_D) = E) &\subseteq (E \subseteq (a \subseteq \{\supset\} * a)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
((a^1 \subseteq S_D) = E) &\subseteq (E = (a \subseteq \{\supset\} * a)) & (T_{M275})(\text{Supersets of the Universal Set}) \\
((a^1 \subseteq S_D) \subseteq (a \subseteq \{\supset\} * a)) &= E & (T_{M094})(\text{Probabilistic Rule}) \\
(a^1 \subseteq S_D) &\subseteq (a \subseteq \{\supset\} * a) & (A_{MX16})(\text{Order Axiom})
\end{aligned}$$

$$(a^1 \subseteq S_D) = (a \subseteq \{\supset\} * a)(a^1 \neq a) \quad (T_{S218}) \text{ Definition of a Discrete Set}$$

$$(a^1 \subseteq S_D) = (a \subseteq \{\subset\} * a)(a^1 \neq a) \quad (T_{S218A}) \text{ Definition of a Discrete Set}$$

Example Proof of (T_{S218})

$$\begin{aligned}
(a^1 \subseteq S_D) &\subseteq (a \subseteq \{\supset\} * a) & (T_{R719})(\text{Eqn1}) \\
(a^1 \subseteq S_D) &\subseteq (a^1 \neq a) & (T_{S625})(\text{Eqn2}) \\
(a^1 \subseteq S_D)(a^1 \subseteq S_D) &\subseteq (a \subseteq \{\supset\} * a)(a^1 \neq a) & (T_{M669})(\text{Combination of Eqns1\&2}) \\
(a^1 \subseteq S_D) &\subseteq (a \subseteq \{\supset\} * a)(a^1 \neq a) & (T_{M667})(\text{Conjunction Tautology})(\text{Eqn3}) \\
(a \subseteq \{\supset\} * a)(a \neq a^1) &\subseteq (a^1 \subseteq S_D) & (D_{R95})(\text{Definition of a Discrete Set})(\text{Eqn4}) \\
(a^1 \subseteq S_D) &= (a \subseteq \{\supset\} * a)(a^1 \neq a) & (T_{M430})(\text{Biconditional Introduction Eqns3\&4})
\end{aligned}$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \subseteq \{\subset\} * a) \quad (T_{S217})$$

$$(a^1 \subseteq S_D) \subseteq (a^1 \subseteq \{\supset\} * a) \quad (T_{S217A})$$

Example Proof of (T_{S217})

$$\begin{aligned}
(a^1 \subseteq S_D) &\subseteq (a \subseteq \{\subset\} * a^1) & (T_{S623})(\text{Eqn1}) \\
\{\subset\} * a^1 &\subseteq \{\supset\} * a^1 & (T_{S563})(\text{Eqn2}) \\
(a^1 \subseteq S_D) &\subseteq (a \subseteq \{\supset\} * a^1) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\
(a^1 \subseteq \{\supset\} * a) &= (a \subseteq \{\supset\} * a^1) & (T_{S030})(\text{Active/Passive Theorem}) \\
(a^1 \subseteq \{\subset\} * a) &= (a \subseteq \{\supset\} * a^1) & (D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set})(\text{Eqn4}) \\
(a^1 \subseteq S_D) &\subseteq (a^1 \subseteq \{\subset\} * a) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})
\end{aligned}$$

$$(a^1 \subseteq S_D) = (a \subseteq \{\subset\} * a)(a^1 \neq a) \quad (T_{R720})$$

$$(a^1 \subseteq S_D) = (a \subseteq \{\supset\} * a)(a^1 \neq a) \quad (T_{R720A})$$

Example Proof of (T_{R720})

$$\begin{aligned}
(a^1 \subseteq S_D) &= (a \subseteq \{\supset\} * a)(a^1 \neq a) & (T_{S218})(\text{Eqn1}) \\
(a \subseteq c * b) &= (b \subseteq \vee c * a) & (T_{R457})(\text{Active/Passive Theorem})(\text{Eqn2}) \\
\text{Let } b &= a & (\text{Eqn3}) \\
(a \subseteq c * a) &= (a \subseteq \vee c * a) & (A_{MX19})(\text{Substitution of Eqn3 into Eqn2})(\text{Eqn4}) \\
\text{Let } c &= \{\subset\} & (\text{Eqn5}) \\
(a \subseteq \{\subset\} * a) &= (a \subseteq \vee \{\subset\} * a) & (A_{MX19})(\text{Substitution of Eqn5 into Eqn4}) \\
(a \subseteq \{\subset\} * a) &= (a \subseteq \{\supset\} * a) & (D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set})(\text{Eqn6}) \\
(a^1 \subseteq S_D) &= (a \subseteq \{\subset\} * a)(a^1 \neq a) & (A_{MX19})(\text{Substitution of Eqn6 into Eqn1})
\end{aligned}$$

$$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\subset\} * a) \quad (T_{S611})$$

$$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\supset\} * a) \quad (T_{S611A})$$

Example Proof of (T_{S611})

$$\begin{aligned}
(a^1 \subseteq S_D) &= (a \subseteq \{\subset\} * a)(a^1 \neq a) & (T_{R720}) \\
(a^1 \subseteq S_D) &\subseteq (a \subseteq \{\subset\} * a) & (T_{M073b})(\text{Single Premise Syllogism})
\end{aligned}$$

$$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\parallel\} * a) \quad (T_{R704})$$

$$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\parallel\} * a) \quad (T_{R704A})$$

Example Proof of (T_{R704})

$$\begin{aligned}
(\{\subset\} * a = \emptyset) &\subseteq (\{\parallel\} * a = \emptyset) & (T_{T882}) \\
(\{\subset\} * a = \sim \emptyset) &\subseteq (\{\parallel\} * a = \sim \emptyset) & (T_{M008})(\text{Inverse of an Equality})
\end{aligned}$$

$(\{\subset\} * a = E) \subseteq (\{\ \} * a = E)$	(T_{M054}) (Negative of Null Set)
$((\{\subset\} * a) \subseteq (\{\ \} * a)) = E$	(T_{M094}) (Probabilistic Rule)
$(\{\subset\} * a) \subseteq (\{\ \} * a)$	(A_{MX16}) (Order Axiom)
$(a \subseteq \{\subset\} * a) \subseteq (a \subseteq \{\ \} * a)$	(T_{M272}) (Deductive Rule)(Eqn1)
$(a^1 \subseteq S_D) = (a \subseteq \{\subset\} * a)(a^1 \neq a)$	(T_{R720})
$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\subset\} * a)$	(T_{M073b}) (Single Premise Syllogism)(Eqn2)
$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\ \} * a)$	(T_{M139}) (Hypothetical Syllogism Eqns2&1)

$$(a^1 \subseteq S_D) = (a \subseteq (\{\supset\} * a)(\{\subset\} * a))(a^1 \neq a) \quad (T_{S219}) \text{ Definition of a Discrete Set}$$

$$(a^1 \subseteq S_D) = (a \subseteq (\{\subset\} * a)(\{\supset\} * a))(a^1 \neq a) \quad (T_{S219A}) \text{ Definition of a Discrete Set}$$

Example Proof of (T_{S219})

$(a^1 \subseteq S_D) = (a \subseteq \{\supset\} * a)(a^1 \neq a)$	(T_{S219}) (Definition of a Discrete Set)
$(a^1 \subseteq S_D) = (a \subseteq \{\supset\} * a)(a \subseteq \{\supset\} * a)(a^1 \neq a)$	(T_{M667}) (Conjunction Tautology)(Eqn1)
$(a \subseteq \{\subset\} * a) = (a \subseteq \{\subset\} * a)$	(D_{M00}) (Definition of $=$)
$(a \subseteq \{\subset\} * a) = (a \subseteq \vee \{\subset\} * a)$	(T_{R457}) (Active/Passive Theorem)
$(a \subseteq \{\subset\} * a) = (a \subseteq \{\supset\} * a)$	(D_{A143}) (Symmetrical Opposite of an Asymmetrical Relationship Set)(Eqn2)
$(a^1 \subseteq S_D) = (a \subseteq \{\subset\} * a)(a \subseteq \{\supset\} * a)(a^1 \neq a)$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)

$$(a^1 \subseteq S_D) \subseteq (\vee Sa. \{\|\}. \vee Qa = \emptyset) \quad (T_{S707})$$

$$(a^1 \subseteq S_D) \subseteq (\vee Sa. \{\|\}. \vee Qa = \emptyset) \quad (T_{S707A})$$

Example Proof of (T_{S707})

$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\ \} * a)$	(T_{R704})
$(a^1 \subseteq S_D) \subseteq (\vee Sa. \{\ \}. \vee Qa = \emptyset)$	(T_{S082})

$$(a^1 \subseteq S_D) \subseteq (\vee Sa. \{\supset\}. \vee Qa = \emptyset) \quad (T_{S705})$$

$$(a^1 \subseteq S_D) \subseteq (\vee Sa. \{\subset\}. \vee Qa = \emptyset) \quad (T_{S705A})$$

$$(a^1 \subseteq S_D) \subseteq (\vee Sa. \{\subset\}. \vee Qa = \emptyset) \quad (T_{S706})$$

$$(a^1 \subseteq S_D) \subseteq (\vee Sa. \{\supset\}. \vee Qa = \emptyset) \quad (T_{S706A})$$

Example Proof of (T_{S705})

$(a^1 \subseteq S_D) = (a \subseteq (\{\supset\} * a)(\{\subset\} * a))(a^1 \neq a)$	(T_{S219}) (Definition of a Discrete Set)
$(a^1 \subseteq S_D) \subseteq (a \subseteq (\{\supset\} * a)(\{\subset\} * a))$	(T_{M073b}) (Single Premise Syllogism)
$(a^1 \subseteq S_D) \subseteq (a \subseteq (\{\supset\} * a))$	(T_{M073}) (Single Premise Syllogism)
$(a^1 \subseteq S_D) \subseteq (\vee Sa. \{\supset\}. \vee Qa = \emptyset)$	(T_{S082})

$$(a^1 \subseteq S_D) \subseteq (\vee Sa. \vee Qa \subseteq \{\sim\}\{\sim\}\{\|\}) \quad (T_{S656})$$

$$(a^1 \subseteq S_D) \subseteq (\vee Sa. \vee Qa \subseteq \{\sim\}\{\sim\}\{\|\}) \quad (T_{S656A})$$

Example Proof of (T_{S656})

$(a^1 \subseteq S_D) \subseteq (\vee Sa. \{\supset\}. \vee Qa = \emptyset)$	(T_{S705}) (Eqn1)
$(a^1 \subseteq S_D) \subseteq (\vee Sa. \{\subset\}. \vee Qa = \emptyset)$	(T_{S706}) (Eqn2)
$(a^1 \subseteq S_D) \subseteq (\vee Sa. \{\ \}. \vee Qa = \emptyset)$	(T_{S707}) (Eqn3)
$(a^1 \subseteq S_D)(a^1 \subseteq S_D)(a^1 \subseteq S_D) \subseteq ((\vee Sa. \{\subset\}. \vee Qa = \emptyset)(\vee Sa. \{\supset\}. \vee Qa = \emptyset)(\vee Sa. \{\ \}. \vee Qa = \emptyset)$	(T_{M669}) (Combination of Eqns1,2&3)
$(a^1 \subseteq S_D) \subseteq ((\vee Sa. \{\subset\}. \vee Qa = \emptyset)(\vee Sa. \{\supset\}. \vee Qa = \emptyset)(\vee Sa. \{\ \}. \vee Qa = \emptyset)$	(T_{M667}) (Conjunction Tautology)
$(a^1 \subseteq S_D) \subseteq (\vee Sa. \{\subset\}. \vee Qa. \vee Sa. \{\supset\}. \vee Qa. \vee Sa. \{\ \}. \vee Qa = \emptyset)$	(T_{M122}) (Probabilistic Conjunction)
$(a^1 \subseteq S_D) \subseteq (\vee Sa. \{\subset\}. \{\supset\}. \{\ \}. \vee Qa = \emptyset)$	(T_{M667}) (Conjunction Tautology)
$(a^1 \subseteq S_D) \subseteq (\vee Sa. \vee Qa \subseteq \{\sim\}\{\sim\}\{\ \})$	(T_{M735}) ($\subseteq$ Existential Form)

$(b^1 \subseteq S_D)(a \subseteq b)(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(T_{S221}) Subsets of Discrete Sets
$(\hat{b}^1 \subseteq S_D)(a \subseteq \hat{b})(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(T_{S221A}) Subsets of Discrete Sets
$(b^1 \subseteq S_D)(a \subseteq b)(a = a^1) \subseteq (a^1 \subseteq S_E)$	(T_{S616}) Members of Discrete Sets
$(\hat{b}^1 \subseteq S_D)(a \subseteq \hat{b})(a = a^1) \subseteq (a^1 \subseteq S_E)$	(T_{S616A}) Members of Discrete Sets

Example Proof of (T_{S221})

Let $(b^1 \subseteq S_D)$	
$(b^1 \subseteq S_D) = E$	(A_{MX16}) (Order Axiom)(Eqn1)
$(b^1 \subseteq S_D) = (b \subseteq S_E)(b \neq b^1)$	(A_{S53}) (Axiom of Discrete Sets and Entities)
$(b^1 \subseteq S_D) \subseteq (b \subseteq S_E)$	(T_{M073b}) (Single Premise Syllogism)(Eqn2)
$E \subseteq (b \subseteq S_E)$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)
$E = (b \subseteq S_E)$	(T_{M275}) (Supersets of the Universal Set)
$b \subseteq S_E$	(A_{MX16}) (Order Axiom) (Eqn3)
Let $a \subseteq b$	(Eqn4)
$a \subseteq S_E$	(T_{M139}) (Hypothetical Syllogism Eqns4&3)
Let $a \neq a^1$	
$a^1 \subseteq S_D$	(A_{S53}) (Axiom of Discrete Sets and Entities)
Thus $(b^1 \subseteq S_D)(a \subseteq b)(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(A_{MX02}) (Conditional Introduction)

$((a_n \subseteq a)(a_n \neq a_n^1) = E) \subseteq ((a^1 \subseteq S_D) \subseteq (a_n^1 \subseteq S_D))$	(T_{T973})
$((a_n \subseteq a)(a_n \neq a_n^1) = E) \subseteq ((a_n^1 \subseteq \sim S_D) \subseteq (a^1 \subseteq \sim S_D))$	(T_{T974})
$((a_n \subseteq a)(a_n \neq a_n^1) = E) \subseteq ((a^1 \subseteq S_D) \subseteq (a_n^1 \subseteq S_D))$	(T_{T973A})
$((a_n \subseteq a)(a_n \neq a_n^1) = E) \subseteq ((a_n^1 \subseteq \sim S_D) \subseteq (a^1 \subseteq \sim S_D))$	(T_{T974A})

Example Proof of (T_{T973}) and (T_{T974})

$(b^1 \subseteq S_D)(a \subseteq b)(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(T_{S221}) (Subsets of Discrete Sets)
$((a \subseteq b)(a \neq a^1) = E) \subseteq ((b^1 \subseteq S_D) \subseteq (a^1 \subseteq S_D))$	(A_{MX35}) (Exportation/Importation) for truth sets
$((a_n \subseteq a)(a_n \neq a_n^1) = E) \subseteq ((a^1 \subseteq S_D) \subseteq (a_n^1 \subseteq S_D))$	Resymbolising
$((a_n \subseteq a)(a_n \neq a_n^1) = E) \subseteq ((a_n^1 \subseteq \sim S_D) \subseteq (a^1 \subseteq \sim S_D))$	(T_{M012}) (Transposition)

$((a + b)^1 \subseteq S_D)(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(T_{S226})
$((a + \hat{b})^1 \subseteq S_D)(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(T_{S226A})
$((a + b)^1 \subseteq S_D)(a = a^1) \subseteq (a^1 \subseteq S_E)$	(T_{S615})
$((a + \hat{b})^1 \subseteq S_D)(a = a^1) \subseteq (a^1 \subseteq S_E)$	(T_{S615A})

Example Proof of (T_{S226})

$(b^1 \subseteq S_D)(a \subseteq b)(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(T_{S221}) (Subsets of Discrete Sets)(Eqn1)
Let $b = a + b$	(Eqn2)
$((a + b)^1 \subseteq S_D)(a \subseteq a + b)(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(A_{MX19}) (Substitution of Eqn1 into Eqn2)
$((a + b)^1 \subseteq S_D)E(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(T_{M001}) (Universality of (T_{M082}) Disjunction Introduction)
$((a + b)^1 \subseteq S_D)(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(A_{MX11}) (Universal Set Conjunction Axiom)

$(b^1 \subseteq S_P)(a \subseteq b)(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(T_{T818}) Subsets of Point Continuum Sets
$(\hat{b}^1 \subseteq S_P)(a \subseteq \hat{b})(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(T_{T818A})
$(b^1 \subseteq S_P)(a \subseteq b)(a = a^1) \subseteq (a^1 \subseteq S_E)$	(T_{S595}) Members of Point Continuum Sets
$(\hat{b}^1 \subseteq S_P)(a \subseteq \hat{b})(a = a^1) \subseteq (a^1 \subseteq S_E)$	(T_{S595A})

Example Proof of (T_{T818})

$(b^1 \subseteq S_D)(a \subseteq b)(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(T_{S221}) (Eqn1)
$S_P \subseteq S_D$	(A_{R53}) (Axiom of Points)
$(S_P \subseteq S_D) = E$	(A_{MX16}) (Order Axiom)

$(b^1 \subseteq S_P) \subseteq (b^1 \subseteq S_D)$	(T_{M099}) (Entity Mode / Set Mode Rule)
$(b^1 \subseteq S_P)(a \subseteq b)(a \neq a^1) \subseteq (b^1 \subseteq S_D)(a \subseteq b)(a \neq a^1)$	(T_{M029}) (Deduction)(Eqn2)
$(b^1 \subseteq S_P)(a \subseteq b)(a \neq a^1) \subseteq (a^1 \subseteq S_D)$	(T_{M139}) (Hypothetical Syllogism Eqns1&2)

$$(a^1 \subseteq S_D)((a + b)^1 \subseteq S_D) \subseteq (a^1 \not\subseteq b) \quad (T_{S230})$$

$$(a^1 \subseteq S_D)((a + \hat{b})^1 \subseteq S_D) \subseteq (a^1 \not\subseteq \hat{b}) \quad (T_{S230A})$$

Example Proof of (T_{S230})

$(a^1 \subseteq S_D)(b^1 \subseteq S_D) \subseteq (a^1 \not\subseteq b)$	(D_{R97}) (Definition of a Discrete Set)(Eqn1)
Let $b = a + b$	(Eqn2)
$(a^1 \subseteq S_D)((a + b)^1 \subseteq S_D) \subseteq (a^1 \not\subseteq a + b)$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)(Eqn3)
$b \subseteq a + b$	(T_{M082}) (Disjunction Introduction)
$(a^1 \subseteq b) \subseteq (a^1 \subseteq a + b)$	(T_{M099}) (Entity Mode / Set Mode Rule)
$(a^1 \not\subseteq a + b) \subseteq (a^1 \not\subseteq b)$	(T_{M012}) (Transposition)(Eqn4)
$(a^1 \subseteq S_D)((a + b)^1 \subseteq S_D) \subseteq (a^1 \not\subseteq b)$	(T_{M139}) (Hypothetical Syllogism Eqns 3&4)

$$(a \neq a^1) \subseteq ((a.\{\supset\}^*a)^1 \subseteq S_D) \quad (T_{S967})$$

$$(a \neq a^1) \subseteq ((a.\{\subset\}^*a)^1 \subseteq S_D) \quad (T_{S967A})$$

Example Proof of (T_{S967})

Let $a \neq a^1$	
$(a \subseteq \{\supset\}^*a) \subseteq (a^1 \subseteq S_D)$	(D_{R95}) (Definition of a Discrete Set) since $a \neq a^1$
$(a = a.\{\supset\}^*a) \subseteq (a^1 \subseteq S_D)$	(A_{MX44}) ($\subseteq$ Equality Form)
$(a^1 = (a.\{\supset\}^*a)^1) \subseteq (a^1 \subseteq S_D)$	(T_{M479})
$(a.\{\supset\}^*a)^1 \subseteq S_D$	(T_{M099}) (Entity Mode / Set Mode Rule) since $a^1 \neq \emptyset$
Thus $(a \neq a^1) \subseteq ((a.\{\supset\}^*a)^1 \subseteq S_D)$	(A_{MX02}) (Conditional Introduction)(Eqn1)

$$(a^1 \subseteq S_D) \subseteq (a.\{\supset\}^*a \neq \emptyset) \quad (T_{S994})$$

$$(a^1 \subseteq S_D) \subseteq (a.\{\subset\}^*a \neq \emptyset) \quad (T_{S994A})$$

Proof of (T_{S994})

$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\supset\}^*a)$	(T_{R719})
$(a^1 \subseteq S_D) \subseteq (a = a.\{\supset\}^*a)$	(A_{MX44}) ($\subseteq$ Equality Form)
$((a^1 \subseteq S_D) \subseteq (a = a.\{\supset\}^*a)) = E$	(A_{MX16}) (Order Axiom)
$((a^1 \subseteq S_D) = E) \subseteq ((a = a.\{\supset\}^*a) = E)$	(T_{M094}) (Probabilistic Rule)
$((a^1 \subseteq S_D) = E) \subseteq ((a \neq \emptyset) = (a.\{\supset\}^*a \neq \emptyset))$	(T_{M523}) (Probabilistic Rule)
$((a^1 \subseteq S_D) = E) \subseteq ((a \neq \emptyset) = (a \neq \emptyset)(a.\{\supset\}^*a \neq \emptyset))$	(T_{M772})
$((a^1 \subseteq S_D) = E) \subseteq ((a \neq \emptyset).E = (a \neq \emptyset)(a.\{\supset\}^*a \neq \emptyset))$	(A_{MX17}) (Universal Set Conjunction Axiom)
$((a^1 \subseteq S_D) = E)((a \neq \emptyset) = E) \subseteq (E = (a.\{\supset\}^*a \neq \emptyset))$	(T_{M444b}) (Exportation from a Biconditional)
$((a^1 \subseteq S_D)(a \neq \emptyset) = E) \subseteq (E = (a.\{\supset\}^*a \neq \emptyset))$	(T_{M108}) (Conjunction Introduction)
$((a^1 \subseteq S_D)(a \neq \emptyset) \subseteq (a.\{\supset\}^*a \neq \emptyset)) = E$	(T_{M094}) (Probabilistic Rule)
$(a^1 \subseteq S_D)(a \neq \emptyset) \subseteq (a.\{\supset\}^*a \neq \emptyset)$	(A_{MX16}) (Order Axiom)(Eqn1)
$(a^1 = \emptyset^1) \subseteq (a^1 \subseteq S_E)$	(T_{S224}) (The Null Set Entity is Discrete)
$(a^1 \not\subseteq S_E) \subseteq (a^1 \neq \emptyset^1)$	(T_{M012}) (Transposition)
$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq \sim S_E) \subseteq (a^1 \neq \emptyset^1))$	(T_{M090}) (Inverse Unit Subset Rule)(Eqn2)
$S_D \subseteq \sim S_E$	(T_{S628})
$(S_D \subseteq \sim S_E) = E$	(A_{MX16}) (Order Axiom)
$(a^1 \subseteq S_D) \subseteq (a^1 \subseteq \sim S_E)$	(T_{M099}) (Entity Mode / Set Mode Rule)(Eqn3)
$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq S_D) \subseteq (a^1 \neq \emptyset^1))$	(T_{M139}) (Hypothetical Syllogism Eqns3&2)
$(a^1 \subseteq S_D)(a^1 \neq \emptyset) \subseteq (a^1 \neq \emptyset^1)$	(A_{MX35}) (Exportation/Importation) for truth sets (Eqn4)
$(a^1 \subseteq S_D) \subseteq (a^1 \neq \emptyset)$	(T_{S626}) (Eqn5)
$(a^1 \subseteq S_D) = (a^1 \subseteq S_D)(a^1 \neq \emptyset)$	(A_{MX44}) ($\subseteq$ Equality Form)(Eqn6)
$(a^1 \subseteq S_D) \subseteq (a^1 \neq \emptyset^1)$	(A_{MX19}) (Substitution of Eqn6 into Eqn4)(Eqn7)
$(a^1 \subseteq S_D)(a^1 \subseteq S_D) \subseteq (a^1 \neq \emptyset)(a^1 \neq \emptyset^1)$	(T_{M669}) (Combination of Eqns5&7)
$(a^1 \subseteq S_D) \subseteq (a^1 \neq \emptyset)(a^1 \neq \emptyset^1)$	(T_{M667}) (Conjunction Tautology)(Eqn8)

$$\begin{aligned}
& (a^1 \neq \emptyset^1). (a^1 \neq \emptyset) \subseteq (a \neq \emptyset) \\
& (a^1 \subseteq S_D) \subseteq (a \neq \emptyset) \\
& (a^1 \subseteq S_D) = (a^1 \subseteq S_D)(a \neq \emptyset) \\
& (a^1 \subseteq S_D) \subseteq (a. \{\supset\} * a \neq \emptyset)
\end{aligned}$$

$$\begin{aligned}
& (T_{M507})(Eqn9) \\
& (T_{M139})(Hypothetical Syllogism Eqns8\&9) \\
& (A_{MX44})(\subseteq \text{Equality Form})(Eqn10) \\
& (A_{MX19})(\text{Substitution of Eqn10 into Eqn1})
\end{aligned}$$

8.4 Theorems for Continuum Sets

$$(a^1 \subseteq S_C) = (a \subseteq S_C) \quad (T_{S604})$$

$$(a^1 \subseteq S_C) = (a \subseteq S_C) \quad (T_{S604A})$$

Example Proof of (T_{S604})

$$\begin{aligned}
& (a^1 + b^1 \subseteq S_C) = (a + b \subseteq S_C) \\
& \text{Let } b^1 = \emptyset \\
& b = \emptyset \\
& (a^1 + \emptyset \subseteq S_C) = (a + \emptyset \subseteq S_C) \\
& (a^1 \subseteq S_C) = (a \subseteq S_C)
\end{aligned}$$

$$\begin{aligned}
& (A_{R07})(\text{Axiom of Continuum Sets})(Eqn1) \\
& (Eqn2) \\
& (A_{MX50})(\text{Components of a Null Entity})(Eqn3) \\
& (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\
& (T_{M668})(\text{Null Set Disjunction})
\end{aligned}$$

$$R_B \subseteq S_C \quad (T_{S420})$$

$$R_B \subseteq S_C \quad (T_{S420A})$$

$$(c^1 \subseteq R_B) \subseteq (c^1 \subseteq S_C) \quad (T_{S421})$$

$$(c^1 \subseteq R_B) \subseteq (c^1 \subseteq S_C) \quad (T_{S421A})$$

Example Proof of (T_{S420}) & (T_{S421})

$$\begin{aligned}
& (X = (\{=\}, \{\subseteq\}, \{\supset\}, \{\|\}, \{\subseteq\}, \{\supset\}) = (X^1 \subseteq R_B) \quad (D_{R94})(\text{Definition of a Membership Relationship Set}) \\
& (X = \{=\}) + (X = \{\subseteq\}) + (X = \{\supset\}) + (X = \{\|\}) + (X = \{\subseteq\}) + (X = \{\supset\}) \subseteq (X^1 \subseteq R_B) \\
& \quad (D_{M15})(\text{Definition of a Variable Set})(Eqn1) \\
& (X = \{\subseteq\}, \{\supset\}, \{\|\}, \{=\}, \{\subseteq\}, \{\supset\}, \{+\}, \{.\}, \{*\}, \{\diamond\}) \subseteq (X^1 \subseteq S_C) \\
& \quad (A_{A08})(\text{Binary Relationship Continuum Axiom}) \\
& (X = \{\subseteq\}) + (X = \{\supset\}) + (X = \{\|\}) + (X = \{=\}) + (X = \{\subseteq\}) + (X = \{\supset\}) + (X = \{+\}) + (X = \{.\}) + (X = \{*\}) + \\
& (X = \{\diamond\}) \subseteq (X^1 \subseteq S_C) \quad (D_{M15})(\text{Definition of a Variable Set})(Eqn2) \\
& (X = \{=\}) + (X = \{\subseteq\}) + (X = \{\supset\}) + (X = \{\|\}) + (X = \{\subseteq\}) + (X = \{\supset\}) \subseteq (X = \{\subseteq\}) + (X = \{\supset\}) + (X = \{\|\}) + \\
& (X = \{=\}) + (X = \{\subseteq\}) + (X = \{\supset\}) + (X = \{+\}) + (X = \{.\}) + (X = \{*\}) + (X = \{\diamond\}) \\
& \quad (T_{M082})(\text{Disjunction Introduction})(Eqn3) \\
& (X^1 \subseteq R_B) \subseteq (X = \{\subseteq\}) + (X = \{\supset\}) + (X = \{\|\}) + (X = \{=\}) + (X = \{\subseteq\}) + (X = \{\supset\}) + (X = \{+\}) + (X = \{.\}) + \\
& (X = \{*\}) + (X = \{\diamond\}) \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn3})(Eqn4) \\
& (X^1 \subseteq R_B) \subseteq (X^1 \subseteq S_C) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns4\&2}) \\
& (c^1 \subseteq R_B) \subseteq (c^1 \subseteq S_C) \quad \text{Resymbolising} \\
& R_B \subseteq S_C \quad (T_{M751})(\text{Entity Mode / Set Mode Rule}) \text{ since } X^1 \neq \emptyset
\end{aligned}$$

$$(a^1 \subseteq S_C) \subseteq (a^1 \neq \emptyset^1) \quad (T_{S396})$$

$$(a^1 \subseteq S_C) \subseteq (a^1 \neq E^1) \quad (T_{S396A})$$

Example Proof of (T_{S396})

$$\begin{aligned}
& \emptyset^1 \subseteq S_E \quad (T_{S355})(Eqn1) \\
& S_E \subseteq \sim S_C \quad (T_{S629})(Eqn2) \\
& \emptyset^1 \subseteq \sim S_C \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1 \& 2}) \\
& S_C \subseteq \sim \emptyset^1 \quad (T_{M012})(\text{Transposition}) \\
& (a^1 \subseteq S_C) \subseteq (a^1 \subseteq \sim \emptyset^1) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule}) \\
& ((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq S_C) \subseteq (a^1 \neq \emptyset^1)) \quad (T_{M090})(\text{Inverse Unit Subset Rule}) \\
& (a^1 \neq \emptyset)(a^1 \subseteq S_C) \subseteq (a^1 \neq \emptyset^1) \quad (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\
& (a^1 \neq \emptyset)(a^1 \subseteq S_C) \subseteq (a^1 \sim \emptyset^1)(a^1 \neq \emptyset^1) \quad (T_{M198}) \\
& (a^1 \neq \emptyset)(a^1 \subseteq S_C) \subseteq (a^1 \neq \emptyset^1) \quad (T_{M073})(\text{Single Premise Syllogism})(Eqn3) \\
& \text{Let } a^1 = \emptyset \quad (Eqn4) \\
& \text{Let } (a^1 \subseteq S_C) \not\subseteq (a^1 \neq \emptyset^1) \quad (Eqn5)
\end{aligned}$$

$$(\emptyset \subseteq S_c) \not\subseteq (\emptyset \neq \emptyset^1)$$

$$E \not\subseteq (\emptyset \neq \emptyset^1)$$

$$E \not\subseteq E$$

$$\sim(E \subseteq E)$$

$$\sim(E = E)$$

$$E \neq E$$

$$\emptyset$$

$$\text{Thus } ((a^1 = \emptyset) = E)((a^1 \subseteq S_c) \not\subseteq (a^1 \neq \emptyset^1)) = \emptyset$$

$$((a^1 = \emptyset) = E) \subseteq ((a^1 \subseteq S_c) \subseteq (a^1 \neq \emptyset^1))$$

$$(a^1 = \emptyset)(a^1 \subseteq S_c) \subseteq (a^1 \neq \emptyset^1)$$

$$(a^1 \neq \emptyset)(a^1 \subseteq S_c) + (a^1 = \emptyset)(a^1 \subseteq S_c) \subseteq (a^1 \neq \emptyset^1) + (a^1 \neq \emptyset^1)$$

$$(a^1 \neq \emptyset)(a^1 \subseteq S_c) + (a^1 = \emptyset)(a^1 \subseteq S_c) \subseteq (a^1 \neq \emptyset^1)$$

$$((a^1 \neq \emptyset) + (a^1 = \emptyset))(a^1 \subseteq S_c) \subseteq (a^1 \neq \emptyset^1)$$

$$E(a^1 \subseteq S_c) \subseteq (a^1 \neq \emptyset^1)$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \neq \emptyset^1)$$

$$(A_{MX19})(\text{Substitution of Eqn4 into Eqn5})$$

$$(T_{M001})(\text{Universality of } (T_{M670})\text{Null Set Theorem})$$

$$(T_{M001})(\text{Universality of } (T_{M780})\text{Existence of } \emptyset^1)$$

$$(T_{M259})(\text{Definition of } \not\subseteq)$$

$$(T_{M275})(\text{Supersets of the Universal Set})$$

$$(T_{M260})(\text{Definition of } \neq)$$

$$(T_{M345})$$

$$(A_{MX02})(\text{Conditional Introduction})$$

$$(T_{M735})(\subseteq \text{Existential Form})$$

$$(A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets (Eqn6)}$$

$$(A_{MX21})(\text{Combination of Eqns3\&6})$$

$$(A_{MX01})(\text{Disjunction Tautology})$$

$$(T_{M407})(\text{Distribution of } . \text{ over } +)$$

$$(T_{M671})(\text{Tertium non Datur})$$

$$(A_{MX11})(\text{Universal Set Conjunction Axiom})$$

Alternative Proof of (T_{S396})

$$(a^1 = \emptyset^1) \subseteq (a^1 \subseteq S_E) \quad (T_{S224})(\text{Eqn1})$$

$$S_E \subseteq \sim S_C \quad (T_{S629})$$

$$(a^1 \subseteq S_E) \subseteq (a^1 \subseteq \sim S_C) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule}) \text{ NB the condition } a^1 \neq \emptyset \text{ holds true. (Eqn2)}$$

$$(a^1 = \emptyset^1) \subseteq (a^1 \subseteq \sim S_C) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$$

$$(a^1 = \emptyset^1) \subseteq (a^1 \not\subseteq S_C) \quad (T_{M090})(\text{Inverse Unit Subset Rule}) \text{ NB the condition } a^1 \neq \emptyset \text{ holds true.}$$

$$(a^1 \subseteq S_C) \subseteq (a^1 \neq \emptyset^1) \quad (T_{M012})(\text{Transposition})$$

$$(a^1 \subseteq S_C) \subseteq (a \neq \emptyset)$$

$$(T_{S459})$$

$$(a^1 \subseteq S_C) \subseteq (a \neq \emptyset)$$

$$(T_{S459A})$$

Example Proof of (T_{S459})

$$(a^1 \subseteq S_C) \subseteq (a^1 \neq \emptyset^1) \quad (T_{S396})$$

$$(a^1 \subseteq S_C) \subseteq (a \neq \emptyset) \quad (T_{M479})$$

$$(a^1 \subseteq S_C) \subseteq (a^1 \neq a)$$

$$(T_{S495})$$

$$(a^1 \subseteq S_C) \subseteq (a \neq a^1)$$

$$(T_{S495A})$$

Example Proof of (T_{S495})

$$(a^1 \subseteq S_E) = (a^1 = a)$$

$$(D_{R03})(\text{Definition of a Discrete Entity})$$

$$(a^1 \not\subseteq S_E) = (a^1 \neq a)$$

$$(T_{M008})(\text{Inverse of an Equality})(\text{Eqn1})$$

$$S_E \subseteq \sim S_C$$

$$(T_{S629})$$

$$S_C \subseteq \sim S_E$$

$$(T_{M012})(\text{Transposition})$$

$$((a \neq \emptyset) = E) \subseteq ((a^1 \subseteq S_C) \subseteq (a^1 \subseteq \sim S_E))$$

$$(T_{M099})(\text{Entity Mode / Set Mode Rule})$$

$$((a \neq \emptyset) = E) \subseteq ((a^1 \subseteq S_C) \subseteq (a^1 \not\subseteq S_E))$$

$$(T_{M090})(\text{Inverse Unit Subset Rule})$$

$$(a^1 \subseteq S_C)(a \neq \emptyset) \subseteq (a^1 \not\subseteq S_E)$$

$$(A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets (Eqn2)}$$

$$(a^1 \subseteq S_C) \subseteq (a \neq \emptyset)$$

$$(T_{S459})$$

$$(a^1 \subseteq S_C) = (a^1 \subseteq S_C)(a \neq \emptyset)$$

$$(A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn3})$$

$$(a^1 \subseteq S_C) \subseteq (a^1 \not\subseteq S_E)$$

$$(A_{MX19})(\text{Substitution of Eqn3 into Eqn2})(\text{Eqn4})$$

$$(a^1 \subseteq S_C) \subseteq (a^1 \neq a)$$

$$(T_{M139})(\text{Hypothetical Syllogism Eqns1\&4})$$

$$(a^1 \subseteq S_C) \subseteq (a \neq \emptyset^1)$$

$$(T_{S403})$$

$$(a^1 \subseteq S_C) \subseteq (a \neq E^1)$$

$$(T_{S403A})$$

Example Proof of (T_{S403})

$$(a^1 \subseteq S_C) \subseteq (a^1 \neq \emptyset^1) \quad (T_{S396})(\text{Eqn1})$$

$$(a = \emptyset^1) \subseteq (a^1 = \emptyset^1) \quad (T_{M499})$$

$$(a^1 \neq \emptyset^1) \subseteq (a \neq \emptyset^1) \quad (T_{M012})(\text{Transposition})(\text{Eqn2})$$

$$(a^1 \subseteq S_c) \subseteq (a \neq \emptyset^1)$$

(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$((a^1 \subseteq S_c) = E) \subseteq ((a^1 = \emptyset) = (a = \emptyset)) \quad (T_{S356})$$

$$((a^1 \subseteq S_c) = E) \subseteq ((a^1 = \emptyset) = (a = \emptyset)) \quad (T_{S356A})$$

Example Proof of (T_{S356})

$$(a = \emptyset) \subseteq (a^1 = \emptyset) + (a^1 = \emptyset^1)$$

(T_{M496})

$$(a = \emptyset)(a^1 \neq \emptyset^1) \subseteq (a^1 = \emptyset)$$

(T_{M468})(Eqn1)

$$(a^1 \subseteq S_c) \subseteq (a^1 \neq \emptyset^1)$$

(T_{S396})

$$(a^1 \subseteq S_c)(a = \emptyset) \subseteq (a = \emptyset)(a^1 \neq \emptyset^1)$$

(T_{M029})(Deduction)(Eqn2)

$$(a^1 \subseteq S_c)(a = \emptyset) \subseteq (a^1 = \emptyset)$$

(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$((a^1 \subseteq S_c) = E) \subseteq ((a = \emptyset) \subseteq (a^1 = \emptyset))$$

(A_{MX35})(Exportation/Importation) for truth sets(Eqn3)

$$(a^1 = \emptyset) \subseteq (a = \emptyset)$$

(A_{MX50})(Components of a Null Entity)

$$((a^1 = \emptyset) \subseteq (a = \emptyset)) = E$$

(A_{MX16})(Order Axiom)(Eqn4)

$$((a^1 \subseteq S_c) = E).E \subseteq ((a = \emptyset) \subseteq (a^1 = \emptyset)).((a^1 = \emptyset) \subseteq (a = \emptyset))$$

(T_{M669})(Combination of Eqns3&4)

$$((a^1 \subseteq S_c) = E).E \subseteq ((a = \emptyset) = (a^1 = \emptyset))$$

(T_{M430})(Biconditional Introduction)

$$((a^1 \subseteq S_c) = E) \subseteq ((a = \emptyset) = (a^1 = \emptyset))$$

(A_{MX17})(Universal Set Conjunction Axiom)

$$((a^1 \subseteq S_c) = E) \subseteq ((a^1 \neq \emptyset) = (a \neq \emptyset)) \quad (T_{S357})$$

$$((a^1 \subseteq S_c) = E) \subseteq ((a^1 \neq \emptyset) = (a \neq \emptyset)) \quad (T_{S357A})$$

Example Proof of (T_{S357})

$$((a^1 \subseteq S_c) = E) \subseteq ((a^1 = \emptyset) = (a = \emptyset))$$

(T_{S356})

$$((a^1 \subseteq S_c) = E) \subseteq ((a^1 \neq \emptyset) = (a \neq \emptyset))$$

(T_{M008})(Inverse of an Equality)

$$(a^1 \subseteq S_c) \subseteq (a^1 \neq \emptyset)$$

(T_{S460})

$$(a^1 \subseteq S_c) \subseteq (a^1 \neq \emptyset)$$

(T_{S460A})

Example Proof of (T_{S460})

$$(a^1 \subseteq S_c) \subseteq (a \neq \emptyset)$$

(T_{S459})

$$(a^1 \subseteq S_c) \subseteq (a^1 \neq \emptyset)$$

(T_{S357})

$$(a^1 \subseteq S_c) \subseteq (a \cdot \sim a^1 = \{\subset\} * a^1)$$

(T_{S366})

$$(a^1 \subseteq S_c) \subseteq (a \cdot \sim a^1 = \{\supset\} * a^1)$$

(T_{S366A})

Example Proof of (T_{S366})

$$(a^1 \subseteq S_c) \subseteq (a = \{\subset\} * a^1)$$

(A_{R02})(Definition of Disaggregation for Continuum Sets)

$$(a^1 \subseteq S_c) \subseteq (a \cdot \sim a^1 = \{\subset\} * a^1 \cdot \sim a^1)$$

(T_{M233})(Deduction)

$$(a^1 \subseteq S_c) \subseteq (a = (\{\{\subset\} + \{=\}\} * a^1 + a^1) \cdot \sim a^1)$$

(D_{A141})(Disjunction with $\{=\}$)

$$(a^1 \subseteq S_c) \subseteq (a = (\{\subset\} * a^1 + \{=\} * a^1) \cdot \sim a^1)$$

(T_{R002c})(Distribution of $+$ over $*$)

$$(a^1 \subseteq S_c) \subseteq (a = (\{\subset\} * a^1 + a^1) \cdot \sim a^1)$$

(D_{A144})(Definition of the Reflexive Relationship Set)

$$(a^1 \subseteq S_c) \subseteq (a = \{\subset\} * a^1 \cdot \sim a^1 + a^1 \cdot \sim a^1)$$

(T_{M407})(Distribution of $\cdot$ over $+$)

$$(a^1 \subseteq S_c) \subseteq (a = \{\subset\} * a^1 \cdot \sim a^1 + \emptyset)$$

(T_{M621})(Complement Conjunction)

$$(a^1 \subseteq S_c) \subseteq (a = \{\subset\} * a^1 \cdot \sim a^1)$$

(T_{M668})(Null Set Disjunction)

$$(a^1 \subseteq S_c) \subseteq (a = \{\subset\} * a^1 \cdot \sim (\{=\} * a^1))$$

(D_{A144})(Definition of the Reflexive Relationship Set)

$$(a^1 \subseteq S_c) \subseteq (a = \{\subset\} * a^1 \cdot \{=\} * a^1)$$

(D_{R04})(Definition of $\cdot$)(Eqn1)

$$\{\subset\} \subseteq \{\neq\}$$

(T_{A103})

$$\{\subset\} * a^1 \subseteq \{\neq\} * a^1$$

(T_{R197})(Relational Deduction)

$$\{\subset\} * a^1 = \{\subset\} * a^1 \cdot \{\neq\} * a^1$$

(A_{MX44})($\subseteq$ Equality Form)

$$(a^1 \neq \emptyset) \subseteq (\{\subset\} * a^1 = \{\subset\} * a^1 \cdot \{=\} * a^1)$$

(T_{R245})(Negative Rule for Single Element Object)(Eqn2)

$$(a^1 \neq \emptyset)(a^1 \subseteq S_c) \subseteq (a = \{\subset\} * a^1)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)

$$\text{Let } a^1 = \emptyset$$

(Eqn4)

$$a = \emptyset$$

(A_{MX50})(Components of a Null Entity)(Eqn5)

$$(a = \emptyset) = E$$

(A_{MX16})(Order Axiom)(Eqn6)

Let $(a^1 \subseteq S_c) \not\subseteq (a = \{\subseteq\} * a^1)$	(Eqn7)
$(\emptyset \subseteq S_c) \not\subseteq (a = \{\subseteq\} * \emptyset)$	(A_{MX19})(Substitution of Eqn5 into Eqn7)
$(\emptyset \subseteq S_c) \not\subseteq (a = \emptyset)$	(T_{S181})(Null Object Theorem)(Eqn8)
$(\emptyset \subseteq S_c) \not\subseteq E$	(A_{MX19})(Substitution of Eqn6 into Eqn8)
$\sim E$	(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)
$\emptyset$	(D_{M02})(Definition of $\emptyset$)
Thus $((a^1 = \emptyset) = E)((a^1 \subseteq S_c) \not\subseteq (a = \{\subseteq\} * a^1)) = \emptyset$	(A_{MX02})(Conditional Introduction)
$((a^1 = \emptyset) = E) \subseteq ((a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a^1))$	(T_{M735})($\subseteq$ Existential Form)
$(a^1 = \emptyset)(a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a^1)$	(A_{MX35})(Exportation/Importation) for truth sets (Eqn9)
$(a^1 = \emptyset)(a^1 \subseteq S_c) + (a^1 \neq \emptyset)(a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a^1) + (a = \{\subseteq\} * a^1)$	(A_{MX21})(Combination of Eqns3 & 9)
$(a^1 = \emptyset)(a^1 \subseteq S_c) + (a^1 \neq \emptyset)(a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a^1)$	(A_{MX01})(Disjunction Tautology)
$((a^1 = \emptyset) + (a^1 \neq \emptyset))(a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a^1)$	(T_{M407})(Distribution of . over +)
$E(a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a^1)$	(T_{M671})(Tertium non Datur)
$(a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a^1)$	(A_{MX11})(Universal Set Conjunction Axiom)

$$(a^1 \subseteq S_c) \subseteq (\{\subseteq\} * a^1 = \{\subseteq\} * a) \quad (T_{S367})$$

$$(a^1 \subseteq S_c) \subseteq (\{\supset\} * a^1 = \{\supset\} * a) \quad (T_{S367A})$$

Example Proof of (T_{S367})

$(a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a^1)$	(A_{R02})(Definition of Disaggregation for Continuum Sets)(Eqn1)
$(a = \{\subseteq\} * a^1) \subseteq (\{\subseteq\} * a = \{\subseteq\} * \{\subseteq\} * a^1)$	(T_{R522})(Relational Deduction)(Eqn2)
$(a^1 \subseteq S_c) \subseteq (\{\subseteq\} * a = \{\subseteq\} * \{\subseteq\} * a^1)$	(T_{M139})(Hypothetical Syllogism Eqns1&2)(Eqn3)
$\{\subseteq\} * \{\subseteq\} * a^1 = \{\subseteq\} * a^1$	(T_{A556})(Eqn4)
$(a^1 \subseteq S_c) \subseteq (\{\subseteq\} * a = \{\subseteq\} * a^1)$	(A_{MX19})(Substitution of Eqn4 into Eqn3)

$$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 = a. \{\subseteq\} * a) \quad (T_{S363})$$

$$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 = a. \{\supset\} * a) \quad (T_{S363A})$$

Example Proof of (T_{S363})

$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 = \{\subseteq\} * a^1)$	(T_{S366})(Eqn1)
$(a. \sim a^1 = \{\subseteq\} * a^1) \subseteq (a. a. \sim a^1 = a. \{\subseteq\} * a^1)$	(T_{M233})(Deduction)(Eqn2)
$(a^1 \subseteq S_c) \subseteq (a. a. \sim a^1 = a. \{\subseteq\} * a^1)$	(T_{M139})(Hypothetical Syllogism Eqns1&2)
$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 = a. \{\subseteq\} * a^1)$	(T_{M667})(Conjunction Tautology)(Eqn3)
$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 = a. \{\subseteq\} * a^1)$	(T_{S367})

$$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 = \{\subseteq\} * a) \quad (T_{S368})$$

$$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 = \{\supset\} * a) \quad (T_{S368A})$$

Example Proof of (T_{S368})

$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 = \{\subseteq\} * a^1)$	(T_{S366})(Eqn1)
$(a^1 \subseteq S_c) \subseteq (\{\subseteq\} * a^1 = \{\subseteq\} * a)$	(T_{S367})(Eqn2)
$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 = \{\subseteq\} * a)$	(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq \{\subseteq\} * a) \quad (T_{S370})$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq \{\supset\} * a) \quad (T_{S370A})$$

Example Proof of (T_{S370})

$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 = \{\subseteq\} * a)$	(T_{S368})
$(a^1 \subseteq S_c) \subseteq (\{\subseteq\} * a \subseteq \sim a^1)$	(T_{M073b})(Single Premise Syllogism)
$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq \{\subseteq\} * a)$	(T_{M012})(Transposition)

$$(a^1 \subseteq S_c) \subseteq (a \subseteq \{\supset\} * a^1) \quad (T_{S371})$$

$$(a^1 \subseteq S_c) \subseteq (a \subseteq \{\subseteq\} * a^1) \quad (T_{S371A})$$

Example Proof of (T_{S371})

$$\begin{aligned} (a^1 \subseteq S_c) &\subseteq (a^1 \subseteq \{\subseteq\} * a) && (T_{S370}) \\ (a^1 \subseteq S_c) &\subseteq (a \subseteq \vee \{\subseteq\} * a^1) && (T_{S030})(\text{Active/Passive Theorem}) \\ (a^1 \subseteq S_c) &\subseteq (a \subseteq \{\supset\} * a^1) && (D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set}) \end{aligned}$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq a) \quad (T_{R716})$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq a) \quad (T_{R716A})$$

Example Proof of (T_{R716})

$$\begin{aligned} (a^1 \subseteq S_c) &\subseteq (a = \{\subseteq\} * a^1) && (A_{R02})(\text{Definition of Disaggregation for Continuum Sets}) \\ (a^1 \subseteq S_c) &\subseteq (a = \{\{\subseteq\} + \{=\}\} * a^1) && (D_{A141})(\text{Disjunction with } \{=\}) \\ (a^1 \subseteq S_c) &\subseteq (a = \{\subseteq\} * a^1 + \{=\} * a^1) && (T_{R002c})(\text{Distribution of } + \text{ over } *) \\ (a^1 \subseteq S_c) &\subseteq (a = \{\subseteq\} * a^1 + a^1) && (D_{A144})(\text{Definition of the Reflexive Relationship Set}) \\ (a^1 \subseteq S_c) &\subseteq (a^1 \subseteq a) && (T_{M072b})(\text{Single Premise Syllogism}) \end{aligned}$$

$$(b^1 \subseteq S_c) \subseteq (c * b^1 \subseteq c * b) \quad (T_{S339})$$

$$(\hat{b}^1 \subseteq S_c) \subseteq (c * \hat{b}^1 \subseteq c * \hat{b}) \quad (T_{S339A})$$

Example Proof of (T_{S339})

$$\begin{aligned} (b^1 \subseteq S_c) &\subseteq (b^1 \subseteq b) && (T_{R716}) \\ (b^1 \subseteq S_c) &\subseteq (c * b^1 \subseteq c * b) && (T_{R196})(\text{Relational Deduction}) \end{aligned}$$

$$(b^1 \subseteq S_c) \subseteq (c * b \subseteq c * b^1) \quad (T_{S340})$$

$$(\hat{b}^1 \subseteq S_c) \subseteq (c * \hat{b} \subseteq c * \hat{b}^1) \quad (T_{S340A})$$

Example Proof of (T_{S340})

$$\begin{aligned} (b^1 \subseteq S_c) &\subseteq (c * b^1 \subseteq c * b) && (T_{S339}) \\ (b^1 \subseteq S_c) &\subseteq (c * b \subseteq c * b^1) && (T_{M012})(\text{Transposition}) \end{aligned}$$

$$(a^1 \subseteq S_c) \subseteq (a^1 = a \cdot \{\subseteq\} * a) \quad (T_{S360})$$

$$(a^1 \subseteq S_c) \subseteq (a^1 = a \cdot \{\supset\} * a) \quad (T_{S360A})$$

Example Proof of (T_{S360})

$$\begin{aligned} (a^1 \subseteq S_c) &\subseteq (a \cdot \sim a^1 = \{\subseteq\} * a) && (T_{S368}) \\ (a^1 \subseteq S_c) &\subseteq (\sim(a \cdot \sim a^1) = \{\subseteq\} * a) && (T_{M008})(\text{Inverse of an Equality}) \\ (a^1 \subseteq S_c) &\subseteq (a^1 + \underline{a} = \{\subseteq\} * a) && (T_{M025})(\text{DeMorgan's Law})(\text{Eqn1}) \\ (a^1 + \underline{a} = \{\subseteq\} * a) &\subseteq ((a^1 + \underline{a}) \cdot a = a \cdot \{\subseteq\} * a) && (T_{M233})(\text{Deduction})(\text{Eqn2}) \\ (a^1 \subseteq S_c) &\subseteq ((a^1 + \underline{a}) \cdot a = a \cdot \{\subseteq\} * a) && (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\ (a^1 \subseteq S_c) &\subseteq (a^1 \cdot a + \underline{a} \cdot a = a \cdot \{\subseteq\} * a) && (T_{M407})(\text{Distribution of } \cdot \text{ over } +) \\ (a^1 \subseteq S_c) &\subseteq (a^1 \cdot a + \emptyset = a \cdot \{\subseteq\} * a) && (T_{M621})(\text{Complement Conjunction}) \\ (a^1 \subseteq S_c) &\subseteq (a^1 \cdot a = a \cdot \{\subseteq\} * a) && (T_{M668})(\text{Null Set Disjunction})(\text{Eqn3}) \\ (a^1 \subseteq S_c) &\subseteq (a^1 \subseteq a) && (T_{R716}) \\ (a^1 \subseteq S_c) &\subseteq (a^1 = a^1 \cdot a) && (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn4}) \\ (a^1 \subseteq S_c) &\subseteq (a^1 = a \cdot \{\subseteq\} * a) && (A_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \end{aligned}$$

$$(a^1 \subseteq S_c) \subseteq (a \subseteq \{\supset\} * a) \quad (T_{S373})$$

$$(a^1 \subseteq S_c) \subseteq (a \subseteq \{\subseteq\} * a) \quad (T_{S373A})$$

Example Proof of (T_{S373})

$$\begin{aligned} (a^1 \subseteq S_c) &\subseteq (a \cdot \sim a^1 = \{\subseteq\} * a) && (T_{S368}) \\ (a^1 \subseteq S_c) &\subseteq (\{\subseteq\} * a \subseteq a) && (T_{M073b})(\text{Single Premise Syllogism}) \\ (a^1 \subseteq S_c) &\subseteq (a \subseteq \{\subseteq\} * a) && (T_{M012})(\text{Transposition}) \\ (a^1 \subseteq S_c) &\subseteq (a \subseteq \vee \{\subseteq\} * a) && (T_{R457})(\text{Active/Passive Theorem}) \\ (a^1 \subseteq S_c) &\subseteq (a \subseteq \{\supset\} * a) && (D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set}) \end{aligned}$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq \{\supset\} * a) \quad (T_{S374})$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq \{\subset\} * a) \quad (T_{S374_A})$$

Example Proof of (T_{S374})

$$(a^1 \subseteq S_c) \subseteq (a \subseteq \{\supset\} * a) \quad (T_{S373})(\text{Eqn1})$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq a) \quad (T_{R716})(\text{Eqn2})$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq \{\supset\} * a) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$$

$$(a^1 \subseteq S_c) \subseteq (a = \{\supset\} * a) \quad (T_{S375})$$

$$(a^1 \subseteq S_c) \subseteq (a = \{\subset\} * a) \quad (T_{S375_A})$$

Example Proof of (T_{S375})

$$(a^1 \subseteq S_c) \subseteq (a \subseteq \{\supset\} * a) \quad (T_{S373})$$

$$(a^1 \subseteq S_c) \subseteq (a = a. \{\supset\} * a) \quad (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn1})$$

$$a = \{=\} * a \quad (D_{A144})(\text{Definition of the Reflexive Relationship Set})$$

$$a = \{=\} * a \quad (T_{M008})(\text{Inverse of an Equality})(\text{Eqn2})$$

$$(a^1 \subseteq S_c) \subseteq (a = \{=\} * a. \{\supset\} * a) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(a^1 \subseteq S_c) \subseteq (a = \sim(\{=\} * a + \{\supset\} * a)) \quad (T_{M026})(\text{DeMorgan's Law})$$

$$(a^1 \subseteq S_c) \subseteq (a = \sim(\{=\} + \{\supset\}) * a) \quad (T_{R002c})(\text{Distribution of } + \text{ over } *)$$

$$(a^1 \subseteq S_c) \subseteq (a = \sim(\{\supset\} * a)) \quad (D_{A141})(\text{Disjunction with } \{=\})$$

$$(a^1 \subseteq S_c) \subseteq (a = \{\supset\} * a) \quad (D_{R04})(\text{Definition of } *)$$

$$(a^1 \subseteq S_c) \subseteq (\{\subset\} * a^1 = \{\supset\} * a) \quad (T_{S237})$$

$$(a^1 \subseteq S_c) \subseteq (\{\supset\} * a^1 = \{\subset\} * a) \quad (T_{S237_A})$$

Example Proof of (T_{S237})

$$(a^1 \subseteq S_c) \subseteq (a = \{\subset\} * a^1) \quad (A_{R02})(\text{Definition of Disaggregation for Continuum Sets})(\text{Eqn1})$$

$$(a^1 \subseteq S_c) \subseteq (a = \{\supset\} * a) \quad (T_{S375})(\text{Eqn2})$$

$$(a^1 \subseteq S_c) \subseteq (\{\subset\} * a^1 = \{\supset\} * a) \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$$

$$(a^1 \subseteq S_c) \subseteq (a = \{\subset\} * \{\supset\} * a) \quad (T_{S458})$$

$$(a^1 \subseteq S_c) \subseteq (a = \{\supset\} * \{\subset\} * a) \quad (T_{S458_A})$$

Proof of (T_{S458})

$$(a^1 \subseteq S_c) \subseteq (\{\subset\} * a^1 = \{\supset\} * a) \quad (T_{S237})$$

$$(a^1 \subseteq S_c) \subseteq (\{\subset\} * \{\subset\} * a^1 = \{\subset\} * \{\supset\} * a) \quad (T_{R196})(\text{Relational Deduction})$$

$$(a^1 \subseteq S_c) \subseteq (\{\subset\} * a^1 = \{\subset\} * \{\supset\} * a) \quad (T_{A817})$$

$$(a^1 \subseteq S_c) \subseteq (a = \{\subset\} * \{\supset\} * a) \quad (A_{R02})(\text{Definition of Disaggregation for Continuum Sets})$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq \{\supset\} * a) \quad (T_{S376})$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq \{\subset\} * a) \quad (T_{S376_A})$$

Example Proof of (T_{S376})

$$(a^1 \subseteq S_c) \subseteq (a = \{\supset\} * a) \quad (T_{S375})(\text{Eqn1})$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq a) \quad (T_{R716})(\text{Eqn2})$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq \{\supset\} * a) \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$$

$$(a^1 \subseteq S_c) \subseteq (a = \{\subset\} * a) \quad (T_{S359})$$

$$(a^1 \subseteq S_c) \subseteq (a = \{\supset\} * a) \quad (T_{S359_A})$$

Example Proof of (T_{S359})

$$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 = \{\subset\} * a) \quad (T_{S368})$$

$$(a^1 \subseteq S_c) \subseteq (\{\subset\} * a \subseteq a) \quad (T_{M073b})(\text{Single Premise Syllogism})$$

$$(a^1 \subseteq S_c) \subseteq (\{\subset\} * a = a. \{\subset\} * a) \quad (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn1})$$

$$\begin{aligned}
a &= a + a.\{\subseteq\} * a & (T_{M063})(\text{Partial Redundancy})(\text{Eqn2}) \\
(a^1 \subseteq S_c) &\subseteq (a = a + \{\subseteq\} * a) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\
(a^1 \subseteq S_c) &\subseteq (a = \{=\} * a + \{\subseteq\} * a) & (D_{A144})(\text{Definition of the Reflexive Relationship Set}) \\
(a^1 \subseteq S_c) &\subseteq (a = (\{=\} + \{\subseteq\}) * a) & (T_{R002c})(\text{Distribution of } + \text{ over } *) \\
(a^1 \subseteq S_c) &\subseteq (a = \{\subseteq\} * a) & (D_{A141})(\text{Disjunction with } \{=\})
\end{aligned}$$

$$S = \{\subseteq\} * S \quad (T_{A286})$$

$$Q = \{\subseteq\} * Q \quad (T_{A287})$$

$$V = \{\subseteq\} * V \quad (T_{A288})$$

$$^v S = \{\subseteq\} * ^v S \quad (T_{A295})$$

$$^v Q = \{\subseteq\} * ^v Q \quad (T_{A296})$$

Example Proof of (T_{A286})

$$\begin{aligned}
(a^1 \subseteq S_c) &\subseteq (a = \{\subseteq\} * a) & (T_{S359})(\text{Eqn1}) \\
\text{Let } a &= S & (\text{Eqn2}) \\
a^1 &= S^1 & (T_{M479})(\text{Eqn3}) \\
(S^1 \subseteq S_c) &\subseteq (S = \{\subseteq\} * S) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})(\text{Eqn4}) \\
(X = S, Q, V, ^v S, ^v Q) &\subseteq (X^1 \subseteq S_c) & (A_{A07})(\text{Sub-relationship Continuum Axiom}) \\
S^1 &\subseteq S_c & \text{Extraction} \\
(S^1 \subseteq S_c) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn5}) \\
E &\subseteq (S = \{\subseteq\} * S) & (A_{MX19})(\text{Substitution of Eqn5 into Eqn4}) \\
E &= (S = \{\subseteq\} * S) & (T_{M275})(\text{Supersets of the Universal Set}) \\
S &= \{\subseteq\} * S & (A_{MX16})(\text{Order Axiom})
\end{aligned}$$

$$\{\subseteq\} = \{\subseteq\} * \{\subseteq\} \quad (T_{A289})$$

$$\{\supset\} = \{\subseteq\} * \{\supset\} \quad (T_{A290})$$

$$\{=\} = \{\subseteq\} * \{=\} \quad (T_{A291})$$

$$\{\|\} = \{\subseteq\} * \{\|\} \quad (T_{A292})$$

$$\{+\} = \{\subseteq\} * \{+\} \quad (T_{A293})$$

$$\{\cdot\} = \{\subseteq\} * \{\cdot\} \quad (T_{A294})$$

Example Proof of (T_{A289})

$$\begin{aligned}
(a^1 \subseteq S_c) &\subseteq (a = \{\subseteq\} * a) & (T_{S359})(\text{Eqn1}) \\
\text{Let } a &= \{\subseteq\} & (\text{Eqn2}) \\
a^1 &= \{\subseteq\}^1 & (T_{M479})(\text{Eqn3}) \\
(\{\subseteq\}^1 \subseteq S_c) &\subseteq (\{\subseteq\} = \{\subseteq\} * \{\subseteq\}) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})(\text{Eqn4}) \\
(X = \{\subseteq\}, \{\supset\}, \{\|\}, \{=\}, \{\subseteq\}, \{\supset\}, \{+\}, \{\cdot\}, \{*\}, \{\diamond\}) &\subseteq (X^1 \subseteq S_c) & (A_{A08})(\text{Binary Relationship Continuum Axiom}) \\
\{\subseteq\}^1 &\subseteq S_c & \text{Extraction} \\
(\{\subseteq\}^1 \subseteq S_c) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn5}) \\
E &\subseteq (\{\subseteq\} = \{\subseteq\} * \{\subseteq\}) & (A_{MX19})(\text{Substitution of Eqn5 into Eqn4}) \\
E &= (\{\subseteq\} = \{\subseteq\} * \{\subseteq\}) & (T_{M275})(\text{Supersets of the Universal Set}) \\
\{\subseteq\} &= \{\subseteq\} * \{\subseteq\} & (A_{MX16})(\text{Order Axiom})
\end{aligned}$$

$$\{\subseteq\} = \{\subseteq\} * \{\subseteq\} \quad (T_{A297})$$

$$\{\supset\} = \{\subseteq\} * \{\supset\} \quad (T_{A298})$$

Example Proof of (T_{A297})

$$\begin{aligned}
&\{\subseteq\} * \{\subseteq\} \\
&\{\subseteq\} * (\{\subseteq\} + \{=\}) & (D_{A141})(\text{Disjunction with } \{=\}) \\
&(\{\subseteq\} * \{\subseteq\}) + (\{\subseteq\} * \{=\}) & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\
&\{\subseteq\} + (\{\subseteq\} * \{=\}) & (T_{A289}) \\
&\{\subseteq\} + \{=\} & (T_{A291}) \\
&\{\subseteq\} & (D_{A141})(\text{Disjunction with } \{=\})
\end{aligned}$$

And since this proof is reversible $\{\subseteq\} = \{\subseteq\} * \{\subseteq\}$

$$(a^1 \subseteq S_c) \subseteq (\{\subseteq\} * a^1 = \{\subseteq\} * a) \quad (T_{S378})$$

$$(a^1 \subseteq S_c) \subseteq (\{\supseteq\} * a^1 = \{\supseteq\} * a) \quad (T_{S378A})$$

Example Proof of (T_{S378})

$$(a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a) \quad (T_{S359})(Eqn1)$$

$$(a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a^1) \quad (A_{R02})(\text{Definition of Disaggregation for Continuum Sets})(Eqn2)$$

$$(a^1 \subseteq S_c) \subseteq (\{\subseteq\} * a^1 = \{\subseteq\} * a) \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$$

$$((a^1 + b^1)^1 \subseteq S_c) = (a^1 + b^1 \subseteq S_c) \quad (T_{S937})$$

$$((a^1 + b^1)^1 \subseteq S_c) = (a^1 + b^1 \subseteq S_c) \quad (T_{S937A})$$

Example Proof of (T_{S937})

$$(a^1 \subseteq S_c) = (a \subseteq S_c) \quad (T_{S604})(Eqn1)$$

$$\text{Let } a = a^1 + b^1 \quad (Eqn2)$$

$$a^1 = (a^1 + b^1)^1 \quad (T_{M479})(Eqn3)$$

$$((a^1 + b^1)^1 \subseteq S_c) = (a^1 + b^1 \subseteq S_c) \quad (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})$$

$$((a + b)^1 \subseteq S_c) = (a + b \subseteq S_c) \quad (T_{S938})$$

$$((a + b)^1 \subseteq S_c) = (a + b \subseteq S_c) \quad (T_{S938A})$$

Example Proof of (T_{S938})

$$(a^1 \subseteq S_c) = (a \subseteq S_c) \quad (T_{S604})(Eqn1)$$

$$\text{Let } a = a + b \quad (Eqn2)$$

$$a^1 = (a + b)^1 \quad (T_{M479})(Eqn3)$$

$$((a + b)^1 \subseteq S_c) = (a + b \subseteq S_c) \quad (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})$$

$$(a^1 + b^1 \subseteq S_c) = ((a + b)^1 \subseteq S_c) \quad (T_{S939})$$

$$(a^1 + b^1 \subseteq S_c) = ((a + b)^1 \subseteq S_c) \quad (T_{S939A})$$

Example Proof of (T_{S939})

$$(a^1 + b^1 \subseteq S_c) = (a + b \subseteq S_c) \quad (A_{R01})(\text{Axiom of Continuum Sets})(Eqn1)$$

$$((a + b)^1 \subseteq S_c) = (a + b \subseteq S_c) \quad (T_{S938})(Eqn2)$$

$$(a^1 + b^1 \subseteq S_c) = ((a + b)^1 \subseteq S_c) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$((a^1 + b^1)^1 \subseteq S_c) = ((a + b)^1 \subseteq S_c) \quad (T_{S940})$$

$$((a^1 + b^1)^1 \subseteq S_c) = ((a + b)^1 \subseteq S_c) \quad (T_{S940A})$$

Example Proof of (T_{S940})

$$(a^1 + b^1 \subseteq S_c) = ((a + b)^1 \subseteq S_c) \quad (T_{S939})(Eqn1)$$

$$((a^1 + b^1)^1 \subseteq S_c) = (a^1 + b^1 \subseteq S_c) \quad (T_{S937})(Eqn2)$$

$$((a^1 + b^1)^1 \subseteq S_c) = ((a + b)^1 \subseteq S_c) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$((a^1 + b^1)^1 \subseteq S_c) = (a + b \subseteq S_c) \quad (T_{S941})$$

$$((a^1 + b^1)^1 \subseteq S_c) = (a + b \subseteq S_c) \quad (T_{S941A})$$

Example Proof of (T_{S941})

$$((a + b)^1 \subseteq S_c) = (a + b \subseteq S_c) \quad (T_{S938})(Eqn1)$$

$$((a^1 + b^1)^1 \subseteq S_c) = ((a + b)^1 \subseteq S_c) \quad (T_{S940})(Eqn2)$$

$$((a^1 + b^1)^1 \subseteq S_c) = (a + b \subseteq S_c) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(a^1 \subseteq S_c)(b^1 \subseteq S_c) = ((a + b)^1 \subseteq S_c) \quad (T_{S942})$$

$$(a^1 \subseteq S_c)(b^1 \subseteq S_c) = ((a + b)^1 \subseteq S_c) \quad (T_{S942A})$$

Example Proof of (T_{S942})

$$\begin{aligned} (a^1 + b^1 \subseteq S_c) &= ((a + b)^1 \subseteq S_c) & (T_{S939}) \\ (a^1 \subseteq S_c)(b^1 \subseteq S_c) &= ((a + b)^1 \subseteq S_c) & (T_{M131b})(\text{Conjunction of Sufficiencies}) \end{aligned}$$

$$((a^1 + b^1)^1 \subseteq S_c) = ((a^1 + b)^1 \subseteq S_c) \quad (T_{S943})$$

$$((a^1 + \hat{b}^1)^1 \subseteq S_c) = ((a^1 + \hat{b}^1)^1 \subseteq S_c) \quad (T_{S943A})$$

$$((a^1 + b^1)^1 \subseteq S_c) = ((a + b^1)^1 \subseteq S_c) \quad (T_{S944})$$

$$((a^1 + \hat{b}^1)^1 \subseteq S_c) = ((a^1 + \hat{b}^1)^1 \subseteq S_c) \quad (T_{S944A})$$

Example Proof of (T_{S943})

$$\begin{aligned} ((a^1 + b^1)^1 \subseteq S_c) &= ((a + b)^1 \subseteq S_c) & (T_{S940})(\text{Eqn1}) \\ \text{Let } a &= a^1 & (\text{Eqn2}) \\ ((a^{11} + b^1)^1 \subseteq S_c) &= ((a^1 + b)^1 \subseteq S_c) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ ((a^1 + b^1)^1 \subseteq S_c) &= ((a^1 + b)^1 \subseteq S_c) & (D_{M17})(\text{Definition of } a^{11}) \text{ since } a^1 \neq \emptyset \end{aligned}$$

$$((a^1 + b^1)^1 \subseteq S_c) \subseteq ((a^1 + b)^1 = (a + b)^1) \quad (T_{S601})$$

$$((a^1 + \hat{b}^1)^1 \subseteq S_c) \subseteq ((a^1 + \hat{b}^1)^1 = (a + \hat{b})^1) \quad (T_{S601A})$$

$$((a^1 + b^1)^1 \subseteq S_c) \subseteq ((a + b^1)^1 = (a + b)^1) \quad (T_{S602})$$

$$((a^1 + \hat{b}^1)^1 \subseteq S_c) \subseteq ((a + \hat{b}^1)^1 = (a + \hat{b})^1) \quad (T_{S602A})$$

Example Proof of (T_{S601})

$$\begin{aligned} \text{Let } (a^1 + b^1)^1 &\subseteq S_c \\ (a^1 + b)^1 &\subseteq S_c & (T_{S943}) \\ a^1 + b^1 &\subseteq S_c & (T_{S937})(\text{Eqn1}) \\ a^1 &\subseteq a^1 + b^1 & (T_{M082})(\text{Disjunction Introduction})(\text{Eqn2}) \\ a^1 &\subseteq S_c & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\ b^1 &\subseteq a^1 + b^1 & (T_{M082})(\text{Disjunction Introduction})(\text{Eqn3}) \\ b^1 &\subseteq S_c & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&3}) \\ (a^1 + b)^1 &= (a^1 + b)^1 & (D_{M00})(\text{Definition of } =) \\ (a^1 + b)^1 &= \{\supset\} \diamond \{a^1 + b\} \cdot \{\supset\} \ast \{\supset\} \diamond \{a^1 + b\} & (D_{R98})(\text{General Definition of Aggregation}) \\ (a^1 + b)^1 &= \{\supset\} \diamond \{\supset\} \ast (a^1 + b) \cdot \{\supset\} \ast \{\supset\} \diamond \{\supset\} \ast (a^1 + b) & (T_{S359}) \text{ Since } (a^1 + b)^1 \subseteq S_c \\ (a^1 + b)^1 &= \{\supset\} \diamond \{\{\supset\} \ast a^1 + \{\supset\} \ast b\} \cdot \{\supset\} \ast \{\supset\} \diamond \{\{\supset\} \ast a^1 + \{\supset\} \ast b\} & (T_{R002b})(\text{Distribution of } \ast \text{ over } +) \\ (a^1 + b)^1 &= \{\supset\} \diamond (a + \{\supset\} \ast b) \cdot \{\supset\} \ast \{\supset\} \diamond (a + \{\supset\} \ast b) & (A_{R02})(\text{Definition of Disaggregation for Continuum Sets}) \\ (a^1 + b)^1 &= \{\supset\} \diamond (a + b) \cdot \{\supset\} \ast \{\supset\} \diamond (a + b) & (T_{S359}) \\ (a^1 + b)^1 &= (a + b)^1 & (D_{R98})(\text{General Definition of Aggregation}) \\ \text{Thus } (a^1 + b^1 \subseteq S_c) &\subseteq ((a^1 + b)^1 = (a + b)^1) & (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$((a^1 + b^1)^1 \subseteq S_c) \subseteq ((a^1 + b)^1 = (a + b)^1) \quad (T_{S603})$$

$$((a^1 + \hat{b}^1)^1 \subseteq S_c) \subseteq ((a^1 + \hat{b}^1)^1 = (a + \hat{b})^1) \quad (T_{S603A})$$

Example Proof of (T_{S603})

$$\begin{aligned} \text{Let } (a^1 + b^1)^1 &\subseteq S_c \\ a^1 + b^1 &\subseteq S_c & (T_{S937})(\text{Eqn1}) \\ a^1 &\subseteq a^1 + b^1 & (T_{M082})(\text{Disjunction Introduction})(\text{Eqn2}) \\ a^1 &\subseteq S_c & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\ b^1 &\subseteq a^1 + b^1 & (T_{M082})(\text{Disjunction Introduction})(\text{Eqn3}) \\ b^1 &\subseteq S_c & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&3}) \\ (a^1 + b^1)^1 &= (a^1 + b^1)^1 & (D_{M00})(\text{Definition of } =) \\ (a^1 + b^1)^1 &= \{\supset\} \diamond (a^1 + b^1) \cdot \{\supset\} \ast \{\supset\} \diamond (a^1 + b^1) & (D_{R98})(\text{General Definition of Aggregation}) \\ (a^1 + b^1)^1 &= \{\supset\} \diamond \{\supset\} \ast (a^1 + b^1) \cdot \{\supset\} \ast \{\supset\} \diamond \{\supset\} \ast (a^1 + b^1) & (T_{S359}) \\ (a^1 + b^1)^1 &= \{\supset\} \diamond (\{\supset\} \ast a^1 + \{\supset\} \ast b^1) \cdot \{\supset\} \ast \{\supset\} \diamond (\{\supset\} \ast a^1 + \{\supset\} \ast b^1) & (T_{R002b})(\text{Distribution of } \ast \text{ over } +) \\ (a^1 + b^1)^1 &= \{\supset\} \diamond (a + b) \cdot \{\supset\} \ast \{\supset\} \diamond (a + b) & (A_{R02})(\text{Definition of Disaggregation for Continuum Sets}) \end{aligned}$$

$$(a^1 + b^1)^1 = (a + b)^1$$

Thus $((a^1 + b^1)^1 \subseteq S_c) \subseteq ((a^1 + b^1)^1 = (a + b)^1)$

(D_{R98})(General Definition of Aggregation)(A_{MX02})(Conditional Introduction)

$$(a^1 \subseteq S_c)(b \subseteq a)(b \neq \emptyset)(b \neq b^1) \subseteq (b^1 \subseteq S_c)$$

(T₇₉₀)

$$(a^1 \subseteq S_c)(\bar{b} \subseteq a)(\bar{b} \neq \emptyset)(\bar{b} \neq \bar{b}^1) \subseteq (\bar{b}^1 \subseteq S_c)$$

(T_{790A})Example Proof of (T₇₉₀)

$$(a^1 + b^1 \subseteq S_c) = ((a + b)^1 \subseteq S_c)$$

(T₅₉₃₉)

$$(a^1 \subseteq S_c)(b^1 \subseteq S_c) = ((a + b)^1 \subseteq S_c)$$

(T_{M131b})(Conjunction of Sufficiencies)(Eqn1)

$$(b^1 \subseteq S_c) \subseteq (b \neq \emptyset)$$

(T₅₄₅₉)

$$(b^1 \subseteq S_c) \subseteq (b \neq b^1)$$

(T₅₄₉₅)

$$(b^1 \subseteq S_c)(b^1 \subseteq S_c) \subseteq (b \neq \emptyset)(b \neq b^1)$$

(T_{M669})(Combination)

$$(b^1 \subseteq S_c) \subseteq (b \neq \emptyset)(b \neq b^1)$$

(T_{M667})(Conjunction Tautology)

$$(b^1 \subseteq S_c) = (b^1 \subseteq S_c)(b \neq \emptyset)(b \neq b^1)$$

(A_{MX44})($\subseteq$ Equality Form)(Eqn2)

$$(a^1 \subseteq S_c)(b^1 \subseteq S_c)(b \neq \emptyset)(b \neq b^1) = ((a + b)^1 \subseteq S_c)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$((a + b)^1 \subseteq S_c) \subseteq (b^1 \subseteq S_c)(b \neq \emptyset)(b \neq b^1)$$

(T_{M073b})(Single Premise Syllogism)

$$((a + b)^1 \subseteq S_c)(b \neq \emptyset)(b \neq b^1) \subseteq (b^1 \subseteq S_c)$$

(T_{M249b})

$$((a + b)^1 \subseteq S_c)E(b \neq \emptyset)(b \neq b^1) \subseteq (b^1 \subseteq S_c)$$

(A_{MX11})(Universal Set Conjunction Axiom)(Eqn3)

$$b \subseteq a + b$$

(T_{M082})(Disjunction Introduction)

$$(b \subseteq a + b) = E$$

(A_{MX16})(Order Axiom)(Eqn4)

$$((a + b)^1 \subseteq S_c)(b \subseteq a + b)(b \neq \emptyset)(b \neq b^1) \subseteq (b^1 \subseteq S_c)$$

(A_{MX19})(Substitution of Eqn4 into Eqn3)(Eqn5)

$$\text{Let } a + b = a$$

(Eqn6)

$$(a^1 \subseteq S_c)(b \subseteq a)(b \neq \emptyset)(b \neq b^1) \subseteq (b^1 \subseteq S_c)$$

(A_{MX19})(Substitution of Eqn6 into Eqn5)

$$(a^1 \subseteq S_c S_u)(b \subseteq a)(b \neq \emptyset)(b \neq b^1) \subseteq (b^1 \subseteq S_c S_u)$$

(T₇₃₀)

$$(a^1 \subseteq S_c S_u)(\bar{b} \subseteq a)(\bar{b} \neq \emptyset)(\bar{b} \neq \bar{b}^1) \subseteq (\bar{b}^1 \subseteq S_c S_u)$$

(T_{730A})Example Proof of (T₇₃₀)

$$(a^1 \subseteq S_c)(b \subseteq a)(b \neq \emptyset)(b \neq b^1) \subseteq (b^1 \subseteq S_c)$$

(T₇₉₀)(Eqn1)

$$(a^1 \subseteq S_u)(b \subseteq a) \subseteq (b^1 \subseteq S_u)$$

(A_{S27})(Subsets of Unordered Sets)(Eqn2)

$$(a^1 \subseteq S_c)(a^1 \subseteq S_u)(b \subseteq a)(b \neq \emptyset)(b \neq b^1) \subseteq (b^1 \subseteq S_c)(b^1 \subseteq S_u)$$

(T_{M669})(Combination of Eqns1&2)

$$(a^1 \subseteq S_c S_u)(b \subseteq a)(b \neq \emptyset)(b \neq b^1) \subseteq (b^1 \subseteq S_c S_u)$$

(T_{M132})(Composition)

$$(a^1 \subseteq S_c) \subseteq (\{\supset\} * a^1 \subseteq \{\subset\} * a)$$

(T_{S382})

$$(a^1 \subseteq S_c) \subseteq (\{\subset\} * a^1 \subseteq \{\supset\} * a)$$

(T_{S382A})Example Proof of (T_{S382})

$$(a \neq \emptyset) \subseteq (\{\supset\} * a^1 \subseteq \{\subset\} * a)$$

(T_{S381})(Eqn1)

$$(a^1 \subseteq S_c) \subseteq (a \neq \emptyset)$$

(T_{S459})(Eqn2)

$$(a^1 \subseteq S_c) \subseteq (\{\supset\} * a^1 \subseteq \{\subset\} * a)$$

(T_{M139})(Hypothetical Syllogism Eqns1&2)(Eqn3)

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq \{\subset\} * a)$$

(T_{S370})(Eqn4)

$$(a^1 \subseteq S_c) \subseteq (\{\supset\} * a^1 + a^1 \subseteq \{\subset\} * a + \{\subset\} * a)$$

(A_{MX21})(Combination of Eqns3&4)

$$(a^1 \subseteq S_c) \subseteq (\{\supset\} * a^1 + a^1 \subseteq \{\subset\} * a)$$

(A_{MX01})(Disjunction Tautology)

$$(a^1 \subseteq S_c) \subseteq (\{\supset\} * a^1 + \{=\} * a^1 \subseteq \{\subset\} * a)$$

(D_{A144})(Definition of the Reflexive Relationship Set)

$$(a^1 \subseteq S_c) \subseteq ((\{\supset\} + \{=\}) * a^1 \subseteq \{\subset\} * a)$$

(T_{R002c})(Distribution of + over *)

$$(a^1 \subseteq S_c) \subseteq (\{\supset\} * a^1 \subseteq \{\subset\} * a)$$

(D_{A141})(Disjunction with {=})

$$(a^1 \subseteq S_c) \subseteq (\{\supset\} * a^1 = \{\supset\} * a)$$

(T_{A094})

$$(a^1 \subseteq S_c) \subseteq (\{\supset\} * a^1 = \{\subset\} * a)$$

(T_{A094A})Example Proof of (T_{A094})

$$(a^1 \subseteq S_c) \subseteq (\{\subset\} * a^1 = \{\supset\} * a)$$

(T_{S237})

$$(a^1 \subseteq S_c) \subseteq (\{\subset\} * a^1 = \{\supset\} * a)$$

(T_{M008})(Inverse of an Equality)

$(a^1 \subseteq S_c)(a^1 \neq \emptyset) \subseteq (\{\subseteq\} * a^1 = \{\supset\} * \underline{a})$	$(TR_{245})(\text{Negative Rule for Single Element Object})(\text{Eqn1})$
$(a^1 \subseteq S_c) \subseteq (a^1 \neq \emptyset)$	(TS_{460})
$(a^1 \subseteq S_c) = (a^1 \subseteq S_c)(a^1 \neq \emptyset)$	$(AMX_{44})(\subseteq \text{Equality Form})(\text{Eqn2})$
$(a^1 \subseteq S_c) \subseteq (\{\subseteq\} * a^1 = \{\supset\} * \underline{a})$	$(AMX_{19})(\text{Substitution of Eqn2 into Eqn1})$

$(a^1 \subseteq S_c) \subseteq (\{\supset\} \diamond a \subseteq \{\subseteq\} * a)$	(TS_{384})
$(a^1 \subseteq S_c) \subseteq (\{\subseteq\} \diamond a \subseteq \{\supset\} * a)$	(TS_{384_A})

Example Proof of (TS_{384})

$(a \neq \emptyset) \subseteq (\{\supset\} \diamond a = \{\supset\} * a^1)$	$(TS_{402})(\text{Eqn1})$
$(a^1 \subseteq S_c) \subseteq (a \neq \emptyset)$	$(TS_{459})(\text{Eqn2})$
$(a^1 \subseteq S_c) \subseteq (\{\supset\} \diamond a = \{\supset\} * a^1)$	$(TM_{139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3})$
$(a^1 \subseteq S_c) \subseteq (\{\supset\} * a^1 \subseteq \{\subseteq\} * a)$	$(TS_{382})(\text{Eqn4})$
$(a^1 \subseteq S_c) \subseteq (\{\supset\} \diamond a \subseteq \{\subseteq\} * a)$	$(AMX_{19})(\text{Substitution of Eqn3 into Eqn4})$

$(a^1 \subseteq S_c) \subseteq (a^1 = a.\{\supset\} \diamond a)$	(TS_{361})
$(a^1 \subseteq S_c) \subseteq (a^1 = a.\{\subseteq\} \diamond a)$	(TS_{361_A})

Example Proof of (TS_{361})

$(a^1 \subseteq S_c) \subseteq (\{\supset\} \diamond a \subseteq \{\subseteq\} * a)$	$(TS_{384})(\text{Eqn1})$
$(a^1 \subseteq S_c) \subseteq (a.\sim a^1 = \{\subseteq\} * a)$	(TS_{368})
$(a^1 \subseteq S_c) \subseteq (\sim(a.\sim a^1) = \{\subseteq\} * a)$	$(TM_{008})(\text{Inverse of an Equality})$
$(a^1 \subseteq S_c) \subseteq (a^1 + \underline{a} = \{\subseteq\} * a)$	$(TM_{025})(\text{DeMorgan's Law})(\text{Eqn2})$
$(a^1 \subseteq S_c) \subseteq (\{\supset\} \diamond a \subseteq \underline{a} + a^1)$	$(AMX_{19})(\text{Substitution of Eqn2 into Eqn1})$
$(a^1 \subseteq S_c) \subseteq (a.\{\supset\} \diamond a \subseteq a^1)$	(TM_{468})
$(a^1 \subseteq S_c) \subseteq (a.\{\supset\} \diamond a \subseteq a^1) + (a.\{\supset\} \diamond a = a^1)$	$(TM_{266})(\text{Definition of } \subseteq)(\text{Eqn3})$
Let $a^1 \subseteq S_c$	
Let $a.\{\supset\} \diamond a \subseteq a^1$	
$(a.\{\supset\} \diamond a = \emptyset)(a^1 \neq \emptyset)$	$(AMX_{40})(\text{Proper Subset of Unit Set Axiom})$
$(a.\{\supset\} \diamond a = \emptyset)(a^1 \neq \emptyset) = E$	$(AMX_{16})(\text{Order Axiom})$
$(a^1 \neq \emptyset) = E$	(TM_{084})
$a^1 \neq \emptyset$	$(AMX_{16})(\text{Order Axiom})(\text{Eqn4})$
$(a.\{\supset\} \diamond a = \emptyset) = E$	(TM_{084})
$a.\{\supset\} \diamond a = \emptyset$	$(AMX_{16})(\text{Order Axiom})(\text{Eqn5})$
$a^1 \subseteq \{\supset\} \diamond a$	(TS_{215})
$a.a^1 \subseteq a.\{\supset\} \diamond a$	$(TM_{029})(\text{Deduction})(\text{Eqn6})$
$a.a^1 \subseteq \emptyset$	$(AMX_{19})(\text{Substitution of Eqn5 into Eqn6})$
$a^1 \subseteq \underline{a}$	$(TM_{735})(\subseteq \text{Existential Form})(\text{Eqn7})$
$a^1 \subseteq a$	$(TR_{716})(\text{Eqn8})$
$a^1 a^1 \subseteq \underline{a} a$	$(TM_{669})(\text{Combination of Eqns7 \& 8})$
$a^1 \subseteq \underline{a} a$	$(TM_{667})(\text{Conjunction Tautology})$
$a^1 \subseteq \emptyset$	$(TM_{627})(\text{Complement Conjunction})$
$a^1 = \emptyset$	$(TM_{052})(\text{Eqn9})$
$\emptyset \neq \emptyset$	$(AMX_{19})(\text{Substitution of Eqn9 into Eqn4})$
$\emptyset$	(TM_{346})
Thus $(a^1 \subseteq S_c)(a.\{\supset\} \diamond a \subseteq a^1) = \emptyset$	$(AMX_{02})(\text{Conditional Introduction})(\text{Eqn10})$
$(a^1 \subseteq S_c) \subseteq \emptyset + (a.\{\supset\} \diamond a = a^1)$	$(AMX_{19})(\text{Substitution of Eqn10 into Eqn3})$
$(a^1 \subseteq S_c) \subseteq (a.\{\supset\} \diamond a = a^1)$	$(TM_{668})(\text{Null Set Disjunction})$

$(a^1 \subseteq S_c) \subseteq (a^1 = \{\supset\} \diamond a.\{\subseteq\} * a)$	(TS_{386})
$(a^1 \subseteq S_c) \subseteq (a^1 = \{\subseteq\} \diamond a.\{\supset\} * a)$	(TS_{386_A})

Example Proof of (TS_{386})

$(a^1 \subseteq S_c) \subseteq (a^1 = a.\{\supset\} \diamond a)$	$(TS_{361})(\text{Eqn1})$
$(a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a)$	$(TS_{359})(\text{Eqn2})$
$(a^1 \subseteq S_c) \subseteq (a^1 = \{\supset\} \diamond a.\{\subseteq\} * a)$	$(AMX_{19})(\text{Substitution of Eqn2 into Eqn1})$

$$(a^1 \subseteq S_c) \subseteq (a^1 = \{\supset\} \diamond a. \{\supset\} * a) \quad (T_{S393}) \text{ Definition of Aggregation for Continuum Sets}$$

$$(a^1 \subseteq S_c) \subseteq (a^1 = \{\subseteq\} \diamond a. \{\subseteq\} * a) \quad (T_{S393A}) \text{ Definition of Abstraction for Continuum Sets}$$

Example Proof of (T_{S393})

$$(a^1 \subseteq S_c) \subseteq (a^1 = a. \{\supset\} \diamond a) \quad (T_{S361})(\text{Eqn1})$$

$$(a^1 \subseteq S_c) \subseteq (a = \{\supset\} * a) \quad (T_{S375})(\text{Eqn2})$$

$$(a^1 \subseteq S_c) \subseteq (a^1 = \{\supset\} \diamond a. \{\supset\} * a) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$((b^1 \subseteq S_c) = E) \subseteq ((a \subseteq b) = (a^1 \subseteq b)) \quad (T_{S394})$$

$$((b^1 \subseteq S_c) = E) \subseteq ((a \subseteq \hat{b}) = (a^1 \subseteq \hat{b})) \quad (T_{S394A})$$

Example Proof of (T_{S394})

Let $a^1 \subseteq S_c$
 Let $a \subseteq b$ (Eqn1)
 $a^1 \subseteq a$ (T_{R716})(Eqn2)
 $a^1 \subseteq b$ (T_{M139})(Hypothetical Syllogism Eqns1&2)
 Thus $(a^1 \subseteq S_c)(a \subseteq b) \subseteq (a^1 \subseteq b)$ (A_{MX02})(Conditional Introduction)(Eqn3)
 $(a^1 \subseteq S_c)(b^1 \subseteq S_c)(a \subseteq b) \subseteq (a^1 \subseteq S_c)(a \subseteq b)$ (T_{M083})(Conjunction Elimination)(Eqn4)
 $(a^1 \subseteq S_c)(b^1 \subseteq S_c)(a \subseteq b) \subseteq (a^1 \subseteq b)$ (T_{M139})(Hypothetical Syllogism Eqns3&4)
 $((a^1 \subseteq S_c)(b^1 \subseteq S_c) = E) \subseteq ((a \subseteq b) \subseteq (a^1 \subseteq b))$ (A_{MX35})(Exportation/Importation) for truth sets(Eqn5)
 Let $b^1 \subseteq S_c$
 Let $a^1 \subseteq b$
 $\{\subseteq\} * a^1 \subseteq \{\subseteq\} * b$ (T_{R196})(Relational Deduction)(Eqn6)
 $b = \{\subseteq\} * b$ (T_{S359})(Eqn7)
 $\{\subseteq\} * a^1 \subseteq b$ (A_{MX19})(Substitution of Eqn7 into Eqn6)(Eqn8)
 $(a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a^1)$ (A_{R02})(Definition of Disaggregation for Continuum Sets)(Eqn9)
 $(a^1 \subseteq S_c) \subseteq (a \subseteq b)$ (A_{MX19})(Substitution of Eqn9 into Eqn8)
 Thus $(a^1 \subseteq S_c)(b^1 \subseteq S_c)(a^1 \subseteq b) \subseteq (a \subseteq b)$ (A_{MX02})(Conditional Introduction)
 $((a^1 \subseteq S_c)(b^1 \subseteq S_c) = E) \subseteq ((a^1 \subseteq b) \subseteq (a \subseteq b))$ (A_{MX35})(Exportation/Importation) for truth sets(Eqn10)
 $((a^1 \subseteq S_c)(b^1 \subseteq S_c) = E).((a^1 \subseteq S_c)(b^1 \subseteq S_c) = E) \subseteq ((a^1 \subseteq b) \subseteq (a \subseteq b)).((a \subseteq b) \subseteq (a^1 \subseteq b))$
 (T_{M669})(Combination of Eqns5&10)
 $((a^1 \subseteq S_c)(b^1 \subseteq S_c) = E) \subseteq ((a^1 \subseteq b) \subseteq (a \subseteq b)).((a \subseteq b) \subseteq (a^1 \subseteq b))$
 (T_{M667})(Conjunction Tautology)
 $((a^1 \subseteq S_c)(b^1 \subseteq S_c) = E) \subseteq ((a^1 \subseteq b) = (a \subseteq b))$ (T_{M430})(Biconditional Introduction)
 $(a^1 \subseteq S_c)(b^1 \subseteq S_c)(a^1 \subseteq b) = (a^1 \subseteq S_c)(b^1 \subseteq S_c)(a \subseteq b)$
 (T_{M444b})(Exportation from a Biconditional)(Eqn11)
 Let $b^1 \subseteq S_c$
 $b \subseteq S_c$ (T_{S604})(Eqn12)
 Let $a^1 \subseteq b$ (Eqn13)
 $a^1 \subseteq S_c$ (T_{M139})(Hypothetical Syllogism Eqns13&12)
 Thus $(b^1 \subseteq S_c)(a^1 \subseteq b) \subseteq (a^1 \subseteq S_c)$ (A_{MX02})(Conditional Introduction)
 $(b^1 \subseteq S_c)(a^1 \subseteq b) = (a^1 \subseteq S_c)(b^1 \subseteq S_c)(a^1 \subseteq b)$ (A_{MX44})($\subseteq$ Equality Form)(Eqn14)
 $(b^1 \subseteq S_c)(a^1 \subseteq b) = (b^1 \subseteq S_c)(a \subseteq b)$ (A_{MX19})(Substitution of Eqn14 into Eqn11)
 $((b^1 \subseteq S_c) = E) \subseteq ((a \subseteq b) = (a^1 \subseteq b))$ (T_{M444b})(Exportation from a Biconditional)

$$((a^1 \subseteq S_c) = E) \subseteq ((a \subseteq b) \subseteq (a^1 \subseteq b)) \quad (T_{S311})$$

$$((a^1 \subseteq S_c) = E) \subseteq ((a \subseteq \hat{b}) \subseteq (a^1 \subseteq \hat{b})) \quad (T_{S311A})$$

Example Proof of (T_{S311})

Let $a^1 \subseteq S_c$
 Let $a \subseteq b$
 $a^1 \subseteq a$ (T_{R716})
 $a^1 \subseteq b$ (T_{M402})
 Thus $(a^1 \subseteq S_c)(a \subseteq b) \subseteq (a^1 \subseteq b)$ (A_{MX02})(Conditional Introduction)
 $((a^1 \subseteq S_c) = E) \subseteq ((a \subseteq b) \subseteq (a^1 \subseteq b))$ (A_{MX35})(Exportation/Importation) for truth sets

Alternative Proof of (T_{S311})

Let $(a^1 \subseteq S_c)$	
$(a^1 \subseteq a)$	(T _{R716})
$(a^1 \subseteq a) = E$	(A _{MX16})(Order Axiom)(Eqn1)
Let $(a \subset b)$	
$(a \subset b) = E$	(A _{MX16})(Order Axiom)(Eqn2)
$(a^1 \subseteq a)(a \subset b) \subseteq (a^1 \subset b)$	(T _{M402})(Eqn3)
$E.E \subseteq (a^1 \subset b)$	(A _{MX19})(Substitution of Eqns1&2 into Eqn3)
$E \subseteq (a^1 \subset b)$	(T _{M667})(Conjunction Tautology)
$E = (a^1 \subset b)$	(T _{M275})(Supersets of the Universal Set)
$a^1 \subset b$	(A _{MX16})(Order Axiom)
Thus $(a^1 \subseteq S_c)(a \subset b) \subseteq (a^1 \subset b)$	(A _{MX02})(Conditional Introduction)
$((a^1 \subseteq S_c) = E) \subseteq ((a \subset b) \subseteq (a^1 \subset b))$	(A _{MX35})(Exportation/Importation) for truth sets

$$((b^1 \subseteq S_c) = E) \subseteq ((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1)) \quad (T_{S324})$$

$$((b^1 \subseteq S_c) = E) \subseteq ((a \subseteq \hat{b}) = (a^1 \subseteq \{\supset\} * \hat{b}^1)) \quad (T_{S324A})$$

Example Proof of (T_{S324})

$((b^1 \subseteq S_c) = E) \subseteq ((a \subseteq b) = (a^1 \subseteq b))$	(T _{S394})(Eqn1)
$(b^1 \subseteq S_c) \subseteq (b = \{\subseteq\} * b^1)$	(A _{R02})(Definition of Disaggregation for Continuum Sets)(Eqn2)
$((b^1 \subseteq S_c) = E) \subseteq ((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))$	(A _{MX19})(Substitution of Eqn2 into Eqn1)

$$((b^1 \subseteq S_c)(a \neq \emptyset) = E) \subseteq ((a \subset b) = (a^1 \subseteq \{\subseteq\} * b^1)) \quad (T_{S312})$$

$$((b^1 \subseteq S_c)(a \neq \emptyset) = E) \subseteq ((a \subset \hat{b}) = (a^1 \subseteq \{\supset\} * \hat{b}^1)) \quad (T_{S312A})$$

Example Proof of (T_{S312})

$((b^1 \subseteq S_c) = E) \subseteq ((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1))$	(T _{S324})(Eqn1)
$((a \subseteq b) = (a^1 \subseteq \{\subseteq\} * b^1)) \subseteq ((a \subseteq b)(a \neq b) = (a^1 \subseteq \{\subseteq\} * b^1)(a \neq b))$	(T _{M029})(Deduction)(Eqn2)
$((b^1 \subseteq S_c) = E) \subseteq ((a \subseteq b)(a \neq b) = (a^1 \subseteq \{\subseteq\} * b^1)(a \neq b))$	(T _{M139})(Hypothetical Syllogism Eqns1&2)
$((b^1 \subseteq S_c) = E) \subseteq (((a \subset b) + (a = b))(a \neq b) = (a^1 \subseteq \{\subseteq\} * b^1)(a \neq b))$	(T _{M266})(Definition of $\subseteq$)
$((b^1 \subseteq S_c) = E) \subseteq ((a \subset b)(a \neq b) + (a = b)(a \neq b) = (a^1 \subseteq \{\subseteq\} * b^1)(a \neq b))$	(T _{M407})(Distribution of . over +)
$((b^1 \subseteq S_c) = E) \subseteq ((a \subset b)(a \neq b) + \emptyset = (a^1 \subseteq \{\subseteq\} * b^1)(a \neq b))$	(T _{M621})(Complement Conjunction)
$((b^1 \subseteq S_c) = E) \subseteq ((a \subset b)(a \neq b) = (a^1 \subseteq \{\subseteq\} * b^1)(a \neq b))$	(T _{M668})(Null Set Disjunction)(Eqn3)
$(a = b)(a \subset b) = \emptyset$	(T _{M037})(Conjunction of Binary Relations)
$(a \subset b) \subseteq (a \neq b)$	(T _{M735})($\subseteq$ Existential Form)
$(a \subset b) = (a \subset b)(a \neq b)$	(A _{MX44})($\subseteq$ Equality Form)(Eqn4)
$((b^1 \subseteq S_c) = E) \subseteq ((a \subset b) = (a^1 \subseteq \{\subseteq\} * b^1)(a \neq b))$	(A _{MX19})(Substitution of Eqn4 into Eqn3)(Eqn5)
$(a = b) = (a^1 = b^1)$	(T _{M479})
$(a \neq b) = (a^1 \neq b^1)$	(T _{M008})(Inverse of an Equality)
$((a^1 \neq \emptyset) = E) \subseteq ((a \neq b) = (a^1 \subseteq \sim\{=\} * b^1))$	(T _{M351})
$((a^1 \neq \emptyset) = E) \subseteq ((a \neq b) = (a^1 \subseteq \sim\{=\} * b^1))$	(D _{A144})(Definition of the Reflexive Relationship Set)
$((a^1 \neq \emptyset) = E) \subseteq ((a \neq b) = (a^1 \subseteq \{=\} * b^1))$	(D _{R04})(Definition of $\simeq$)
$((a^1 \neq \emptyset) = E) \subseteq ((a \neq b) = (b^1 \subseteq \{=\} * a^1))$	(T _{R614})(Active/Passive Theorem)
$((a^1 \neq \emptyset) = E) \subseteq ((a \neq b) = (b^1 \subseteq \{=\} * a^1))$	(T _{A485})
$((a^1 \neq \emptyset) = E) \subseteq ((a \neq b) = (b^1 \subseteq \{ \neq \} * a^1))$	(T _{R245})(Negative Rule for Single Element Object)
$((a^1 \neq \emptyset) = E) \subseteq ((a \neq b) = (a^1 \subseteq \{ \neq \} * b^1))$	(T _{R519})(Active/Passive Theorem)
$((a^1 \neq \emptyset) = E) \subseteq ((a \neq b) = (a^1 \subseteq \{ \neq \} * b^1))$	(T _{A491})(Eqn6)
$((b^1 \subseteq S_c) = E)((a^1 \neq \emptyset) = E) \subseteq ((a \subset b) = (a^1 \subseteq \{\subseteq\} * b^1)(a^1 \subseteq \{ \neq \} * b^1))$	(A _{MX19})(Substitution of Eqn6 into Eqn5)
$((b^1 \subseteq S_c) = E)((a^1 \neq \emptyset) = E) \subseteq ((a \subset b) = (a^1 \subseteq (\{\subseteq\} * b^1) . (\{ \neq \} * b^1)))$	(T _{M132})(Composition)

$$((b^1 \subseteq S_c) = E)((a^1 \neq \emptyset) = E) \subseteq ((a \subset b) = (a^1 \subseteq (\{\subseteq\} \cdot \{\neq\}) * b^1))$$

(T_{R239})(Distribution of $\cdot$ over $*$ for Entities)

$$((b^1 \subseteq S_c) = E)((a^1 \neq \emptyset) = E) \subseteq ((a \subset b) = (a^1 \subseteq \{\subset\} * b^1))$$

(T_{A227})

$$(c^1 \subseteq S_c) = (S_c \subseteq S_c) \quad (T_{S442})$$

$$(c^1 \subseteq S_c) = ((S_c)^1 \subseteq S_c) \quad (T_{S442A})$$

$$(c^1 \subseteq S_c) = (Qc \subseteq S_c) \quad (T_{S443})$$

$$(c^1 \subseteq S_c) = ((Qc)^1 \subseteq S_c) \quad (T_{S443A})$$

$$(c^1 \subseteq S_c) = (S_c + Qc \subseteq S_c) \quad (T_{S441})$$

$$(c^1 \subseteq S_c) = ((S_c + Qc)^1 \subseteq S_c) \quad (T_{S441A})$$

Example Proof of (T_{S441}), (T_{S442}) & (T_{S443})

$$(X = S, Q, V, {}^vS, {}^vQ):((c^1 \subseteq S_c) = ((X * c)^1 \subseteq S_c)) \quad (A_{R07})(\text{Continuum Matching Axiom})$$

$$(c^1 \subseteq S_c) = ((S_c)^1 \subseteq S_c) \quad \text{Extraction (Eqn1)}$$

$$((S_c)^1 \subseteq S_c) = (S_c \subseteq S_c) \quad (T_{S604})(\text{Eqn2})$$

$$(c^1 \subseteq S_c) = (S_c \subseteq S_c) \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})(\text{Eqn3})$$

$$(X = S, Q, V, {}^vS, {}^vQ):((c^1 \subseteq S_c) = ((X * c)^1 \subseteq S_c)) \quad (A_{R07})(\text{Continuum Matching Axiom})$$

$$(c^1 \subseteq S_c) = ((Qc)^1 \subseteq S_c) \quad \text{Extraction (Eqn4)}$$

$$((Qc)^1 \subseteq S_c) = (Qc \subseteq S_c) \quad (T_{S604})(\text{Eqn5})$$

$$(c^1 \subseteq S_c) = (S_c \subseteq S_c) \quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn5})(\text{Eqn6})$$

$$(c^1 \subseteq S_c) \cdot (c^1 \subseteq S_c) = (S_c \subseteq S_c) \cdot (Qc \subseteq S_c) \quad (T_{M231})(\text{Combination of Eqns3\&6})$$

$$(c^1 \subseteq S_c) = (S_c \subseteq S_c) \cdot (Qc \subseteq S_c) \quad (T_{M667})(\text{Conjunction Tautology})$$

$$(c^1 \subseteq S_c) = (S_c + Qc \subseteq S_c) \quad (T_{M131b})(\text{Conjunction of Sufficiencies})$$

$$(c^1 \subseteq S_c)(a^1 \subseteq S_c + Qc) \subseteq (a^1 \subseteq S_c) \quad (T_{S423})$$

$$(c^1 \subseteq S_c)(a^1 \subseteq S_c + Qc) \subseteq (a^1 \subseteq S_c) \quad (T_{S423A})$$

Example Proof of (T_{S423})

$$(c^1 \subseteq S_c) = (S_c + Qc \subseteq S_c) \quad (T_{S441})$$

$$((c^1 \subseteq S_c)(a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq S_c + Qc) \subseteq (a^1 \subseteq S_c)) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})$$

$$(c^1 \subseteq S_c)(a^1 \neq \emptyset)(a^1 \subseteq S_c + Qc) \subseteq (a^1 \subseteq S_c) \quad (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets (Eqn1)}$$

Let $c^1 \subseteq S_c$

Let $a^1 = \emptyset$ (Eqn2)

Let $(a^1 \subseteq S_c + Qc) \not\subseteq (a^1 \subseteq S_c)$ (Eqn3)

$$(\emptyset \subseteq S_c + Qc) \not\subseteq (\emptyset \subseteq S_c) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn3})$$

$$E \not\subseteq E \quad (T_{M001})(\text{Universality of } (T_{M670}) \text{ Null Set Theorem})$$

$$(E \sim \subset E)(E \neq E) \quad (T_{M198})$$

$$(E \sim \subset E)\emptyset \quad (T_{M345})$$

$$\emptyset \quad (T_{M050})(\text{Null Set Conjunction})$$

Thus $((c^1 \subseteq S_c)(a^1 = \emptyset) = E)((a^1 \subseteq S_c + Qc) \not\subseteq (a^1 \subseteq S_c)) = \emptyset$

$$(c^1 \subseteq S_c)(a^1 = \emptyset)(a^1 \subseteq S_c + Qc) \subseteq (a^1 \subseteq S_c) \quad (A_{MX02})(\text{Conditional Introduction})$$

$$(c^1 \subseteq S_c)(a^1 \neq \emptyset)(a^1 \subseteq S_c + Qc) + (c^1 \subseteq S_c)(a^1 = \emptyset)(a^1 \subseteq S_c + Qc) \subseteq (a^1 \subseteq S_c) + (a^1 \subseteq S_c) \quad (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets (Eqn4)}$$

$$(c^1 \subseteq S_c)(a^1 \neq \emptyset)(a^1 \subseteq S_c + Qc) + (c^1 \subseteq S_c)(a^1 = \emptyset)(a^1 \subseteq S_c + Qc) \subseteq (a^1 \subseteq S_c) \quad (A_{MX21})(\text{Combination of Eqns1\&4})$$

$$(c^1 \subseteq S_c)(a^1 \neq \emptyset)(a^1 \subseteq S_c + Qc) + (c^1 \subseteq S_c)(a^1 = \emptyset)(a^1 \subseteq S_c + Qc) \subseteq (a^1 \subseteq S_c) \quad (A_{MX01})(\text{Disjunction Tautology})$$

$$(c^1 \subseteq S_c)(a^1 \subseteq S_c + Qc)((a^1 \neq \emptyset) + (a^1 = \emptyset)) \subseteq (a^1 \subseteq S_c) \quad (T_{M407})(\text{Distribution of } \cdot \text{ over } +)$$

$$(c^1 \subseteq S_c)(a^1 \subseteq S_c + Qc)E \subseteq (a^1 \subseteq S_c) \quad (T_{M671})(\text{Tertium non Datur})$$

$$(c^1 \subseteq S_c)(a^1 \subseteq S_c + Qc) \subseteq (a^1 \subseteq S_c) \quad (A_{MX11})(\text{Universal Set Conjunction Axiom})$$

$$((c^1 \subseteq S_c)(a^1 \subseteq S_c + Qc) = E) \subseteq ((a^1 = \emptyset) = (a = \emptyset)) \quad (T_{S424})$$

$$((c^1 \subseteq S_c)(a^1 \subseteq S_c + Qc) = E) \subseteq ((a^1 = \emptyset) = (a = \emptyset)) \quad (T_{S424A})$$

$$((c^1 \subseteq S_c)(a^1 \subseteq S_c + Qc) = E) \subseteq ((a^1 \neq \emptyset) = (a \neq \emptyset)) \quad (T_{S425})$$

$$((c^1 \subseteq S_c)(a^1 \subseteq S_c + Qc) = E) \subseteq ((a^1 \neq \emptyset) = (a \neq \emptyset)) \quad (T_{S425A})$$

Example Proof of (T_{S424}) & (T_{S425})

$$\begin{aligned} (c^1 \subseteq S_c)(a^1 \subseteq S_c + Qc) &\subseteq (a^1 \subseteq S_c) & (T_{S423}) \\ ((c^1 \subseteq S_c)(a^1 \subseteq S_c + Qc) = E) &\subseteq ((a^1 \subseteq S_c) = E) & (T_{M094})(\text{Probabilistic Rule})(\text{Eqn1}) \\ ((a^1 \subseteq S_c) = E) &\subseteq ((a^1 = \emptyset) = (a = \emptyset)) & (T_{S356})(\text{Eqn2}) \\ ((c^1 \subseteq S_c)(a^1 \subseteq S_c + Qc) = E) &\subseteq ((a^1 = \emptyset) = (a = \emptyset)) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\ ((c^1 \subseteq S_c)(a^1 \subseteq S_c + Qc) = E) &\subseteq ((a^1 \neq \emptyset) = (a \neq \emptyset)) & (T_{M008})(\text{Inverse of an Equality}) \end{aligned}$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq \{\supset\} \diamond (a \sim a^1)) \quad (T_{S472})$$

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq \{\subset\} \diamond (a \sim a^1)) \quad (T_{S472A})$$

Example Proof of (T_{S472})

$$\begin{aligned} (a^1 \subseteq S_c) &\subseteq (a \sim a^1 = \{\subset\} * a^1) & (T_{S366})(\text{Eqn1}) \\ (a \sim a^1 = \{\subset\} * a^1) &\subseteq (a \sim a^1 \subseteq \{\subset\} * a^1) & (T_{M425})(\text{Biconditional Elimination})(\text{Eqn2}) \\ (a^1 \subseteq S_c) &\subseteq (a \sim a^1 \subseteq \{\subset\} * a^1) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\ ((a^1 \subseteq S_c) \subseteq (a \sim a^1 \subseteq \{\subset\} * a^1)) &= E & (A_{MX16})(\text{Order Axiom}) \\ ((a^1 \subseteq S_c) = E) &\subseteq ((a \sim a^1 \subseteq \{\subset\} * a^1) = E) & (T_{M094})(\text{Probabilistic Rule})(\text{Eqn1}) \\ (((a \sim a^1) \neq \emptyset)(a^1 \neq \emptyset) = E) + ((a^1 = \emptyset)((a \sim a^1) = \emptyset) = E) &= ((a^1 \subseteq \{\supset\} \diamond (a \sim a^1)) = ((a \sim a^1) \subseteq \vee \{\supset\} * a^1)) & (T_{S028})(\text{Active/Passive Theorem}) \\ (((a \sim a^1) \neq \emptyset)(a^1 \neq \emptyset) = E) + ((a^1 = \emptyset)((a \sim a^1) = \emptyset) = E) &= ((a^1 \subseteq \{\supset\} \diamond (a \sim a^1)) = ((a \sim a^1) \subseteq \{\subset\} * a^1)) & (D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set}) \\ (((a \sim a^1) \neq \emptyset)(a^1 \neq \emptyset) = E) &\subseteq ((a^1 \subseteq \{\supset\} \diamond (a \sim a^1)) = ((a \sim a^1) \subseteq \{\subset\} * a^1)) & (T_{M072})(\text{Single Premise Syllogism})(\text{Eqn3}) \\ a \sim a^1 &\subseteq a & (T_{M083})(\text{Conjunction Elimination}) \\ (a \sim a^1 \subseteq a) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\ (a \sim a^1 \neq \emptyset) &\subseteq (a \neq \emptyset) & (T_{M096})(\text{Probabilistic Rule}) \\ (a \sim a^1 \neq \emptyset) &= (a \sim a^1 \neq \emptyset)(a \neq \emptyset) & (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn5}) \\ (((a \sim a^1) \neq \emptyset) = E) &\subseteq ((a^1 \subseteq \{\supset\} \diamond (a \sim a^1)) = ((a \sim a^1) \subseteq \{\subset\} * a^1)) & (A_{MX19})(\text{Substitution of Eqn5 into Eqn3})(\text{Eqn6}) \\ (a^1 \subseteq S_c) &= (a \subseteq S_c) & (T_{S604})(\text{Eqn7}) \\ (a \sim a^1 \subseteq a)(a \subseteq S_c) &\subseteq (a \sim a^1 \subseteq S_c) & (T_{M139})(\text{Hypothetical Syllogism})(\text{Eqn8}) \\ E(a \subseteq S_c) &\subseteq (a \sim a^1 \subseteq S_c) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn8}) \\ (a \subseteq S_c) &\subseteq (a \sim a^1 \subseteq S_c) & (A_{MX17})(\text{Universal Set Conjunction Axiom})(\text{Eqn9}) \\ (a \sim a^1 \subseteq S_c) &\subseteq ((a \sim a^1) \neq \emptyset) & (T_{S460})(\text{Eqn10}) \\ (a^1 \subseteq S_c) &\subseteq ((a \sim a^1) \neq \emptyset) & (T_{M139})(\text{Hypothetical Syllogism Eqns9\&10}) \\ ((a^1 \subseteq S_c) \subseteq ((a \sim a^1) \neq \emptyset)) &= E & (A_{MX16})(\text{Order Axiom}) \\ ((a^1 \subseteq S_c) = E) &\subseteq (((a \sim a^1) \neq \emptyset) = E) & (T_{M094})(\text{Probabilistic Rule})(\text{Eqn11}) \\ ((a^1 \subseteq S_c) = E) &\subseteq ((a^1 \subseteq \{\supset\} \diamond (a \sim a^1)) = ((a \sim a^1) \subseteq \{\subset\} * a^1)) & (T_{M139})(\text{Hypothetical Syllogism Eqns6\&11})(\text{Eqn12}) \\ ((a^1 \subseteq S_c) = E) &\subseteq ((a^1 \subseteq \{\supset\} \diamond (a \sim a^1)) = E) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn12}) \\ ((a^1 \subseteq S_c) \subseteq (a^1 \subseteq \{\supset\} \diamond (a \sim a^1))) &= E & (T_{M094})(\text{Probabilistic Rule}) \\ (a^1 \subseteq S_c) &\subseteq (a^1 \subseteq \{\supset\} \diamond (a \sim a^1)) & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$(a^1 \subseteq S_c) \subseteq (\{\|\} * a^1 \subseteq \{\|\} * a) \quad (T_{S476})$$

$$(a^1 \subseteq S_c) \subseteq (\{\|\} * a^1 \subseteq \{\|\} * a) \quad (T_{S476A})$$

Example Proof of (T_{S476})

$$\begin{aligned} (a^1 \subseteq S_c) &\subseteq (a^1 \subseteq a) & (T_{R716}) \\ (a^1 \subseteq S_c) &\subseteq (\{\|\} * a^1 \subseteq \{\|\} * a) & (T_{R196})(\text{Relational Deduction}) \end{aligned}$$

$$(a^1 \subseteq S_c) \subseteq (a \sim a^1 = (\{\supset\} * a).(\{\subset\} * a)) \quad (T_{R703})$$

$$(a^1 \subseteq S_c) \subseteq (a \cdot \sim a^1 = (\{\subset\} * a) \cdot (\{\supset\} * a)) \quad (T_{R703A})$$

Example Proof of (T_{R703})

$$\begin{aligned} (a^1 \subseteq S_c) &= (a \cdot \sim a^1 = \{\subset\} * a) & (D_{R62})(\text{Definition of a Continuum Set Part2}) \\ (a^1 \subseteq S_c) &\subseteq (a \cdot \sim a^1 = a \cdot \{\subset\} * a) & (T_{M772})(\text{Eqn1}) \\ (a^1 \subseteq S_c) &= (a = (\{\supset\} * a) \cdot (\{\supset\} * a)) & (D_{R59})(\text{Definition of a Continuum Set Part3}) \\ (a^1 \subseteq S_c) &\subseteq (a = \{\supset\} * a) & (T_{M073b})(\text{Single Premise Syllogism})(\text{Eqn2}) \\ (a^1 \subseteq S_c) &\subseteq (a \cdot \sim a^1 = (\{\supset\} * a) \cdot (\{\subset\} * a)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(a^1 \subseteq S_c) \subseteq (a \cdot \sim a^1 \subseteq \{\|\} * a) \quad (T_{S474})$$

$$(a^1 \subseteq S_c) \subseteq (a \cdot \sim a^1 \subseteq \{\|\} * a) \quad (T_{S474A})$$

Example Proof of (T_{S474})

$$\begin{aligned} (a^1 \subseteq S_c) &= (a \cdot \sim a^1 = \{\|\} * a \cdot \{\|\} * a^1) & (D_{R58})(\text{Definition of a Continuum Set Part1}) \\ (a^1 \subseteq S_c) &\subseteq (a \cdot \sim a^1 \subseteq \{\|\} * a) & (T_{M073b})(\text{Single Premise Syllogism}) \end{aligned}$$

$$(a^1 \subseteq S_c) \subseteq (a \cdot \sim a^1 = a \cdot \sim a^1 \cdot \{\|\} * a) \quad (T_{S364})$$

$$(a^1 \subseteq S_c) \subseteq (a \cdot \sim a^1 = a \cdot \sim a^1 \cdot \{\|\} * a) \quad (T_{S364A})$$

Example Proof of (T_{S364})

$$\begin{aligned} (a^1 \subseteq S_c) &\subseteq (a \cdot \sim a^1 \subseteq \{\|\} * a) & (T_{S474}) \\ (a^1 \subseteq S_c) &\subseteq (a \cdot \sim a^1 = a \cdot \sim a^1 \cdot \{\|\} * a) & (A_{MX44})(\subseteq \text{Equality Form}) \end{aligned}$$

$$(a^1 \subseteq S_c) \subseteq (\{\subset\} * a \subseteq \{\|\} * a) \quad (T_{S492})$$

$$(a^1 \subseteq S_c) \subseteq (\{\supset\} * a \subseteq \{\|\} * a) \quad (T_{S492A})$$

Example Proof of (T_{S492})

$$\begin{aligned} (a^1 \subseteq S_c) &\subseteq (a \cdot \sim a^1 \subseteq \{\|\} * a) & (T_{S474}) \\ (a^1 \subseteq S_c) &\subseteq (\{\subset\} * a \subseteq \{\|\} * a) & (T_{S368}) \end{aligned}$$

$$(a^1 \subseteq S_c) \subseteq ((\{\subset\} * a) \cdot (\{\supset\} * a) \subseteq \{\|\} * a) \quad (T_{S467})$$

$$(a^1 \subseteq S_c) \subseteq ((\{\supset\} * a) \cdot (\{\subset\} * a) \subseteq \{\|\} * a) \quad (T_{S467A})$$

Example Proof of (T_{S467})

$$\begin{aligned} (a^1 \subseteq S_c) &= (a \cdot \sim a^1 = (\{\|\} * a) \cdot (\{\|\} * a^1)) & (D_{R58})(\text{Definition of a Continuum Set Part 1}) \\ (a^1 \subseteq S_c) &\subseteq (a \cdot \sim a^1 \subseteq (\{\|\} * a)) & (T_{M073b})(\text{Single Premise Syllogism})(\text{Eqn1}) \\ (a^1 \subseteq S_c) &\subseteq (a \cdot \sim a^1 = (\{\supset\} * a) \cdot (\{\subset\} * a)) & (T_{R703})(\text{Eqn2}) \\ (a^1 \subseteq S_c) &\subseteq ((\{\subset\} * a) \cdot (\{\supset\} * a) \subseteq \{\|\} * a) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(E^1 \subseteq S_c) \subseteq ((\{\subset\} * E) \cdot (\{\supset\} * E) \subseteq \{\|\} * E) \quad (T_{S344})$$

$$(\emptyset^1 \subseteq S_c) \subseteq ((\{\supset\} * E) \cdot (\{\subset\} * E) \subseteq \{\|\} * E) \quad (T_{S344A})$$

Example Proof of (T_{S344})

$$\begin{aligned} (a^1 \subseteq S_c) &\subseteq ((\{\subset\} * a) \cdot (\{\supset\} * a) \subseteq \{\|\} * a) & (T_{S467})(\text{Eqn1}) \\ \text{Let } a &= E & (\text{Eqn2}) \\ a^1 &= E^1 & (T_{M488})(\text{Eqn3}) \\ (E^1 \subseteq S_c) &\subseteq ((\{\subset\} * E) \cdot (\{\supset\} * E) \subseteq \{\|\} * E) & (A_{MX27})(\text{Substitution of Eqns2\&3 into Eqn1})(\text{Eqn4}) \end{aligned}$$

$$(E^1 \subseteq S_c) \subseteq (\{\|\} * E \subseteq (\{\subset\} * E) + (\{\supset\} * E)) \quad (T_{S347})$$

$$(\emptyset^1 \subseteq S_c) \subseteq (\{\|\} * E \subseteq (\{\supset\} * E) + (\{\subset\} * E)) \quad (T_{S347A})$$

Example Proof of (T_{S347})

$$\begin{aligned} (E^1 \subseteq S_c) &\subseteq ((\{\subset\} * E) \cdot (\{\supset\} * E) \subseteq \{\|\} * E) & (T_{S344}) \\ (E^1 \subseteq S_c) &\subseteq (\{\|\} * E \subseteq \sim(\{\subset\} * E) \cdot (\{\supset\} * E)) & (T_{M012})(\text{Transposition}) \\ (E^1 \subseteq S_c) &\subseteq (\{\|\} * E \subseteq (\{\subset\} * E) + (\{\supset\} * E)) & (T_{M025})(\text{DeMorgan's Law}) \end{aligned}$$

$$(E^1 \subseteq S_c) \subseteq (S\{\neg\}.S\{\supset\} \subseteq S\{\parallel\}) \quad (T_{S345})$$

$$(\emptyset^1 \subseteq S_c) \subseteq (S\{\supset\}.S\{\neg\} \subseteq S\{\parallel\}) \quad (T_{S345A})$$

Example Proof of (T_{S345})

$$(E^1 \subseteq S_c) \subseteq ((\{\neg\} * E).(\{\supset\} * E) \subseteq \{\parallel\} * E) \quad (T_{S344})$$

$$(E^1 \subseteq S_c) \subseteq (S\{\neg\}.S\{\supset\} \subseteq S\{\parallel\}) \quad (T_{R210})$$

$$(E^1 \subseteq S_c) \subseteq (\sim S\{\parallel\} \subseteq \sim S\{\neg\} + \sim S\{\supset\}) \quad (T_{S348})$$

$$(\emptyset^1 \subseteq S_c) \subseteq (\sim S\{\parallel\} \subseteq \sim S\{\supset\} + \sim S\{\neg\}) \quad (T_{S348A})$$

Example Proof of (T_{S348})

$$(E^1 \subseteq S_c) \subseteq (S\{\neg\}.S\{\supset\} \subseteq S\{\parallel\}) \quad (T_{S345})$$

$$(E^1 \subseteq S_c) \subseteq (\sim S\{\parallel\} \subseteq \sim (S\{\neg\}.S\{\supset\})) \quad (T_{M012})(\text{Transposition})$$

$$(E^1 \subseteq S_c) \subseteq (\sim S\{\parallel\} \subseteq \sim S\{\neg\} + \sim S\{\supset\}) \quad (T_{M025})(\text{DeMorgan's Law})$$

$$(a^1 \subseteq S_c) \subseteq ((\{\parallel\} * a) \subseteq (\{\neg\} * a) + (\{\supset\} * a)) \quad (T_{S346})$$

$$(a^1 \subseteq S_c) \subseteq ((\{\parallel\} * a) \subseteq (\{\supset\} * a) + (\{\neg\} * a)) \quad (T_{S346A})$$

Example Proof of (T_{S346})

$$(a^1 \subseteq S_c) \subseteq ((\{\neg\} * a).(\{\supset\} * a) \subseteq \{\parallel\} * a) \quad (T_{S467})$$

$$(a^1 \subseteq S_c) \subseteq (\{\parallel\} * a \subseteq \sim (\{\neg\} * a).(\{\supset\} * a)) \quad (T_{M012})(\text{Transposition})$$

$$(a^1 \subseteq S_c) \subseteq (\{\parallel\} * a \subseteq (\{\neg\} * a) + (\{\supset\} * a)) \quad (T_{M025})(\text{DeMorgan's Law})$$

$$(a^1 \subseteq S_c) \subseteq (a.(\{\neg\} * a).(\{\supset\} * a) \subseteq a.(\{\parallel\} * a)) \quad (T_{R702})$$

$$(a^1 \subseteq S_c) \subseteq (a.(\{\supset\} * a).(\{\neg\} * a) \subseteq a.(\{\parallel\} * a)) \quad (T_{R702A})$$

Example Proof of (T_{R702})

$$(a^1 \subseteq S_c) \subseteq ((\{\neg\} * a).(\{\supset\} * a) \subseteq \{\parallel\} * a) \quad (T_{S467})$$

$$(a^1 \subseteq S_c) \subseteq (a.(\{\neg\} * a).(\{\supset\} * a) \subseteq a.(\{\parallel\} * a)) \quad (T_{M029})(\text{Deduction})$$

$$((a^1 \subseteq S_c) = E) \subseteq ((x^1 \subseteq a.(\{\neg\} * a).(\{\supset\} * a)) \subseteq (x^1 \subseteq a.(\{\parallel\} * a))) \quad (T_{S343})$$

$$((a^1 \subseteq S_c) = E) \subseteq ((x^1 \subseteq a.(\{\supset\} * a).(\{\neg\} * a)) \subseteq (x^1 \subseteq a.(\{\parallel\} * a))) \quad (T_{S343A})$$

Example Proof of (T_{S343})

$$(a^1 \subseteq S_c) \subseteq (a.(\{\neg\} * a).(\{\supset\} * a) \subseteq a.(\{\parallel\} * a)) \quad (T_{R702})$$

$$((a^1 \subseteq S_c)(x^1 \neq \emptyset) = E) \subseteq ((x^1 \subseteq a.(\{\neg\} * a).(\{\supset\} * a)) \subseteq (x^1 \subseteq a.(\{\parallel\} * a))) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})$$

$$(a^1 \subseteq S_c)(x^1 \neq \emptyset)(x^1 \subseteq a.(\{\neg\} * a).(\{\supset\} * a)) \subseteq (x^1 \subseteq a.(\{\parallel\} * a)) \quad (A_{MX35})(\text{Exportation/Importation for truth sets Eqn1})$$

$$\text{Let } a^1 \subseteq S_c \quad (Eqn2)$$

$$\text{Let } x^1 = \emptyset \quad (Eqn2)$$

$$\text{Let } (x^1 \subseteq a.(\{\neg\} * a).(\{\supset\} * a)) \not\subseteq (x^1 \subseteq a.(\{\parallel\} * a)) \quad (Eqn3)$$

$$(\emptyset \subseteq a.(\{\neg\} * a).(\{\supset\} * a)) \not\subseteq (\emptyset \subseteq a.(\{\parallel\} * a)) \quad (A_{MX21})(\text{Substitution of Eqn2 into Eqn3})$$

$$E \not\subseteq E \quad (T_{M001})(\text{Universality of } (T_{M670}) \text{ Null Set Theorem})$$

$$(E \sim E)(E \neq E) \quad (T_{M198})$$

$$(E \sim E) \emptyset \quad (T_{M345})$$

$$\emptyset \quad (T_{M050})(\text{Null Set Conjunction})$$

$$\text{Thus } ((a^1 \subseteq S_c)(x^1 = \emptyset) = E)((x^1 \subseteq a.(\{\neg\} * a).(\{\supset\} * a)) \not\subseteq (x^1 \subseteq a.(\{\parallel\} * a))) = \emptyset \quad (A_{MX02})(\text{Conditional Introduction})$$

$$((a^1 \subseteq S_c)(x^1 = \emptyset) = E) \subseteq ((x^1 \subseteq b.(\{\neg\} * b).(\{\supset\} * b)) \subseteq (x^1 \subseteq b.(\{\parallel\} * b))) \quad (T_{M735})(\subseteq \text{Existential Form})$$

$$(a^1 \subseteq S_c)(x^1 = \emptyset)(x^1 \subseteq a.\{\{\subset\}^*a\}.\{\{\supset\}^*a\}) \subseteq (x^1 \subseteq a.\{\{\|\}^*a\})$$

(*AMX35*)(Exportation/Importation) for truth sets
(Eqn4)

$$(a^1 \subseteq S_c)(x^1 = \emptyset)(x^1 \subseteq a.\{\{\subset\}^*a\}.\{\{\supset\}^*a\}) + (a^1 \subseteq S_c)(x^1 \neq \emptyset)(x^1 \subseteq a.\{\{\subset\}^*a\}.\{\{\supset\}^*a\}) \subseteq (x^1 \subseteq a.\{\{\|\}^*a\}) + (x^1 \subseteq a.\{\{\|\}^*a\})$$

(*AMX27*)(Combination of Eqns1&4)

$$(a^1 \subseteq S_c)(x^1 = \emptyset)(x^1 \subseteq a.\{\{\subset\}^*a\}.\{\{\supset\}^*a\}) + (a^1 \subseteq S_c)(x^1 \neq \emptyset)(x^1 \subseteq a.\{\{\subset\}^*a\}.\{\{\supset\}^*a\}) \subseteq (x^1 \subseteq a.\{\{\|\}^*a\})$$

(*AMX07*)(Disjunction Tautology)

$$((a^1 \subseteq S_c)(x^1 = \emptyset) + (a^1 \subseteq S_c)(x^1 \neq \emptyset))(x^1 \subseteq a.\{\{\subset\}^*a\}.\{\{\supset\}^*a\}) \subseteq (x^1 \subseteq a.\{\{\|\}^*a\})$$

(*TM407*)(Distribution of . over +)

$$((a^1 \subseteq S_c)((x^1 = \emptyset) + (x^1 \neq \emptyset))(x^1 \subseteq a.\{\{\subset\}^*a\}.\{\{\supset\}^*a\}) \subseteq (x^1 \subseteq a.\{\{\|\}^*a\})$$

(*TM407*)(Distribution of . over +)

$$(a^1 \subseteq S_c)E(x^1 \subseteq a.\{\{\subset\}^*a\}.\{\{\supset\}^*a\}) \subseteq (x^1 \subseteq a.\{\{\|\}^*a\})$$

(*TM671*)(Tertium non Datur)

$$(a^1 \subseteq S_c)(x^1 \subseteq a.\{\{\subset\}^*a\}.\{\{\supset\}^*a\}) \subseteq (x^1 \subseteq a.\{\{\|\}^*a\})$$

(*AMX17*)(Universal Set Conjunction Axiom)

$$(a^1 \subseteq S_c)(x^1 \subseteq a.\{\{\subset\}^*a\}).(x^1 \subseteq a.\{\{\supset\}^*a\}) \subseteq (x^1 \subseteq a.\{\{\|\}^*a\})$$

(*TM132*)(Composition)

$$((a^1 \subseteq S_c) = E) \subseteq ((x^1 \subseteq a.\{\{\subset\}^*a\}).(x^1 \subseteq a.\{\{\supset\}^*a\}) \subseteq (x^1 \subseteq a.\{\{\|\}^*a\}))$$

(*AMX35*)(Exportation/Importation) for truth sets

$$(E^1 \subseteq S_c) = (\sim E^1 \subseteq \{\|\}^*E)$$

(*TS352*)

$$(\emptyset^1 \subseteq S_c) = (\sim \emptyset^1 \subseteq \{\|\}^*E)$$

(*TS352A*)

Example Proof of (*TS352*)

$$(a^1 \subseteq S_c) = (a.\sim a^1 \subseteq \{\|\}^*a)$$

(*TS474*)(Eqn1)

$$\text{Let } a = E$$

(Eqn2)

$$a^1 = E^1$$

(*TM488*)(Eqn3)

$$(E^1 \subseteq S_c) = (E.\sim E^1 \subseteq \{\|\}^*E)$$

(*AMX19*)(Substitution of Eqns2&3 into Eqn1)

$$(E^1 \subseteq S_c) = (\sim E^1 \subseteq \{\|\}^*E)$$

(*AMX17*)(Universal Set Conjunction Axiom)

$$(a^1 \subseteq S_c) \subseteq (a^1 \subset a)$$

(*TS666*)

$$(a^1 \subseteq S_c) \subseteq (a^1 \subset a)$$

(*TS666A*)

Example Proof of (*TS666*)

$$(a^1 \subseteq S_c) \subseteq (a^1 \subseteq a)$$

(*TR716*)

$$(a^1 \subseteq S_c) \subseteq (a^1 \subset a) + (a^1 = a)$$

(*TM266*)(Definition of $\subseteq$)(Eqn1)

$$(a^1 \subseteq S_c) \subseteq (a \neq a^1)$$

(*TS495*)(Eqn2)

$$(a^1 \subseteq S_c)(a^1 \subseteq S_c) \subseteq (a \neq a^1)((a^1 \subset a) + (a^1 = a))$$

(*TM669*)(Combination of Eqns1&2)

$$(a^1 \subseteq S_c) \subseteq (a \neq a^1)((a^1 \subset a) + (a^1 = a))$$

(*TM667*)(Conjunction Tautology)

$$(a^1 \subseteq S_c) \subseteq (a \neq a^1)(a^1 \subset a) + (a \neq a^1)(a^1 = a)$$

(*TM407*)(Distribution of . over +)

$$(a^1 \subseteq S_c) \subseteq (a \neq a^1)(a^1 \subset a) + \emptyset$$

(*TM627*)(Complement Conjunction)

$$(a^1 \subseteq S_c) \subseteq (a \neq a^1)(a^1 \subset a)$$

(*TM668*)(Null Set Disjunction)

$$(a^1 \subseteq S_c) \subseteq (a^1 \subset a)$$

(*TM073*)(Single Premise Syllogism)

$$(a^1 \subseteq S_c) \subseteq (a.\sim a^1 \neq \emptyset)$$

(*TS667*)

$$(a^1 \subseteq S_c) \subseteq (a.\sim a^1 \neq \emptyset)$$

(*TS667A*)

Example Proof of (*TS667*)

$$(a^1 \subseteq S_c) \subseteq (a^1 \subset a)$$

(*TS666*)(Eqn1)

$$(a^1 \subset a) = (a^1 a = \emptyset)(\sim a^1.a \neq \emptyset)$$

(*TM718*)(Definition of $\subset$)

$$(a^1 \subset a) \subseteq (\sim a^1.a \neq \emptyset)$$

(*TM073b*)(Single Premise Syllogism)(Eqn2)

$$(a^1 \subseteq S_c) \subseteq (a.\sim a^1 \neq \emptyset)$$

(*TM139*)(Hypothetical Syllogism Eqns1&2)

$$(a^1 \subseteq S_c) \subseteq (\forall Sa.\{\subset\}.\forall Qa \neq \emptyset)$$

(*TS708*)

$$(a^1 \subseteq S_c) \subseteq (\forall Sa.\{\supset\}.\forall Qa \neq \emptyset)$$

(*TS708A*)

Example Proof of (*TS708*)

$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 = (\{\supset\} * a). (\{\subset\} * a))$	(T_{R703})
$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 \subseteq \{\subset\} * a)$	$(T_{M073b})(\text{Single Premise Syllogism})$
$(a^1 \subseteq S_c) \subseteq (a. a. \sim a^1 \subseteq a. \{\subset\} * a)$	$(T_{M029})(\text{Deduction})$
$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 \subseteq a. \{\subset\} * a)$	$(T_{M667})(\text{Conjunction Tautology})$
$((a^1 \subseteq S_c) \subseteq (a. \sim a^1 \subseteq a. \{\subset\} * a)) = E$	$(A_{MX16})(\text{Order Axiom})$
$((a^1 \subseteq S_c) = E) \subseteq ((a. \sim a^1 \subseteq a. \{\subset\} * a) = E)$	$(T_{M094})(\text{Probabilistic Rule})$
$((a^1 \subseteq S_c) = E) \subseteq ((a. \sim a^1 \neq \emptyset) \subseteq (a. \{\subset\} * a \neq \emptyset))$	$(T_{M096})(\text{Probabilistic Rule})$
$(a^1 \subseteq S_c)(a. \sim a^1 \neq \emptyset) \subseteq (a. \{\subset\} * a \neq \emptyset)$	$(A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets (Eqn1)}$
$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 \neq \emptyset)$	(T_{S667})
$(a^1 \subseteq S_c) = (a^1 \subseteq S_c)(a. \sim a^1 \neq \emptyset)$	$(A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn2})$
$(a^1 \subseteq S_c) \subseteq (a. \{\subset\} * a \neq \emptyset)$	$(A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$
$(a^1 \subseteq S_c) \subseteq (a \not\subseteq \{\subset\} * a)$	$(T_{M735})(\subseteq \text{Existential Form}) \ \& \ (T_{M008})(\text{Inverse of an Equality})$
$(a^1 \subseteq S_c) \subseteq (\forall Sa. \{\subset\}. \forall Qa \neq \emptyset)$	(T_{S081})

$$(a^1 \subseteq S_c) \subseteq (\forall Sa. \{\supset\}. \forall Qa \neq \emptyset) \quad (T_{S709})$$

$$(a^1 \subseteq S_c) \subseteq (\forall Sa. \{\subset\}. \forall Qa \neq \emptyset) \quad (T_{S709A})$$

Example Proof of (T_{S709})

$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 = (\{\supset\} * a). (\{\subset\} * a))$	(T_{R703})
$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 \subseteq \{\supset\} * a)$	$(T_{M073b})(\text{Single Premise Syllogism})$
$(a^1 \subseteq S_c) \subseteq (a. a. \sim a^1 \subseteq a. \{\supset\} * a)$	$(T_{M029})(\text{Deduction})$
$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 \subseteq a. \{\supset\} * a)$	$(T_{M667})(\text{Conjunction Tautology})$
$((a^1 \subseteq S_c) \subseteq (a. \sim a^1 \subseteq a. \{\supset\} * a)) = E$	$(A_{MX16})(\text{Order Axiom})$
$((a^1 \subseteq S_c) = E) \subseteq ((a. \sim a^1 \subseteq a. \{\supset\} * a) = E)$	$(T_{M094})(\text{Probabilistic Rule})$
$((a^1 \subseteq S_c) = E) \subseteq ((a. \sim a^1 \neq \emptyset) \subseteq (a. \{\supset\} * a \neq \emptyset))$	$(T_{M096})(\text{Probabilistic Rule})$
$(a^1 \subseteq S_c)(a. \sim a^1 \neq \emptyset) \subseteq (a. \{\supset\} * a \neq \emptyset)$	$(A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets (Eqn1)}$
$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 \neq \emptyset)$	(T_{S667})
$(a^1 \subseteq S_c) = (a^1 \subseteq S_c)(a. \sim a^1 \neq \emptyset)$	$(A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn2})$
$(a^1 \subseteq S_c) \subseteq (a. \{\supset\} * a \neq \emptyset)$	$(A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$
$(a^1 \subseteq S_c) \subseteq (a \not\subseteq \{\supset\} * a)$	$(T_{M735})(\subseteq \text{Existential Form}) \ \& \ (T_{M008})(\text{Inverse of an Equality})$
$(a^1 \subseteq S_c) \subseteq (\forall Sa. \{\supset\}. \forall Qa \neq \emptyset)$	(T_{S081})

$$(a^1 \subseteq S_c) \subseteq (\forall Sa. \{\|\}. \forall Qa \neq \emptyset) \quad (T_{S665})$$

$$(a^1 \subseteq S_c) \subseteq (\forall Sa. \{\|\}. \forall Qa \neq \emptyset) \quad (T_{S665A})$$

Example Proof of (T_{S665})

$(a^1 \subseteq S_c) = (a. \sim a^1 \subseteq \{\ \} * a)$	$(T_{S474})(\text{Definition of a Continuum Set})$
$(a^1 \subseteq S_c) \subseteq (a. a. \sim a^1 \subseteq a. \{\ \} * a)$	$(T_{M029})(\text{Deduction})$
$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 \subseteq a. \{\ \} * a)$	$(T_{M667})(\text{Conjunction Tautology})$
$((a^1 \subseteq S_c) \subseteq (a. \sim a^1 \subseteq a. \{\ \} * a)) = E$	$(A_{MX16})(\text{Order Axiom})$
$((a^1 \subseteq S_c) = E) \subseteq ((a. \sim a^1 \subseteq a. \{\ \} * a) = E)$	$(T_{M094})(\text{Probabilistic Rule})$
$((a^1 \subseteq S_c) = E) \subseteq ((a. \sim a^1 \neq \emptyset) \subseteq (a. \{\ \} * a \neq \emptyset))$	$(T_{M096})(\text{Probabilistic Rule})$
$(a^1 \subseteq S_c)(a. \sim a^1 \neq \emptyset) \subseteq (a. \{\ \} * a \neq \emptyset)$	$(A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets (Eqn1)}$
$(a^1 \subseteq S_c) \subseteq (a. \sim a^1 \neq \emptyset)$	(T_{S667})
$(a^1 \subseteq S_c) = (a^1 \subseteq S_c)(a. \sim a^1 \neq \emptyset)$	$(A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn2})$
$(a^1 \subseteq S_c) \subseteq (a. \{\ \} * a \neq \emptyset)$	$(A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$
$(a^1 \subseteq S_c) \subseteq (a \not\subseteq \{\ \} * a)$	$(T_{M735})(\subseteq \text{Existential Form}) \ \& \ (T_{M008})(\text{Inverse of an Equality})$
$(a^1 \subseteq S_c) \subseteq (\forall Sa. \{\ \}. \forall Qa \neq \emptyset)$	(T_{S081})

$$(a^1 \subseteq S_c) \subseteq (\forall Sa. \{+\}. \forall Qa \neq \emptyset) \quad (T_{S669})$$

$$(a^1 \subseteq S_c) \subseteq (\forall Sa. \{+\}. \forall Qa \neq \emptyset) \quad (T_{S669A})$$

Example Proof of (T_{S669})

$(a^1 \subseteq S_c) \subseteq (\forall Sa. \{\ \}. \forall Qa \neq \emptyset)$	$(T_{S665})(\text{Eqn1})$
$\{\ \} \subseteq \{+\}$	(T_{A105})

$\forall \text{Sa}.\{\ \}. \forall \text{Qa} \subseteq \forall \text{Sa}.\{\neq\}. \forall \text{Qa}$	(T_{M029})(Deduction)
$(\forall \text{Sa}.\{\ \}. \forall \text{Qa} \subseteq \forall \text{Sa}.\{\neq\}. \forall \text{Qa}) = \text{E}$	($AMX16$)(Order Axiom)
$(\forall \text{Sa}.\{\ \}. \forall \text{Qa} \neq \emptyset) \subseteq (\forall \text{Sa}.\{\neq\}. \forall \text{Qa} \neq \emptyset)$	(T_{M096})(Probabilistic Rule)(Eqn2)
$(a^1 \subseteq \text{Sc}) \subseteq (\forall \text{Sa}.\{\neq\}. \forall \text{Qa} \neq \emptyset)$	(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$(a^1 \subseteq \text{Sc}) \subseteq (a \neq a^1)(a \neq \emptyset) \quad (T_{S668})$$

$$(a^1 \subseteq \text{Sc}) \subseteq (a \neq a^1)(a \neq \emptyset) \quad (T_{S668A})$$

Example Proof of (T_{S668})

$(a^1 \subseteq \text{Sc}) \subseteq (a^1 \subset a)$	(T_{S666})(Eqn1)
$(a^1 \subset a) \subseteq (a \neq \emptyset)$	(T_{M540})(Eqn2)
$(a^1 \subseteq \text{Sc}) \subseteq (a \neq \emptyset)$	(T_{M139})(Hypothetical Syllogism Eqns1&2)(Eqn3)
$(a^1 \subseteq \text{Sc}) \subseteq (a^1 \subset a)$	(T_{S666})(Eqn1)
$(a^1 = a)(a^1 \subset a) = \emptyset$	(T_{M037})(Conjunction of Binary Relations)
$(a^1 \subset a) \subseteq (a \neq a^1)$	(T_{M735})($\subseteq$ Existential Form)(Eqn4)
$(a^1 \subseteq \text{Sc}) \subseteq (a \neq a^1)$	(T_{M139})(Hypothetical Syllogism Eqns1&4)(Eqn5)
$(a^1 \subseteq \text{Sc})(a^1 \subseteq \text{Sc}) \subseteq (a \neq a^1)(a \neq \emptyset)$	(T_{M669})(Combination of Eqns3&5)
$(a^1 \subseteq \text{Sc}) \subseteq (a \neq a^1)(a \neq \emptyset)$	(T_{M667})(Conjunction Tautology)

$$(a^1 \subseteq \text{Sc}) \subseteq (a^1 \subseteq \{\supset\} * a) \quad (T_{S491})$$

$$(a^1 \subseteq \text{Sc}) \subseteq (a^1 \subseteq \{\subset\} * a) \quad (T_{S491A})$$

Example Proof of (T_{S491})

$(a^1 \subseteq \text{Sc}) = (a = \{\supset\} * a)$	($DR59$)(Definition of a Continuum Set Part3)
$(a^1 \subseteq \text{Sc}) \subseteq (a^1 \subseteq a)$	(T_{R716})(Eqn2)
$(a^1 \subseteq \text{Sc}) \subseteq (a^1 \subseteq \{\supset\} * a)$	($AMX19$)(Substitution of Eqn1 into Eqn2)

$$(a^1 \subseteq \text{Sc}) \subseteq (a = a.\{\supset\} * a) \quad (T_{S362})$$

$$(a^1 \subseteq \text{Sc}) \subseteq (a = a.\{\subset\} * a) \quad (T_{S362A})$$

Example Proof of (T_{S362})

$(a^1 \subseteq \text{Sc}) = (a = (\{\supset\} * a)(\{\supset\} * a))$	($DR59$)(Definition of a Continuum Set Part3)
$(a^1 \subseteq \text{Sc}) \subseteq (a = \{\supset\} * a)$	(T_{M073b})(Single Premise Syllogism)(Eqn1)
$(a = \{\supset\} * a) \subseteq (a.a = a.\{\supset\} * a)$	(T_{M029})(Deduction)(Eqn2)
$(a^1 \subseteq \text{Sc}) \subseteq (a.a = a.\{\supset\} * a)$	($AMX19$)(Substitution of Eqn1 into Eqn2)
$(a^1 \subseteq \text{Sc}) \subseteq (a = a.\{\supset\} * a)$	(T_{M667})(Conjunction Tautology)

$$(b^1 \subseteq \text{Sc}) \subseteq (\{\supset\} * \{\supset\} * b = \{\supset\} * b) \quad (T_{A438}) \text{ Super-Entity Function Rule for Continuum Sets}$$

$$(b^1 \subseteq \text{Sc}) \subseteq (\{\subset\} * \{\subset\} * b = \{\subset\} * b) \quad (T_{A438A}) \text{ Super-Entity Function Rule for Continuum Sets}$$

Example Proof of (T_{A438})

$(c^1 \subseteq R_T) \subseteq (c * c * b \subseteq c * b)$	(T_{S177})(Definition of a Transitive Relationship Set)(Eqn1)
Let $c = \{\supset\}$	(Eqn2)
$\{\supset\}^1 = c^1$	(T_{M479})(Eqn3)
$(\{\supset\}^1 \subseteq R_T) \subseteq (\{\supset\} * \{\supset\} * b \subseteq \{\supset\} * b)$	($AMX19$)(Substitution of Eqns2&3 into Eqn1)(Eqn4)
$\{\supset\}^1 \subseteq R_T$	(T_{A247})
$(\{\supset\}^1 \subseteq R_T) = \text{E}$	($AMX16$)(Order Axiom)(Eqn5)
$\text{E} \subseteq (\{\supset\} * \{\supset\} * b \subseteq \{\supset\} * b)$	($AMX19$)(Substitution of Eqn5 into Eqn4)(Eqn6)
$(b^1 \subseteq \text{Sc}) \subseteq (b = b.\{\supset\} * b)$	(T_{S362})
$(b^1 \subseteq \text{Sc}) \subseteq (b \subseteq \{\supset\} * b)$	($AMX44$)($\subseteq$ Equality Form)
$(b^1 \subseteq \text{Sc}) \subseteq (\{\supset\} * b \subseteq \{\supset\} * \{\supset\} * b)$	(T_{R196})(Relational Deduction)(Eqn7)
$(b^1 \subseteq \text{Sc}) \text{E} \subseteq (\{\supset\} * b \subseteq \{\supset\} * \{\supset\} * b)(\{\supset\} * \{\supset\} * b \subseteq \{\supset\} * b)$	(T_{M669})(Combination of Eqns6&7)
$(b^1 \subseteq \text{Sc}) \subseteq (\{\supset\} * b \subseteq \{\supset\} * \{\supset\} * b)(\{\supset\} * \{\supset\} * b \subseteq \{\supset\} * b)$	($AMX11$)(Universal Set Conjunction Axiom)

$$(b^1 \subseteq S_c) \subseteq (\{\supset\} * b = \{\supset\} * \{\supset\} * b)$$

(T_{M430})(Biconditional Introduction)

$$(b^1 \subseteq S_c) \subseteq (\{\supset\} * \{\supset\} * b = \{\supset\} * b)$$

(T_{S899}) Sub-Entity Function Rule for Continuum Sets

$$(\hat{b}^1 \subseteq S_c) \subseteq (\{\supset\} * \{\supset\} * \hat{b} = \{\supset\} * \hat{b})$$

(T_{S899A}) Sub-Entity Function Rule for Continuum Sets

Example Proof of (T_{S899})

$$(c^1 \subseteq R_T) \subseteq (c * c * b \subseteq c * b)$$

(T_{S177})(Definition of a Transitive Relationship Set)(Eqn1)

$$\text{Let } c = \{\supset\}$$

(Eqn2)

$$\{\supset\}^1 = c^1$$

(T_{M479})(Eqn3)

$$(\{\supset\}^1 \subseteq R_T) \subseteq (\{\supset\} * \{\supset\} * b \subseteq \{\supset\} * b)$$

(A_{MX19})(Substitution of Eqns2&3 into Eqn1)(Eqn4)

$$\{\supset\}^1 \subseteq R_T$$

(T_{A241})

$$(\{\supset\}^1 \subseteq R_T) = E$$

(A_{MX16})(Order Axiom)(Eqn5)

$$E \subseteq (\{\supset\} * \{\supset\} * b \subseteq \{\supset\} * b)$$

(A_{MX19})(Substitution of Eqn5 into Eqn4)(Eqn6)

$$(b^1 \subseteq S_c) \subseteq (\{\supset\} * b \subseteq \{\supset\} * \{\supset\} * b)$$

(A_{R60})(Sub-Entity Function Axiom for Continuum Sets)(Eqn7)

$$(b^1 \subseteq S_c)E \subseteq (\{\supset\} * b \subseteq \{\supset\} * \{\supset\} * b)(\{\supset\} * \{\supset\} * b \subseteq \{\supset\} * b)$$

(T_{M669})(Combination of Eqns6&7)

$$(b^1 \subseteq S_c) \subseteq (\{\supset\} * b \subseteq \{\supset\} * \{\supset\} * b)(\{\supset\} * \{\supset\} * b \subseteq \{\supset\} * b)$$

(A_{MX11})(Universal Set Conjunction Axiom)

$$(b^1 \subseteq S_c) \subseteq (\{\supset\} * b = \{\supset\} * \{\supset\} * b)$$

(T_{M430})(Biconditional Introduction)

$$(b^1 \subseteq S_c) \subseteq (\{\supset\} * \{\supset\} * b^1 = \{\supset\} * b^1)$$

(T_{S900}) Sub-Entity Function Rule for Continuum Sets

$$(\hat{b}^1 \subseteq S_c) \subseteq (\{\supset\} * \{\supset\} * \hat{b}^1 = \{\supset\} * \hat{b}^1)$$

(T_{S900A}) Sub-Entity Function Rule for Continuum Sets

Example Proof of (T_{S900})

$$(b^1 \subseteq S_c) \subseteq (\{\supset\} * \{\supset\} * b = \{\supset\} * b)$$

(T_{S899})(Sub-Entity Function Rule for Continuum Sets)

$$(b^1 \subseteq S_c) \subseteq (\{\supset\} * \{\supset\} * b^1 = \{\supset\} * b^1)$$

(T_{S367})

$$((b^1 \subseteq S_c) = E) \subseteq ((b_r^1 \subseteq b) \subseteq (\{\supset\} * \{\supset\} * b_r^1 = \{\supset\} * b_r^1))$$

(T_{S831})

$$((\hat{b}^1 \subseteq S_c) = E) \subseteq ((\hat{b}_r^1 \subseteq \hat{b}) \subseteq (\{\supset\} * \{\supset\} * \hat{b}_r^1 = \{\supset\} * \hat{b}_r^1))$$

(T_{S831A})

Example Proof of (T_{S831})

$$\text{Let } b^1 \subseteq S_c$$

$$b \subseteq S_c$$

(T_{S604})(Eqn1)

$$\text{Let } b_r^1 \subseteq b$$

(Eqn2)

$$b_r^1 \subseteq S_c$$

(T_{M139})(Hypothetical Syllogism Eqns2&1)

$$\{\supset\} * \{\supset\} * b_r^1 = \{\supset\} * b_r^1$$

(T_{S900})(Sub-Entity Function Rule for Continuum Sets)

$$\text{Thus } (b^1 \subseteq S_c)(b_r^1 \subseteq b) \subseteq (\{\supset\} * \{\supset\} * b_r^1 = \{\supset\} * b_r^1)$$

(A_{MX02})(Conditional Introduction)

$$((b^1 \subseteq S_c) = E) \subseteq ((b_r^1 \subseteq b) \subseteq (\{\supset\} * \{\supset\} * b_r^1 = \{\supset\} * b_r^1))$$

(A_{MX35})(Exportation/Importation) for truth sets

$$((b^1 \subseteq S_c) = E) \subseteq ((b_r^1 \subseteq b) \subseteq (\{\supset\} * \{\supset\} * b_r^1 = \{\supset\} * b_r^1))$$

(T_{S832})

$$((\hat{b}^1 \subseteq S_c) = E) \subseteq ((\hat{b}_r^1 \subseteq \hat{b}) \subseteq (\{\supset\} * \{\supset\} * \hat{b}_r^1 = \{\supset\} * \hat{b}_r^1))$$

(T_{S832A})

Example Proof of (T_{S832})

$$\text{Let } b^1 \subseteq S_c$$

$$b \subseteq S_c$$

(T_{S604})(Eqn1)

$$\text{Let } b_r^1 \subseteq b$$

(Eqn2)

$$b_r^1 \subseteq S_c$$

(T_{M139})(Hypothetical Syllogism Eqns2&1)

$$\{\supset\} * \{\supset\} * b_r^1 = \{\supset\} * b_r^1$$

(T_{A438})(Super-Entity Function Rule for Continuum Sets)

$$\text{Thus } (b^1 \subseteq S_c)(b_r^1 \subseteq b) \subseteq (\{\supset\} * \{\supset\} * b_r^1 = \{\supset\} * b_r^1)$$

(A_{MX02})(Conditional Introduction)

$$((b^1 \subseteq S_c) = E) \subseteq ((b_r^1 \subseteq b) \subseteq (\{\supset\} * \{\supset\} * b_r^1 = \{\supset\} * b_r^1))$$

(A_{MX35})(Exportation/Importation) for truth sets

Chapter 9 Proofs of Relationship Theorems – Part 6 – Mapping, Aspect Symmetry & Further Theorems of Binary Relations

9.1 Theorems of Mapped Relationships

$$\mathbf{o}_x * \mathbf{a} = {}^v \mathbf{X} * \mathbf{o} * \mathbf{X} * \mathbf{a} \quad (T_{R571})$$

$$\mathbf{o}_{x*} \mathbf{a} = {}^v \mathbf{X} * \mathbf{o}_* \mathbf{X} * \mathbf{a} \quad (T_{R572})$$

$$\mathbf{o}_x \diamond \mathbf{a} = {}^v \mathbf{X} * \mathbf{o} \diamond \mathbf{X} * \mathbf{a} \quad (T_{R573})$$

$$\mathbf{o}_{x\diamond} \mathbf{a} = {}^v \mathbf{X} * \mathbf{o}_{\diamond} \mathbf{X} * \mathbf{a} \quad (T_{R574})$$

Example Proof of (T_{R571})

$$\begin{aligned} (\mathbf{p} = \{*, \{*\}, \{\diamond\}, \{\diamond\}) : (\mathbf{o}_x \langle \mathbf{p} \rangle \mathbf{a} = {}^v \mathbf{X} * \mathbf{o} \langle \mathbf{p} \rangle \mathbf{X} * \mathbf{a}) & \quad (D_{R34}) \text{(Definition of a Mapped Relationship)} \\ \mathbf{o}_x * \mathbf{a} = {}^v \mathbf{X} * \mathbf{o} * \mathbf{X} * \mathbf{a} & \quad \text{Extraction} \end{aligned}$$

$$(\mathbf{X} = \{=\}) = (\mathbf{o}_x * \mathbf{b} = \mathbf{o} * \mathbf{b}) \quad (T_{S418})$$

Proof of (T_{S418})

$$\begin{aligned} \mathbf{o}_x * \mathbf{b} &= {}^v \mathbf{X} * \mathbf{o} * \mathbf{X} * \mathbf{b} & (T_{R571}) \text{(Definition of a mapped relationship)} \\ \text{Let } \mathbf{X} &= \{=\} & (\text{Eqn2}) \\ \mathbf{o}_x * \mathbf{b} &= {}^v \{=\} * \mathbf{o} * \{=\} * \mathbf{b} & (A_{MX19}) \text{(Substitution of Eqn2 into Eqn1)} \\ \mathbf{o}_x * \mathbf{b} &= \{=\} * \mathbf{o} * \{=\} * \mathbf{b} & (T_{A485}) \\ \mathbf{o}_x * \mathbf{b} &= \{=\} * \mathbf{o} * \mathbf{b} & (T_{A080}) \\ \mathbf{o}_x * \mathbf{b} &= \mathbf{o} * \mathbf{b} & (T_{A080}) \\ \text{Thus } (\mathbf{X} = \{=\}) &= (\mathbf{o}_x * \mathbf{b} = \mathbf{o} * \mathbf{b}) & (A_{MX02}) \text{(Conditional Introduction) since this proof is reversible} \end{aligned}$$

$$(\mathbf{X} = \{=\}) = (\{=\}_x * \mathbf{b} = \{=\} * \mathbf{b}) \quad (T_{S419})$$

Proof of (T_{S419})

$$\begin{aligned} (\mathbf{X} = \{=\}) &= (\mathbf{o}_x * \mathbf{b} = \mathbf{o} * \mathbf{b}) & (T_{S418}) \text{(Eqn1)} \\ \text{Let } \mathbf{o} &= \{=\} & (\text{Eqn2}) \\ (\mathbf{X} = \{=\}) &= (\{=\}_x * \mathbf{b} = \{=\} * \mathbf{b}) & (A_{MX19}) \text{(Substitution of Eqn2 into Eqn1)} \end{aligned}$$

$$(\mathbf{X} = \{=\}) = (\{=\}_x = \{=\}) \quad (T_{S448})$$

Proof of (T_{S448})

$$\begin{aligned} (\mathbf{X} = \{=\}) &= (\mathbf{o}_x = \mathbf{o}) & (D_{R91}) \text{(Definition of Reflexive Mapping)(Eqn1)} \\ \text{Let } \mathbf{o} &= \{=\} & (\text{Eqn2}) \\ (\mathbf{X} = \{=\}) &= (\{=\}_x = \{=\}) & (A_{MX19}) \text{(Substitution of Eqn2 into Eqn1)} \end{aligned}$$

9.2 Theorems of Aspect Symmetry

A. Theorems using the Axioms and Definitions of Relationship Theory Only

$${}^v \mathbf{A} \mathbf{E} = \mathbf{E} \quad (T_{R832})$$

Proof of (T_{R832})

$$\begin{aligned} {}^A \mathbf{E} &= \mathbf{E} & (A_{S87}) \text{(Aspect Opposite of E)} \\ {}^A \mathbf{E} &= \mathbf{E} & (D_{S18}) \text{(Simplification of the Abstract Function)} \end{aligned}$$

$${}^v\mathcal{A}E = E \quad (\mathcal{A}_{S78})(\text{Relationship Symmetry Axiom})$$

$$(a^1 \subseteq S_D) = (a = (\{\supset\}_{\mathcal{A}} * a^1) (\{\subset\}_{\mathcal{A}} * \{\supset\}_{\mathcal{A}} * a^1)) \quad (\mathcal{T}_{S539}) \text{ Definition of Reverse Abstraction for a Discrete Set}$$

Proof of ($\mathcal{T}_{S539}$)

$$(a^1 \subseteq S_D) = (a = (\{\subset\}_{\mathcal{A}} * a^1) (\{\supset\}_{\mathcal{A}} * \{\subset\}_{\mathcal{A}} * a^1))$$

($\mathcal{D}_{R93}$)(Definition of Disaggregation for Discrete Sets)

$$(a^1 \subseteq S_D) = (a = (\{\supset\}_{\mathcal{A}} * a^1) (\{\subset\}_{\mathcal{A}} * \{\supset\}_{\mathcal{A}} * a^1))$$

($\mathcal{A}_{R23}$)(Reverse Abstract Relationships Axiom)

$${}^v\mathcal{A} = \mathcal{A} \quad (\mathcal{T}_{T877})$$

Proof of ($\mathcal{T}_{T877}$)

$$\mathcal{A}^1 \subseteq R_P R_S R_U \quad (\mathcal{A}_{S86})(\text{Nature of } \mathcal{A})$$

$$\mathcal{A}^1 \subseteq R_S \quad \text{Extraction}$$

$${}^v\mathcal{A} = \mathcal{A} \quad (\mathcal{D}_{A94})(\text{Definition of a Symmetrical Relationship Set})$$

B. Theorems using the Axioms and Definitions of Relationship Theory and Membership Theory Only

$$(a = a_1^1 +++ a_n^1) = (\mathcal{A}a = \mathcal{A}a_1^1 +++ \mathcal{A}a_n^1) \quad (\mathcal{T}_{T988})$$

$$(a = a_1^1 +++ a_n^1) = (\mathcal{A}a = (\sim a_1)^1 +++ (\sim a_n)^1) \quad (\mathcal{T}_{T821})$$

Proof of ($\mathcal{T}_{T988}$) & ($\mathcal{T}_{T821}$)

$$\text{Let } a = a_1^1 +++ a_n^1$$

$$\mathcal{A}a = \mathcal{A}(a_1^1 +++ a_n^1) \quad (\mathcal{T}_{R535})$$

$$\mathcal{A}a = \mathcal{A}a_1^1 +++ \mathcal{A}a_n^1 \quad (\mathcal{A}_{S64})(\text{Distribution of } * \text{ over } +)$$

$$\text{Thus } (a = a_1^1 +++ a_n^1) = (\mathcal{A}a = \mathcal{A}a_1^1 +++ \mathcal{A}a_n^1) \quad (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \text{ since this proof is reversible.}$$

$$(a = a_1^1 +++ a_n^1) = (\mathcal{A}a = (\sim a_1)^1 +++ (\sim a_n)^1) \quad (\mathcal{D}_{S21})(\text{Aspect Opposite of } a^1)$$

$$\mathcal{A}E^1 = \emptyset^1 \quad (\mathcal{T}_{T803}) \text{ Aspect Opposite of } E^1$$

Proof of ($\mathcal{T}_{T803}$)

$$\mathcal{A}a^1 = (\sim a)^1 \quad (\mathcal{D}_{S21})(\text{Aspect Opposite of } a^1)(\text{Eqn1})$$

$$\text{Let } a = E \quad (\text{Eqn2})$$

$$\mathcal{A}E^1 = (\sim E)^1 \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$\mathcal{A}E^1 = \emptyset^1 \quad (\mathcal{D}_{M02})(\text{Definition of } \emptyset)$$

$$\mathcal{A}\emptyset^1 = E^1 \quad (\mathcal{T}_{S450}) \text{ Aspect Opposite of } \emptyset^1$$

Proof of ($\mathcal{T}_{S450}$)

$$\mathcal{A}a^1 = (\sim a)^1 \quad (\mathcal{D}_{S21})(\text{Aspect Opposite of } a^1)$$

$$\text{Let } a = \emptyset \quad (\text{Eqn2})$$

$$\mathcal{A}\emptyset^1 = (\sim \emptyset)^1 \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$\mathcal{A}\emptyset^1 = E^1 \quad (\mathcal{T}_{M054})(\text{Negative of Null Set})$$

$$(a \neq S, {}^vS, Q, {}^vQ, \{.\}, \{+\}, \{\triangleleft\}) : (\mathcal{A}a^1 = \mathcal{A}({}^va)^1) \quad (\mathcal{T}_{T824})$$

Proof of ($\mathcal{T}_{T824}$)

$$(a \neq S, {}^vS, Q, {}^vQ, \{.\}, \{+\}, \{\triangleleft\}) :$$

$$\mathcal{A}a^1 = \mathcal{A}a^1 \quad (\mathcal{D}_{M00})(\text{Definition of } =)$$

$$\mathcal{A}a^1 = \mathcal{A}({}^va)^1 \quad (\mathcal{A}_{S78})(\text{Relationship Symmetry Axiom})$$

$$(\mathcal{A}a \neq S, {}^vS, Q, {}^vQ, \{.\}, \{+\}, \{\triangleleft\}) : (\mathcal{A}a^1 = {}^v(\mathcal{A}a^1)) \quad (\mathcal{T}_{T825})$$

Proof of (T₈₂₅)

$(\mathcal{A}a \neq S, {}^vS, Q, {}^vQ, \{.\}, \{+\}, \{<\}):$

$\mathcal{A}a^1 = \mathcal{A}a^1$

(D_{M00})(Definition of =)

$\mathcal{A}a^1 = {}^v(\mathcal{A}a)^1$

(A_{S78})(Relationship Symmetry Axiom)

$c_{\mathcal{A}} * a = {}^v \mathcal{A} * c * \mathcal{A} * a$

(T_{S505})

$c_{\mathcal{A}} * a = {}^v \mathcal{A} * c * \mathcal{A} * a$

(T_{S506})

$c_{\mathcal{A}} \diamond a = {}^v \mathcal{A} * c \diamond \mathcal{A} * a$

(T_{S507})

$c_{\mathcal{A}} \diamond a = {}^v \mathcal{A} * c \diamond \mathcal{A} * a$

(T_{S508})

Example Proof of (T_{S505})

$(p = \{.\}, \{.\}, \{\diamond\}, \{\diamond\}): (o_x \langle p \rangle a = {}^v \mathcal{X} * o \langle p \rangle \mathcal{X} * a)$

(D_{R34})(Definition of a Mapped Relationship)

$o_{\mathcal{X}} * a = {}^v \mathcal{X} * o * \mathcal{X} * a$

Extraction (Eqn1)

Let $\mathcal{X} = \mathcal{A}$

(Eqn2)

Let $o = c$

(Eqn3)

$c_{\mathcal{A}} * a = {}^v \mathcal{A} * c * \mathcal{A} * a$

(A_{MX19})(Substitution of Eqns2&3 into Eqn1)

$(o = \{.\}, \{.\}, \{\diamond\}, \{\|\}, \{\subseteq\}, \{\supseteq\}) \subseteq (o_{\mathcal{A}} = {}^v o)$

(T_{S514})

Proof of (T_{S514})

$(o = \{.\}, \{.\}, \{\diamond\}, \{\|\}, \{\subseteq\}, \{\supseteq\}) = (o^1 \subseteq R_B)$

(D_{R94})(Definition of a Membership Relationship Set) (Eqn1)

$(o^1 \subseteq R_B) \subseteq (o_{\mathcal{A}} = {}^v o)$

(A_{R23})(Reverse Abstract Relationships Axiom)(Eqn2)

$(o = \{.\}, \{.\}, \{\diamond\}, \{\|\}, \{\subseteq\}, \{\supseteq\}) \subseteq (o_{\mathcal{A}} = {}^v o)$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

C. Higher Tier Proofs

$\mathcal{A}a = a$

(T₈₁₆) Aspect Closure Theorem

Proof of (T₈₁₆)

Let $a = a_1^1 +++ a_n^1$

(Eqn1)

$\mathcal{A}a = (\sim a_1)^1 +++ (\sim a_n)^1$

(T₈₂₁)

$\mathcal{A}a = \mathcal{A}((\sim a_1)^1 +++ (\sim a_n)^1)$

(T_{R535})

$\mathcal{A}a = \mathcal{A}(\sim a_1)^1 +++ \mathcal{A}(\sim a_n)^1$

(A_{S64})(Distribution of * over +)

$\mathcal{A}a = (\sim \sim a_1)^1 +++ (\sim \sim a_n)^1$

(D_{S21})(Aspect Opposite of a^1)

$\mathcal{A}a = a_1^1 +++ a_n^1$

(A_{MX13})(Double Complement Axiom)(Eqn2)

$\mathcal{A}a = a$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

$\mathcal{A}\emptyset = \emptyset$

(T₈₀₁) Aspect Opposite of $\emptyset$

Proof of (T₈₀₁)

$c * \emptyset = \emptyset$

(T_{S181})(Null Object Theorem)(Eqn1)

Let $c = \mathcal{A}$

(Eqn2)

$\mathcal{A} * \emptyset = \emptyset$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$\mathcal{A}\emptyset = \emptyset$

(D_{S18})(Simplification of the Abstract Function)

$(a \neq \emptyset) = (\mathcal{A}a \neq \emptyset)$

(T₉₅₄)

Proof of (T₉₅₄)

Let $a = \emptyset$

$\mathcal{A}a = \mathcal{A}\emptyset$

(T_{R196})(Relational Deduction)

$\mathcal{A}a = \emptyset$

(T₈₀₁)(Aspect Opposite of $\emptyset$)

Thus $(a = \emptyset) \subseteq (\mathcal{A}a = \emptyset)$	$(\mathbf{A}_{MX02})$ (Conditional Introduction)(Eqn1)
Let $\mathcal{A}a = \emptyset$	
$\mathcal{A}\mathcal{A}a = \mathcal{A}\emptyset$	$(\mathbf{T}_{R196})$ (Relational Deduction)
$a = \mathcal{A}\emptyset$	$(\mathbf{T}_{T816})$ (Aspect Closure Theorem)
$a = \emptyset$	$(\mathbf{T}_{T801})$ (Aspect Opposite of $\emptyset$)
Thus $(\mathcal{A}a = \emptyset) \subseteq (a = \emptyset)$	$(\mathbf{A}_{MX02})$ (Conditional Introduction)(Eqn2)
$(a = \emptyset) = (\mathcal{A}a = \emptyset)$	$(\mathbf{T}_{M430})$ (Biconditional Introduction Eqns1&2)
$(a \neq \emptyset) = (\mathcal{A}a \neq \emptyset)$	$(\mathbf{T}_{M008})$ (Inverse of an Equality)

$\{=\}_{\mathcal{A}} * a = \{=\} * a$	$(\mathbf{T}_{R660})$
$\{\supset\}_{\mathcal{A}} * a = \{\supset\} * a$	$(\mathbf{T}_{S509})$
$\{\subset\}_{\mathcal{A}} * a = \{\subset\} * a$	$(\mathbf{T}_{S510})$
$\{\parallel\}_{\mathcal{A}} * a = \{\parallel\} * a$	$(\mathbf{T}_{S511})$

Example Proof of $(\mathbf{T}_{R660})$

$(o = \{=\}, \{\subset\}, \{\supset\}, \{\parallel\}, \{\subseteq\}, \{\supseteq\}) \subseteq (o_{\mathcal{A}} = {}^v o)$	$(\mathbf{T}_{S514})$
$\{=\}_{\mathcal{A}} = {}^v \{=\}$	$(\mathbf{A}_{MX19})$ (Substitution)
$\{=\}_{\mathcal{A}} = \{=\}$	$(\mathbf{T}_{A485})$
$\{=\}_{\mathcal{A}} * a = \{=\} * a$	$(\mathbf{T}_{R522b})$ (Relational Deduction)

$\{\supset\}_{\mathcal{A}} * a = \{\supseteq\} * a$	$(\mathbf{T}_{R661})$
$\{\subseteq\}_{\mathcal{A}} * a = \{\supset\} * a$	$(\mathbf{T}_{R662})$

Example Proof of $(\mathbf{T}_{R661})$

$\{\supset\}_{\mathcal{A}} * a = {}^v \mathcal{A} * \{\supset\} * \mathcal{A} * a$	$(\mathbf{T}_{S505})$ (Eqn1)
$\{\supset\}_{\mathcal{A}} * a = {}^v \mathcal{A} * (\{\supset\} + \{=\}) * \mathcal{A} * a$	$(\mathbf{D}_{A141})$ (Disjunction with $\{=\}$)
$\{\supset\}_{\mathcal{A}} * a = {}^v \mathcal{A} * (\{\supset\} * \mathcal{A} * a + \{=\} * \mathcal{A} * a)$	$(\mathbf{T}_{R002c})$ (Distribution of $+$ over $*$)
$\{\supset\}_{\mathcal{A}} * a = {}^v \mathcal{A} * \{\supset\} * \mathcal{A} * a + {}^v \mathcal{A} * \{=\} * \mathcal{A} * a$	$(\mathbf{T}_{R002b})$ (Distribution of $*$ over $+$)
$\{\supset\}_{\mathcal{A}} * a = \{\supset\}_{\mathcal{A}} * a + \{=\}_{\mathcal{A}} * a$	$(\mathbf{T}_{S505})$
$\{\supset\}_{\mathcal{A}} * a = \{\subset\} * a + \{=\} * a$	$(\mathbf{A}_{R23})$ (Reverse Abstract Relationships Axiom) or $(\mathbf{T}_{S509})$ & $(\mathbf{T}_{R660})$
$\{\supset\}_{\mathcal{A}} * a = (\{\subset\} + \{=\}) * a$	$(\mathbf{T}_{R002c})$ (Distribution of $+$ over $*$)
$\{\supset\}_{\mathcal{A}} * a = \{\supseteq\} * a$	$(\mathbf{D}_{A141})$ (Disjunction with $\{=\}$)

$\{=\}_{\mathcal{A}} * a = \{=\}_{\mathcal{A}} * a$	$(\mathbf{T}_{S512})$
$\{\supset\}_{\mathcal{A}} * a = \{\supset\}_{\mathcal{A}} * a$	$(\mathbf{T}_{R604})$
$\{\subset\}_{\mathcal{A}} * a = \{\supset\}_{\mathcal{A}} * a$	$(\mathbf{T}_{R690})$
$\{\parallel\}_{\mathcal{A}} * a = \{\parallel\}_{\mathcal{A}} * a$	$(\mathbf{T}_{S513})$
$\{\supset\}_{\mathcal{A}} * a = \{\supseteq\}_{\mathcal{A}} * a$	$(\mathbf{T}_{R658})$
$\{\subseteq\}_{\mathcal{A}} * a = \{\supset\}_{\mathcal{A}} * a$	$(\mathbf{T}_{R685})$

Example Proof of $(\mathbf{T}_{R690})$

$\{\subset\}_{\mathcal{A}} * a = \{\supset\} * a$	$(\mathbf{T}_{S510})$
$\{\subset\}_{\mathcal{A}} * a = \{\supset\}_{\mathcal{A}} * a$	$(\mathbf{T}_{M008})$ (Inverse of an Equality)

$\{\subseteq\} \circ E = \emptyset^1$	$(\mathbf{T}_{T845})$

Proof of $(\mathbf{T}_{T845})$

$\mathcal{A}\emptyset^1 = E^1$	$(\mathbf{T}_{S450})$ (Aspect Opposite of $\emptyset^1$)

$$\begin{aligned}
\mathcal{A} * \emptyset^1 &= \mathbf{E}^1 & (D_{S18}) (\text{Simplification of the Abstract Function}) \\
\emptyset^1 &= {}^v \mathcal{A} * \mathbf{E}^1 & (T_{S531}) \text{ since } \emptyset^1 \text{ and } \mathbf{E}^1 \text{ are closed with respect to } \mathcal{A} \text{ (Eqn1)} \\
\mathbf{E}^1 &= \{\supset\} \diamond \mathbf{E} & (T_{A643}) (\text{Eqn2}) \\
\emptyset^1 &= {}^v \mathcal{A} * \{\supset\} \diamond \mathbf{E} & (A_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\
\emptyset^1 &= {}^v \mathcal{A} * \{\supset\} \diamond \mathcal{A} * \mathbf{E} & (A_{S87}) (\text{Aspect Opposite of } \mathbf{E}) \\
\emptyset^1 &= \{\supset\} \mathcal{A} \diamond \mathbf{E} & (T_{S507}) \\
\emptyset^1 &= \{\subseteq\} \diamond \mathbf{E} & (T_{R661}) \\
\{\subseteq\} \diamond \mathbf{E} &= \emptyset^1 & (T_{M197}) (\text{Commutativity of } =)
\end{aligned}$$

$$\{\subseteq\} * \mathbf{E} = \sim \emptyset^1 \quad (T_{S566})$$

Proof of (T_{S566})

$$\begin{aligned}
\{\subseteq\} * \mathbf{E} &= \{\subseteq\} \diamond \mathbf{E} & (A_{R10}) (\text{Axiom of Complementary Relationship Sets}) (\text{Eqn1}) \\
\{\subseteq\} \diamond \mathbf{E} &= \emptyset^1 & (T_{R845}) \\
\{\subseteq\} \diamond \mathbf{E} &= \sim \emptyset^1 & (T_{M008}) (\text{Inverse of an Equality}) (\text{Eqn2}) \\
\{\subseteq\} * \mathbf{E} &= \sim \emptyset^1 & (A_{MX19}) (\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$\mathcal{A} * \{\subseteq\} * \mathbf{a}^1 = \{\supset\} * \mathcal{A} * \mathbf{a}^1 \quad (T_{T828})$$

Proof of (T_{T828})

$$\begin{aligned}
\{\subseteq\} * \mathbf{a}^1 &= \{\supset\} \mathcal{A} * \mathbf{a}^1 & (T_{R661}) \\
\{\subseteq\} * \mathbf{a}^1 &= {}^v \mathcal{A} * \{\supset\} * \mathcal{A} * \mathbf{a}^1 & (T_{S505}) \\
\mathcal{A} * \{\subseteq\} * \mathbf{a}^1 &= \mathcal{A} * {}^v \mathcal{A} * \{\supset\} * \mathcal{A} * \mathbf{a}^1 & (T_{R522}) (\text{Relational Deduction}) \\
\mathcal{A} * \{\subseteq\} * \mathbf{a}^1 &= \mathcal{A} * \mathcal{A} * \{\supset\} * \mathcal{A} * \mathbf{a}^1 & (A_{S78}) (\text{Relationship Symmetry Axiom}) \\
\mathcal{A} * \{\subseteq\} * \mathbf{a}^1 &= \{\supset\} * \mathcal{A} * \mathbf{a}^1 & (T_{T816}) (\text{Aspect Closure Theorem})
\end{aligned}$$

$${}^v \mathcal{A} * \{\supset\} * \mathbf{b}^1 = \{\subseteq\} * {}^v \mathcal{A} * \mathbf{b}^1 \quad (T_{T829})$$

Proof of (T_{T829})

$$\begin{aligned}
\{\supset\} * \mathbf{b}^1 &= \{\subseteq\} \mathcal{A} * \mathbf{b}^1 & (T_{R662}) \\
\{\supset\} * \mathbf{b}^1 &= {}^v \mathcal{A} * \{\subseteq\} * \mathcal{A} * \mathbf{b}^1 & (T_{S505}) \\
\{\supset\} * \mathbf{b}^1 &= \mathcal{A} * \{\subseteq\} * {}^v \mathcal{A} * \mathbf{b}^1 & (A_{S78}) (\text{Relationship Symmetry Axiom}) \\
{}^v \mathcal{A} * \{\supset\} * \mathbf{b}^1 &= {}^v \mathcal{A} * \mathcal{A} * \{\subseteq\} * {}^v \mathcal{A} * \mathbf{b}^1 & (T_{R522}) (\text{Relational Deduction}) \\
{}^v \mathcal{A} * \{\supset\} * \mathbf{b}^1 &= \mathcal{A} * \mathcal{A} * \{\subseteq\} * {}^v \mathcal{A} * \mathbf{b}^1 & (A_{S78}) (\text{Relationship Symmetry Axiom}) \\
{}^v \mathcal{A} * \{\supset\} * \mathbf{b}^1 &= \{\subseteq\} * {}^v \mathcal{A} * \mathbf{b}^1 & (T_{T816}) (\text{Aspect Closure Theorem})
\end{aligned}$$

$$\mathcal{A} * \{\supset\} * \mathbf{a}^1 = \{\subseteq\} * \mathcal{A} * \mathbf{a}^1 \quad (T_{T830})$$

Proof of (T_{T830})

$$\begin{aligned}
\{\supset\} * \mathbf{a}^1 &= \{\subseteq\} \mathcal{A} * \mathbf{a}^1 & (T_{R662}) \\
\{\supset\} * \mathbf{a}^1 &= {}^v \mathcal{A} * \{\subseteq\} * \mathcal{A} * \mathbf{a}^1 & (T_{S505}) \\
\mathcal{A} * \{\supset\} * \mathbf{a}^1 &= \mathcal{A} * {}^v \mathcal{A} * \{\subseteq\} * \mathcal{A} * \mathbf{a}^1 & (T_{R522}) (\text{Relational Deduction}) \\
\mathcal{A} * \{\supset\} * \mathbf{a}^1 &= \mathcal{A} * \mathcal{A} * \{\subseteq\} * \mathcal{A} * \mathbf{a}^1 & (A_{S78}) (\text{Relationship Symmetry Axiom}) \\
\mathcal{A} * \{\supset\} * \mathbf{a}^1 &= \{\subseteq\} * \mathcal{A} * \mathbf{a}^1 & (T_{T816}) (\text{Aspect Closure Theorem})
\end{aligned}$$

$${}^v \mathcal{A} * \{\subseteq\} * \mathbf{b}^1 = \{\supset\} * {}^v \mathcal{A} * \mathbf{b}^1 \quad (T_{T831})$$

Proof of (T_{T831})

$$\begin{aligned}
\{\subseteq\} * \mathbf{b}^1 &= \{\supset\} \mathcal{A} * \mathbf{b}^1 & (T_{R661}) \\
\{\subseteq\} * \mathbf{b}^1 &= {}^v \mathcal{A} * \{\supset\} * \mathcal{A} * \mathbf{b}^1 & (T_{S505}) \\
\{\subseteq\} * \mathbf{b}^1 &= \mathcal{A} * \{\supset\} * {}^v \mathcal{A} * \mathbf{b}^1 & (A_{S78}) (\text{Relationship Symmetry Axiom}) \\
{}^v \mathcal{A} * \{\subseteq\} * \mathbf{b}^1 &= {}^v \mathcal{A} * \mathcal{A} * \{\supset\} * {}^v \mathcal{A} * \mathbf{b}^1 & (T_{R522}) (\text{Relational Deduction}) \\
{}^v \mathcal{A} * \{\subseteq\} * \mathbf{b}^1 &= \mathcal{A} * \mathcal{A} * \{\supset\} * {}^v \mathcal{A} * \mathbf{b}^1 & (A_{S78}) (\text{Relationship Symmetry Axiom})
\end{aligned}$$

$$\forall \mathcal{A} * \{\subseteq\} * \mathcal{b}^1 = \{\supset\} * \forall \mathcal{A} * \mathcal{b}^1 \quad (\mathcal{T}_{7816}) (\text{Aspect Closure Theorem})$$

$$(\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1) = (\mathcal{A}\mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A}\mathbf{b}^1) \quad (\mathcal{T}_{538})$$

Proof of ($\mathcal{T}_{538}$)

Let $\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1$

$$\mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A}\mathbf{b}^1 \quad (\mathcal{T}_{509})$$

$$\mathbf{a}^1 \subseteq \forall \mathcal{A} * \{\supset\} * \mathcal{A} * \mathbf{b}^1 \quad (\mathcal{T}_{505})$$

$$\mathcal{A} * \mathbf{a}^1 \subseteq \mathcal{A} * \forall \mathcal{A} * \{\supset\} * \mathcal{A} * \mathbf{b}^1 \quad (\mathcal{T}_{196}) (\text{Relational Deduction})$$

$$\mathcal{A} * \mathbf{a}^1 \subseteq \mathcal{A} * \mathcal{A} * \{\supset\} * \mathcal{A} * \mathbf{b}^1 \quad (\mathcal{A}_{578}) (\text{Relationship Symmetry Axiom})$$

$$\mathcal{A} * \mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A} * \mathbf{b}^1 \quad (\mathcal{T}_{816}) (\text{Aspect Closure Theorem})$$

Thus $(\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1) = (\mathcal{A}\mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A}\mathbf{b}^1)$ ($\mathcal{A}_{MX02}$) (Conditional Introduction) Since this proof can be reversed.

$$(\mathbf{a}^1 \subseteq \{\supset\} * \mathbf{b}^1) = (\mathcal{A}\mathbf{a}^1 \subseteq \{\subseteq\} * \mathcal{A}\mathbf{b}^1) \quad (\mathcal{T}_{953})$$

Proof of ($\mathcal{T}_{953}$)

$$(\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1) = (\mathcal{A}\mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A}\mathbf{b}^1) \quad (\mathcal{T}_{538}) (\text{Eqn1})$$

$$\text{Let } \mathbf{a}^1 = \mathcal{A}\mathbf{a}^1 \quad (\text{Eqn2})$$

$$\text{Let } \mathbf{b}^1 = \mathcal{A}\mathbf{b}^1 \quad (\text{Eqn3})$$

$$(\mathcal{A}\mathbf{a}^1 \subseteq \{\subseteq\} * \mathcal{A}\mathbf{b}^1) = (\mathcal{A}\mathcal{A}\mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A}\mathcal{A}\mathbf{b}^1) \quad (\mathcal{A}_{MX19}) (\text{Substitution of Eqns2\&3 into Eqn1})$$

$$(\mathcal{A}\mathbf{a}^1 \subseteq \{\subseteq\} * \mathcal{A}\mathbf{b}^1) = (\mathbf{a}^1 \subseteq \{\supset\} * \mathbf{b}^1) \quad (\mathcal{T}_{816}) (\text{Aspect Closure Theorem})$$

$$(\mathbf{a}.\{\supset\} * \mathbf{b} \neq \emptyset) = (\mathcal{A}\mathbf{a}.\{\subseteq\} * \mathcal{A}\mathbf{b} \neq \emptyset) \quad (\mathcal{T}_{952})$$

$$(\mathbf{a}.\{\supset\} * \mathbf{b} \neq \emptyset) = (\mathcal{A}\mathbf{a}.\{\subseteq\} * \mathcal{A}\mathbf{b} \neq \emptyset) \quad (\mathcal{T}_{951})$$

Example Proof of ($\mathcal{T}_{951}$)

$$(\mathcal{A}\mathbf{a}_r^1 \subseteq \{\subseteq\} * \mathcal{A}\mathbf{b}_r^1) = (\mathbf{a}_r^1 \subseteq \{\supset\} * \mathbf{b}_r^1) \quad (\mathcal{T}_{953}) (\text{Eqn1})$$

$$(\mathcal{A}\mathbf{a}_1^1 \subseteq \{\subseteq\} * \mathcal{A}\mathbf{b}_1^1) \dots (\mathcal{A}\mathbf{a}_n^1 \subseteq \{\subseteq\} * \mathcal{A}\mathbf{b}_n^1) = (\mathbf{a}_1^1 \subseteq \{\supset\} * \mathbf{b}_1^1) \dots (\mathbf{a}_n^1 \subseteq \{\supset\} * \mathbf{b}_n^1) \quad (\mathcal{T}_{M231}) (\text{Combination of Eqn1 for } r = 1 \text{ to } n)$$

$$(\mathcal{A}\mathbf{a}_1^1 + \dots + \mathcal{A}\mathbf{a}_n^1 \subseteq \{\subseteq\} * \mathcal{A}\mathbf{b}_1^1 + \dots + \mathcal{A}\mathbf{b}_n^1) = (\mathbf{a}_1^1 + \dots + \mathbf{a}_n^1 \subseteq \{\supset\} * \mathbf{b}_1^1 + \dots + \mathbf{b}_n^1) \quad (\mathcal{A}_{MX21}) (\text{Combination Axiom})$$

$$(\mathcal{A}(\mathbf{a}_1^1 + \dots + \mathbf{a}_n^1) \subseteq \{\subseteq\} * \mathcal{A}(\mathbf{b}_1^1 + \dots + \mathbf{b}_n^1)) = (\mathbf{a}_1^1 + \dots + \mathbf{a}_n^1 \subseteq \{\supset\} * (\mathbf{b}_1^1 + \dots + \mathbf{b}_n^1)) \quad (\mathcal{A}_{S64}) (\text{Distribution of } * \text{ over } +) (\text{Eqn1})$$

$$\text{Let } \mathbf{a}_1^1 + \dots + \mathbf{a}_n^1 = \mathbf{a} \quad (\text{Eqn2})$$

$$\text{Let } \mathbf{b}_1^1 + \dots + \mathbf{b}_n^1 = \mathbf{b} \quad (\text{Eqn3})$$

$$(\mathcal{A}\mathbf{a} \subseteq \{\subseteq\} * \mathcal{A}\mathbf{b}) = (\mathbf{a} \subseteq \{\supset\} * \mathbf{b}) \quad (\mathcal{A}_{MX19}) (\text{Substitution of Eqns2\&3 into Eqn1})$$

$$(\mathcal{A}\mathbf{a}.\{\subseteq\} * \mathcal{A}\mathbf{b} = \emptyset) = (\mathbf{a}.\{\supset\} * \mathbf{b} = \emptyset) \quad (\mathcal{T}_{M735}) (\subseteq \text{ Existential Form})$$

$$(\mathcal{A}\mathbf{a}.\{\subseteq\} * \mathcal{A}\mathbf{b} \neq \emptyset) = (\mathbf{a}.\{\supset\} * \mathbf{b} \neq \emptyset) \quad (\mathcal{T}_{M008}) (\text{Inverse of an Equality})$$

$$(\mathbf{a} \neq \emptyset) \subseteq (\mathbf{a}^1 = (\{\subseteq\}_{\mathcal{A}} \diamond \mathbf{a}) (\{\subseteq\}_{\mathcal{A}^*} \{\subseteq\}_{\mathcal{A}} \diamond \mathbf{a})) \quad (\mathcal{T}_{5470}) \text{ General Definition of Abstraction}$$

Proof of ($\mathcal{T}_{5470}$)

$$(\mathbf{a} \neq \emptyset) \subseteq (\mathbf{a}^1 = (\{\supset\} \diamond \mathbf{a}).(\{\supset\} * \{\supset\} \diamond \mathbf{a})) \quad (\mathcal{D}_{R98}) (\text{General Definition of Aggregation})$$

$$(\mathbf{a} \neq \emptyset) \subseteq (\mathbf{a}^1 = (\{\subseteq\}_{\mathcal{A}} \diamond \mathbf{a}) (\{\subseteq\}_{\mathcal{A}^*} \{\subseteq\}_{\mathcal{A}} \diamond \mathbf{a})) \quad (\mathcal{T}_{5510}) \& (\mathcal{T}_{R662})$$

$$(\mathbf{a}^1 \subseteq \mathbf{S}_c) = (\mathbf{a}^1 = (\{\subseteq\}_{\mathcal{A}} \diamond \mathbf{a}).(\{\subseteq\}_{\mathcal{A}^*} \mathbf{a})) \quad (\mathcal{T}_{5469}) \text{ Definition of Abstraction for a Continuum Set}$$

Proof of ($\mathcal{T}_{5469}$)

$$(\mathbf{a}^1 \subseteq \mathbf{S}_c) \subseteq (\mathbf{a}^1 = \{\supset\} \diamond \mathbf{a}.\{\supset\} * \mathbf{a}) \quad (\mathcal{T}_{5393}) (\text{Definition of Aggregation for Continuum Sets})$$

$$(\mathbf{a}^1 \subseteq \mathbf{S}_c) = (\mathbf{a}^1 = (\{\subseteq\}_{\mathcal{A}} \diamond \mathbf{a}).(\{\subseteq\}_{\mathcal{A}^*} \mathbf{a})) \quad (\mathcal{T}_{R661}) \& (\mathcal{T}_{R662})$$

$$(\mathbf{a}^1 \subseteq \mathbf{S}_c) = (\mathbf{a} = \{\supset\}_{\mathcal{A}} * \mathbf{a}^1) \quad (\mathcal{T}_{5471}) \text{ Definition of Reverse Abstraction for Continuum Sets}$$

Proof of (T_{S471})

$$(a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a^1)$$

$$(a^1 \subseteq S_c) = (a = \{\supseteq\}_{\mathcal{A}} * a^1)$$

(A_{R02})(Definition of Disaggregation for Continuum Sets)
(T_{R661})

$$a \subseteq \{\supseteq\}_{\mathcal{A}} * a^1 \quad (T_{S331})$$

Proof of (T_{S331})

$$a \subseteq \{\subseteq\} * a^1$$

$$a \subseteq \{\supseteq\}_{\mathcal{A}} * a^1$$

(T_{S581})(General Definition of Disaggregation)
(T_{R661})

$$a^1 \subseteq \{\supseteq\}_{\mathcal{A}} * a^1 \quad (T_{S332})$$

Proof of (T_{S332})

$$a^1 \subseteq a^1 + \{\subseteq\} * a^1$$

$$a^1 \subseteq \{=\} * a^1 + \{\subseteq\} * a^1$$

$$a^1 \subseteq (\{=\} + \{\subseteq\}) * a^1$$

$$a^1 \subseteq \{\subseteq\} * a^1$$

$$a^1 \subseteq \{\supseteq\}_{\mathcal{A}} * a^1$$

(T_{M082})(Disjunction Introduction)
(D_{A144})(Definition of the Reflexive Relationship Set)
(T_{R002c})(Distribution of + over *)
(D_{A141})(Disjunction with {=})
(T_{R661})

$$c * b \subseteq c * \{\supseteq\}_{\mathcal{A}} b^1 \quad (T_{S333})$$

Proof of (T_{S333})

$$b \subseteq \{\subseteq\} * b^1$$

$$c * b \subseteq c * \{\subseteq\} * b^1$$

$$c * b \subseteq c * \{\supseteq\}_{\mathcal{A}} b^1$$

(T_{S581})(General Definition of Disaggregation)
(T_{R196})(Relational Deduction)
(T_{R661})

$$c * \{\supseteq\}_{\mathcal{A}} b^1 \subseteq c * b \quad (T_{S334})$$

Proof of (T_{S334})

$$c * b \subseteq c * \{\supseteq\}_{\mathcal{A}} b^1$$

$$c * \{\supseteq\}_{\mathcal{A}} b^1 \subseteq c * b$$

(T_{S333})
(T_{M012})(Transposition)

$$c * b^1 \subseteq c * \{\supseteq\}_{\mathcal{A}} b^1 \quad (T_{S335})$$

Proof of (T_{S335})

$$b^1 \subseteq b^1 + \{\subseteq\} * b^1$$

$$b^1 \subseteq \{=\} * b^1 + \{\subseteq\} * b^1$$

$$b^1 \subseteq (\{=\} + \{\subseteq\}) * b^1$$

$$b^1 \subseteq \{\subseteq\} * b^1$$

$$c * b^1 \subseteq c * \{\subseteq\} * b^1$$

$$c * b^1 \subseteq c * \{\supseteq\}_{\mathcal{A}} b^1$$

(T_{M082})(Disjunction Introduction)
(D_{A144})(Definition of the Reflexive Relationship Set)
(T_{R002c})(Distribution of + over *)
(D_{A141})(Disjunction with {=})
(T_{R196})(Relational Deduction)
(T_{R661})

$$c * \{\supseteq\}_{\mathcal{A}} b^1 \subseteq c * b^1 \quad (T_{S336})$$

Proof of (T_{S336})

$$c * b^1 \subseteq c * \{\supseteq\}_{\mathcal{A}} b^1$$

$$c * \{\supseteq\}_{\mathcal{A}} b^1 \subseteq c * b^1$$

(T_{S335})
(T_{M012})(Transposition)

$$E = S\mathcal{A} \quad (T_{T833})$$

Proof of (T_{T833})

$$^{\mathcal{A}}E = E$$

$$\mathcal{A}E = E$$

$$S\mathcal{A} = E$$

(A_{S87})(Aspect Opposite of E)
(D_{S18})(Simplification of the Abstract Function)
(T_{R210})

$$E = Q\mathcal{A} \quad (T_{7834})$$

Proof of (T₇₈₃₄)

$$\forall \mathcal{A} E = E \quad (T_{7832})$$

$$Q\mathcal{A} = E \quad (T_{R211})$$

$$a \subseteq S\mathcal{A} \quad (T_{7835})$$

Proof of (T₇₈₃₅)

$$a \subseteq E \quad (T_{M744})(\text{Universal Set Theorem})$$

$$a \subseteq S\mathcal{A} \quad (T_{7833})$$

$$a \subseteq Q\mathcal{A} \quad (T_{7836})$$

Proof of (T₇₈₃₆)

$$a \subseteq E \quad (T_{M744})(\text{Universal Set Theorem})$$

$$a \subseteq Q\mathcal{A} \quad (T_{7834})$$

$$(a = b) = (\mathcal{A}a = \mathcal{A}b) \quad (T_{7876})$$

Proof of (T₇₈₇₆)

$$((\forall c)^1 \subseteq R_o)(a \subseteq Qc)(b \subseteq Qc)(o^1 \subseteq R_b) \subseteq ((a \langle o \rangle b) = (c * a \langle o \rangle c * b))$$

(T_{R535})(Eqn1)

$$\text{Let } c = \mathcal{A}$$

(Eqn2)

$$\text{Let } o = \{=\}$$

(Eqn3)

$$((\forall \mathcal{A})^1 \subseteq R_o)(a \subseteq Q\mathcal{A})(b \subseteq Q\mathcal{A})(\{=\}^1 \subseteq R_b) \subseteq ((a \langle \{=\} \rangle b) = (\mathcal{A} * a \langle \{=\} \rangle \mathcal{A} * b))$$

(A_{MX19})(Substitution of Eqns2&3 into Eqn1)

$$((\forall \mathcal{A})^1 \subseteq R_o)(a \subseteq Q\mathcal{A})(b \subseteq Q\mathcal{A})(\{=\}^1 \subseteq R_b) \subseteq ((a = b) = (\mathcal{A} * a = \mathcal{A} * b))$$

(D_{A229})(Definition of $\langle \rangle$)

$$((\forall \mathcal{A})^1 \subseteq R_o)EE(\{=\}^1 \subseteq R_b) \subseteq ((a = b) = (\mathcal{A} * a = \mathcal{A} * b)) \quad (T_{M001})(\text{Universality of (T}_{7836}\text{)})(\text{Eqn4})$$

$$(o = \{=\}, \{ \subseteq \}, \{ \supseteq \}, \{ \parallel \}, \{ \subseteq \}, \{ \supseteq \}) = (o^1 \subseteq R_b)$$

(D_{R94})(Definition of a Membership Relationship Set)

$$\{=\}^1 \subseteq R_b$$

Extraction

$$(\{=\}^1 \subseteq R_b) = E$$

(A_{MX16})(Order Axiom)(Eqn5)

$$(\mathcal{A}^1 \subseteq R_p) = (\mathcal{A}^1 \subseteq R_o).((\forall \mathcal{A})^1 \subseteq R_o)$$

(D_{R31})(Definition of a Pairing Relationship Set)(Eqn6)

$$\mathcal{A}^1 \subseteq R_p R_s R_u$$

(A_{S86})(Nature of $\mathcal{A}$)

$$\mathcal{A}^1 \subseteq R_p$$

Extraction

$$(\mathcal{A}^1 \subseteq R_p) = E$$

(A_{MX16})(Order Axiom)(Eqn7)

$$E = (\mathcal{A}^1 \subseteq R_o).((\forall \mathcal{A})^1 \subseteq R_o)$$

(A_{MX19})(Substitution of Eqn7 into Eqn6)

$$((\forall \mathcal{A})^1 \subseteq R_o) = E$$

(T_{M084})(Eqn8)

$$EEEE \subseteq ((a = b) = (\mathcal{A} * a = \mathcal{A} * b))$$

(A_{MX19})(Substitution of Eqns5&8 into Eqn4)

$$E \subseteq ((a = b) = (\mathcal{A} * a = \mathcal{A} * b))$$

(A_{MX11})(Universal Set Conjunction Axiom)

$$E = ((a = b) = (\mathcal{A} * a = \mathcal{A} * b))$$

(T_{M275})(Supersets of the Universal Set)

$$(a = b) = (\mathcal{A} * a = \mathcal{A} * b)$$

(A_{MX16})(Order Axiom)

$$(\mathcal{A}b + b)^1 \subseteq D * \mathcal{A}^1$$

(T₇₈₈₀)

Proof of (T₇₈₈₀)

$$((\forall \mathcal{A})^1 \subseteq R_o)(b \subseteq Q\mathcal{A}) \subseteq ((\mathcal{A} * b + b)^1 \subseteq D * \mathcal{A}^1)$$

(A_{R25})(One-to-One Relationship Closure Axiom)

$$((\forall \mathcal{A})^1 \subseteq R_o)E \subseteq ((\mathcal{A} * b + b)^1 \subseteq D * \mathcal{A}^1)$$

(T_{M001})(Universality of (T₇₈₃₆))(Eqn1)

$$(\mathcal{A}^1 \subseteq R_p) = (\mathcal{A}^1 \subseteq R_o).((\forall \mathcal{A})^1 \subseteq R_o)$$

(D_{R31})(Definition of a Pairing Relationship Set)(Eqn2)

$$\mathcal{A}^1 \subseteq R_p R_s R_u$$

(A_{S86})(Nature of $\mathcal{A}$)

$$\mathcal{A}^1 \subseteq R_p$$

Extraction

$$(\mathcal{A}^1 \subseteq R_p) = E$$

(A_{MX16})(Order Axiom)(Eqn3)

$$E = (\mathcal{A}^1 \subseteq R_o).((\forall \mathcal{A})^1 \subseteq R_o)$$

(A_{MX19})(Substitution of Eqn3 into Eqn2)

$$\begin{aligned}
((\forall \mathcal{A})^1 \subseteq \mathbf{R}_0) &= \mathbf{E} & (\mathcal{T}_{M084})(\text{Eqn4}) \\
\mathbf{E} \subseteq ((\mathcal{A} * \mathbf{b} + \mathbf{b})^1 \subseteq \mathbf{D} * \mathcal{A}^1) & & (\mathcal{A}_{MX19})(\text{Substitution of Eqn4 into Eqn1}) \\
\mathbf{E} \subseteq ((\mathcal{A} * \mathbf{b} + \mathbf{b})^1 \subseteq \mathbf{D} * \mathcal{A}^1) & & (\mathcal{A}_{MX11})(\text{Universal Set Conjunction Axiom}) \\
\mathbf{E} = ((\mathcal{A} * \mathbf{b} + \mathbf{b})^1 \subseteq \mathbf{D} * \mathcal{A}^1) & & (\mathcal{T}_{M275})(\text{Supersets of the Universal Set}) \\
(\mathcal{A} * \mathbf{b} + \mathbf{b})^1 \subseteq \mathbf{D} * \mathcal{A}^1 & & (\mathcal{A}_{MX16})(\text{Order Axiom}) \\
(\mathcal{A} * \mathbf{b} + \mathbf{b})^1 \subseteq \mathbf{D} * \mathcal{A}^1 & & (\mathcal{D}_{S18})(\text{Simplification of the Abstract Function})
\end{aligned}$$

$$\mathbf{b} = \mathcal{A} * \forall \mathcal{A} * \mathbf{b} \quad (\mathcal{T}_{T878})$$

Proof of ($\mathcal{T}_{T878}$)

$$\begin{aligned}
((\mathbf{c} * \mathbf{b} + \mathbf{b})^1 \subseteq \mathbf{D} * \mathbf{c}^1) &= (\mathbf{b} = \forall \mathbf{c} * \mathbf{c} * \mathbf{b}) & (\mathcal{T}_{S526})(\text{Eqn1}) \\
\text{Let } \mathbf{c} &= \mathcal{A} & (\text{Eqn2}) \\
((\mathcal{A} * \mathbf{b} + \mathbf{b})^1 \subseteq \mathbf{D} * \mathcal{A}^1) &= (\mathbf{b} = \forall \mathcal{A} * \mathcal{A} * \mathbf{b}) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\
(\mathcal{A} * \mathbf{b} + \mathbf{b})^1 &\subseteq \mathbf{D} * \mathcal{A}^1 & (\mathcal{T}_{T880}) \\
((\mathcal{A} * \mathbf{b} + \mathbf{b})^1 \subseteq \mathbf{D} * \mathcal{A}^1) &= \mathbf{E} & (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\
\mathbf{E} &= (\mathbf{b} = \forall \mathcal{A} * \mathcal{A} * \mathbf{b}) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\
\mathbf{b} &= \forall \mathcal{A} * \mathcal{A} * \mathbf{b} & (\mathcal{A}_{MX16})(\text{Order Axiom}) \\
\mathbf{b} &= \mathcal{A} * \forall \mathcal{A} * \mathbf{b} & (\mathcal{T}_{T877})
\end{aligned}$$

$$(\mathbf{a} \subseteq \mathbf{b}) = (\mathcal{A} \mathbf{a} \subseteq \mathcal{A} \mathbf{b}) \quad (\mathcal{T}_{T429})$$

Proof of ($\mathcal{T}_{T429}$)

$$\begin{aligned}
\text{Let } \mathbf{a} &\subseteq \mathbf{b} \\
\mathcal{A} \mathbf{a} &\subseteq \mathcal{A} \mathbf{b} & (\mathcal{T}_{R196})(\text{Relational Deduction}) \\
\text{Thus } (\mathbf{a} \subseteq \mathbf{b}) &\subseteq (\mathcal{A} \mathbf{a} \subseteq \mathcal{A} \mathbf{b}) & (\mathcal{A}_{MX02})(\text{Conditional Introduction})(\text{Eqn1}) \\
\text{Let } \mathcal{A} \mathbf{a} &\subseteq \mathcal{A} \mathbf{b} \\
\forall \mathcal{A} \mathcal{A} \mathbf{a} &\subseteq \forall \mathcal{A} \mathcal{A} \mathbf{b} & (\mathcal{T}_{R196})(\text{Relational Deduction}) \\
\mathcal{A}^{\forall} \mathcal{A} \mathbf{a} &\subseteq \mathcal{A}^{\forall} \mathcal{A} \mathbf{b} & (\mathcal{T}_{T877}) \\
\mathbf{a} &\subseteq \mathbf{b} & (\mathcal{T}_{T878}) \\
\text{Thus } (\mathcal{A} \mathbf{a} \subseteq \mathcal{A} \mathbf{b}) &\subseteq (\mathbf{a} \subseteq \mathbf{b}) & (\mathcal{A}_{MX02})(\text{Conditional Introduction})(\text{Eqn2}) \\
(\mathbf{a} \subseteq \mathbf{b}) &= (\mathcal{A} \mathbf{a} \subseteq \mathcal{A} \mathbf{b}) & (\mathcal{T}_{M430})(\text{Biconditional Introduction Eqns1\&2})
\end{aligned}$$

$$((\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \subseteq \mathbf{b}) = (\mathbf{a}^1 \subseteq \{\supset\} * \mathbf{b}^1)) \quad (\mathcal{T}_{T422})$$

Proof of ($\mathcal{T}_{T422}$)

$$\begin{aligned}
((\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) &\subseteq ((\mathbf{a} \subseteq \mathbf{b}) = (\mathbf{a}^1 \subseteq \{\supset\} * \mathbf{b}^1)) & (\mathcal{T}_{S412}) \\
((\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) &\subseteq ((\mathcal{A} \mathbf{a} \subseteq \mathcal{A} \mathbf{b}) = (\mathcal{A} \mathbf{a}^1 \subseteq \mathcal{A} * \{\supset\} * \mathbf{b}^1)) & (\mathcal{T}_{T429}) \\
((\mathcal{A} \mathbf{b}^1 \neq \mathcal{A} \emptyset) = \mathbf{E}) &\subseteq ((\mathcal{A} \mathbf{a} \subseteq \mathcal{A} \mathbf{b}) = (\mathcal{A} \mathbf{a}^1 \subseteq \mathcal{A} * \{\supset\} * \mathbf{b}^1)) & (\mathcal{T}_{T876}) \& (\mathcal{T}_{M008}) \\
((\mathcal{A} \mathbf{b}^1 \neq \mathcal{A} \emptyset) = \mathbf{E}) &\subseteq ((\mathcal{A} \mathbf{a} \subseteq \mathcal{A} \mathbf{b}) = (\mathcal{A} \mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A} * \mathbf{b}^1)) & (\mathcal{T}_{T828}) \\
((\mathcal{A} \mathbf{b}^1 \neq \emptyset) = \mathbf{E}) &\subseteq ((\mathcal{A} \mathbf{a} \subseteq \mathcal{A} \mathbf{b}) = (\mathcal{A} \mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A} * \mathbf{b}^1)) & (\mathcal{T}_{S181})(\text{Null Object Theorem}) \\
((\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) &\subseteq ((\mathbf{a} \subseteq \mathbf{b}) = (\mathbf{a}^1 \subseteq \{\supset\} * \mathbf{b}^1)) & (\mathcal{D}_{S30})(\text{Definition of } \mathbf{a} \text{ Symbolism})
\end{aligned}$$

$$(\mathbf{a} \neq \emptyset) \subseteq (\mathcal{A} \mathbf{a}^1 = (\{\supset\} \diamond \mathcal{A} \mathbf{a}) (\{\supset\} * \{\supset\} \diamond \mathcal{A} \mathbf{a})) \quad (\mathcal{T}_{T879})$$

Proof of ($\mathcal{T}_{T879}$)

$$\begin{aligned}
(\mathbf{a} \neq \emptyset) &\subseteq (\mathbf{a}^1 = (\{\supset\} \diamond \mathcal{A} \mathbf{a}) (\{\supset\} * \{\supset\} \diamond \mathcal{A} \mathbf{a})) & (\mathcal{T}_{S470})(\text{General Definition of Abstraction}) \\
(\mathbf{a} \neq \emptyset) &\subseteq (\mathbf{a}^1 = (\forall \mathcal{A} * \{\supset\} \diamond \mathcal{A} * \mathbf{a}) (\{\supset\} * \forall \mathcal{A} * \{\supset\} \diamond \mathcal{A} * \mathbf{a})) & (\mathcal{T}_{S507}) \\
(\mathbf{a} \neq \emptyset) &\subseteq (\mathbf{a}^1 = (\forall \mathcal{A} * \{\supset\} \diamond \mathcal{A} * \mathbf{a}) (\forall \mathcal{A} * \{\supset\} * \mathcal{A} * \{\supset\} \diamond \mathcal{A} * \mathbf{a})) & (\mathcal{T}_{S506}) \\
(\mathbf{a} \neq \emptyset) &\subseteq (\mathbf{a}^1 = (\forall \mathcal{A} * \{\supset\} \diamond \mathcal{A} * \mathbf{a}) (\forall \mathcal{A} * \{\supset\} * \{\supset\} \diamond \mathcal{A} * \mathbf{a})) & (\mathcal{T}_{T878}) \\
(\mathbf{a} \neq \emptyset) &\subseteq (\mathcal{A} \mathbf{a}^1 = \mathcal{A} ((\forall \mathcal{A} * \{\supset\} \diamond \mathcal{A} * \mathbf{a}) (\forall \mathcal{A} * \{\supset\} * \{\supset\} \diamond \mathcal{A} * \mathbf{a}))) & (\mathcal{T}_{T876}) \\
(\mathbf{a} \neq \emptyset) &\subseteq (\mathcal{A} \mathbf{a}^1 = ((\mathcal{A} * \forall \mathcal{A} * \{\supset\} \diamond \mathcal{A} * \mathbf{a}) (\mathcal{A} * \forall \mathcal{A} * \{\supset\} * \{\supset\} \diamond \mathcal{A} * \mathbf{a}))) & (\mathcal{A}_{R35})(\text{One-to-One Relationship}) \\
& & \text{Distribution Axiom}) \\
(\mathbf{a} \neq \emptyset) &\subseteq (\mathcal{A} \mathbf{a}^1 = (\{\supset\} \diamond \mathcal{A} * \mathbf{a}) (\{\supset\} * \{\supset\} \diamond \mathcal{A} * \mathbf{a})) & (\mathcal{T}_{T878})
\end{aligned}$$

$$(a \neq \emptyset) \subseteq (\mathcal{A}a^1 = (\{\subseteq\} \diamond^{\mathcal{A}} a)(\{\subseteq\} * \{\subseteq\} \diamond^{\mathcal{A}} a))$$

(D_{S18})(Simplification of the Abstract Function)

$$(a^1 \subseteq S_D) \subseteq (a^1 = (a.(\{\supset\} * a))^1)$$

(T_{823})

Proof of (T_{823})

$$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\supset\} * a)$$

(T_{R719})

$$(a^1 \subseteq S_D) \subseteq (a = a.(\{\supset\} * a))$$

($AMX44$)($\subseteq$ Equality Form)

$$(a^1 \subseteq S_D) \subseteq (a^1 = (a.(\{\supset\} * a))^1)$$

(T_{M479})

$$\mathcal{A}a^1 = (\mathcal{A}a)^1$$

(T_{438})

Proof of (T_{438})

$$((\mathcal{X}E)^1 \subseteq D_M + P_M) = ((\mathcal{X}a)^1 = \mathcal{X}a^1)$$

(A_{S05})(Physical Dimension Aggregation Axiom)(Eqn1)

$$\text{Let } \mathcal{X} = \mathcal{A}$$

(Eqn2)

$$((\mathcal{A}E)^1 \subseteq D_M + P_M) = ((\mathcal{A}a)^1 = \mathcal{A}a^1)$$

($AMX19$)(Substitution of Eqn2 into Eqn1)(Eqn3)

$$(X = \mathcal{A}, Ob^1, \mathcal{L}_{XYZ}b^1) \subseteq ((\mathcal{X}E)^1 \subseteq P_M)$$

(A_{R89})(Axiom of Polydimensions)

$$(\mathcal{A}E)^1 \subseteq P_M$$

Extraction

$$(\mathcal{A}E)^1 \subseteq D_M + P_M$$

(T_{M150})(Single Premise Syllogism)

$$((\mathcal{A}E)^1 \subseteq D_M + P_M) = E$$

($AMX16$)(Order Axiom)(Eqn4)

$$E = ((\mathcal{A}a)^1 = \mathcal{A}a^1)$$

($AMX19$)(Substitution of Eqn4 into Eqn3)

$$(\mathcal{A}a)^1 = \mathcal{A}a^1$$

($AMX16$)(Order Axiom)

$$(a^1 \subseteq S_D) \subseteq (a^1 = (a.(\{\subseteq\} * a))^1)$$

(T_{450})

Proof of (T_{450})

$$(a^1 \subseteq S_D) \subseteq (a^1 = (a.(\{\supset\} * a))^1)$$

(T_{823})

$$(\mathcal{A}a^1 \subseteq S_D) \subseteq (a^1 = (a.(\{\supset\} * a))^1)$$

(A_{S73})(Axiom of Abstract Discrete Sets)

$$(\mathcal{A}a^1 \subseteq S_D) \subseteq (\mathcal{A}a^1 = \mathcal{A}(a.(\{\supset\} * a))^1)$$

(T_{876})

$$(\mathcal{A}a^1 \subseteq S_D) \subseteq (\mathcal{A}a^1 = (\mathcal{A}(a.(\{\supset\} * a)))^1)$$

(T_{438})

$$(\mathcal{A}a^1 \subseteq S_D) \subseteq (\mathcal{A}a^1 = (\mathcal{A}a. \mathcal{A}(\{\supset\} * a))^1)$$

(A_{R35})(One-to-One Relationship Distribution Axiom)

$$(\mathcal{A}a^1 \subseteq S_D) \subseteq (\mathcal{A}a^1 = (\mathcal{A}a. \mathcal{A}(\{\subseteq\} * \mathcal{A}a))^1)$$

(T_{R690})

$$(\mathcal{A}a^1 \subseteq S_D) \subseteq (\mathcal{A}a^1 = (\mathcal{A}a. \mathcal{A}(\vee \mathcal{A}(\{\subseteq\} * \mathcal{A}a))^1)$$

(T_{R572})

$$(\mathcal{A}a^1 \subseteq S_D) \subseteq (\mathcal{A}a^1 = (\mathcal{A}a.(\{\subseteq\} * \mathcal{A}a))^1)$$

(T_{878})(Eqn1)

$$\text{Let } a = \mathcal{A}a$$

(Eqn2)

$$\text{Let } a^1 = \mathcal{A}a^1$$

(Eqn3)

$$(a^1 \subseteq S_D) \subseteq (a^1 = (a.(\{\subseteq\} * a))^1)$$

($AMX19$)(Substitution of Eqns2&3 into Eqn1)

$$(b^1 \subseteq \{\subseteq\} * a^1) = (\mathcal{B}^1 \subseteq \{\supset\} * a^1)$$

(T_{804})

$$(b^1 \subseteq \{\supset\} * a^1) = (\mathcal{B}^1 \subseteq \{\subseteq\} * a^1)$$

(T_{790})

Example Proof of (T_{804})

$$\text{Let } b^1 \subseteq \{\subseteq\} * a^1$$

$$b^1 \subseteq \{\supset\} * \mathcal{A}a^1$$

(T_{S509})

$$b^1 \subseteq \vee \mathcal{A} * \{\supset\} * \mathcal{A}a^1$$

(T_{S505})

$$\mathcal{A} * b^1 \subseteq \mathcal{A} * \vee \mathcal{A} * \{\supset\} * \mathcal{A}a^1$$

(T_{R522})(Relational Deduction)

$$\mathcal{A} * b^1 \subseteq \{\supset\} * \mathcal{A}a^1$$

(T_{878})(Eqn1)

$$\text{Let } a^1 = \mathcal{A}a^1$$

(Eqn2)

$$\text{Let } \mathcal{B}^1 = \mathcal{A}b^1$$

(Eqn3)

$$\mathcal{B}^1 \subseteq \{\supset\} * a^1$$

($AMX19$)(Substitution of Eqns2&3 into Eqn1)

$$\text{Thus } (b^1 \subseteq \{\subseteq\} * a^1) = (\mathcal{B}^1 \subseteq \{\supset\} * a^1)$$

($AMX02$)(Conditional Introduction) since this proof is reversible

$$(a^1 \neq \emptyset) = (\mathcal{A}a^1 \neq \emptyset)$$

(T_{436})

Proof of (T₇₄₃₆)

$$\begin{aligned}
 (a \neq \emptyset) &= (\mathcal{A}a \neq \emptyset) & (T_{7954})(\text{Eqn1}) \\
 \text{Let } a &= a^1 & (\text{Eqn2}) \\
 (a^1 \neq \emptyset) &= (\mathcal{A}a^1 \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
 (a^1 \neq \emptyset) &= (\mathcal{A}a^1 \neq \emptyset) & (D_{S18})(\text{Simplification of the Abstract Function})
 \end{aligned}$$

$$a^1 = \mathcal{A} * a^1 \quad (T_{7947})$$

$$a^1 = {}^v \mathcal{A} * a^1 \quad (T_{7946})$$

Proof of (T₇₉₄₇) & (T₇₉₄₆)

$$\begin{aligned}
 a &= \mathcal{A} * a & (D_{S30})(\text{Definition of } a \text{ Symbolism}) \\
 a^1 &= (\mathcal{A} * a)^1 & (T_{M479}) \\
 a^1 &= \mathcal{A} * a^1 & (T_{7438}) \\
 {}^v \mathcal{A} * a^1 &= {}^v \mathcal{A} * \mathcal{A} * a^1 & (T_{R522})(\text{Relational Deduction}) \\
 {}^v \mathcal{A} * a^1 &= \mathcal{A} * {}^v \mathcal{A} * a^1 & (T_{7877}) \\
 {}^v \mathcal{A} * a^1 &= a^1 & (T_{7878})
 \end{aligned}$$

$$(a \neq \emptyset) \subseteq (a^1 = (\{\subseteq\} \diamond a) \cdot (\{\subseteq\} * \{\subseteq\} \diamond a)) \quad (T_{7945}) \text{ General Definition of Abstraction}$$

Proof of (T₇₉₄₅)

$$\begin{aligned}
 (a \neq \emptyset) &\subseteq (\mathcal{A}a^1 = (\{\subseteq\} \diamond \mathcal{A}a)(\{\subseteq\} * \{\subseteq\} \diamond \mathcal{A}a)) & (T_{7879}) \\
 (a \neq \emptyset) &\subseteq (a^1 = (\{\subseteq\} \diamond \mathcal{A}a)(\{\subseteq\} * \{\subseteq\} \diamond \mathcal{A}a)) & (T_{7947}) \\
 (a \neq \emptyset) &\subseteq (a^1 = (\{\subseteq\} \diamond a)(\{\subseteq\} * \{\subseteq\} \diamond a)) & (D_{S30})(\text{Definition of } a \text{ Symbolism})(\text{Eqn1}) \\
 (a \neq \emptyset) &= (\mathcal{A}a \neq \emptyset) & (T_{7954}) \\
 (a \neq \emptyset) &= (a \neq \emptyset) & (D_{S30})(\text{Definition of } a \text{ Symbolism})(\text{Eqn2}) \\
 (a \neq \emptyset) &\subseteq (a^1 = (\{\subseteq\} \diamond a)(\{\subseteq\} * \{\subseteq\} \diamond a)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
 \end{aligned}$$

$$(a^1 \subseteq S_c) \subseteq ((a \cdot (\{\supset\} * a))^1 = \emptyset^1) \quad (T_{7891})$$

$$(a^1 \subseteq S_c) \subseteq ((a \cdot (\{\subseteq\} * a))^1 = E^1) \quad (T_{7891A})$$

Example Proof of (T₇₈₉₁)

$$\begin{aligned}
 (a^1 \subseteq S_c) &= (a = (\{\supset\} * a)(\{\supset\} * a)) & (D_{R59})(\text{Definition of a Continuum Set Part3}) \\
 (a^1 \subseteq S_c) &\subseteq (a = \{\supset\} * a) & (T_{M073b})(\text{Single Premise Syllogism}) \\
 (a^1 \subseteq S_c) &\subseteq (a \cdot (\{\supset\} * a) = (\{\supset\} * a) \cdot (\{\supset\} * a)) & (T_{M233})(\text{Deduction}) \\
 (a^1 \subseteq S_c) &\subseteq (a \cdot (\{\supset\} * a) = \emptyset) & (T_{M621})(\text{Complement Conjunction}) \\
 (a^1 \subseteq S_c) &\subseteq ((a \cdot (\{\supset\} * a))^1 = \emptyset^1) & (T_{M479})
 \end{aligned}$$

$$(a^1 \subseteq S_c) \subseteq ((a \cdot (\{\subseteq\} * a))^1 = E^1) \quad (T_{7850})$$

Proof of (T₇₈₅₀)

$$\begin{aligned}
 (a^1 \subseteq S_c) &\subseteq ((a \cdot (\{\supset\} * a))^1 = \emptyset^1) & (T_{7891}) \\
 (a^1 \subseteq S_c) &\subseteq (\mathcal{A}(a \cdot (\{\supset\} * a))^1 = \mathcal{A}\emptyset^1) & (T_{R522})(\text{Relational Deduction}) \\
 (a^1 \subseteq S_c) &\subseteq (\mathcal{A}(a \cdot (\{\supset\} * a))^1 = E^1) & (T_{S450})(\text{Aspect Opposite of } \emptyset^1) \\
 (a^1 \subseteq S_c) &\subseteq ((\mathcal{A}(a \cdot (\{\supset\} * a)))^1 = E^1) & (T_{7438}) \\
 (a^1 \subseteq S_c) &\subseteq ((\mathcal{A}a \cdot \mathcal{A}(\{\supset\} * a))^1 = E^1) & (A_{R35})(\text{One-to-One Relationship Distribution Axiom}) \\
 (a^1 \subseteq S_c) &\subseteq ((\mathcal{A}a \cdot \mathcal{A}(\{\subseteq\} * a))^1 = E^1) & (T_{R690}) \\
 (a^1 \subseteq S_c) &\subseteq ((\mathcal{A}a \cdot \mathcal{A}^v \mathcal{A}(\{\subseteq\} * a))^1 = E^1) & (T_{R572}) \\
 (a^1 \subseteq S_c) &\subseteq ((\mathcal{A}a \cdot \{\subseteq\} * \mathcal{A}a)^1 = E^1) & (T_{7878}) \\
 (a^1 \subseteq S_c) &\subseteq ((a \cdot \{\subseteq\} * a)^1 = E^1) & (D_{S30})(\text{Definition of } a \text{ Symbolism})
 \end{aligned}$$

9.3 Further Theorems of Binary Operators

Further theorems of binary operators are proven below. They were not proven in the earlier chapter because they rely on (T_{7879}) above for their proof.

$$\emptyset^1 = (\{\subseteq\} \diamond E)(\{\subset\} \ast \{\subseteq\} \diamond E) \quad (T_{S463}) \text{ Definition of Null Set Entity}$$

Proof of (T_{S463})

$$\begin{aligned} (a \neq \emptyset) &\subseteq ({}^A a^1 = (\{\subseteq\} \diamond {}^A a)(\{\subset\} \ast \{\subseteq\} \diamond {}^A a)) && (T_{7879})(\text{Eqn1}) \\ \text{Let } a &= E && (\text{Eqn2}) \\ a^1 &= E^1 && (T_{M479})(\text{Eqn3}) \\ (E \neq \emptyset) &\subseteq ({}^A E^1 = (\{\subseteq\} \diamond {}^A E)(\{\subset\} \ast \{\subseteq\} \diamond {}^A E)) && (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\ (E \neq \emptyset) &\subseteq ({}^A E^1 = (\{\subseteq\} \diamond E)(\{\subset\} \ast \{\subseteq\} \diamond E)) && (A_{S87})(\text{Aspect Opposite of } E) \\ (E \neq \emptyset) &\subseteq (\emptyset^1 = (\{\subseteq\} \diamond E)(\{\subset\} \ast \{\subseteq\} \diamond E)) && (T_{7803})(\text{Aspect Opposite of } E^1) \\ E &\subseteq (\emptyset^1 = (\{\subseteq\} \diamond E)(\{\subset\} \ast \{\subseteq\} \diamond E)) && (T_{M344}) \\ E &= (\emptyset^1 = (\{\subseteq\} \diamond E)(\{\subset\} \ast \{\subseteq\} \diamond E)) && (T_{M275})(\text{Supersets of the Universal Set}) \\ \emptyset^1 &= (\{\subseteq\} \diamond E)(\{\subset\} \ast \{\subseteq\} \diamond E) && (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$\{\supset\} \ast \emptyset^1 = \sim \emptyset^1 \quad (T_{A646})$$

Proof of (T_{A646})

$$\begin{aligned} \emptyset^1 &= (\{\subseteq\} \diamond E)(\{\subset\} \ast \{\subseteq\} \diamond E) && (T_{S463}) \text{ Definition of Null Set Entity} \\ \emptyset^1 &\subseteq (\{\subseteq\} \diamond E) && (T_{M073b})(\text{Single Premise Syllogism}) \\ E &\subseteq {}^V \{\subseteq\} \diamond \emptyset^1 && (T_{R293})(\text{Active/Passive Theorem}) \\ E &= {}^V \{\subseteq\} \diamond \emptyset^1 && (T_{M275})(\text{Supersets of the Universal Set}) \\ E &= \{\supset\} \diamond \emptyset^1 && (T_{7741})(\text{Definition of } {}^V \{\subseteq\}) \\ E &= \{\supset\} \ast \emptyset^1 && (T_{R232})(\text{Interchangeability of } \ast \text{ and } \diamond \text{ for Entities}) \\ E &= (\{\supset\} + \{=\}) \ast \emptyset^1 && (D_{A141})(\text{Disjunction with } \{=\}) \\ E &= \{\supset\} \ast \emptyset^1 + \{=\} \ast \emptyset^1 && (T_{R002c})(\text{Distribution of } + \text{ over } \ast)(\text{Eqn1}) \\ \{\supset\} &\subseteq \{+\} && (T_{A104}) \\ \{\supset\} \ast \emptyset^1 &\subseteq \{+\} \ast \emptyset^1 && (T_{R197})(\text{Relational Deduction}) \\ \{\supset\} \ast \emptyset^1 &\subseteq \{=\} \ast \emptyset^1 && (T_{R245})(\text{Negative Rule for Single Element Object}) \\ E &= \{\supset\} \ast \emptyset^1 + \{=\} \ast \emptyset^1 && (T_{M064})(\subseteq \text{ Universal Form})(\text{Eqn2}) \\ E.E &= (\{\supset\} \ast \emptyset^1 + \{=\} \ast \emptyset^1).(\{\supset\} \ast \emptyset^1 + \{=\} \ast \emptyset^1) && (T_{M669})(\text{Combination of Eqns1\&2}) \\ E &= (\{\supset\} \ast \emptyset^1 + \{=\} \ast \emptyset^1).(\{\supset\} \ast \emptyset^1 + \{=\} \ast \emptyset^1) && (T_{M667})(\text{Conjunction Tautology}) \\ E &= (\{\supset\} \ast \emptyset^1).(\{\supset\} \ast \emptyset^1) + (\{=\} \ast \emptyset^1).(\{\supset\} \ast \emptyset^1) + (\{=\} \ast \emptyset^1).(\{=\} \ast \emptyset^1) && (T_{M407})(\text{Distribution of } . \text{ over } +) \\ E &= \emptyset + (\{=\} \ast \emptyset^1).(\{\supset\} \ast \emptyset^1) + (\{\supset\} \ast \emptyset^1).(\{=\} \ast \emptyset^1) + \emptyset && (T_{M621})(\text{Complement Conjunction}) \\ E &= (\{=\} \ast \emptyset^1).(\{\supset\} \ast \emptyset^1) + (\{\supset\} \ast \emptyset^1).(\{=\} \ast \emptyset^1) && (T_{M668})(\text{Null Set Disjunction}) \\ \{=\} \ast \emptyset^1 &= \{\supset\} \ast \emptyset^1 && (T_{M068})(\text{Material Equivalence}) \\ \sim \emptyset^1 &= \{\supset\} \ast \emptyset^1 && (D_{A144})(\text{Definition of the Reflexive Relationship Set}) \end{aligned}$$

$$\{\supset\} \ast E = \sim \emptyset^1 \quad (T_{A233})$$

Proof of (T_{A233})

$$\begin{aligned} \{\supset\} \ast \{\supset\} \ast \emptyset^1 &\subseteq \{\supset\} \ast \emptyset^1 && (T_{A283})(\text{Eqn1}) \\ \{\supset\} \ast \emptyset^1 &= \sim \emptyset^1 && (T_{A646})(\text{Eqn2}) \\ \{\supset\} \ast \sim \emptyset^1 &\subseteq \{\supset\} \ast \emptyset^1 && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \{\supset\} \ast \sim \emptyset^1 &= \{\supset\} \ast \sim \emptyset^1. \{\supset\} \ast \emptyset^1 && (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn3}) \\ \{\supset\} \ast E &= \{\supset\} \ast E && (D_{M00})(\text{Definition of } =) \\ \{\supset\} \ast E &= \{\supset\} \ast (\emptyset^1 + \sim \emptyset^1) && (T_{M671})(\text{Tertium non Datur}) \\ \{\supset\} \ast E &= \{\supset\} \ast \emptyset^1 + \{\supset\} \ast \sim \emptyset^1 && (T_{R002b})(\text{Distribution of } \ast \text{ over } +)(\text{Eqn4}) \\ \{\supset\} \ast E &= \{\supset\} \ast \emptyset^1 + \{\supset\} \ast \sim \emptyset^1. \{\supset\} \ast \emptyset^1 && (A_{MX19})(\text{Substitution of Eqn3 into Eqn4}) \\ \{\supset\} \ast E &= \{\supset\} \ast \emptyset^1 && (T_{M063})(\text{Partial Redundancy})(\text{Eqn5}) \\ \{\supset\} \ast E &= \sim \emptyset^1 && (A_{MX19})(\text{Substitution of Eqn2 into Eqn5}) \end{aligned}$$

$$\{\supset\} \ast E = E \quad (T_{A219})$$

Proof of (T_{A219})

$$\begin{array}{ll}
 \{\supset\} * E & \\
 (\{\supset\} + \{=\}) * E & (D_{A141})(\text{Disjunction with } \{=\}) \\
 \{\supset\} * E + \{=\} * E & (T_{R002c})(\text{Distribution of } + \text{ over } *) \\
 \sim \emptyset^1 + \{=\} * E & (T_{A233}) \\
 \sim \emptyset^1 + E & (T_{A080}) \\
 E & (T_{M055})(\text{Universal Set Disjunction})
 \end{array}$$

$$S\{\supset\} = E \quad (T_{A267})$$

Proof of (T_{A267})

$$\begin{array}{ll}
 \{\supset\} * E = E & (T_{A219}) \\
 S\{\supset\}.^VQE = E & (A_{R27})(\text{Function Axiom}) \\
 S\{\supset\}.E = E & (T_{R116})(\text{Sub-relationships with the Universal Set}) \\
 S\{\supset\} = E & (A_{MX11})(\text{Universal Set Conjunction Axiom})
 \end{array}$$

$$Q\{\subseteq\} = E \quad (T_{A270})$$

Proof of (T_{A270})

$$\begin{array}{ll}
 S\{\supset\} = E & (T_{A267})(\text{Eqn1}) \\
 \{\supset\} = ^V\{\subseteq\} & (T_{T741})(\text{Definition of } ^V\{\subseteq\}) (\text{Eqn2}) \\
 S^V\{\subseteq\} = E & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
 Q\{\subseteq\} = E & (T_{R016})(\text{Reversibility of Matching Entities})
 \end{array}$$

$$S\{\supset\} = \sim \emptyset^1 \quad (T_{A261})$$

Proof of (T_{A261})

$$\begin{array}{ll}
 \{\supset\} * E = \sim \emptyset^1 & (T_{A233}) \\
 S\{\supset\}.^VQE = \sim \emptyset^1 & (A_{R27})(\text{Function Axiom}) \\
 S\{\supset\}.E = \sim \emptyset^1 & (T_{R116})(\text{Sub-relationships with the Universal Set}) \\
 S\{\supset\} = \sim \emptyset^1 & (A_{MX11})(\text{Universal Set Conjunction Axiom})
 \end{array}$$

$$Q\{\subset\} = \sim \emptyset^1 \quad (T_{A262})$$

Proof of (T_{A262})

$$\begin{array}{ll}
 S\{\supset\} = \sim \emptyset^1 & (T_{A261})(\text{Eqn1}) \\
 \{\supset\} = ^V\{\subset\} & (D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set})(\text{Eqn2}) \\
 S^V\{\subset\} = \sim \emptyset^1 & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
 Q\{\subset\} = \sim \emptyset^1 & (T_{R016})(\text{Reversibility of Matching Entities})
 \end{array}$$

$$\{\nexists\} \diamond E = \emptyset \quad (T_{A274})$$

Proof of (T_{A274})

$$\begin{array}{ll}
 \{\supset\} * E = E & (T_{A219}) \\
 \{\nexists\} \diamond E = E & (T_{A199}) \\
 \{\nexists\} \diamond E = \sim E & (T_{M008})(\text{Inverse of an Equality}) \\
 \{\nexists\} \diamond E = \emptyset & (D_{M02})(\text{Definition of } \emptyset)
 \end{array}$$

$$\{\supset\} \times E = \emptyset^1 \quad (T_{R639})$$

Proof of (T_{R639})

$$\begin{array}{ll}
 \{\supset\} * E = \sim \emptyset^1 & (T_{A233}) \\
 \{\supset\} \times E = \emptyset^1 & (T_{M008})(\text{Inverse of an Equality})
 \end{array}$$

$$\{\supset\} * \sim \emptyset^1 = \sim \emptyset^1 \quad (T_{S461})$$

Proof of (T_{S461})

$\{\supset\} * \{\supset\} * \emptyset^1 \subseteq \{\supset\} * \emptyset^1$	(T_{A283})(Eqn1)
$\{\supset\} * \emptyset^1 = \sim \emptyset^1$	(T_{A646})(Eqn2)
$\{\supset\} * \sim \emptyset^1 \subseteq \sim \emptyset^1$	(A_{MX19})(Substitution of Eqn2 into Eqn1)
$\{\supset\} * \sim \emptyset^1 \subseteq \{\supset\} * \sim \emptyset^1, \sim \emptyset^1$	(A_{MX44})($\subseteq$ Equality Form)(Eqn3)
$\{\supset\} * \sim \emptyset^1 = (\{\supset\} + \{=\}) * \sim \emptyset^1$	(D_{A141})(Disjunction with $\{=\}$)
$\{\supset\} * \sim \emptyset^1 = \{\supset\} * \sim \emptyset^1 + \{=\} * \sim \emptyset^1$	(T_{R002c})(Distribution of $+$ over $*$)
$\{\supset\} * \sim \emptyset^1 = \{\supset\} * \sim \emptyset^1 + \sim \emptyset^1$	(T_{A080})(Eqn4)
$\{\supset\} * \sim \emptyset^1 = \{\supset\} * \sim \emptyset^1, \sim \emptyset^1 + \sim \emptyset^1$	(A_{MX19})(Substitution of Eqn3 into Eqn4)
$\{\supset\} * \sim \emptyset^1 = \sim \emptyset^1$	(T_{M063})(Partial Redundancy)

$$\{\supset\} * \emptyset^1 = E \quad (T_{A149})$$

Proof of (T_{A149})

$\{\supset\} * \emptyset^1 = \{\supset\} * \emptyset^1$	(D_{M00})(Definition of $=$)
$\{\supset\} * \emptyset^1 = (\{\supset\} + \{=\}) * \emptyset^1$	(D_{A141})(Disjunction with $\{=\}$)
$\{\supset\} * \emptyset^1 = \{\supset\} * \emptyset^1 + \{=\} * \emptyset^1$	(T_{R002c})(Distribution of $+$ over $*$)
$\{\supset\} * \emptyset^1 = \sim \emptyset^1 + \{=\} * \emptyset^1$	(T_{A646}) i.e. $\{\supset\} * \emptyset^1 = \sim \emptyset^1$
$\{\supset\} * \emptyset^1 = \sim \emptyset^1 + \emptyset^1$	(D_{A144})(Definition of the Reflexive Relationship Set)
$\{\supset\} * \emptyset^1 = E$	(T_{M671})(Tertium non Datur)

$$\{\supset\} \diamond \emptyset^1 = E \quad (T_{A150})$$

Proof of (T_{A150})

$\{\supset\} * \emptyset^1 = E$	(T_{A149})
$\{\supset\} \diamond \emptyset^1 = E$	(T_{R232})

$$(a^1 \neq \emptyset) \subseteq (\emptyset^1 \subseteq \{\subseteq\} * a^1) \quad (T_{B46})$$

Proof of (T_{B46})

$\emptyset^1 = (\{\subseteq\} \diamond E).(\{\subseteq\} * \{\subseteq\} \diamond E)$	(T_{S463})(Definition of Null Set Entity)
$\emptyset^1 \subseteq \{\subseteq\} \diamond E$	(T_{M073b})(Single Premise Syllogism)(Eqn1)
$a^1 \subseteq E$	(T_{M744})(Universal Set Theorem)
$(a^1 \neq \emptyset) \subseteq (\{\subseteq\} \diamond E \subseteq \{\subseteq\} \diamond a^1)$	(T_{R262})(Relational Deduction)(Eqn2)
$(a^1 \neq \emptyset) \subseteq (\emptyset^1 \subseteq \{\subseteq\} \diamond a^1)$	(T_{M139})(Hypothetical Syllogism Eqns1&2)
$(a^1 \neq \emptyset) \subseteq (\emptyset^1 \subseteq \{\subseteq\} * a^1)$	(T_{R232})(Interchangeability of $*$ and $\diamond$ for Entities)

$$(a^1 \subseteq S_c) \subseteq (\emptyset^1 \subseteq a) \quad (T_{S834})$$

$$(a^1 \subseteq S_c) \subseteq (E^1 \subseteq a) \quad (T_{S834A})$$

$$(a^1 \subseteq S_c) \subseteq (\emptyset^1 = a.\emptyset^1) \quad (T_{S833})$$

$$(a^1 \subseteq S_c) \subseteq (E^1 = a.E^1) \quad (T_{S833A})$$

Example Proof of (T_{S834}) & (T_{S833})

$(a^1 \neq \emptyset) \subseteq (\emptyset^1 \subseteq \{\subseteq\} * a^1)$	(T_{B46})(Eqn1)
$(a^1 \subseteq S_c) \subseteq (a = \{\subseteq\} * a^1)$	(A_{R02})(Definition of Disaggregation for Continuum Sets)(Eqn2)
$(a^1 \subseteq S_c)(a^1 \neq \emptyset) \subseteq (\emptyset^1 \subseteq a)$	(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)
$(a^1 \subseteq S_c) \subseteq (a^1 \neq \emptyset)$	(T_{S460})
$(a^1 \subseteq S_c) = (a^1 \subseteq S_c)(a^1 \neq \emptyset)$	(A_{MX44})($\subseteq$ Equality Form)(Eqn4)
$(a^1 \subseteq S_c) \subseteq (\emptyset^1 \subseteq a)$	(A_{MX19})(Substitution of Eqn4 into Eqn3)
$(a^1 \subseteq S_c) \subseteq (\emptyset^1 = a.\emptyset^1)$	(A_{MX44})($\subseteq$ Equality Form)

$$(a + \emptyset^1)^1 = a^1 \quad (T_{S222}) \text{ Aggregation with } \emptyset^1$$

Proof of (T_{S222})

Let $a \neq \emptyset$	(Eqn1)

$(a \neq \emptyset) = E$	(A_{MX16}) (Order Axiom)(Eqn2)
$(a \neq \emptyset) + (\emptyset^1 \neq \emptyset) = (a + \emptyset^1 \neq \emptyset)$	(T_{M238}) (Probabilistic Disjunction)
$(a \neq \emptyset) \subseteq (a + \emptyset^1 \neq \emptyset)$	(T_{M072}) (Single Premise Syllogism) (Eqn3)
$E \subseteq (a + \emptyset^1 \neq \emptyset)$	(A_{MX19}) (Substitution of Eqn2 into Eqn3)
$E = (a + \emptyset^1 \neq \emptyset)$	(T_{M275}) (Supersets of the Universal Set)
$a + \emptyset^1 \neq \emptyset$	(A_{MX16}) (Order Axiom)(Eqn4)
$\{\supset\} \diamond (a + \emptyset^1) = \{\supset\} \diamond (a + \emptyset^1)$	(D_{M00}) (Definition of $=$)
$\{\supset\} \diamond (a + \emptyset^1) = (\{\supset\} \diamond a) . (\{\supset\} \diamond \emptyset^1)$	(T_{R235}) (Non Distribution of $\diamond$ over $+$)
$\{\supset\} \diamond (a + \emptyset^1) = (\{\supset\} \diamond a) . (\{\supset\} * \emptyset^1)$	(T_{R232}) (Interchangeability of $*$ and $\diamond$ for Entities)
$\{\supset\} \diamond (a + \emptyset^1) = (\{\supset\} \diamond a) . E$	(T_{A149})
$\{\supset\} \diamond (a + \emptyset^1) = \{\supset\} \diamond a$	(A_{MX11}) (Universal Set Conjunction Axiom)(Eqn5)
$a^1 = (\{\supset\} \diamond a) . (\{\supset\} * \{\supset\} \diamond a)$	(D_{R98}) (General Definition of Aggregation)(Eqn6)
$a^1 = (\{\supset\} \diamond (a + \emptyset^1)) . (\{\supset\} * \{\supset\} \diamond (a + \emptyset^1))$	(A_{MX19}) (Substitution of Eqn5 into Eqn6)(Eqn7)
$(a + \emptyset^1)^1 = (\{\supset\} \diamond (a + \emptyset^1)) . (\{\supset\} * \{\supset\} \diamond (a + \emptyset^1))$	(D_{R98}) (General Definition of Aggregation Eqn1)(Eqn8)
$(a + \emptyset^1)^1 = a^1$	(A_{MX19}) (Substitution of Eqn7 into Eqn8)

$$a + \emptyset^1 = a \quad (T_{T420})$$

Proof of (T_{T420})

$$(a + \emptyset^1)^1 = a^1 \quad (T_{S222}) \text{(Aggregation with } \emptyset^1)$$

$$a + \emptyset^1 = a \quad (T_{M479})$$

Chapter 10 Proofs of Theorems of Natural Language

10.1 Proofs

$$(a \subseteq b) = (\forall Sa.\{=\}. \forall Qb = \emptyset) \quad (T_{7894})$$

$$(a \subseteq b) = (\forall Sa.\{=\}. \forall Qb = \emptyset) \quad (T_{7895})$$

Example Proof of (T₇₈₉₄)

$$\begin{aligned} a &\subseteq b \\ \{=\} * a &\subseteq b & (T_{A080}) \\ Q(\{=\}. \forall Sa) &\subseteq b & (T_{R020})(\text{Function Theorem}) \\ Q(\{=\}. \forall Sa) &\subseteq b & (T_{A485}) \\ \{=\}. \forall Sa &\subseteq \forall Qb & (T_{S240}) \\ \forall Sa.\{=\} &\subseteq \forall Qb & (T_{M404})(\text{Commutative .}) \\ \forall Sa.\{=\}. \sim \forall Qb &= \emptyset & (T_{M735})(\subseteq \text{Existential Form}) \\ \forall Sa.\{=\}. \forall Qb &= \emptyset & (T_{R037})(\text{Sub-Relationship Negative Theorem}) \\ \text{Thus } (a \subseteq b) &= (\forall Sa.\{=\}. \forall Qb = \emptyset) & (A_{MX02})(\text{Conditional Introduction}) \text{ since this proof is reversible.} \end{aligned}$$

$$(a \not\subseteq b) = (\forall Sa.\{=\}. \forall Qb \neq \emptyset) \quad (T_{7896})$$

$$(a \not\subseteq b) = (\forall Sa.\{=\}. \forall Qb \neq \emptyset) \quad (T_{7897})$$

Example Proof of (T₇₈₉₆)

$$\begin{aligned} (a \subseteq b) &= (\forall Sa.\{=\}. \forall Qb = \emptyset) & (T_{7894}) \\ (a \not\subseteq b) &= (\forall Sa.\{=\}. \forall Qb \neq \emptyset) & (T_{M008})(\text{Inverse of an Equality}) \end{aligned}$$

$$\forall S\{\subseteq\}a^1 \subseteq \{\subseteq\} * \forall Sa^1 \quad (T_{S618})$$

$$(x^1 \subseteq \{\subseteq\}a^1) \subseteq (\forall Sx^1 \subseteq \{\subseteq\} * \forall Sa^1) \quad (T_{S607})$$

Proof of (T_{S618}) & (T_{S607})

$$\begin{aligned} \forall S\{\subseteq\}a^1 &\subseteq E & (T_{M744})(\text{Universal Set Theorem})(\text{Eqn1}) \\ (X = \forall S, \forall Q) &\subseteq (\{\subseteq\} * \forall Sa^1 = E) & (A_{R29})(\text{Universal Relationships Axiom}) \\ \{\subseteq\} * \forall Sa^1 &= E & \text{Extraction (Eqn2)} \\ \forall S\{\subseteq\}a^1 &\subseteq \{\subseteq\} * \forall Sa^1 & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (\forall Sx^1 \subseteq \forall S\{\subseteq\}a^1) &\subseteq (\forall Sx^1 \subseteq \{\subseteq\} * \forall Sa^1) & (T_{M272})(\text{Deductive Rule}) \\ (x^1 \subseteq \{\subseteq\}a^1) &\subseteq (\forall Sx^1 \subseteq \{\subseteq\} * \forall Sa^1) & (T_{R023d})(\text{Sub-Relationship Deduction}) \end{aligned}$$

$$\forall Q\{\subseteq\}b^1 \subseteq \{\subseteq\} * \forall Qb^1 \quad (T_{S609})$$

$$(y^1 \subseteq \{\subseteq\}b^1) \subseteq (\forall Qy^1 \subseteq \{\subseteq\} * \forall Qb^1) \quad (T_{S608})$$

Proof of (T_{S609}) & (T_{S608})

$$\begin{aligned} \forall Q\{\subseteq\}b^1 &\subseteq E & (T_{M744})(\text{Universal Set Theorem})(\text{Eqn1}) \\ (X = \forall S, \forall Q) &\subseteq (\{\subseteq\} * \forall Qa^1 = E) & (A_{R29})(\text{Universal Relationships Axiom}) \\ \{\subseteq\} * \forall Qa^1 &= E & \text{Extraction (Eqn2)} \\ \forall Q\{\subseteq\}b^1 &\subseteq \{\subseteq\} * \forall Qb^1 & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (\forall Qy^1 \subseteq \forall Q\{\subseteq\}b^1) &\subseteq (\forall Qy^1 \subseteq \{\subseteq\} * \forall Qa^1) & (T_{M272})(\text{Deductive Rule}) \\ (y^1 \subseteq \{\subseteq\}b^1) &\subseteq (\forall Qy^1 \subseteq \{\subseteq\} * \forall Qa^1) & (T_{R024d})(\text{Sub-Relationship Deduction}) \end{aligned}$$

$$\forall Sa. \forall Qb \subseteq \{\subseteq\} * (\forall Sa^1. \forall Qb^1) \quad (T_{S612})$$

Proof of (T_{S612})

$$\begin{aligned} a &\subseteq \{\subseteq\}a^1 & (T_{S581})(\text{General Definition of Disaggregation}) \\ \forall Sa &\subseteq \forall S\{\subseteq\}a^1 & (T_{R023d})(\text{Sub-Relationship Deduction})(\text{Eqn1}) \\ b &\subseteq \{\subseteq\}b^1 & (T_{S581})(\text{General Definition of Disaggregation}) \\ \forall Qb &\subseteq \forall Q\{\subseteq\}b^1 & (T_{R024d})(\text{Sub-Relationship Deduction})(\text{Eqn2}) \end{aligned}$$

$$\begin{aligned}
& \forall \text{Sa} . \forall \text{Qb} \subseteq \forall \text{S}\{\subseteq\} \text{a}^1 . \forall \text{Q}\{\subseteq\} \text{b}^1 & (T_{M669})(\text{Combination of Eqns1\&2})(\text{Eqn3}) \\
& \forall \text{S}\{\subseteq\} \text{a}^1 . \forall \text{Q}\{\subseteq\} \text{b}^1 \subseteq \{\subseteq\} * (\forall \text{Sa}^1 . \forall \text{Qb}^1) & (A_{R32})(\text{Eqn4}) \\
& \forall \text{Sa} . \forall \text{Qb} \subseteq \{\subseteq\} * (\forall \text{Sa}^1 . \forall \text{Qb}^1) & (T_{M139})(\text{Hypothetical Syllogism Eqns3\&4})
\end{aligned}$$

$$(\text{x}^1 \subseteq \{\subseteq\} \text{a}^1)(\text{y}^1 \subseteq \{\subseteq\} \text{b}^1) \subseteq (\forall \text{Sx}^1 . \forall \text{Qy}^1 \subseteq \{\subseteq\} * (\forall \text{Sa}^1 . \forall \text{Qb}^1)) \quad (T_{S610})$$

Proof of (T_{S610})

$$\begin{aligned}
& \text{Let } \text{x}^1 \subseteq \{\subseteq\} \text{a}^1 & \\
& \forall \text{Sx}^1 \subseteq \forall \text{S}\{\subseteq\} \text{a}^1 & (T_{R023d})(\text{Sub-Relationship Deduction})(\text{Eqn1}) \\
& \text{Let } \text{y}^1 \subseteq \{\subseteq\} \text{b}^1 & \\
& \forall \text{Qy}^1 \subseteq \forall \text{Q}\{\subseteq\} \text{b}^1 & (T_{R024d})(\text{Sub-Relationship Deduction})(\text{Eqn2}) \\
& \forall \text{Sx}^1 . \forall \text{Qy}^1 \subseteq \forall \text{S}\{\subseteq\} \text{a}^1 . \forall \text{Q}\{\subseteq\} \text{b}^1 & (T_{M669})(\text{Combination of Eqns1\&2})(\text{Eqn3}) \\
& \forall \text{S}\{\subseteq\} \text{a}^1 . \forall \text{Q}\{\subseteq\} \text{b}^1 \subseteq \{\subseteq\} * (\forall \text{Sa}^1 . \forall \text{Qb}^1) & (A_{R32})(\text{Eqn4}) \\
& \forall \text{Sx}^1 . \forall \text{Qy}^1 \subseteq \{\subseteq\} * (\forall \text{Sa}^1 . \forall \text{Qb}^1) & (T_{M139})(\text{Hypothetical Syllogism Eqns3\&4}) \\
& \text{Thus } (\text{x}^1 \subseteq \{\subseteq\} \text{a}^1)(\text{y}^1 \subseteq \{\subseteq\} \text{b}^1) \subseteq (\forall \text{Sx}^1 . \forall \text{Qy}^1 \subseteq \{\subseteq\} * (\forall \text{Sa}^1 . \forall \text{Qb}^1)) & \\
& & (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$(\text{a}(\text{c} * \text{b}) \neq \emptyset) = (\forall \text{Sa} . \text{c} . \forall \text{Qb} \neq \emptyset) \quad (T_{R227})$$

Proof of (T_{R227})

$$\begin{aligned}
& (\text{a}(\text{c} * \text{b}) = \emptyset) = (\forall \text{Sa} . \text{c} . \forall \text{Qb} = \emptyset) & (T_{R228}) \\
& (\text{a}(\text{c} * \text{b}) \neq \emptyset) = (\forall \text{Sa} . \text{c} . \forall \text{Qb} \neq \emptyset) & (T_{M008})(\text{Inverse of an Equality})
\end{aligned}$$

$$\forall (\forall \text{Sa}^1 . \text{c} . \forall \text{Qb}^1) = (\forall \text{Sb}^1 . \forall \text{c} . \forall \text{Qa}^1) \quad (T_{S210})$$

Proof of (T_{S210})

$$\begin{aligned}
& \forall (\forall \text{Sa}^1 . \text{c} . \forall \text{Qb}^1) & \\
& \forall (\forall \text{Sa}^1) . \forall \text{c} . \forall (\forall \text{Qb}^1) & (T_{R012a})(\text{Distribution of } \forall * \text{ over } .) \\
& \forall \text{Qa}^1 . \forall \text{c} . \forall (\forall \text{Qb}^1) & (T_{R019}) \\
& \forall \text{Qa}^1 . \forall \text{c} . \forall \text{Sb}^1 & (A_{R13}) \\
& \forall \text{Sb}^1 . \forall \text{c} . \forall \text{Qa}^1 & (T_{M404})(\text{Commutative } .) \\
& \text{Thus } \forall (\forall \text{Sa}^1 . \text{c} . \forall \text{Qb}^1) = (\forall \text{Sb}^1 . \forall \text{c} . \forall \text{Qa}^1) & (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$(\forall \text{Sa} . \text{c} . \forall \text{Qb} = \emptyset) = (\forall \text{Sb} . \forall \text{c} . \forall \text{Qa} = \emptyset) \quad (T_{S281})$$

Proof of (T_{S281})

$$\begin{aligned}
& (\forall \text{Sa} . \text{c} . \forall \text{Qb} = \emptyset) = (\text{a} \subseteq \text{c} * \text{b}) & (T_{S082}) \\
& (\forall \text{Sa} . \text{c} . \forall \text{Qb} = \emptyset) = (\text{b} \subseteq \forall \text{c} * \text{a}) & (T_{R457})(\text{Active/Passive Theorem}) \\
& (\forall \text{Sa} . \text{c} . \forall \text{Qb} = \emptyset) = (\forall \text{Sb} . \forall \text{c} . \forall \text{Qa} = \emptyset) & (T_{S082})
\end{aligned}$$

$$(\forall \text{Sa} . \text{c} . \forall \text{Qb} \neq \emptyset) = (\forall \text{Sb} . \forall \text{c} . \forall \text{Qa} \neq \emptyset) \quad (T_{S280})$$

Proof of (T_{S280})

$$\begin{aligned}
& (\forall \text{Sa} . \text{c} . \forall \text{Qb} = \emptyset) = (\forall \text{Sb} . \forall \text{c} . \forall \text{Qa} = \emptyset) & (T_{S281}) \\
& (\forall \text{Sa} . \text{c} . \forall \text{Qb} \neq \emptyset) = (\forall \text{Sb} . \forall \text{c} . \forall \text{Qa} \neq \emptyset) & (T_{M008})(\text{Inverse of an Equality})
\end{aligned}$$

$$\forall \text{Sa} . \text{c} . \forall \text{Qb} = \forall \text{Sb} . \forall \text{c} . \forall \text{Qa} \quad (T_{S617})$$

Proof of (T_{S617})

$$\begin{aligned}
& (\forall \text{Sa} . \text{c} . \forall \text{Qb} \neq \emptyset) = (\forall \text{Sb} . \forall \text{c} . \forall \text{Qa} \neq \emptyset) & (T_{S280}) \\
& (\forall \text{Sa} . \text{c} . \forall \text{Qb} = \forall \text{Sb} . \forall \text{c} . \forall \text{Qa}) = \text{E} & (T_{M523})(\text{Probabilistic Rule}) \\
& \forall \text{Sa} . \text{c} . \forall \text{Qb} = \forall \text{Sb} . \forall \text{c} . \forall \text{Qa} & (A_{MX16})(\text{Order Axiom})
\end{aligned}$$

$$\forall \text{Sa} . \forall \text{Qb} = \forall \text{Qa} . \forall \text{Sb} \quad (T_{S613})$$

Proof of (T_{S613})

$$\begin{aligned}
& \forall \text{Sa} . \text{c} . \forall \text{Qb} = \forall \text{Sb} . \forall \text{c} . \forall \text{Qa} & (T_{S617})(\text{Eqn1}) \\
& \text{Let } \text{c} = \text{E} & (\text{Eqn2})
\end{aligned}$$

$$\begin{aligned} \forall \text{Sa}. \text{E}. \forall \text{Qb} &= \forall \text{Sb}. \text{E}. \forall \text{Qa} & (\text{A}_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\ \forall \text{Sa}. \text{E}. \forall \text{Qb} &= \forall \text{Sb}. \text{E}. \forall \text{Qa} & (\text{T}_{793}) (\text{Relationship Opposite of E}) \\ \forall \text{Sa}. \forall \text{Qb} &= \forall \text{Sb}. \forall \text{Qa} & (\text{A}_{MX17}) (\text{Universal Set Conjunction Axiom}) \end{aligned}$$

$$(\forall \text{Sa}^1. \text{c}. \forall \text{Qb}^1 = \emptyset) = (\forall \text{Sb}^1. \forall \text{c}. \forall \text{Qa}^1 = \emptyset) \quad (\text{T}_{R616})$$

Proof of (T_{R616})

$$\begin{aligned} (\forall \text{Sa}. \text{c}. \forall \text{Qb} = \emptyset) &= (\forall \text{Sb}. \forall \text{c}. \forall \text{Qa} = \emptyset) & (\text{T}_{S281}) (\text{Eqn1}) \\ \text{Let } \text{a} &= \text{a}^1 & (\text{Eqn2}) \\ \text{Let } \text{b} &= \text{b}^1 & (\text{Eqn3}) \\ (\forall \text{Sa}^1. \text{c}. \forall \text{Qb}^1 = \emptyset) &= (\forall \text{Sb}^1. \forall \text{c}. \forall \text{Qa}^1 = \emptyset) & (\text{A}_{MX19}) (\text{Substitution of Eqns2 \& 3 into Eqn1}) \end{aligned}$$

$$(\forall \text{Sa}^1. \text{c}. \forall \text{Qb}^1 \neq \emptyset) = (\forall \text{Sb}^1. \forall \text{c}. \forall \text{Qa}^1 \neq \emptyset) \quad (\text{T}_{R615})$$

Proof of (T_{R615})

$$\begin{aligned} (\forall \text{Sa}. \text{c}. \forall \text{Qb} \neq \emptyset) &= (\forall \text{Sb}. \forall \text{c}. \forall \text{Qa} \neq \emptyset) & (\text{T}_{S280}) (\text{Eqn1}) \\ \text{Let } \text{a} &= \text{a}^1 & (\text{Eqn2}) \\ \text{Let } \text{b} &= \text{b}^1 & (\text{Eqn3}) \\ (\forall \text{Sa}^1. \text{c}. \forall \text{Qb}^1 \neq \emptyset) &= (\forall \text{Sb}^1. \forall \text{c}. \forall \text{Qa}^1 \neq \emptyset) & (\text{A}_{MX19}) (\text{Substitution of Eqns2 \& 3 into Eqn1}) \end{aligned}$$

$$(\text{a}^1 \not\subseteq \text{c} * \text{b}) = (\forall \text{Sa}^1. \text{c}. \forall \text{Qb} \neq \emptyset) \quad (\text{T}_{S089})$$

Proof of (T_{S089})

$$\begin{aligned} (\text{a} \not\subseteq \text{c} * \text{b}) &= (\forall \text{Sa}. \text{c}. \forall \text{Qb} \neq \emptyset) & (\text{T}_{S081}) (\text{Eqn1}) \\ \text{Let } \text{a} &= \text{a}^1 & (\text{Eqn2}) \\ (\text{a}^1 \not\subseteq \text{c} * \text{b}) &= (\forall \text{Sa}^1. \text{c}. \forall \text{Qb} \neq \emptyset) & (\text{A}_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(\text{a}^1 \subseteq \text{c} * \text{b}) = (\forall \text{Sa}^1. \text{c}. \forall \text{Qb} = \emptyset) \quad (\text{T}_{S090})$$

$$((\text{a}^1 \neq \emptyset) = \text{E}) \subseteq ((\text{a}^1 \not\subseteq \text{c} * \text{b}) = (\forall \text{Sa}^1. \text{c}. \forall \text{Qb} = \emptyset)) \quad (\text{T}_{S088})$$

Proof of (T_{S090}) \& (T_{S088})

$$\begin{aligned} (\text{a}^1 \not\subseteq \text{c} * \text{b}) &= (\forall \text{Sa}^1. \text{c}. \forall \text{Qb} \neq \emptyset) & (\text{T}_{S089}) \\ (\text{a}^1 \subseteq \text{c} * \text{b}) &= (\forall \text{Sa}^1. \text{c}. \forall \text{Qb} = \emptyset) & (\text{T}_{M008}) (\text{Inverse of an Equality}) \\ ((\text{a}^1 \neq \emptyset) = \text{E}) &\subseteq ((\text{a}^1 \not\subseteq \text{c} * \text{b}) = (\forall \text{Sa}^1. \text{c}. \forall \text{Qb} = \emptyset)) & (\text{T}_{M090}) (\text{Inverse Unit Subset Rule}) \end{aligned}$$

$$(\forall \text{Sa}. \sim(\text{c}. \forall \text{Qb}) = \emptyset) \subseteq (\text{a} \subseteq \text{c} * \text{b}) \quad (\text{T}_{S079})$$

Proof of (T_{S079})

$$\begin{aligned} \text{Let } \forall \text{Sa}. \sim(\text{c}. \forall \text{Qb}) &= \emptyset \\ \forall \text{Sa} &\subseteq (\text{c}. \forall \text{Qb}) & (\text{T}_{M735}) (\subseteq \text{Existential Form}) \\ \text{S} \forall \text{Sa} &\subseteq \text{S}(\text{c}. \forall \text{Qb}) & (\text{T}_{R026a}) (\text{Sub-Relationship Deduction}) \\ \text{a} &\subseteq \text{S}(\text{c}. \forall \text{Qb}) & (\text{T}_{R012m}) (\text{Sub-relationship Closure Theorem}) \\ \text{a} &\subseteq \text{c} * \text{b} & (\text{A}_{R27}) (\text{Function Axiom}) \\ \text{Thus } (\forall \text{Sa}. \sim(\text{c}. \forall \text{Qb}) = \emptyset) &\subseteq (\text{a} \subseteq \text{c} * \text{b}) & (\text{A}_{MX02}) (\text{Conditional Introduction}) \end{aligned}$$

$$(\text{a} \not\subseteq \text{c} * \text{b}) \subseteq (\forall \text{Sa}. \sim(\text{c}. \forall \text{Qb}) \neq \emptyset) \quad (\text{T}_{S080})$$

Proof of (T_{S080})

$$\begin{aligned} (\forall \text{Sa}. \sim(\text{c}. \forall \text{Qb}) = \emptyset) &\subseteq (\text{a} \subseteq \text{c} * \text{b}) & (\text{T}_{S079}) \\ (\text{a} \not\subseteq \text{c} * \text{b}) &\subseteq (\forall \text{Sa}. \sim(\text{c}. \forall \text{Qb}) \neq \emptyset) & (\text{T}_{M012}) (\text{Transposition}) \end{aligned}$$

$$((\text{a} \neq \emptyset) = \text{E}) \subseteq ((\text{a} \subseteq \text{c} * \text{b}) \subseteq (\forall \text{Sa}. \sim(\text{c}. \forall \text{Qb}) \neq \emptyset)) \quad (\text{T}_{S086})$$

Proof of (T_{S086})

$$\begin{aligned} ((\text{a} \neq \emptyset) = \text{E}) &\subseteq ((\text{a} \subseteq \text{c} * \text{b}) \subseteq (\text{a} \not\subseteq \text{c} * \text{b})) & (\text{T}_{M175}) (\text{Negation Introduction}) (\text{Eqn1}) \\ (\text{a} \not\subseteq \text{c} * \text{b}) &= (\forall \text{Sa}. \sim(\text{c}. \forall \text{Qb}) \neq \emptyset) & (\text{T}_{S080}) (\text{Eqn2}) \end{aligned}$$

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq c*b) \subseteq (\forall Sa. \sim(c. \forall Qb) \neq \emptyset)) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$((a \neq \emptyset) = E) \subseteq ((\forall Sa. \sim(c. \forall Qb) = \emptyset) \subseteq (a \not\subseteq c*b)) \quad (T_{S085})$$

Proof of (T_{S085})

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq c*b) \subseteq (\forall Sa. \sim(c. \forall Qb) \neq \emptyset)) \quad (T_{S086})$$

$$((a \neq \emptyset) = E) \subseteq ((\forall Sa. \sim(c. \forall Qb) = \emptyset) \subseteq (a \not\subseteq c*b)) \quad (T_{M012})(\text{Transposition})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b) \subseteq (\forall Sa^1. \sim(c. \forall Qb) \neq \emptyset)) \quad (T_{S091})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq c*b) \subseteq (\forall Sa^1. \sim(c. \forall Qb) \neq \emptyset)) \quad (T_{S093})$$

Proof of (T_{S091}) & (T_{S093})

$$((a \neq \emptyset) = E) \subseteq ((a \subseteq c*b) \subseteq (\forall Sa. \sim(c. \forall Qb) \neq \emptyset)) \quad (T_{S086})(\text{Eqn1})$$

$$\text{Let } a = a^1 \quad (\text{Eqn2})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b) \subseteq (\forall Sa^1. \sim(c. \forall Qb) \neq \emptyset)) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq c*b) \subseteq (\forall Sa^1. \sim(c. \forall Qb) \neq \emptyset)) \quad (T_{M090})(\text{Inverse Unit Subset Rule})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((\forall Sa^1. \sim(c. \forall Qb) = \emptyset) \subseteq (a^1 \not\subseteq c*b)) \quad (T_{S092})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((\forall Sa^1. \sim(c. \forall Qb) = \emptyset) \subseteq (a^1 \subseteq c*b)) \quad (T_{S094})$$

Proof of (T_{S092}) & (T_{S094})

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b) \subseteq (\forall Sa^1. \sim(c. \forall Qb) \neq \emptyset)) \quad (T_{S091})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((\forall Sa^1. \sim(c. \forall Qb) = \emptyset) \subseteq (a^1 \not\subseteq c*b)) \quad (T_{M012})(\text{Transposition})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((\forall Sa^1. \sim(c. \forall Qb) = \emptyset) \subseteq (a^1 \subseteq c*b)) \quad (T_{M090})(\text{Inverse Unit Subset Rule})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b^1) = (\forall Sa^1. c. \forall Qb^1 \neq \emptyset)) \quad (T_{R608})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq c*b^1) = (\forall Sa^1. c. \forall Qb^1 = \emptyset)) \quad (T_{R609})$$

Proof of (T_{R608}) & (T_{R609})

$$(a \not\subseteq c*b) = (\forall Sa. c. \forall Qb \neq \emptyset) \quad (T_{S081})$$

$$(a^1 \not\subseteq c*b^1) = (\forall Sa^1. c. \forall Qb^1 \neq \emptyset) \quad \text{Resymbolising (Eqn1)}$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq c*b^1) = (a^1 \subseteq c*b^1)) \quad (T_{M090})(\text{Inverse Unit Subset Rule})(\text{Eqn2})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b^1) = (\forall Sa^1. c. \forall Qb^1 \neq \emptyset)) \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq c*b^1) = (\forall Sa^1. c. \forall Qb^1 = \emptyset)) \quad (T_{M008})(\text{Inverse of an Equality})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq c*b^1) = (\forall Sa^1. c. \forall Qb^1 \neq \emptyset)) \quad (T_{R610})$$

Proof of (T_{R610})

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b^1) = (\forall Sa^1. c. \forall Qb^1 \neq \emptyset)) \quad (T_{R608})(\text{Eqn1})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq c*b^1) = (a^1 \subseteq c*b^1)) \quad (T_{M090})(\text{Inverse Unit Subset Rule})(\text{Eqn2})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq c*b^1) = (\forall Sa^1. c. \forall Qb^1 \neq \emptyset)) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b^1) = (\forall Sa^1. c. \forall Qb^1 = \emptyset)) \quad (T_{R611})$$

Proof of (T_{R611})

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq c*b^1) = (\forall Sa^1. c. \forall Qb^1 = \emptyset)) \quad (T_{R609})(\text{Eqn1})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq c*b^1) = (a^1 \subseteq c*b^1)) \quad (T_{M090})(\text{Inverse Unit Subset Rule})(\text{Eqn2})$$

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \subseteq c*b^1) = (\forall Sa^1. c. \forall Qb^1 = \emptyset)) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c*b) = E) = E) = (((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \not\subseteq (\forall Sx^1. c. \forall Qy^1 = \emptyset)))$$

(T_{S272})

Proof of (T_{S272})

Let $a \subseteq c*b$

Let $x^1 \neq \emptyset$

Let $y^1 \neq \emptyset$

$$(x^1 \subseteq a) \subseteq (x^1 \subseteq c * b)$$

(T_{M099})(Entity Mode / Set Mode Rule)

$$(x^1 \subseteq a) \subseteq (x^1 \not\subseteq c * b)$$

(T_{M090})(Inverse Unit Subset Rule)

$$(x^1 \subseteq a) \subseteq (b \not\subseteq {}^v c * x^1)$$

(T_{S031})(Active/Passive Theorem)

$$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \not\subseteq (y^1 \subseteq {}^v c * x^1))$$

(T_{M751})(Entity Mode / Set Mode Rule) &

(T_{M008})(Inverse of an Equality)

$$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \not\subseteq (y^1 \not\subseteq {}^v c * x^1))$$

(T_{M090})(Inverse Unit Subset Rule)

$$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \not\subseteq (x^1 \not\subseteq c * y^1))$$

(T_{R612})(Active/Passive Theorem)

$$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \not\subseteq ({}^v Sx^1.c. {}^v Qy^1 = \emptyset))$$

(T_{R609})

$$\text{Thus } (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \subseteq c * b) = E) = E) = (((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \not\subseteq ({}^v Sx^1.c. {}^v Qy^1 = \emptyset)))$$

(A_{MX02})(Conditional Introduction) since this proof is reversible

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) = E) = E \subseteq$$

$$(((a \subseteq c * b) = E) = E) = (((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \not\subseteq ({}^v Sx^1.c. {}^v Qy^1 = \emptyset)))$$

(T_{M444c})(Biconditional Exportation)

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) = E) = E \subseteq (((a \not\subseteq c * b) = E) = E) = (((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \not\subseteq ({}^v Sx^1.c. {}^v Qy^1 = \emptyset)))$$

(T_{S273})

Proof of (T_{S273})

Let $a \not\subseteq c * b$

Let $x^1 \neq \emptyset$

Let $y^1 \neq \emptyset$

$$(x^1 \subseteq a) \not\subseteq (x^1 \subseteq c * b)$$

(T_{M751})(Entity Mode / Set Mode Rule) &

(T_{M008})(Inverse of an Equality)

$$(x^1 \subseteq a) \not\subseteq (x^1 \not\subseteq c * b)$$

(T_{M090})(Inverse Unit Subset Rule)

$$(x^1 \subseteq a) \not\subseteq (b \not\subseteq {}^v c * x^1)$$

(T_{S031})(Active/Passive Theorem)

$$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \not\subseteq (y^1 \subseteq {}^v c * x^1))$$

(T_{M751})(Entity Mode / Set Mode Rule) &

(T_{M008})(Inverse of an Equality)

$$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \not\subseteq (y^1 \not\subseteq {}^v c * x^1))$$

(T_{M090})(Inverse Unit Subset Rule)

$$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \not\subseteq (x^1 \not\subseteq c * y^1))$$

(T_{R612})(Active/Passive Theorem)

$$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \not\subseteq ({}^v Sx^1.c. {}^v Qy^1 = \emptyset))$$

(T_{R609})

$$\text{Thus } (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \not\subseteq c * b) = E) = E) = (((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \not\subseteq ({}^v Sx^1.c. {}^v Qy^1 = \emptyset)))$$

(A_{MX02})(Conditional Introduction) since this proof is reversible

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) = E) = E \subseteq$$

$$(((a \not\subseteq c * b) = E) = E) = (((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \not\subseteq ({}^v Sx^1.c. {}^v Qy^1 = \emptyset)))$$

(T_{M444c})(Biconditional Exportation)

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) = E) = E \subseteq (((a \not\subseteq c * b) = E) = E) = (((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq ({}^v Sx^1.c. {}^v Qy^1 = \emptyset)))$$

(T_{S274})

Proof of (T_{S274})

Let $a \not\subseteq c * b$

Let $x^1 \neq \emptyset$

Let $y^1 \neq \emptyset$

$$(x^1 \subseteq a) \not\subseteq (x^1 \subseteq c * b)$$

(T_{M751})(Entity Mode / Set Mode Rule) &

(T_{M008})(Inverse of an Equality)

$$(x^1 \subseteq a) \not\subseteq (b \subseteq {}^v c * x^1)$$

(T_{S030})(Active/Passive Theorem)

$$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (y^1 \subseteq {}^v c * x^1))$$

(T_{M099})(Entity Mode / Set Mode Rule)

$$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (y^1 \not\subseteq {}^v c * x^1))$$

(T_{M090})(Inverse Unit Subset Rule)

$$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (x^1 \not\subseteq c * y^1))$$

(T_{R612})(Active/Passive Theorem)

$$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq ({}^v Sx^1.c. {}^v Qy^1 = \emptyset))$$

(T_{R609})

$$\text{Thus } (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \not\subseteq c * b) = E) = E) = (((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq ({}^v Sx^1.c. {}^v Qy^1 = \emptyset)))$$

(A_{MX02})(Conditional Introduction) since this proof is reversible

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) = E) = E \subseteq$$

$$(((a \not\subseteq c * b) = E) = E) = (((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq ({}^v Sx^1.c. {}^v Qy^1 = \emptyset)))$$

(T_{M444c})(Biconditional Exportation)

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) = E) = E \subseteq (((a \subseteq c * b) = E) = E) = ((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 = \emptyset)))$$

(T_{S275})

Proof of (T_{S275})

Let $a \subseteq c * b$

Let $x^1 \neq \emptyset$

Let $y^1 \neq \emptyset$

$(x^1 \subseteq a) \subseteq (x^1 \subseteq c * b)$

(T_{M099})(Entity Mode / Set Mode Rule)

$(x^1 \subseteq a) \subseteq (b \subseteq \forall c * x^1)$

(T_{S030})(Active/Passive Theorem)

$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (y^1 \subseteq \forall c * x^1))$

(T_{M099})(Entity Mode / Set Mode Rule)

$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (y^1 \not\subseteq \forall c * x^1))$

(T_{M090})(Inverse Unit Subset Rule)

$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (x^1 \not\subseteq c * y^1))$

(T_{R612})(Active/Passive Theorem)

$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 = \emptyset))$

(T_{R609})

Thus $((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \subseteq c * b) = E) = E \subseteq (((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 = \emptyset)))$

(A_{MX02})(Conditional Introduction) since this proof is reversible

$((((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) = E) = E) \subseteq$

$((((a \subseteq c * b) = E) = E) = ((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 = \emptyset)))$

(T_{M444c})(Biconditional Exportation)

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) = E \subseteq$$

$$(((a \subseteq c \diamond b) = E) = E) = ((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset)))$$

(T_{S270})

Proof of (T_{S270})

Let $a \subseteq c \diamond b$

Let $a \neq \emptyset$

Let $b \neq \emptyset$

Let $x^1 \neq \emptyset$

Let $y^1 \neq \emptyset$

$(x^1 \subseteq a) \subseteq (x^1 \subseteq c \diamond b)$

(T_{M099})(Entity Mode / Set Mode Rule)

$(x^1 \subseteq a) \subseteq (b \subseteq \forall c \diamond x^1)$

(T_{R293})(Active/Passive Theorem) since $(a \neq \emptyset)(b \neq \emptyset)$

$(x^1 \subseteq a) \subseteq (b \subseteq \forall c * x^1)$

(T_{R232})(Interchangeability of * and $\diamond$ for Entities)

$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (y^1 \subseteq \forall c * x^1))$

(T_{M099})(Entity Mode / Set Mode Rule)

$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (x^1 \subseteq c * y^1))$

(T_{R519})(Active/Passive Theorem)

$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))$

(T_{R608})

Thus $((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset)(a \subseteq c \diamond b) = E) = E =$

$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))$

(A_{MX02})(Conditional Introduction) since this proof is reversible

$((((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) = E) \subseteq$

$((((a \subseteq c \diamond b) = E) = E) = ((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset)))$

(T_{M444c})(Biconditional Exportation)

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) = E \subseteq$$

$$(((a \not\subseteq c \diamond b) = E) = E) = ((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset)))$$

(T_{S271})

Proof of (T_{S271})

Let $a \not\subseteq c \diamond b$

Let $a \neq \emptyset$

Let $b \neq \emptyset$

Let $x^1 \neq \emptyset$

Let $y^1 \neq \emptyset$

$(x^1 \subseteq a) \not\subseteq (x^1 \subseteq c \diamond b)$

(T_{M751})(Entity Mode / Set Mode Rule) &

(T_{M008})(Inverse of an Equality)

$(x^1 \subseteq a) \not\subseteq (b \subseteq \forall c \diamond x^1)$

(T_{R293})(Active/Passive Theorem) since $(a \neq \emptyset)(b \neq \emptyset)$

$(x^1 \subseteq a) \not\subseteq (b \subseteq \forall c * x^1)$

(T_{R232})(Interchangeability of * and $\diamond$ for Entities)

$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (y^1 \subseteq \forall c * x^1))$

(T_{M099})(Entity Mode / Set Mode Rule)

$$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (x^1 \subseteq c * y^1)) \quad (T_{R519})(\text{Active/Passive Theorem})$$

$$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)) \quad (T_{R608})$$

$$\text{Thus } (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset)(a \not\subseteq c \diamond b) = E) = E) =$$

$$(((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)))$$

(A_{MX02})(Conditional Introduction) since this proof is reversible

$$((((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) = E) \subseteq$$

$$(((a \not\subseteq c \diamond b) = E) = E) = (((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)))$$

(T_{M444c})(Biconditional Exportation)

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq$$

$$(((a \not\subseteq c \diamond b) = E) = E) = (((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset))) \quad (T_{S269})$$

Proof of (T_{S269})

Let $a \not\subseteq c \diamond b$

Let $a \neq \emptyset$

Let $b \neq \emptyset$

Let $x^1 \neq \emptyset$

Let $y^1 \neq \emptyset$

$$(x^1 \subseteq a) \not\subseteq (x^1 \subseteq c \diamond b)$$

(T_{M751})(Entity Mode / Set Mode Rule) &

(T_{M008})(Inverse of an Equality)

(T_{M090})(Inverse Unit Subset Rule)

$$(x^1 \subseteq a) \not\subseteq (x^1 \not\subseteq c \diamond b)$$

(T_{R691})(Active/Passive Theorem) since $(a \neq \emptyset)(b \neq \emptyset)$

$$(x^1 \subseteq a) \not\subseteq (b \not\subseteq \forall c \diamond x^1)$$

(T_{R232})(Interchangeability of * and $\diamond$ for Entities)

$$(x^1 \subseteq a) \not\subseteq (b \not\subseteq \forall c * x^1)$$

$$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (y^1 \subseteq \forall c * x^1))$$

(T_{M751})(Entity Mode / Set Mode Rule) &

(T_{M008})(Inverse of an Equality)

$$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (x^1 \subseteq c * y^1))$$

(T_{R519})(Active/Passive Theorem)

$$((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)) \quad (T_{R608})$$

$$\text{Thus } (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset)(a \not\subseteq c \diamond b) = E) = E) =$$

$$(((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)))$$

(A_{MX02})(Conditional Introduction) since this proof is reversible

$$((((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) = E) \subseteq$$

$$(((a \not\subseteq c \diamond b) = E) = E) = (((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)))$$

(T_{M444c})(Biconditional Exportation)

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq$$

$$(((a \not\subseteq c \diamond b) = E) = E) = (((x^1 \subseteq a) = E) \not\subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset))) \quad (T_{S268})$$

Proof of (T_{S268})

Let $a \subseteq c \diamond b$

Let $a \neq \emptyset$

Let $b \neq \emptyset$

Let $x^1 \neq \emptyset$

Let $y^1 \neq \emptyset$

$$(x^1 \subseteq a) \subseteq (x^1 \subseteq c \diamond b)$$

(T_{M099})(Entity Mode / Set Mode Rule)

$$(x^1 \subseteq a) \subseteq (x^1 \not\subseteq c \diamond b)$$

(T_{M090})(Inverse Unit Subset Rule)

$$(x^1 \subseteq a) \subseteq (b \not\subseteq \forall c \diamond x^1)$$

(T_{R691})(Active/Passive Theorem) since $(a \neq \emptyset)(b \neq \emptyset)$

$$(x^1 \subseteq a) \subseteq (b \not\subseteq \forall c * x^1)$$

(T_{R232})(Interchangeability of * and $\diamond$ for Entities)

$$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (y^1 \subseteq \forall c * x^1))$$

(T_{M751})(Entity Mode / Set Mode Rule) &

(T_{M008})(Inverse of an Equality)

$$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (x^1 \subseteq c * y^1))$$

(T_{R519})(Active/Passive Theorem)

$$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)) \quad (T_{R608})$$

$$\text{Thus } (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset)(a \subseteq c \diamond b) = E) = E) =$$

$$(((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)))$$

(A_{MX02})(Conditional Introduction) since this proof is reversible

$$((((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) = E) \subseteq$$

$$(((a \subseteq c \diamond b) = E) = E) = (((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)))$$

(T_{M444c})(Biconditional Exportation)

$$(((x^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b) = E) = ((x^1 \subseteq a) \subseteq (\forall Sx^1.c. \forall Qb = \emptyset))) \quad (T_{S262})$$

Proof of (T_{S262})Let $a \subseteq c * b$ Let $x^1 \neq \emptyset$

$$(x^1 \subseteq a) \subseteq (x^1 \subseteq c * b) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})$$

$$(x^1 \subseteq a) \subseteq (\forall Sx^1.c. \forall Qb = \emptyset) \quad (T_{S082})$$

$$\text{Thus } ((a \subseteq c * b)(x^1 \neq \emptyset) = E) = ((x^1 \subseteq a) \subseteq (\forall Sx^1.c. \forall Qb = \emptyset))$$

(A_{MX02})(Conditional Introduction) since this proof is reversible.

$$(((x^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b) = E) = ((x^1 \subseteq a) \subseteq (\forall Sx^1.c. \forall Qb = \emptyset)))$$

(T_{M444c})(Biconditional Exportation)

$$(((x^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b) = E) = ((x^1 \subseteq a) \subseteq (\forall Sx^1.c. \forall Qb \neq \emptyset))) \quad (T_{S261})$$

Proof of (T_{S261})Let $a \subseteq c * b$ Let $x^1 \neq \emptyset$

$$(x^1 \subseteq a) \subseteq (x^1 \subseteq c * b) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})$$

$$(x^1 \subseteq a) \subseteq (x^1 \not\subseteq c * b) \quad (T_{M090})(\text{Inverse Unit Subset Rule})$$

$$(x^1 \subseteq a) \subseteq (\forall Sx^1.c. \forall Qb \neq \emptyset) \quad (T_{S081})$$

$$\text{Thus } ((x^1 \neq \emptyset)(a \subseteq c * b) = E) = ((x^1 \subseteq a) \subseteq (\forall Sx^1.c. \forall Qb \neq \emptyset))$$

(A_{MX02})(Conditional Introduction) since this proof is reversible.

$$(((x^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b) = E) = ((x^1 \subseteq a) \subseteq (\forall Sx^1.c. \forall Qb \neq \emptyset)))$$

(T_{M444c})(Biconditional Exportation)

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b) = E) = ((x^1 \subseteq a)(y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 = \emptyset))) \quad (T_{S263})$$

Proof of (T_{S263})Let $a \subseteq c * b$ Let $x^1 \neq \emptyset$ Let $y^1 \neq \emptyset$

$$(x^1 \subseteq a) \subseteq (x^1 \subseteq c * b) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})$$

$$(x^1 \subseteq a) \subseteq (b \subseteq \forall c * x^1) \quad (T_{S030})(\text{Active/Passive Theorem})$$

$$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (y^1 \subseteq \forall c * x^1)) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})$$

$$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (x^1 \subseteq c * y^1)) \quad (T_{R614})(\text{Active/Passive Theorem})$$

$$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 = \emptyset)) \quad (T_{S082})$$

$$(x^1 \subseteq a)(y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 = \emptyset) \quad (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets}$$

$$\text{Thus } ((a \subseteq c * b)(x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) = ((x^1 \subseteq a)(y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 = \emptyset))$$

(A_{MX02})(Conditional Introduction) since this proof is reversible.

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b) = E) = ((x^1 \subseteq a)(y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 = \emptyset)))$$

(T_{M444c})(Biconditional Exportation)

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq (((a \subseteq c \diamond b) = E) = ((x^1 \subseteq a)(y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset))) \quad (T_{S260})$$

Proof of (T_{S260})Let $a \subseteq c \diamond b$ Let $a \neq \emptyset$ Let $b \neq \emptyset$ Let $x^1 \neq \emptyset$ Let $y^1 \neq \emptyset$

$$(x^1 \subseteq a) \subseteq (x^1 \subseteq c \diamond b) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})$$

$$(x^1 \subseteq a) \subseteq (b \subseteq \forall c \diamond x^1) \quad (T_{R293})(\text{Active/Passive Theorem}) \text{ since } (a \neq \emptyset)(b \neq \emptyset)$$

$$(x^1 \subseteq a) \subseteq (b \subseteq \forall c * x^1) \quad (T_{R232})(\text{Interchangeability of } * \text{ and } \diamond \text{ for Entities})$$

$$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (y^1 \subseteq \forall c * x^1)) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})$$

$$((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (x^1 \subseteq c * y^1)) \quad (T_{R519})(\text{Active/Passive Theorem})$$

$$\begin{aligned}
& ((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (x^1 \not\subseteq c * y^1)) & (T_{M090})(\text{Inverse Unit Subset Rule}) \\
& ((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)) & (T_{S081}) \\
& ((x^1 \subseteq a)(y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)) & (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\
& \text{Thus } ((a \subseteq c \diamond b)(x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = \\
& ((x^1 \subseteq a)(y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)) & (A_{MX02})(\text{Conditional Introduction}) \text{ since this proof is reversible.} \\
& (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq (((a \subseteq c \diamond b) = E) = ((x^1 \subseteq a)(y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset))) \\
& & (T_{M444c})(\text{Biconditional Exportation})
\end{aligned}$$

$$((x^1 \neq \emptyset) = E) \subseteq ((a \not\subseteq c * b) = (x^1 \subseteq a)(\forall Sx^1.c. \forall Qb \neq \emptyset)) \quad (T_{S264})$$

Proof of (T_{S264})

$$\begin{aligned}
& (((x^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b) = E) = ((x^1 \subseteq a) \subseteq (\forall Sx^1.c. \forall Qb \neq \emptyset))) \quad (T_{S262}) \\
& (((x^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b) = E) = ((x^1 \not\subseteq a) + (\forall Sx^1.c. \forall Qb \neq \emptyset) = E)) \\
& & (T_{M064})(\subseteq \text{Universal Form}) \\
& ((x^1 \neq \emptyset) = E) \subseteq ((a \subseteq c * b) = (x^1 \not\subseteq a) + (\forall Sx^1.c. \forall Qb \neq \emptyset)) & (T_{M094})(\text{Probabilistic Rule}) \\
& ((x^1 \neq \emptyset) = E) \subseteq ((a \not\subseteq c * b) = \sim((x^1 \not\subseteq a) + (\forall Sx^1.c. \forall Qb \neq \emptyset))) & (T_{M008})(\text{Inverse of an Equality}) \\
& ((x^1 \neq \emptyset) = E) \subseteq ((a \not\subseteq c * b) = (x^1 \subseteq a)(\forall Sx^1.c. \forall Qb \neq \emptyset)) & (T_{M026})(\text{DeMorgan's Law})
\end{aligned}$$

$$((x^1 \neq \emptyset) = E) \subseteq ((a \not\subseteq c * b) = (x^1 \subseteq a)(\forall Sx^1.c. \forall Qb \neq \emptyset)) \quad (T_{S265})$$

Proof of (T_{S265})

$$\begin{aligned}
& (((x^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b) = E) = ((x^1 \subseteq a) \subseteq (\forall Sx^1.c. \forall Qb \neq \emptyset))) \quad (T_{S261}) \\
& (((x^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b) = E) = ((x^1 \not\subseteq a) + (\forall Sx^1.c. \forall Qb \neq \emptyset) = E)) \\
& & (T_{M064})(\subseteq \text{Universal Form}) \\
& ((x^1 \neq \emptyset) = E) \subseteq ((a \subseteq c * b) = (x^1 \not\subseteq a) + (\forall Sx^1.c. \forall Qb \neq \emptyset)) & (T_{M094})(\text{Probabilistic Rule}) \\
& ((x^1 \neq \emptyset) = E) \subseteq ((a \not\subseteq c * b) = \sim((x^1 \not\subseteq a) + (\forall Sx^1.c. \forall Qb \neq \emptyset))) & (T_{M008})(\text{Inverse of an Equality}) \\
& ((x^1 \neq \emptyset) = E) \subseteq ((a \not\subseteq c * b) = (x^1 \subseteq a)(\forall Sx^1.c. \forall Qb \neq \emptyset)) & (T_{M026})(\text{DeMorgan's Law})
\end{aligned}$$

$$((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) \subseteq ((a \not\subseteq c * b) = (x^1 \subseteq a)(y^1 \subseteq b)(\forall Sx^1.c. \forall Qy^1 \neq \emptyset)) \quad (T_{S266})$$

Proof of (T_{S266})

$$\begin{aligned}
& (((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b) = E) = ((x^1 \subseteq a)(y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset))) \\
& & (T_{S263}) \\
& (((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b) = E) = (\sim((x^1 \subseteq a)(y^1 \subseteq b)) + (\forall Sx^1.c. \forall Qy^1 \neq \emptyset) = E)) \\
& & (T_{M064})(\subseteq \text{Universal Form}) \\
& ((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) \subseteq ((a \subseteq c * b) = \sim((x^1 \subseteq a)(y^1 \subseteq b)) + (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)) \\
& & (T_{M094})(\text{Probabilistic Rule}) \\
& ((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) \subseteq ((a \not\subseteq c * b) = \sim(\sim((x^1 \subseteq a)(y^1 \subseteq b)) + (\forall Sx^1.c. \forall Qy^1 \neq \emptyset))) \\
& & (T_{M008})(\text{Inverse of an Equality}) \\
& ((x^1 \neq \emptyset)(y^1 \neq \emptyset) = E) \subseteq ((a \not\subseteq c * b) = (x^1 \subseteq a)(y^1 \subseteq b)(\forall Sx^1.c. \forall Qy^1 \neq \emptyset)) \quad (T_{M026})(\text{DeMorgan's Law})
\end{aligned}$$

$$((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) \subseteq ((a \not\subseteq c \diamond b) = (x^1 \subseteq a)(y^1 \subseteq b)(\forall Sx^1.c. \forall Qy^1 \neq \emptyset)) \quad (T_{S267})$$

Proof of (T_{S267})

$$\begin{aligned}
& (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq \\
& (((a \subseteq c \diamond b) = E) = ((x^1 \subseteq a)(y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset))) & (T_{S260}) \\
& (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq \\
& (((a \subseteq c \diamond b) = E) = (\sim((x^1 \subseteq a)(y^1 \subseteq b)) + (\forall Sx^1.c. \forall Qy^1 \neq \emptyset) = E)) & (T_{M064})(\subseteq \text{Universal Form}) \\
& ((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) \subseteq \\
& ((a \subseteq c \diamond b) = \sim((x^1 \subseteq a)(y^1 \subseteq b)) + (\forall Sx^1.c. \forall Qy^1 \neq \emptyset)) & (T_{M094})(\text{Probabilistic Rule}) \\
& ((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) \subseteq \\
& ((a \not\subseteq c \diamond b) = \sim(\sim((x^1 \subseteq a)(y^1 \subseteq b)) + (\forall Sx^1.c. \forall Qy^1 \neq \emptyset))) & (T_{M008})(\text{Inverse of an Equality}) \\
& ((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) \subseteq \\
& ((a \not\subseteq c \diamond b) = (x^1 \subseteq a)(y^1 \subseteq b)(\forall Sx^1.c. \forall Qy^1 \neq \emptyset)) & (T_{M026})(\text{DeMorgan's Law})
\end{aligned}$$

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq (((x^1 \subseteq c \diamond b) = E) = ((y^1 \subseteq b) \subseteq (\forall Sx^1.c. \forall Qy^1 \neq \emptyset))) \quad (T_{R670})$$

Proof of (TR670)

$$\begin{aligned}
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq \\
 & (((a \subseteq c \circ b) = E) = ((x^1 \subseteq a)(y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \quad (TS260)(Eqn1) \\
 & \text{Let } a = x^1 \quad (Eqn2) \\
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq \\
 & (((x^1 \subseteq c \circ b) = E) = ((x^1 \subseteq x^1)(y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \quad (AMX19)(\text{Substitution of Eqn2 into Eqn1}) \\
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq \\
 & (((x^1 \subseteq c \circ b) = E) = (((x^1 \subseteq x^1) + (x^1 = x^1))(y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \\
 & \quad (TM266)(\text{Definition of } \subseteq) \\
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq \\
 & (((x^1 \subseteq c \circ b) = E) = (((x^1 \subseteq x^1) + E)(y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \\
 & \quad (TM001)(\text{Universality of } (D_{M00}) \text{ Definition of } =) \\
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq (((x^1 \subseteq c \circ b) = E) = (E(y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \\
 & \quad (TM055)(\text{Universal Set Disjunction}) \\
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq (((x^1 \subseteq c \circ b) = E) = ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \\
 & \quad (AMX11)(\text{Universal Set Conjunction Axiom})
 \end{aligned}$$

$$(((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq (((x^1 \subseteq c \circ b) = E) = ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \quad (TR671)$$

Proof of (TR671)

$$\begin{aligned}
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) = E \subseteq \\
 & (((a \subseteq c \circ b) = E) = E) = (((x^1 \subseteq a) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \\
 & \quad (TS268)(Eqn1) \\
 & \text{Let } a = x^1 \quad (Eqn2) \\
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) = E \subseteq \\
 & (((x^1 \subseteq c \circ b) = E) = E) = (((x^1 \subseteq x^1) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \\
 & \quad (AMX19)(\text{Substitution of Eqn2 into Eqn1}) \\
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) = E \subseteq \\
 & (((x^1 \subseteq c \circ b) = E) = E) = (((x^1 \subseteq x^1) + (x^1 = x^1)) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \\
 & \quad (TM266)(\text{Definition of } \subseteq) \\
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) = E \subseteq \\
 & (((x^1 \subseteq c \circ b) = E) = E) = (((x^1 \subseteq x^1) + E) = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \\
 & \quad (TM001)(\text{Universality of } (D_{M00}) \text{ Definition of } =) \\
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) = E \subseteq \\
 & (((x^1 \subseteq c \circ b) = E) = E) = ((E = E) \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \\
 & \quad (TM055)(\text{Universal Set Disjunction}) \\
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) = E \subseteq \\
 & (((x^1 \subseteq c \circ b) = E) = E) = (E \subseteq ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \\
 & \quad (TM342) \\
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) = E \subseteq \\
 & (((x^1 \subseteq c \circ b) = E) = E) = (E = ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \\
 & \quad (TM275)(\text{Supersets of the Universal Set}) \\
 & (((x^1 \neq \emptyset)(y^1 \neq \emptyset)(a \neq \emptyset)(b \neq \emptyset) = E) = E) \subseteq (((x^1 \subseteq c \circ b) = E) = ((y^1 \subseteq b) \subseteq (\forall Sx^1.c.\forall Qy^1 \neq \emptyset))) \\
 & \quad (TM094)(\text{Probabilistic Rule})
 \end{aligned}$$

$$(((x^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b^1) = E) = ((x^1 \subseteq a) \subseteq (\forall Sx^1.c.\forall Qb^1 \neq \emptyset))) \quad (TS276)$$

Proof of (TR276)

$$\begin{aligned}
 & (((x^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b) = E) = ((x^1 \subseteq a) \subseteq (\forall Sx^1.c.\forall Qb \neq \emptyset))) \\
 & \quad (TS261)(Eqn1) \\
 & \text{Let } b = b^1 \quad (Eqn2) \\
 & (((x^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b^1) = E) = ((x^1 \subseteq a) \subseteq (\forall Sx^1.c.\forall Qb^1 \neq \emptyset))) \\
 & \quad (AMX19)(\text{Substitution of Eqn2 into Eqn1})
 \end{aligned}$$

$$(((x^1 \neq \emptyset) = E) = E) \subseteq (((a \subseteq c * b^1) = E) = ((x^1 \subseteq a) \subseteq (\forall Sx^1.c.\forall Qb^1 \neq \emptyset))) \quad (TS277)$$

Proof of (TR277)

Let $a \notin c * b^1$

Let $x^1 \neq \emptyset$

$(x^1 \subseteq a) \notin (x^1 \subseteq c * b^1)$

(T_{M751})(Entity Mode / Set Mode Rule) &

(T_{M008})(Inverse of an Equality)

$(x^1 \subseteq a) \notin (\forall Sx^1.c. \forall Qb^1 \neq \emptyset)$

(T_{R608})

Thus $((x^1 \neq \emptyset)(a \notin c * b^1) = E) = ((x^1 \subseteq a) \notin (\forall Sx^1.c. \forall Qb^1 \neq \emptyset))$

(A_{MX02})(Conditional Introduction) since this proof is reversible.

$((x^1 \neq \emptyset) = E) = E \subseteq (((a \notin c * b^1) = E) = ((x^1 \subseteq a) \notin (\forall Sx^1.c. \forall Qb^1 \neq \emptyset)))$

(T_{M444c})(Biconditional Exportation)

$(a \notin c * b^1) = (\forall Sa.c. \forall Qb^1 \neq \emptyset)$

(T_{S278})

Proof of (T_{S278})

$(a \notin c * b) = (\forall Sa.c. \forall Qb \neq \emptyset)$

(T_{S081})(Eqn1)

Let $b = b^1$

(Eqn2)

$(a \notin c * b^1) = (\forall Sa.c. \forall Qb^1 \neq \emptyset)$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$(a \subseteq c * b^1) = (\forall Sa.c. \forall Qb^1 = \emptyset)$

(T_{S279})

Proof of (T_{S279})

$(a \subseteq c * b) = (\forall Sa.c. \forall Qb = \emptyset)$

(T_{S082})(Eqn1)

Let $b = b^1$

(Eqn2)

$(a \subseteq c * b^1) = (\forall Sa.c. \forall Qb^1 = \emptyset)$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$(\{\supset\}_{\mathcal{A}} a^1 \notin c * b^1) = (\forall S\{\supset\}_{\mathcal{A}} a^1.c. \forall Qb^1 \neq \emptyset)$

(T_{S103})

Proof of (T_{S103})

$(a \notin c * b) = (\forall Sa.c. \forall Qb \neq \emptyset)$

(T_{S081})

$(\{\supset\}_{\mathcal{A}} a^1 \notin c * b^1) = (\forall S\{\supset\}_{\mathcal{A}} a^1.c. \forall Qb^1 \neq \emptyset)$

Resymbolising

$(\{\supset\}_{\mathcal{A}} a^1 \subseteq c * b^1) = (\forall S\{\supset\}_{\mathcal{A}} a^1.c. \forall Qb^1 = \emptyset)$

(T_{S104})

Proof of (T_{S104})

$(\{\supset\}_{\mathcal{A}} a^1 \notin c * b^1) = (\forall S\{\supset\}_{\mathcal{A}} a^1.c. \forall Qb^1 \neq \emptyset)$

(T_{S103})

$(\{\supset\}_{\mathcal{A}} a^1 \subseteq c * b^1) = (\forall S\{\supset\}_{\mathcal{A}} a^1.c. \forall Qb^1 = \emptyset)$

(T_{M008})(Inverse of an Equality)

$(a^1 \notin c * \{\supset\}_{\mathcal{A}} b^1) = (\forall Sa^1.c. \forall Q\{\supset\}_{\mathcal{A}} b^1 \neq \emptyset)$

(T_{S105})

Proof of (T_{S105})

$(a \notin c * b) = (\forall Sa.c. \forall Qb \neq \emptyset)$

(T_{S081})

$(a^1 \notin c * \{\supset\}_{\mathcal{A}} b^1) = (\forall Sa^1.c. \forall Q\{\supset\}_{\mathcal{A}} b^1 \neq \emptyset)$

Resymbolising

$(a^1 \subseteq c * \{\supset\}_{\mathcal{A}} b^1) = (\forall Sa^1.c. \forall Q\{\supset\}_{\mathcal{A}} b^1 = \emptyset)$

(T_{S106})

Proof of (T_{S106})

$(a^1 \notin c * \{\supset\}_{\mathcal{A}} b^1) = (\forall Sa^1.c. \forall Q\{\supset\}_{\mathcal{A}} b^1 \neq \emptyset)$

(T_{S105})

$(a^1 \subseteq c * \{\supset\}_{\mathcal{A}} b^1) = (\forall Sa^1.c. \forall Q\{\supset\}_{\mathcal{A}} b^1 = \emptyset)$

(T_{M008})(Inverse of an Equality)

$(\{\supset\}_{\mathcal{A}} a^1 \notin c * \{\supset\}_{\mathcal{A}} b^1) = (\forall S\{\supset\}_{\mathcal{A}} a^1.c. \forall Q\{\supset\}_{\mathcal{A}} b^1 \neq \emptyset)$

(T_{S107})

Proof of (T_{S107})

$(a \notin c * b) = (\forall Sa.c. \forall Qb \neq \emptyset)$

(T_{S081})

$(\{\supset\}_{\mathcal{A}} a^1 \notin c * \{\supset\}_{\mathcal{A}} b^1) = (\forall S\{\supset\}_{\mathcal{A}} a^1.c. \forall Q\{\supset\}_{\mathcal{A}} b^1 \neq \emptyset)$ Resymbolising

$$(\{\supset\}_{\mathcal{A}}a^1 \subseteq c * \{\supset\}_{\mathcal{A}}b^1) = (\forall S\{\supset\}_{\mathcal{A}}a^1.c.\forall Q\{\supset\}_{\mathcal{A}}b^1 = \emptyset) \quad (T_{S108})$$

Proof of (T_{S108})

$$(\{\supset\}_{\mathcal{A}}a^1 \not\subseteq c * \{\supset\}_{\mathcal{A}}b^1) = (\forall S\{\supset\}_{\mathcal{A}}a^1.c.\forall Q\{\supset\}_{\mathcal{A}}b^1 \neq \emptyset) \quad (T_{S107})$$

$$(\{\supset\}_{\mathcal{A}}a^1 \subseteq c * \{\supset\}_{\mathcal{A}}b^1) = (\forall S\{\supset\}_{\mathcal{A}}a^1.c.\forall Q\{\supset\}_{\mathcal{A}}b^1 = \emptyset) \quad (T_{M008})(\text{Inverse of an Equality})$$

$$(\mathcal{A}c^1 \subseteq \{\supset\}_{\mathcal{A}}b^1) = (\{\supset\}_{\mathcal{A}}*b^1 \subseteq \{\supset\}_{\mathcal{A}}*c^1) \quad (T_{R607})$$

Proof of (T_{R607})

$$\text{Let } \mathcal{A}*c^1 \subseteq \{\supset\}_{\mathcal{A}}*\mathcal{A}*b^1$$

$$\mathcal{A}*b^1 \subseteq \{\supset\}_{\mathcal{A}}*\mathcal{A}*c^1 \quad (T_{R519})(\text{Active/Passive Theorem})$$

$$\{\supset\}_{\mathcal{A}}*\mathcal{A}*b^1 \subseteq \{\supset\}_{\mathcal{A}}*\{\supset\}_{\mathcal{A}}*\mathcal{A}*c^1 \quad (T_{R196})(\text{Relational Deduction})$$

$$\{\supset\}_{\mathcal{A}}*\mathcal{A}*b^1 \subseteq \{\supset\}_{\mathcal{A}}*\mathcal{A}*c^1 \quad (T_{A711})$$

$$\forall \mathcal{A}*\{\supset\}_{\mathcal{A}}*\mathcal{A}*b^1 \subseteq \forall \mathcal{A}*\{\supset\}_{\mathcal{A}}*\mathcal{A}*c^1 \quad (T_{R196})(\text{Relational Deduction})$$

$$\{\supset\}_{\mathcal{A}}*b^1 \subseteq \{\supset\}_{\mathcal{A}}*c^1 \quad (T_{S505})$$

Thus $(\mathcal{A}c^1 \subseteq \{\supset\}_{\mathcal{A}}b^1) = (\{\supset\}_{\mathcal{A}}*b^1 \subseteq \{\supset\}_{\mathcal{A}}*c^1)$ (A_{MX02})(Conditional Introduction) since this proof is reversible.

$$(c^1 \subseteq \{\supset\}_{\mathcal{A}}b^1) = (\forall \mathcal{A}\{\supset\}_{\mathcal{A}}b^1 \subseteq \forall \mathcal{A}\{\supset\}_{\mathcal{A}}c^1) \quad (T_{S727})$$

Proof of (T_{S727})

$$(\mathcal{A}c^1 \subseteq \{\supset\}_{\mathcal{A}}b^1) = (\{\supset\}_{\mathcal{A}}*b^1 \subseteq \{\supset\}_{\mathcal{A}}*c^1) \quad (T_{R607})$$

$$(\mathcal{A}c^1 \subseteq \{\supset\}_{\mathcal{A}}b^1) = (\forall \mathcal{A}\{\supset\}_{\mathcal{A}}b^1 \subseteq \forall \mathcal{A}\{\supset\}_{\mathcal{A}}c^1) \quad (T_{S505})(\text{Eqn1})$$

$$\text{Let } b^1 = \mathcal{A}b^1 \quad (\text{Eqn2})$$

$$\text{Let } c^1 = \mathcal{A}c^1 \quad (\text{Eqn3})$$

$$(c^1 \subseteq \{\supset\}_{\mathcal{A}}b^1) = (\forall \mathcal{A}\{\supset\}_{\mathcal{A}}b^1 \subseteq \forall \mathcal{A}\{\supset\}_{\mathcal{A}}c^1) \quad (A_{MX19})(\text{Substitution of Eqns1\&2 into Eqn3})$$

$$(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) \subseteq (b^1 \subseteq \forall c*a^1).(a^1 \subseteq c*b^1) \quad (T_{S620})$$

Proof of (T_{S620})

$$(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) = (\forall Sa^1.c.\forall Qb^1 \neq \emptyset) \quad (D_{M00})(\text{Definition of } =)$$

$$(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) = (\forall Sa^1.c.\forall Qb^1 \neq \emptyset)(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) \quad (T_{M667})(\text{Conjunction Tautology})$$

$$(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) = (\forall Sb^1.\forall c.\forall Qa^1 \neq \emptyset).(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) \quad (T_{R615})$$

$$((b^1 \neq \emptyset) = E) \subseteq ((\forall Sa^1.c.\forall Qb^1 \neq \emptyset) = (b^1 \subseteq \forall c*a^1).(\forall Sa^1.c.\forall Qb^1 \neq \emptyset)) \quad (T_{R608})$$

$$((a^1 \neq \emptyset) = E).((b^1 \neq \emptyset) = E) \subseteq ((\forall Sa^1.c.\forall Qb^1 \neq \emptyset) = (b^1 \subseteq \forall c*a^1).(a^1 \subseteq c*b^1)) \quad (T_{R608})$$

$$((a^1 \neq \emptyset).(b^1 \neq \emptyset) = E) \subseteq ((\forall Sa^1.c.\forall Qb^1 \neq \emptyset) = (b^1 \subseteq \forall c*a^1).(a^1 \subseteq c*b^1)) \quad (T_{M108})(\text{Conjunction Introduction})$$

$$((a^1 \neq \emptyset).(b^1 \neq \emptyset).(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) = E) \subseteq (E = (b^1 \subseteq \forall c*a^1).(a^1 \subseteq c*b^1)) \quad (T_{M444c})(\text{Biconditional Exportation})$$

$$((a^1 \neq \emptyset).(b^1 \neq \emptyset).(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) \subseteq (b^1 \subseteq \forall c*a^1).(a^1 \subseteq c*b^1)) = E \quad (T_{M094})(\text{Probabilistic Rule})$$

$$(a^1 \neq \emptyset).(b^1 \neq \emptyset).(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) \subseteq (b^1 \subseteq \forall c*a^1).(a^1 \subseteq c*b^1) \quad (A_{MX16})(\text{Order Axiom})$$

$$(\forall Sa^1.\forall Qb^1 \neq \emptyset).(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) \subseteq (b^1 \subseteq \forall c*a^1).(a^1 \subseteq c*b^1) \quad (A_{R11})(\text{Inter-relationship Axiom})(\text{Eqn1})$$

$$\forall Sa^1.c.\forall Qb^1 \subseteq \forall Sa^1.\forall Qb^1 \quad (T_{M083})(\text{Conjunction Elimination})$$

$$(\forall Sa^1.c.\forall Qb^1 \subseteq \forall Sa^1.\forall Qb^1) = E \quad (A_{MX16})(\text{Order Axiom})$$

$$(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) \subseteq (\forall Sa^1.\forall Qb^1 \neq \emptyset) \quad (T_{M096})(\text{Probabilistic Rule})$$

$$(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) = (\forall Sa^1.\forall Qb^1 \neq \emptyset)(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) \quad (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn2})$$

$$(\forall Sa^1.c.\forall Qb^1 \neq \emptyset) \subseteq (b^1 \subseteq \forall c*a^1).(a^1 \subseteq c*b^1) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(\forall Sa.c.\forall Qb \neq \emptyset) \subseteq (a \neq \emptyset) \quad (T_{T363})$$

$$(\forall \text{Sa.c.} \forall \text{Qb} \neq \emptyset) \subseteq (\text{b} \neq \emptyset) \quad (\text{T}_{7364})$$

$$(\forall \text{Sa.c.} \forall \text{Qb} \neq \emptyset) \subseteq (\text{c} \neq \emptyset) \quad (\text{T}_{7365})$$

Example Proof of (T₇₃₆₃)

Let $\text{a} = \emptyset$

$$\forall \text{Sa} = \emptyset \quad (\text{T}_{5304})$$

$$\forall \text{Sa.c.} \forall \text{Qb} = \emptyset \quad (\text{T}_{M350})$$

$$\text{Thus } (\text{a} = \emptyset) \subseteq (\forall \text{Sa.c.} \forall \text{Qb} = \emptyset) \quad (\text{A}_{MX02}) (\text{Conditional Introduction})$$

$$(\forall \text{Sa.c.} \forall \text{Qb} \neq \emptyset) \subseteq (\text{a} \neq \emptyset) \quad (\text{T}_{M012}) (\text{Transposition})$$

$$\text{S}(\forall \text{Sa.c.} \forall \text{Qb}) \subseteq \text{a} \quad (\text{T}_{7366})$$

$$\text{Q}(\forall \text{Sa.c.} \forall \text{Qb}) \subseteq \text{b} \quad (\text{T}_{7367})$$

Example Proof of (T₇₃₆₆)

$$\forall \text{Sa.c.} \forall \text{Qb} \subseteq \forall \text{Sa} \quad (\text{T}_{M083}) (\text{Conjunction Elimination})$$

$$\text{S}(\forall \text{Sa.c.} \forall \text{Qb}) \subseteq \text{S} \forall \text{Sa} \quad (\text{T}_{R196}) (\text{Relational Deduction})$$

$$\text{S}(\forall \text{Sa.c.} \forall \text{Qb}) \subseteq \text{a} \quad (\text{T}_{R012m}) (\text{Sub-Relationship Closure Theorem})$$

$$(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1 \neq \emptyset) = (\text{S}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) = \text{a}^1) \quad (\text{T}_{7359})$$

$$(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1 \neq \emptyset) = (\text{Q}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) = \text{b}^1) \quad (\text{T}_{7360})$$

Example Proof of (T₇₃₅₉)

$$\text{Let } \text{x}^1 = \forall \text{Sa}^1.x^1. \forall \text{Qb}^1 \quad (\text{Eqn1})$$

$$\text{Let } \forall \text{Sa}^1.x^1. \forall \text{Qb}^1 \neq \emptyset$$

$$\text{b}^1 \neq \emptyset$$

$$\text{S}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) \neq \emptyset \quad (\text{T}_{7364})$$

$$\sim(\text{S}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) \subseteq \text{a}^1) \quad (\text{T}_{R048}) (\text{Triad Theorem})$$

$$\sim(\text{S}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) \subseteq \text{a}^1) = \text{E} \quad (\text{T}_{M348})$$

$$(\text{S}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) \subseteq \text{a}^1) = \text{E} \quad (\text{A}_{MX16}) (\text{Order Axiom})$$

$$(\text{S}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) \subseteq \text{a}^1) = \text{E} \quad (\text{T}_{M008}) (\text{Inverse of an Equality})$$

$$(\text{S}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) \subseteq \text{a}^1) = \emptyset \quad (\text{D}_{M02}) (\text{Definition of } \emptyset) (\text{Eqn2})$$

$$\text{x}^1 \subseteq \forall \text{Sa}^1. \forall \text{Qb}^1 \quad (\text{A}_{MX44}) (\subseteq \text{Equality Form})$$

$$\text{Sx}^1 \subseteq \text{S}(\forall \text{Sa}^1. \forall \text{Qb}^1) \quad (\text{T}_{R196}) (\text{Relational Deduction})$$

$$\text{Sx}^1 \subseteq \text{a}^1 \quad (\text{T}_{S248}) \text{ since } \text{b}^1 \neq \emptyset \text{ (Eqn3)}$$

$$\text{S}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) \subseteq \text{a}^1 \quad (\text{A}_{MX19}) (\text{Substitution of Eqn1 into Eqn3})$$

$$(\text{S}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) = \text{a}^1) + (\text{S}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) \subseteq \text{a}^1) \quad (\text{T}_{M266}) (\text{Definition of } \subseteq) (\text{Eqn4})$$

$$(\text{S}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) = \text{a}^1) + \emptyset \quad (\text{A}_{MX19}) (\text{Substitution of Eqn2 into Eqn4})$$

$$\text{S}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) = \text{a}^1 \quad (\text{T}_{M668}) (\text{Null Set Disjunction})$$

$$\text{Thus } (\forall \text{Sa}^1.x^1. \forall \text{Qb}^1 \neq \emptyset) = (\text{S}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) = \text{a}^1) \quad (\text{A}_{MX02}) (\text{Conditional Introduction})$$

$$((\text{a}^1 \neq \emptyset)(\text{b}^1 \neq \emptyset) = \text{E}) \subseteq ((\text{a} \subseteq \text{b}) = (\forall \text{Sa}^1. \{\subseteq\}. \forall \text{Qb}^1 \neq \emptyset)) \quad (\text{T}_{7391})$$

$$((\text{a}^1 \neq \emptyset)(\text{b}^1 \neq \emptyset) = \text{E}) \subseteq ((\text{a} \not\subseteq \text{b}) = (\forall \text{Sa}^1. \{\not\subseteq\}. \forall \text{Qb}^1 \neq \emptyset)) \quad (\text{T}_{7392})$$

Example Proof of (T₇₃₉₁)

$$((\text{a}^1 \neq \emptyset)(\text{b}^1 \neq \emptyset) = \text{E}) \subseteq ((\text{a} \langle \text{o} \rangle \text{b}) = (\forall \text{Sa}^1. \text{o}. \forall \text{Qb}^1 \neq \emptyset)) \quad (\text{D}_{A00}) (\text{Definition of an Equation}) (\text{Eqn1})$$

$$\text{Let } \text{o} = \{\subseteq\} \quad (\text{Eqn2})$$

$$((\text{a}^1 \neq \emptyset)(\text{b}^1 \neq \emptyset) = \text{E}) \subseteq ((\text{a} \subseteq \text{b}) = (\forall \text{Sa}^1. \{\subseteq\}. \forall \text{Qb}^1 \neq \emptyset)) \quad (\text{A}_{MX19}) (\text{Substitution of Eqn2 into Eqn1})$$

$$((\text{a}^1 \neq \emptyset) = \text{E}) \subseteq ((\text{a} = \emptyset) = (\forall \text{Sa}^1. \{=\}. \forall \text{Q} \emptyset^1 \neq \emptyset)) \quad (\text{T}_{7393})$$

$$((\text{a}^1 \neq \emptyset) = \text{E}) \subseteq ((\text{a} \neq \emptyset) = (\forall \text{Sa}^1. \{\neq\}. \forall \text{Q} \emptyset^1 \neq \emptyset)) \quad (\text{T}_{7394})$$

Example Proof of (T₇₃₉₃)

$$((\text{a}^1 \neq \emptyset)(\text{b}^1 \neq \emptyset) = \text{E}) \subseteq ((\text{a} \langle \text{o} \rangle \text{b}) = (\forall \text{Sa}^1. \text{o}. \forall \text{Qb}^1 \neq \emptyset))$$

$$(\text{D}_{A00}) (\text{Definition of a Membership Equation}) (\text{Eqn1})$$

$$\text{Let } \text{o} = \{=\} \quad (\text{Eqn2})$$

$$\text{Let } \text{b}^1 = \emptyset^1 \quad (\text{Eqn3})$$

$$\begin{aligned}
&\text{Let } \mathbf{b} = \emptyset && (\text{Eqn4}) \\
&((\mathbf{a}^1 \neq \emptyset)(\emptyset^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \subseteq \emptyset) = (\forall \mathbf{Sa}^1. \{=\}. \forall \mathbf{Q}\emptyset^1 \neq \emptyset)) && (\mathbf{A}_{MX19})(\text{Substitution of Eqns2,3\&4 into Eqn1}) \\
&((\mathbf{a}^1 \neq \emptyset)\mathbf{E} = \mathbf{E}) \subseteq ((\mathbf{a} \subseteq \emptyset) = (\forall \mathbf{Sa}^1. \{=\}. \forall \mathbf{Q}\emptyset^1 \neq \emptyset)) && (\mathbf{T}_{M007})(\text{Universality of } (\mathbf{T}_{M780}) \text{ Existence of } \emptyset^1) \\
&((\mathbf{a}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} \subseteq \emptyset) = (\forall \mathbf{Sa}^1. \{=\}. \forall \mathbf{Q}\emptyset^1 \neq \emptyset)) && (\mathbf{A}_{MX11})(\text{Universal Set Conjunction Axiom})
\end{aligned}$$

$$((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1) = (\mathcal{A}\mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A}\mathbf{b}^1)) \quad (\mathbf{T}_{T902})$$

Proof of ($\mathbf{T}_{T902}$)

$$\begin{aligned}
&\text{Let } (\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) \\
&(\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1) = (\mathcal{A}\mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A}\mathbf{b}^1) && (\mathbf{T}_{S538})(\text{Eqn1}) \\
&(\mathbf{a} = \mathbf{b}) = (\mathcal{A}\mathbf{a} = \mathcal{A}\mathbf{b}) && (\mathbf{T}_{T876}) \\
&(\mathbf{a}^1 \subseteq \{=\} * \mathbf{b}^1) = (\mathcal{A}\mathbf{a}^1 \subseteq \{=\} * \mathcal{A}\mathbf{b}^1) && (\mathbf{T}_{S308})(\text{Eqn2}) \\
&(\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1) + (\mathbf{a}^1 \subseteq \{=\} * \mathbf{b}^1) = (\mathcal{A}\mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A}\mathbf{b}^1) + (\mathcal{A}\mathbf{a}^1 \subseteq \{=\} * \mathcal{A}\mathbf{b}^1) && (\mathbf{T}_{M230})(\text{Combination of Eqns1\&2}) \\
&(\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1 + \{=\} * \mathbf{b}^1) = (\mathcal{A}\mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A}\mathbf{b}^1 + \{=\} * \mathcal{A}\mathbf{b}^1) && (\mathbf{T}_{M170})(\text{Disjunction of Necessities}) \\
&(\mathbf{a}^1 \subseteq (\{\subseteq\} + \{=\}) * \mathbf{b}^1) = (\mathcal{A}\mathbf{a}^1 \subseteq (\{\supset\} + \{=\}) * \mathcal{A}\mathbf{b}^1) && (\mathbf{T}_{R002c})(\text{Distribution of } + \text{ over } *) \\
&(\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1) = (\mathcal{A}\mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A}\mathbf{b}^1) && (\mathbf{D}_{A141})(\text{Disjunction with } \{=\}) \\
&\text{Thus } ((\mathbf{a}^1 \neq \emptyset)(\mathbf{b}^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a}^1 \subseteq \{\subseteq\} * \mathbf{b}^1) = (\mathcal{A}\mathbf{a}^1 \subseteq \{\supset\} * \mathcal{A}\mathbf{b}^1)) && (\mathbf{A}_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$((\mathbf{a} \neq \emptyset) = \mathbf{E}) \subseteq (\mathcal{A} * \mathbf{a}^1 \subseteq \{\subseteq\} \diamond \mathcal{A}\mathbf{a}) \quad (\mathbf{T}_{T903})$$

Proof of ($\mathbf{T}_{T903}$)

$$\begin{aligned}
&\text{Let } \mathbf{a} \neq \emptyset \\
&\mathbf{a}^1 = (\{\subseteq\} \diamond \mathbf{a})(\{\subseteq\} \mathcal{A} * \{\subseteq\} \diamond \mathbf{a}) && (\mathbf{T}_{S470})(\text{General Definition of Abstraction}) \\
&\mathbf{a}^1 \subseteq \{\subseteq\} \diamond \mathbf{a} && (\mathbf{T}_{M073b})(\text{Single Premise Syllogism}) \\
&\mathbf{a}^1 \subseteq \forall \mathcal{A} * \{\subseteq\} \diamond \mathcal{A} * \mathbf{a} && (\mathbf{T}_{S507}) \\
&\mathcal{A} * \mathbf{a}^1 \subseteq \mathcal{A} * \forall \mathcal{A} * \{\subseteq\} \diamond \mathcal{A} * \mathbf{a} && (\mathbf{T}_{R196})(\text{Relational Deduction}) \\
&\mathcal{A} * \mathbf{a}^1 \subseteq \{\subseteq\} \diamond \mathcal{A} * \mathbf{a} && (\mathbf{T}_{T878}) \\
&\text{Thus } ((\mathbf{a} \neq \emptyset) = \mathbf{E}) \subseteq (\mathcal{A} * \mathbf{a}^1 \subseteq \{\subseteq\} \diamond \mathcal{A}\mathbf{a}) && (\mathbf{A}_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$\{\subseteq\} \diamond \mathbf{a} = \{\supset\} \mathcal{A} \diamond \mathbf{a} \quad (\mathbf{T}_{T904})$$

$$\{\supset\} \diamond \mathbf{a} = \{\subseteq\} \mathcal{A} \diamond \mathbf{a} \quad (\mathbf{T}_{T905})$$

Example Proof of ($\mathbf{T}_{T904}$)

$$\begin{aligned}
&\text{Let } \mathbf{a} = \mathbf{a}_1^1 + \dots + \mathbf{a}_n^1 && (\text{Eqn1}) \\
&\{\supset\} \mathcal{A} * \mathbf{a}_r^1 = \{\subseteq\} * \mathbf{a}_r^1 && (\mathbf{T}_{R661})(\text{Eqn2}) \\
&(\{\supset\} \mathcal{A} * \mathbf{a}_1^1) \dots (\{\supset\} \mathcal{A} * \mathbf{a}_n^1) = (\{\subseteq\} * \mathbf{a}_1^1) \dots (\{\subseteq\} * \mathbf{a}_n^1) && (\mathbf{T}_{M231})(\text{Combination of Eqn2 for } r = 1 \text{ to } n) \\
&\{\supset\} \mathcal{A} \diamond (\mathbf{a}_1^1 + \dots + \mathbf{a}_n^1) = \{\subseteq\} \diamond (\mathbf{a}_1^1 + \dots + \mathbf{a}_n^1) && (\mathbf{D}_{S15})(\text{Definition of the Intersection Function Operator})(\text{Eqn3}) \\
&\{\supset\} \mathcal{A} \diamond \mathbf{a} = \{\subseteq\} \diamond \mathbf{a} && (\mathbf{A}_{MX19})(\text{Substitution of Eqn1 into Eqn3}) \\
&\{\subseteq\} \diamond \mathbf{a} = \{\supset\} \mathcal{A} \diamond \mathbf{a} && (\mathbf{T}_{M197})(\text{Commutativity of } =)
\end{aligned}$$

$$((\mathbf{a} \neq \emptyset) = \mathbf{E}) \subseteq (\{\subseteq\} * \mathcal{A}\mathbf{a}^1 = \{\subseteq\} \diamond \mathcal{A}\mathbf{a}) \quad (\mathbf{T}_{T906})$$

Proof of ($\mathbf{T}_{T906}$)

$$\begin{aligned}
&\text{Let } \mathbf{a} \neq \emptyset \\
&\{\supset\} * \mathbf{a}^1 = \{\supset\} \diamond \mathbf{a} && (\mathbf{T}_{S402}) \\
&\{\subseteq\} \mathcal{A} * \mathbf{a}^1 = \{\supset\} \diamond \mathbf{a} && (\mathbf{T}_{R662}) \\
&\{\subseteq\} \mathcal{A} * \mathbf{a}^1 = \{\subseteq\} \mathcal{A} \diamond \mathbf{a} && (\mathbf{T}_{T905}) \\
&\forall \mathcal{A} * \{\subseteq\} * \mathcal{A} * \mathbf{a}^1 = \{\subseteq\} \mathcal{A} \diamond \mathbf{a} && (\mathbf{T}_{R571}) \\
&\forall \mathcal{A} * \{\subseteq\} * \mathcal{A} * \mathbf{a}^1 = \forall \mathcal{A} * \{\subseteq\} \diamond \mathcal{A} * \mathbf{a} && (\mathbf{T}_{R573})
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}^* \vee \mathcal{A}^* \{\subseteq\}^* \mathcal{A}^* a^1 &= \mathcal{A}^* \vee \mathcal{A}^* \{\subseteq\} \diamond \mathcal{A}^* a & (T_{7876}) \\
\{\subseteq\}^* \mathcal{A}^* a^1 &= \{\subseteq\} \diamond \mathcal{A}^* a & (T_{7878}) \\
\text{Thus } ((a \neq \emptyset) = E) &\subseteq ((\{\subseteq\}^* \mathcal{A}^* a^1 = \{\subseteq\} \diamond \mathcal{A}^* a) & (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$((a \neq \emptyset)(c * b \neq \emptyset) = E) \subseteq ((a \subseteq c * b) = (\mathcal{A}^*(c * b)^1 \subseteq \{\subseteq\} \diamond \mathcal{A}a)) \quad (T_{7907})$$

Proof of (T₇₉₀₇)

$$\begin{aligned}
&\text{Let } a \neq \emptyset \\
&\text{Let } c * b \neq \emptyset \\
&\text{Let } a \subseteq c * b \\
&a^1 \subseteq \{\subseteq\}^*(c * b)^1 & (T_{S412}) \\
&(c * b)^1 \subseteq \vee \{\subseteq\}^* a^1 & (T_{R519})(\text{Active/Passive Theorem}) \\
&(c * b)^1 \subseteq \{\supseteq\}^* a^1 & (T_{7741})(\text{Definition of } \vee \{\subseteq\}) \\
&\mathcal{A}^*(c * b)^1 \subseteq \mathcal{A}^* \{\supseteq\}^* a^1 & (T_{R196})(\text{Relational Deduction}) \\
&\mathcal{A}^*(c * b)^1 \subseteq \{\subseteq\}^* \mathcal{A}^* a^1 & (T_{7830})(\text{Eqn1}) \\
&\mathcal{A}^* a^1 \subseteq \{\subseteq\} \diamond \mathcal{A}a & (T_{7903}) \\
&\{\subseteq\}^* \mathcal{A}^* a^1 \subseteq \{\subseteq\}^* \{\subseteq\} \diamond \mathcal{A}a & (T_{R196})(\text{Relational Deduction}) \\
&\{\subseteq\}^* \mathcal{A}^* a^1 \subseteq \{\subseteq\} \diamond \mathcal{A}a & (T_{S407})(\text{Eqn2}) \\
&\mathcal{A}^*(c * b)^1 \subseteq \{\subseteq\} \diamond \mathcal{A}a & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\
&\text{Thus } (a \subseteq c * b) \subseteq (\mathcal{A}^*(c * b)^1 \subseteq \{\subseteq\} \diamond \mathcal{A}a) & (A_{MX02})(\text{Conditional Introduction})(\text{Eqn3}) \\
&\text{Let } \mathcal{A}^*(c * b)^1 \subseteq \{\subseteq\} \diamond \mathcal{A}a & (\text{Eqn4}) \\
&\{\subseteq\}^* \mathcal{A}^* a^1 = \{\subseteq\} \diamond \mathcal{A}a & (T_{7906})(\text{Eqn5}) \\
&\mathcal{A}^*(c * b)^1 \subseteq \{\subseteq\}^* \mathcal{A}^* a^1 & (A_{MX19})(\text{Substitution of Eqn5 into Eqn4}) \\
&\mathcal{A}^*(c * b)^1 \subseteq \mathcal{A}^* \{\supseteq\}^* a^1 & (T_{7830}) \\
&\vee \mathcal{A}^* \mathcal{A}^*(c * b)^1 \subseteq \vee \mathcal{A}^* \mathcal{A}^* \{\supseteq\}^* a^1 & (T_{R196})(\text{Relational Deduction}) \\
&(c * b)^1 \subseteq \{\supseteq\}^* a^1 & (T_{7877}) \& (T_{7816})(\text{Aspect Closure Theorem}) \\
&c * b \supseteq a & (T_{S412}) \\
&a \subseteq c * b & (T_{M196})(\subseteq \text{ Reverse of } \supseteq) \\
&\text{Thus } (\mathcal{A}^*(c * b)^1 \subseteq \{\subseteq\} \diamond \mathcal{A}a) \subseteq (a \subseteq c * b) & (A_{MX02})(\text{Conditional Introduction})(\text{Eqn6}) \\
&(a \subseteq c * b) = (\mathcal{A}^*(c * b)^1 \subseteq \{\subseteq\} \diamond \mathcal{A}a) & (T_{M430})(\text{Biconditional Introduction Eqns3\&6}) \\
&\text{Thus } ((a \neq \emptyset)(c * b \neq \emptyset) = E) \subseteq ((a \subseteq c * b) = (\mathcal{A}^*(c * b)^1 \subseteq \{\subseteq\} \diamond \mathcal{A}a)) & (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$((a \neq \emptyset)(c * b \neq \emptyset) = E) \subseteq ((a \subseteq c * b) = (\mathcal{A}^*(c * b)^1 \subseteq \{\subseteq\}^* \mathcal{A}a^1)) \quad (T_{S597})$$

Proof of (T_{S597})

$$\begin{aligned}
&((a \neq \emptyset)(c * b \neq \emptyset) = E) \subseteq ((a \subseteq c * b) = (\mathcal{A}^*(c * b)^1 \subseteq \{\subseteq\} \diamond \mathcal{A}a)) & (T_{7907})(\text{Eqn1}) \\
&((a \neq \emptyset) = E) \subseteq (\{\subseteq\}^* \mathcal{A}a^1 = \{\subseteq\} \diamond \mathcal{A}a) & (T_{7906})(\text{Eqn2}) \\
&((a \neq \emptyset)(c * b \neq \emptyset) = E) \subseteq ((a \subseteq c * b) = (\mathcal{A}^*(c * b)^1 \subseteq \{\subseteq\}^* \mathcal{A}a^1)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

Chapter 11 Proofs of Theorems of Ordered Sets

11.1 Introductory Theorems

A. Theorems for Sets of Points

$$\{\subset\} * S_P = \emptyset \quad (T_{722})$$

$$\{\supset\} * S_P = \emptyset \quad (T_{722A})$$

Example Proof of (T₇₂₂)

$$\{\subset\} * E^1 = \emptyset \quad (T_{725})$$

$$\{\subset\} * S_P = \emptyset \quad (A_{77})(2^{\text{nd}} \text{ Axiom of Points})$$

$$\{\subseteq\} S_P = S_P \quad (T_{716})$$

$$\{\supseteq\} S_P = S_P \quad (T_{716A})$$

Example Proof of (T₇₁₆)

$$\{\subseteq\} S_P = \{\subset\} S_P + \{=\} S_P \quad (D_{141}) \text{ (Disjunction with } \{=\})$$

$$\{\subseteq\} S_P = \emptyset + \{=\} S_P \quad (T_{722})$$

$$\{\subseteq\} S_P = \{=\} S_P \quad (M_{668}) \text{ (Null Set Disjunction)}$$

$$\{\subseteq\} S_P = S_P \quad (A_{080})$$

$$S_M \subseteq \sim S_P \quad (T_{423})$$

$$S_M \subseteq \sim S_P \quad (T_{423A})$$

Example Proof of (T₄₂₃)

$$S_M \subseteq \sim S_D \quad (A_{526}) \text{ (Mixed Sets are not Discrete)}$$

$$S_D \subseteq \sim S_M \quad (M_{012}) \text{ (Transposition) (Eqn1)}$$

$$S_P \subseteq S_D \quad (A_{53}) \text{ (Axiom of Points) (Eqn2)}$$

$$S_P \subseteq \sim S_M \quad (M_{139}) \text{ (Hypothetical Syllogism Eqns2 and 1)}$$

$$S_M \subseteq \sim S_P \quad (M_{012}) \text{ (Transposition)}$$

B. Introductory Theorems for Non-ordered Sets

The following theorems state that if a^1 is not an ordered set then $\{\prec\} * a = \emptyset$, $\{\succ\} * a = \emptyset$, $Ba = \emptyset$, $Ca = \emptyset$, $Ba^1 = \emptyset$ and $Ca^1 = \emptyset$. Furthermore, if $(a^1 + b^1)^1$ is not a ordered set then $(a^1 > b^1) = \emptyset$ and $(a^1 < b^1) = \emptyset$.

Many of the theorems below are true not only for ordered sets but also for sets which are not ordered. This can be proven by substituting the null set for these expressions when they occur in the equation. Thus, it is not necessary to state that the variables in the theorem form an ordered set. Also, variables can be expressed using the general symbolism a rather than a , except where the axiom, definition or theorem has the condition $a^1 \subseteq S_0$.

$$((X = \{\prec\}, \{\succ\}, B, C) = E) \subseteq ((a^1 \subseteq \sim S_0) \subseteq (Xa^1 = \emptyset)) \quad (T_{968}) \text{ Theorem of Non-ordered Sets}$$

$$((X = \{\prec\}, \{\succ\}, B, C) = E) \subseteq ((a^1 \subseteq \sim S_0) \subseteq (Xa^1 = \emptyset)) \quad (T_{968A}) \text{ Theorem of Non-ordered Sets}$$

Example Proof of (T₉₆₈)

$$((X = \{\prec\}, \{\succ\}, B, C) = E) \subseteq ((a^1 \subseteq \sim S_0) \subseteq (a^1 \not\subseteq QX)(a \not\subseteq QX))$$

$$(A_{905}) \text{ (Axiom of Non-ordered Sets)}$$

$$((X = \{\prec\}, \{\succ\}, B, C) = E) \subseteq ((a^1 \subseteq \sim S_0) \subseteq (a^1 \not\subseteq QX))$$

$$(M_{073}) \text{ (Single Premise Syllogism) (Eqn1)}$$

$$(Xa^1 \neq \emptyset) \subseteq (a^1 \subseteq QX)$$

(T_{S410b})

$$(a^1 \not\subseteq QX) \subseteq (Xa^1 = \emptyset)$$

(T_{M012})(Transposition)(Eqn2)

$$((X = \{<\}, \{>\}, B, C) = E) \subseteq ((a^1 \subseteq \sim S_0) \subseteq (Xa^1 = \emptyset))$$

(T_{M139})(Hypothetical Syllogism Eqns1&2)

From which by extraction:

$$(a^1 \subseteq \sim S_0) \subseteq (\{<\} * a^1 = \emptyset) \quad (T_{T966})$$

$$(a^1 \subseteq \sim S_0) \subseteq (\{<\} * a^1 = \emptyset) \quad (T_{T966A})$$

$$(a^1 \subseteq \sim S_0) \subseteq (\{>\} * a^1 = \emptyset) \quad (T_{T967})$$

$$(a^1 \subseteq \sim S_0) \subseteq (\{>\} * a^1 = \emptyset) \quad (T_{T967A})$$

$$(a^1 \subseteq \sim S_0) \subseteq (Ba^1 = \emptyset) \quad (T_{S715})$$

$$(a^1 \subseteq \sim S_0) \subseteq (Ba^1 = \emptyset) \quad (T_{S715A})$$

$$(a^1 \subseteq \sim S_0) \subseteq (Ca^1 = \emptyset) \quad (T_{S716})$$

$$(a^1 \subseteq \sim S_0) \subseteq (Ca^1 = \emptyset) \quad (T_{S716A})$$

$$((X = \{<\}, \{>\}, B, C) = E) \subseteq ((a^1 \subseteq \sim S_0) \subseteq (a \not\subseteq QX)) \quad (T_{T969}) \text{ Theorem of Non-ordered Sets}$$

$$((X = \{<\}, \{>\}, B, C) = E) \subseteq ((a^1 \subseteq \sim S_0) \subseteq (a \not\subseteq QX)) \quad (T_{T969A}) \text{ Theorem of Non-ordered Sets}$$

Example Proof of (T_{T969})

$$((X = \{<\}, \{>\}, B, C) = E) \subseteq ((a^1 \subseteq \sim S_0) \subseteq (a^1 \not\subseteq QX)(a \not\subseteq QX))$$

(A_{R05})(Axiom of Non-ordered Sets)

$$((X = \{<\}, \{>\}, B, C) = E) \subseteq ((a^1 \subseteq \sim S_0) \subseteq (a \not\subseteq QX)) \quad (T_{M073}) \text{ (Single Premise Syllogism)(Eqn1)}$$

From which by extraction:

$$(a^1 \subseteq \sim S_0) \subseteq (a \not\subseteq Q\{<\}) \quad (T_{T958})$$

$$(a^1 \subseteq \sim S_0) \subseteq (a \not\subseteq Q\{<\}) \quad (T_{T958A})$$

$$(a^1 \subseteq \sim S_0) \subseteq (a \not\subseteq Q\{>\}) \quad (T_{T959})$$

$$(a^1 \subseteq \sim S_0) \subseteq (a \not\subseteq Q\{>\}) \quad (T_{T959A})$$

$$(a^1 \subseteq \sim S_0) \subseteq (a \not\subseteq QB) \quad (T_{T960})$$

$$(a^1 \subseteq \sim S_0) \subseteq (a \not\subseteq QB) \quad (T_{T960A})$$

$$(a^1 \subseteq \sim S_0) \subseteq (a \not\subseteq QC) \quad (T_{T961})$$

$$(a^1 \subseteq \sim S_0) \subseteq (a \not\subseteq QC) \quad (T_{T961A})$$

$$(a^1 \subseteq QB) \subseteq (a^1 \subseteq S_0) \quad (T_{S717})$$

$$(a^1 \subseteq QC) \subseteq (a^1 \subseteq S_0) \quad (T_{S712})$$

Example Proof of (T_{S717})

$$(a^1 \subseteq \sim S_0) \subseteq (a \not\subseteq QB) \quad (T_{T960}) \text{ (Eqn1)}$$

$$(a^1 \not\subseteq S_0) \subseteq (a^1 \subseteq \sim S_0) \quad (T_{M604}) \text{ (Eqn2)}$$

$$(a^1 \not\subseteq S_0) \subseteq (a \not\subseteq QB) \quad (T_{M139}) \text{ (Hypothetical Syllogism Eqns2&1)}$$

$$(a^1 \subseteq QB) \subseteq (a^1 \subseteq S_0) \quad (T_{M012}) \text{ (Transposition)}$$

$$(a^1 \subseteq QB) \subseteq (a^1 \subseteq S_0) \quad \text{Resymbolising}$$

$$(a^1 \subseteq S_u) \subseteq (Ba = \emptyset) \quad (T_{S978})$$

$$(a^1 \subseteq S_U) \subseteq (Ba = \emptyset) \quad (T_{S978A})$$

$$(a^1 \subseteq S_U) \subseteq (Ca = \emptyset) \quad (T_{S979})$$

$$(a^1 \subseteq S_U) \subseteq (Ca = \emptyset) \quad (T_{S979A})$$

$$(a^1 \subseteq S_U) \subseteq (Ba^1 = \emptyset) \quad (T_{S621})$$

$$(a^1 \subseteq S_U) \subseteq (Ba^1 = \emptyset) \quad (T_{S621A})$$

$$(a^1 \subseteq S_U) \subseteq (Ca^1 = \emptyset) \quad (T_{S622})$$

$$(a^1 \subseteq S_U) \subseteq (Ca^1 = \emptyset) \quad (T_{S622A})$$

Example Proof of (T_{S978})

$$((X = \{\<\}, \{\>\}, B, C) = E) \subseteq ((a^1 \subseteq S_U) \subseteq (a^1 \subseteq \sim QX)(a \subseteq \sim QX)) \quad (A_{S70})(\text{Axiom of Unordered Sets})$$

$$(a^1 \subseteq S_U) \subseteq (a^1 \subseteq \sim QB)(a \subseteq \sim QB)$$

Extraction

$$(a^1 \subseteq S_U) \subseteq (a \subseteq \sim QB)$$

(T_{M073})(Single Premise Syllogism)(Eqn1)

$$(a \subseteq \sim QB) \subseteq (Ba = \emptyset)$$

(T_{S970})(Eqn2)

$$(a^1 \subseteq S_U) \subseteq (Ba = \emptyset)$$

(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$((o^1 \subseteq R_L)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 \langle o \rangle b^1) = (\forall Sa^1.o.\forall Qb^1 \neq \emptyset)) \quad (T_{S890}) \text{ Definition of an Ordering Equation}$$

Proof of (T_{S890})

Let $o^1 \subseteq R_L$

$$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a \langle o \rangle b) = (\forall Sa^1.o.\forall Qb^1 \neq \emptyset)) \quad (D_{A00})(\text{Definition of an Equation})$$

$$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 \langle o \rangle b^1) = (\forall Sa^1.o.\forall Qb^1 \neq \emptyset)) \quad (A_{S93})(\text{Relative Axiom})$$

$$\text{Thus } (((o^1 \subseteq R_L) = E) = E) \subseteq (((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 \langle o \rangle b^1) = (\forall Sa^1.o.\forall Qb^1 \neq \emptyset)))$$

(A_{MX02})(Conditional Introduction)

$$((o^1 \subseteq R_L)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 \langle o \rangle b^1) = (\forall Sa^1.o.\forall Qb^1 \neq \emptyset))$$

(A_{MX35})(Exportation/Importation) for truth sets

$$((o^1 \subseteq R_L)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 \langle o \rangle b^1) = (a^1 \subseteq o * b^1)) \quad (T_{S411}) \text{ Definition of an Ordering Relation}$$

Proof of (T_{S411})

$$((o^1 \subseteq R_L)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 \langle o \rangle b^1) = (\forall Sa^1.o.\forall Qb^1 \neq \emptyset))$$

(T_{S890})(Def. of an Ordering Eqn.)

$$((o^1 \subseteq R_L)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 \langle o \rangle b^1) = (\forall Sa^1 \not\subseteq \sim(o.\forall Qb^1)))$$

(T_{M735})($\subseteq$ Existential Form)

$$((o^1 \subseteq R_L)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq (a^1 \langle o \rangle b^1) = (a^1 \not\subseteq \sim S(o.\forall Qb^1))$$

(T_{S120})

$$((o^1 \subseteq R_L)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 \langle o \rangle b^1) = (a^1 \subseteq S(o.\forall Qb^1)))$$

(T_{M090})(Inverse Unit Subset Rule)

$$((o^1 \subseteq R_L)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 \langle o \rangle b^1) = (a^1 \subseteq o * b^1))$$

(A_{R27})(Function Axiom)

$$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 > b^1) = (a^1 \subseteq \{>\} * b^1)) \quad (T_{T962})$$

$$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 < b^1) = (a^1 \subseteq \{<\} * b^1)) \quad (T_{T963})$$

Example Proof of (T_{T962})

$$((o^1 \subseteq R_L)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 \langle o \rangle b^1) = (a^1 \subseteq o * b^1))$$

(T_{S411})(Definition of an Ordering Relation)(Eqn1)

$$(o = \{=\}, \{<\}, \{>\}, \{\leq\}, \{\geq\}) = (o^1 \subseteq R_L)$$

(D_{R96})(Definition of an Ordering Relationship Set)(Eqn2)

$$((o = \{=\}, \{<\}, \{>\}, \{\leq\}, \{\geq\})(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 \langle o \rangle b^1) = (a^1 \subseteq o * b^1))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 > b^1) = (a^1 \subseteq \{>\} * b^1)) \quad \text{Extraction}$$

$$(((a^1 + b^1)^1 \subseteq \sim S_o)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 > b^1) = \emptyset) \quad (T_{T964})$$

$$(((a^1 + b^1)^1 \subseteq \sim S_o)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 > b^1) = \emptyset) \quad (T_{T964A})$$

$$(((a^1 + b^1)^1 \subseteq \sim S_o)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 < b^1) = \emptyset) \quad (T_{T965})$$

$$(((a^1 + b^1)^1 \subseteq \sim S_o)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((a^1 < b^1) = \emptyset) \quad (T_{T965A})$$

Example Proof of (T₉₆₄)

Let $(a^1 + b^1)^1 \subseteq \sim S_0$

Let $a^1 \neq \emptyset$

$(a^1 \neq \emptyset) = E$

(A_{MX16})(Order Axiom)

$(a^1 = \emptyset) = \sim E$

(T_{M008})(Inverse of an Equality)

$(a^1 = \emptyset) = \emptyset$

(D_{M02})(Definition of $\emptyset$)(Eqn1)

Let $b^1 \neq \emptyset$

$(a^1 > b^1) = (a^1 \subseteq \{>\} * b^1)$

(T₉₆₂)

$(a^1 > b^1) = (a^1 \subseteq \emptyset)$

(T₉₆₆)

$(a^1 > b^1) = (a^1 = \emptyset)$

(T_{M052})(Subsets of the Null Set)(Eqn2)

$(a^1 > b^1) = \emptyset$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

Thus $((a^1 + b^1)^1 \subseteq \sim S_0)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E \subseteq ((a^1 > b^1) = \emptyset)$

(A_{MX02})(Conditional Introduction)

C. Introductory Theorems for Subsets of Ordered Sets

The following theorems can be used to convert equations in entire ordered sets to equations in subsets of ordered sets. The theorem (T₉₇₁) implies that if an equation is true for the entire ordered set a^1 , then, providing a_n^1 is not null or singular, it is also true for the subset a_n^1 . The theorem (T₉₇₃) implies that, if an equation is true for the entire discrete set a^1 , then it is also true for the subset a_n^1 , which is also discrete set. The theorem (T₉₇₆) implies that, if an equation is true for the entire continuum set a^1 , then, providing a_n^1 is not null or singular, it is also true for the subset a_n^1 , which is also continuum set.

The theorem (T₉₇₅) is a combination of two of the above, i.e., if an equation is true for the entire discrete ordered set a^1 , then, providing a_n^1 is not null or singular, it is also true for the subset a_n^1 which is also a discrete ordered set.

The theorem (T₉₇₇) is also a combination of two of the above, i.e., if an equation is true for the entire continuum ordered set a^1 , then, providing a_n^1 is not null or singular, it is also true for the subset a_n^1 which is also a continuum ordered set.

$$((a_n \subseteq a)(a_n \neq \emptyset)(a_n \neq a_n^1) = E) \subseteq ((a^1 \subseteq S_0) \subseteq (a_n^1 \subseteq S_0)) \quad (T_{971})$$

$$((a_n \subseteq a)(a_n \neq \emptyset)(a_n \neq a_n^1) = E) \subseteq ((a_n^1 \subseteq \sim S_0) \subseteq (a^1 \subseteq \sim S_0)) \quad (T_{972A})$$

Proof of (T₉₇₁) and (T₉₇₂)

$(b^1 \subseteq S_0)(a \subseteq b)(a \neq \emptyset)(a \neq a^1) \subseteq (a^1 \subseteq S_0) \quad (A_{S48})(\text{Subsets of Ordered Sets})$

$((a \subseteq b)(a \neq \emptyset)(a \neq a^1) = E) \subseteq ((b^1 \subseteq S_0) \subseteq (a^1 \subseteq S_0)) \quad (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets}$

$((a_n \subseteq a)(a_n \neq \emptyset)(a_n \neq a_n^1) = E) \subseteq ((a^1 \subseteq S_0) \subseteq (a_n^1 \subseteq S_0))$

Resymbolising

$((a_n \subseteq a)(a_n \neq \emptyset)(a_n \neq a_n^1) = E) \subseteq ((a_n^1 \subseteq \sim S_0) \subseteq (a^1 \subseteq \sim S_0))$

(T_{M012})(Transposition)

$$(\beta^1 \subseteq S_0 S_0)(\alpha \subseteq \beta)(\alpha \neq \emptyset)(\alpha \neq \alpha^1) \subseteq (\alpha^1 \subseteq S_0 S_0) \quad (T_{957})$$

Proof of (T₉₅₇)

$(\beta^1 \subseteq S_0)(\alpha \subseteq \beta)(\alpha \neq \emptyset)(\alpha \neq \alpha^1) \subseteq (\alpha^1 \subseteq S_0) \quad (A_{S48})(\text{Subsets of Ordered Sets}) \text{ Resymbolised (Eqn1)}$

$(\beta^1 \subseteq S_0)(\alpha \subseteq \beta)(\alpha \neq \alpha^1) \subseteq (\alpha^1 \subseteq S_0) \quad (T_{S221})(\text{Subsets of Discrete Sets}) \text{ Resymbolised (Eqn2)}$

$(\beta^1 \subseteq S_0)(\alpha \subseteq \beta)(\alpha \neq \emptyset)(\alpha \neq \alpha^1)(\beta^1 \subseteq S_0)(\alpha \subseteq \beta)(\alpha \neq \alpha^1) \subseteq (\alpha^1 \subseteq S_0)(\alpha^1 \subseteq S_0)$

(T_{M669})(Combination of Eqns1&2)

$(\beta^1 \subseteq S_0)(\alpha \subseteq \beta)(\alpha \neq \emptyset)(\alpha \neq \alpha^1)(\beta^1 \subseteq S_0) \subseteq (\alpha^1 \subseteq S_0)(\alpha^1 \subseteq S_0)$

(T_{M667})(Conjunction Tautology)

$(\beta^1 \subseteq S_0 S_0)(\alpha \subseteq \beta)(\alpha \neq \emptyset)(\alpha \neq \alpha^1) \subseteq (\alpha^1 \subseteq S_0 S_0)$

(T_{M132})(Composition)

$$((\alpha_n \subseteq \alpha)(\alpha_n \neq \emptyset)(\alpha_n \neq \alpha_n^1) = E) \subseteq ((\alpha^1 \subseteq S_D S_O) \subseteq (\alpha_n^1 \subseteq S_D S_O)) \quad (T_{T975})$$

Proof of (T₉₇₅)

$$(\beta^1 \subseteq S_D S_O)(\alpha \subseteq \beta)(\alpha \neq \emptyset)(\alpha \neq \alpha^1) \subseteq (\alpha^1 \subseteq S_D S_O) \quad (T_{T957})$$

$$((\alpha \subseteq \beta)(\alpha \neq \emptyset)(\alpha \neq \alpha^1) = E) \subseteq ((\beta^1 \subseteq S_D S_O) \subseteq (\alpha^1 \subseteq S_D S_O))$$

(A_{MX35})(Exportation/Importation) for truth sets

$$((\alpha_n \subseteq \alpha)(\alpha_n \neq \emptyset)(\alpha_n \neq \alpha_n^1) = E) \subseteq ((\alpha^1 \subseteq S_D S_O) \subseteq (\alpha_n^1 \subseteq S_D S_O))$$

Resymbolising

$$((a_n \subseteq a)(a_n \neq \emptyset)(a_n \neq a_n^1) = E) \subseteq ((a^1 \subseteq S_C) \subseteq (a_n^1 \subseteq S_C)) \quad (T_{T976})$$

Proof of (T₉₇₆)

Let $a^1 \subseteq S_C$

$a \subseteq S_C$

(T_{S604})(Eqn1)

Let $b \subseteq a$

(Eqn2)

$b \subseteq S_C$

(T_{M139})(Hypothetical Syllogism Eqns2&1)

$b^1 \subseteq S_C$

(T_{S604})

Thus $(a^1 \subseteq S_C)(b \subseteq a) \subseteq (b^1 \subseteq S_C)$

(A_{MX02})(Conditional Introduction)(Eqn3)

$(b^1 \subseteq S_C) \subseteq (b \neq \emptyset)$

(T_{S459})(Eqn4)

$(b^1 \subseteq S_C) \subseteq (b \neq b^1)$

(T_{S495})(Eqn5)

$(b^1 \subseteq S_C) \subseteq (b \neq \emptyset)(b \neq b^1)$

(T_{M132})(Composition Eqns4&5)

$(b^1 \subseteq S_C) = (b^1 \subseteq S_C)(b \neq \emptyset)(b \neq b^1)$

(A_{MX44})($\subseteq$ Equality Form)(Eqn6)

$(a^1 \subseteq S_C)(b \subseteq a) \subseteq (b^1 \subseteq S_C)(b \neq \emptyset)(b \neq b^1)$

(A_{MX19})(Substitution of Eqn6 into Eqn3)

$(a^1 \subseteq S_C)(b \subseteq a)(b \neq \emptyset)(b \neq b^1) \subseteq (b^1 \subseteq S_C)$

(T_{M249b})

$((b \subseteq a)(b \neq \emptyset)(b \neq b^1) = E) \subseteq ((a^1 \subseteq S_C) \subseteq (b^1 \subseteq S_C))$

(A_{MX35})(Exportation/Importation) for truth sets

$((a_n \subseteq a)(a_n \neq \emptyset)(a_n \neq a_n^1) = E) \subseteq ((a^1 \subseteq S_C) \subseteq (a_n^1 \subseteq S_C))$ Resymbolising

$$((a_n \subseteq a)(a_n \neq \emptyset)(a_n \neq a_n^1) = E) \subseteq ((a^1 \subseteq S_C S_O) \subseteq (a_n^1 \subseteq S_C S_O)) \quad (T_{T977})$$

Proof of (T₉₇₇)

$$((a_n \subseteq a)(a_n \neq \emptyset)(a_n \neq a_n^1) = E) \subseteq ((a^1 \subseteq S_C) \subseteq (a_n^1 \subseteq S_C)) \quad (T_{T976})(Eqn1)$$

(T₉₇₆)(Eqn1)

$$((a_n \subseteq a)(a_n \neq \emptyset)(a_n \neq a_n^1) = E) \subseteq ((a^1 \subseteq S_O) \subseteq (a_n^1 \subseteq S_O)) \quad (T_{T971})(Eqn2)$$

(T₉₇₁)(Eqn2)

$$((a_n \subseteq a)(a_n \neq \emptyset)(a_n \neq a_n^1) = E) \subseteq ((a^1 \subseteq S_C) \subseteq (a_n^1 \subseteq S_C)).((a^1 \subseteq S_O) \subseteq (a_n^1 \subseteq S_O))$$

(T_{M132})(Composition Eqns1&2)

$$((a_n \subseteq a)(a_n \neq \emptyset)(a_n \neq a_n^1) = E) \subseteq ((a^1 \subseteq S_C)(a^1 \subseteq S_O) \subseteq (a_n^1 \subseteq S_C)(a_n^1 \subseteq S_O))$$

(T_{M669})(Combination)

$$((a_n \subseteq a)(a_n \neq \emptyset)(a_n \neq a_n^1) = E) \subseteq ((a^1 \subseteq S_C S_O) \subseteq (a_n^1 \subseteq S_C S_O)) \quad (T_{M132})(Composition)$$

(T_{M132})(Composition)

11.2 Proofs of Theorems of Ordering Relationship Sets

A. Ordering Relationship Sets without Arguments

$$\{>\}^1 \subseteq R_T R_A \quad (T_{A249})$$

$$\{<\}^1 \subseteq R_T R_A \quad (T_{T921})$$

Example Proof of (T_{A249})

$$(X = \{<\}, \{>\}) \subseteq (X^1 \subseteq R_T R_A)$$

(A_{R62})(Ordering Relationships Axiom)

$$\{>\}^1 \subseteq R_T R_A$$

Extraction

$$\{\leq\}^1 \subseteq R_T \quad (T_{A248})$$

$$\{\geq\}^1 \subseteq R_T \quad (T_{R696})$$

Example Proof of (T_{A248})

$$\begin{aligned} (\{<\} + \{=\})^1 &\subseteq R_T & (A_{R22})(\text{Transitive Relationships Disjunction Axiom}) \\ \{<\}^1 &\subseteq R_T & (D_{A141})(\text{Disjunction with } \{=\}) \end{aligned}$$

$$\{<\} \subseteq \{#\}\{\sim\}\{\sim<\}\{>\} \quad (T_{S677})$$

$$\{>\} \subseteq \{#\}\{\sim\}\{\sim<\}\{<\} \quad (T_{S678})$$

The theorems (T_{S677}) and (T_{S678}) are proven by extraction from $(X = \{<\}, \{>\}) \subseteq (X \subseteq \{#\}\{\sim\}\{\sim<\}\{>\} \sim^v X)$ (A_{R61})
2nd Ordering Relationship Sets Axiom

$$\{>\} + \{<\} \subseteq \{#\}\{\sim\}\{\sim<\} \quad (T_{S644})$$

Proof of (T_{S644})

$$\begin{aligned} \{<\} &\subseteq \{#\}\{\sim\}\{\sim<\}\{>\} & (T_{S677}) \\ \{<\} &\subseteq \{#\}\{\sim\}\{\sim<\} & (T_{M073})(\text{Single Premise Syllogism})(\text{Eqn1}) \\ \{>\} &\subseteq \{#\}\{\sim\}\{\sim<\}\{<\} & (T_{S678}) \\ \{>\} &\subseteq \{#\}\{\sim\}\{\sim<\} & (T_{M073})(\text{Single Premise Syllogism})(\text{Eqn2}) \\ \{>\} + \{<\} &\subseteq \{#\}\{\sim\}\{\sim<\} + \{#\}\{\sim\}\{\sim<\} & (A_{MX21})(\text{Combination of Eqns1\&2}) \\ \{>\} + \{<\} &\subseteq \{#\}\{\sim\}\{\sim<\} & (A_{MX01})(\text{Disjunction Tautology}) \end{aligned}$$

$$\{<\} \subseteq \{#\} \quad (T_{S643})$$

$$\{<\} \subseteq \{\sim\} \quad (T_{S679})$$

$$\{<\} \subseteq \{\sim<\} \quad (T_{S680})$$

$$\{<\} \subseteq \{>\} \quad (T_{S681})$$

$$\{>\} \subseteq \{#\} \quad (T_{S642})$$

$$\{>\} \subseteq \{\sim\} \quad (T_{S682})$$

$$\{>\} \subseteq \{\sim<\} \quad (T_{S683})$$

$$\{>\} \subseteq \{<\} \quad (T_{S684})$$

$$\{<\} = \{<\}\{#\} \quad (T_{S685})$$

$$\{<\} = \{<\}\{\sim\} \quad (T_{S686})$$

$$\{<\} = \{<\}\{\sim<\} \quad (T_{S687})$$

$$\{<\} = \{<\}\{>\} \quad (T_{S688})$$

$$\{>\} = \{>\}\{#\} \quad (T_{S689})$$

$$\{>\} = \{>\}\{\sim\} \quad (T_{S690})$$

$$\{>\} = \{>\}\{\sim<\} \quad (T_{S691})$$

$$\{>\} = \{>\}\{<\} \quad (T_{S692})$$

$$\{<\}\{=\} = \emptyset \quad (T_{S693})$$

$$\{<\}\{\supset\} = \emptyset \quad (T_{S694})$$

$$\{<\}\{\subset\} = \emptyset \quad (T_{S695})$$

$$\{<\}\{>\} = \emptyset \quad (T_{S696})$$

$$\{>\}\{=\} = \emptyset \quad (T_{S697})$$

$$\{>\}\{\supset\} = \emptyset \quad (T_{S698})$$

$$\{>\}\{\subset\} = \emptyset \quad (T_{S699})$$

Example Proof of (T_{S643}) (T_{S685}) & (T_{S693})

$$\begin{aligned} \{<\} &\subseteq \{#\}\{\sim\}\{\sim<\}\{>\} & (T_{S677}) \\ \{<\} &\subseteq \{#\} & (T_{M073})(\text{Single Premise Syllogism}) \\ \{<\} &= \{<\}\{#\} & (A_{MX44})(\subseteq \text{Equality Form}) \\ \{<\}\{=\} &= \emptyset & (T_{M735})(\subseteq \text{Existential Form}) \end{aligned}$$

$$\{\leq\} \subseteq \{\succ\} \quad (TS829)$$

$$\{\geq\} \subseteq \{\prec\} \quad (TS830)$$

Example Proof of (TS829)

$$\begin{aligned} \{\succ\} &\subseteq \{\neq\} & (TS642) \\ \{=\} &\subseteq \{\succ\} & (TM012)(\text{Transposition})(\text{Eqn1}) \\ \{\succ\} &\subseteq \{\prec\} & (TS684) \\ \{\prec\} &\subseteq \{\succ\} & (TM012)(\text{Transposition})(\text{Eqn2}) \\ \{\prec\} + \{=\} &\subseteq \{\succ\} + \{\succ\} & (AMX21)(\text{Combination of Eqns1\&2}) \\ \{\prec\} + \{=\} &\subseteq \{\succ\} & (AMX01)(\text{Disjunction Tautology}) \\ \{\leq\} &\subseteq \{\succ\} & (DA141)(\text{Disjunction with } \{=\}) \end{aligned}$$

B. Ordering Relationship Sets with Arguments

These equations are true even if $\mathbf{a}$ is not an ordered set. This is because if $\mathbf{a}$ is not ordered, then, for example, $\{\succ\} * \mathbf{a} = \emptyset$. The general symbolism $\mathbf{a}$ is used, therefore rather than $\mathbf{a}$.

$$(\mathbf{a}.\{\succ\} * \mathbf{a}) \neq \emptyset = (\mathbf{a}.\{\prec\} * \mathbf{a}) \neq \emptyset \quad (TS647)$$

Proof of (TS647)

$$\begin{aligned} (\mathbf{a} \subseteq \{\succ\} * \mathbf{a}) &= (\mathbf{a} \subseteq \vee \{\succ\} * \mathbf{a}) & (TR457)(\text{Active/Passive Theorem}) \\ (\mathbf{a} \subseteq \{\succ\} * \mathbf{a}) &= (\mathbf{a} \subseteq \{\prec\} * \mathbf{a}) & (DA143)(\text{Symmetrical Opposite of an Asymmetrical Relationship Set}) \\ (\mathbf{a}.\{\succ\} * \mathbf{a}) = \emptyset &= (\mathbf{a}.\{\prec\} * \mathbf{a}) = \emptyset & (TM735)(\subseteq \text{Existential Form}) \\ (\mathbf{a}.\{\succ\} * \mathbf{a}) \neq \emptyset &= (\mathbf{a}.\{\prec\} * \mathbf{a}) \neq \emptyset & (TM008)(\text{Inverse of an Equality}) \end{aligned}$$

$$\mathbf{a}^1 \subseteq \{\prec\} * \mathbf{a} . \{\succ\} * \mathbf{a} \quad (TS671)$$

Proof of (TS671)

$$\begin{aligned} (\mathbf{a} \neq \emptyset) &\subseteq (\mathbf{a}^1 = (\{\supseteq\} \diamond \mathbf{a}).(\{\supseteq\} * \{\supseteq\} \diamond \mathbf{a})) & (DR98)(\text{General Definition of Aggregation}) \\ (\mathbf{a} \neq \emptyset) &\subseteq (\mathbf{a}^1 \subseteq \{\supseteq\} \diamond \mathbf{a}) & (TM073b)(\text{Single Premise Syllogism})(\text{Eqn1}) \\ (\mathbf{a} \neq \emptyset) &= (\{\neq\} * \mathbf{a} = \{\supseteq\} \diamond \mathbf{a}) & (AR10)(\text{Axiom of Complementary Relationship Sets}) \\ (\mathbf{a} \neq \emptyset) &= (\{\neq\} * \mathbf{a} = \{\supseteq\} \diamond \mathbf{a}) & (TM008)(\text{Inverse of an Equality})(\text{Eqn2}) \\ (\mathbf{a} \neq \emptyset) &\subseteq (\mathbf{a}^1 \subseteq \{\neq\} * \mathbf{a}) & (AMX19)(\text{Substitution of Eqn2 into Eqn1}) \\ (\mathbf{a} \neq \emptyset) &\subseteq (\{\neq\} * \mathbf{a} \subseteq \sim \mathbf{a}^1) & (TM012)(\text{Transposition}) \\ (\mathbf{a} \neq \emptyset) &\subseteq ((\{\sim\} \{\neq\}) * \mathbf{a} \subseteq \sim \mathbf{a}^1) & (TA059)(\text{Eqn3}) \\ \{\prec\} + \{\succ\} &\subseteq \{\neq\} \{\sim\} \{\sim\} & (TS644) \\ \{\prec\} + \{\succ\} &\subseteq \{\sim\} \{\neq\} & (TM073)(\text{Single Premise Syllogism}) \\ (\{\prec\} + \{\succ\}) * \mathbf{a} &\subseteq (\{\sim\} \{\neq\}) * \mathbf{a} & (TR197)(\text{Relational Deduction})(\text{Eqn4}) \\ (\mathbf{a} \neq \emptyset) &\subseteq ((\{\prec\} + \{\succ\}) * \mathbf{a} \subseteq \sim \mathbf{a}^1) & (TM139)(\text{Hypothetical Syllogism Eqns3\&4}) \\ (\mathbf{a} \neq \emptyset) &\subseteq (\mathbf{a}^1 \subseteq (\{\prec\} + \{\succ\}) * \mathbf{a}) & (TM012)(\text{Transposition}) \\ (\mathbf{a} \neq \emptyset) &\subseteq (\mathbf{a}^1 \subseteq \sim((\{\prec\} + \{\succ\}) * \mathbf{a})) & (DR04)(\text{Definition of } *) \\ (\mathbf{a} \neq \emptyset) &\subseteq (\mathbf{a}^1 \subseteq \sim(\{\prec\} * \mathbf{a} + \{\succ\} * \mathbf{a})) & (TR002c)(\text{Distribution of } + \text{ over } *) \\ (\mathbf{a} \neq \emptyset) &\subseteq (\mathbf{a}^1 \subseteq \{\prec\} * \mathbf{a} . \{\succ\} * \mathbf{a}) & (TM026)(\text{DeMorgan's Law})(\text{Eqn5}) \\ \text{Let } \mathbf{a} &= \emptyset & (\text{Eqn6}) \\ \text{Let } \mathbf{a}^1 &\notin \{\prec\} * \mathbf{a} . \{\succ\} * \mathbf{a} & (\text{Eqn7}) \\ \mathbf{a}^1 &\notin \{\prec\} * \emptyset . \{\succ\} * \emptyset & (AMX19)(\text{Substitution of Eqn6 into Eqn7}) \\ \mathbf{a}^1 &\notin \sim \emptyset . \sim \emptyset & (TS187)(\text{Null Object Theorem}) \\ \mathbf{a}^1 &\notin \mathbf{E} . \mathbf{E} & (TM054)(\text{Negative of Null Set}) \\ \mathbf{a}^1 &\notin \mathbf{E} & (TM667)(\text{Conjunction Tautology}) \\ (\mathbf{a}^1 \notin \mathbf{E}) &= \mathbf{E} & (AMX16)(\text{Order Axiom})(\text{Eqn8}) \\ (\mathbf{a}^1 \subseteq \mathbf{E}) &= \mathbf{E} & (TM007)(\text{Universality of } (TM744) \text{ Universal Set Theorem}) \\ (\mathbf{a}^1 \notin \mathbf{E}) &= \sim \mathbf{E} & (TM008)(\text{Inverse of an Equality}) \\ (\mathbf{a}^1 \notin \mathbf{E}) &= \emptyset & (DM02)(\text{Definition of } \emptyset)(\text{Eqn9}) \end{aligned}$$

$\emptyset = E$	(A_{MX19})(Substitution of Eqn9 into Eqn8)
$\emptyset$	(T_{M347})
Thus $(a = \emptyset)(a^1 \notin \{<\}_*a .\{>\}_*a) = \emptyset$	(A_{MX02})(Conditional Introduction)
$(a = \emptyset) \subseteq (a^1 \subseteq \{<\}_*a .\{>\}_*a)$	(T_{M735})($\subseteq$ Existential Form)(Eqn10)
$(a \neq \emptyset) + (a = \emptyset) \subseteq (a^1 \subseteq \{<\}_*a .\{>\}_*a) + (a^1 \subseteq \{<\}_*a .\{>\}_*a)$	(A_{MX21})(Combination of Eqns5&10)
$(a \neq \emptyset) + (a = \emptyset) \subseteq (a^1 \subseteq \{<\}_*a .\{>\}_*a)$	(A_{MX01})(Disjunction Tautology)
$E \subseteq (a^1 \subseteq \{<\}_*a .\{>\}_*a)$	(T_{M671})(Tertium non Datur)
$E = (a^1 \subseteq \{<\}_*a .\{>\}_*a)$	(T_{M275})(Supersets of the Universal Set)
$a^1 \subseteq \{<\}_*a .\{>\}_*a$	(A_{MX16})(Order Axiom)

$$a \subseteq \{<\}_*a^1 .\{>\}_*a^1 \quad (T_{S672})$$

Proof of (T_{S672})

$a^1 \subseteq \{<\}_*a .\{>\}_*a$	(T_{S671})
$a^1 \subseteq \{<\}_*a$	(T_{M073})(Single Premise Syllogism)
$a \subseteq \vee \{<\}_*a^1$	(T_{R457})(Active/Passive Theorem)
$a \subseteq \{>\}_*a^1$	(D_{A143})(Symmetrical Opposite of an Asymmetrical Relationship Set)(Eqn1)
$a^1 \subseteq \{>\}_*a$	(T_{M073})(Single Premise Syllogism)
$a \subseteq \vee \{>\}_*a^1$	(T_{R457})(Active/Passive Theorem)
$a \subseteq \{<\}_*a^1$	(D_{A143})(Symmetrical Opposite of an Asymmetrical Relationship Set)(Eqn2)
$a.a \subseteq \{<\}_*a^1 .\{>\}_*a^1$	(T_{M669})(Combination of Eqns1&2)
$a \subseteq \{<\}_*a^1 .\{>\}_*a^1$	(T_{M667})(Conjunction Tautology)

$$\{<\}_*a^1 \subseteq \{<\}_*a^1 \quad (T_{T118})$$

$$\{<\}_*a^1 \subseteq \{>\}_*a^1 \quad (T_{T120})$$

$$\{>\}_*a^1 \subseteq \{<\}_*a^1 \quad (T_{T121})$$

$$\{>\}_*a^1 \subseteq \{>\}_*a^1 \quad (T_{T122})$$

Example Proof of (T_{T118})

$\{<\} \subseteq \{\sim\}$	(T_{S680})
$\{<\}_*a^1 \subseteq \{\sim\}_*a^1$	(T_{R197})(Relational Deduction)
$(a^1 \neq \emptyset) \subseteq (\{<\}_*a^1 \subseteq \{<\}_*a^1)$	(T_{R245})(Negative Rule for Single Element Object)(Eqn1)
Let $a^1 = \emptyset$	(Eqn2)
Let $(\{<\}_*a^1 \notin \{<\}_*a^1)$	(Eqn3)
$\{<\}_*\emptyset \notin \{<\}_*\emptyset$	(A_{MX19})(Substitution of Eqn2 into Eqn3)
$\emptyset \notin \sim\emptyset$	(T_{S181})(Null Object Theorem)
$\emptyset \notin E$	(T_{M054})(Negative of Null Set)
$\sim E$	(T_{M001})(Universality of (T_{M670}) Null Set Theorem)
$\emptyset$	(D_{M02})(Definition of $\emptyset$)
Thus $(a^1 = \emptyset)(\{<\}_*a^1 \notin \{<\}_*a^1) = \emptyset$	(A_{MX02})(Conditional Introduction)
$(a^1 = \emptyset) \subseteq (\{<\}_*a^1 \subseteq \{<\}_*a^1)$	(T_{M735})($\subseteq$ Existential Form)(Eqn4)
$(a^1 = \emptyset) + (a^1 \neq \emptyset) \subseteq (\{<\}_*a^1 \subseteq \{<\}_*a^1) + (\{<\}_*a^1 \subseteq \{<\}_*a^1)$	(A_{MX21})(Combination of Eqns1&4)
$(a^1 = \emptyset) + (a^1 \neq \emptyset) \subseteq (\{<\}_*a^1 \subseteq \{<\}_*a^1)$	(A_{MX01})(Disjunction Tautology)
$E \subseteq (\{<\}_*a^1 \subseteq \{<\}_*a^1)$	(T_{M671})(Tertium non Datur)
$E = (\{<\}_*a^1 \subseteq \{<\}_*a^1)$	(T_{M275})(Supersets of the Universal Set)
$\{<\}_*a^1 \subseteq \{<\}_*a^1$	(A_{MX16})(Order Axiom)

$$a^1 \subseteq \{<\}_*a^1 \quad (T_{T119})$$

$$a^1 \subseteq \{>\}_*a^1 \quad (T_{T123})$$

Example Proof of (T_{T119})

$\{<\} \subseteq \{+\}$	(T_{S643})
$\{=\} \subseteq \{<\}$	(T_{M012})(Transposition)

$\{=\} * a^1 \subseteq \{<\} * a^1$	(T_{R197})(Relational Deduction)
$a^1 \subseteq \{<\} * a^1$	(D_{A144})(Definition of the Reflexive Relationship Set)
$(a^1 \neq \emptyset) \subseteq (a^1 \subseteq \{<\} * a^1)$	(T_{R245})(Negative Rule for Single Element Object)(Eqn1)
Let $a^1 = \emptyset$	(Eqn2)
Let $a^1 \not\subseteq \{<\} * a^1$	(Eqn3)
$\emptyset \not\subseteq \{<\} * \emptyset$	(A_{MX19})(Substitution of Eqn2 into Eqn3)
$\emptyset \not\subseteq \sim \emptyset$	(T_{S187})(Null Object Theorem)
$\emptyset \not\subseteq E$	(T_{M054})(Negative of Null Set)
$\sim E$	(T_{M001})(Universality of (T_{M670}) Null Set Theorem)
$\emptyset$	(D_{M02})(Definition of $\emptyset$)
Thus $(a^1 = \emptyset)(a^1 \not\subseteq \{<\} * a^1) = \emptyset$	(A_{MX02})(Conditional Introduction)
$(a^1 = \emptyset) \subseteq (a^1 \subseteq \{<\} * a^1)$	(T_{M735})($\subseteq$ Existential Form)(Eqn4)
$(a^1 = \emptyset) + (a^1 \neq \emptyset) \subseteq (a^1 \subseteq \{<\} * a^1) + (a^1 \subseteq \{<\} * a^1)$	(A_{MX21})(Combination of Eqns1&4)
$(a^1 = \emptyset) + (a^1 \neq \emptyset) \subseteq (a^1 \subseteq \{<\} * a^1)$	(A_{MX01})(Disjunction Tautology)
$E \subseteq (a^1 \subseteq \{<\} * a^1)$	(T_{M671})(Tertium non Datur)
$E = (a^1 \subseteq \{<\} * a^1)$	(T_{M275})(Supersets of the Universal Set)
$a^1 \subseteq \{<\} * a^1$	(A_{MX16})(Order Axiom)

$a^1 = a^1.(\{\supset\} * \{<\} * a^1)$	(T_{T124})
$a^1 = a^1.(\{\supset\} * \{>\} * a^1)$	(T_{T125})
$a^1 = a^1.(\{\subseteq\} * \{<\} * a^1)$	(T_{T126})
$a^1 = a^1.(\{\subseteq\} * \{>\} * a^1)$	(T_{T127})

Example Proof of (T_{T124})

$a^1 \subseteq \{<\} * a^1$	(T_{T119})(Eqn1)
$\{<\} * a^1 \subseteq \{\subseteq\} * a^1$	(T_{T118})
$\{<\} * a^1 \subseteq \vee \{\supset\} * a^1$	(D_{A143})(Symmetrical Opposite of an Asymmetrical Relationship Set)
$a^1 \subseteq \{\supset\} * \{<\} * a^1$	(T_{S030})(Active/Passive Theorem)(Eqn2)
$a^1.a^1 \subseteq (\{\supset\} * \{<\} * a^1).(\{<\} * a^1)$	(T_{M669})(Combination of Eqns1&2)
$a^1 \subseteq (\{\supset\} * \{<\} * a^1).(\{<\} * a^1)$	(T_{M667})(Conjunction Tautology)
$a^1 \subseteq \sim(\{\supset\} * \{<\} * a^1 + \{<\} * a^1)$	(T_{M026})(DeMorgan's Law)
$a^1 \subseteq \sim(\{\supset\} * \{<\} * a^1 + \{=\} * \{<\} * a^1)$	(T_{A080})
$a^1 \subseteq \sim(\{\supset\} + \{=\}) * \{<\} * a^1$	(T_{R002c})(Distribution of $+$ over $*$)
$a^1 \subseteq (\{\supset\} + \{=\}) * \{<\} * a^1$	(D_{M03})(Definition of $\sim$)
$a^1 \subseteq \{\supset\} * \{<\} * a^1$	(D_{A141})(Disjunction with $\{=\}$)
$a^1 = a^1.(\{\supset\} * \{<\} * a^1)$	(A_{MX44})($\subseteq$ Equality Form)

$a^1 \subseteq \{\supset\} * (\{<\} * a^1 + \{>\} * a^1)$	(T_{T111})
$a^1 \subseteq \{\subseteq\} * (\{<\} * a^1 + \{>\} * a^1)$	(T_{T112})

Example Proof of (T_{T111})

$\{>\} + \{<\} \subseteq \{\neq\} \{ \sim \supset \} \{ \sim \subseteq \}$	(T_{S644})
$\{>\} + \{<\} \subseteq \{\neq\} \{ \sim \subseteq \}$	(T_{M073})(Single Premise Syllogism)
$(\{>\} + \{<\}) * a^1 \subseteq (\{\neq\} \{ \sim \subseteq \}) * a^1$	(T_{R196})(Relational Deduction)
$(\{>\} + \{<\}) * a^1 \subseteq \{\neq\} * a^1$	(T_{A059})
$\{>\} * a^1 + \{<\} * a^1 \subseteq \{\neq\} * a^1$	(T_{R002c})(Distribution of $+$ over $*$)
$(a^1 \neq \emptyset) \subseteq (\{<\} * a^1 + \{>\} * a^1) \subseteq \{\neq\} * a^1$	(T_{R245})(Negative Rule for Single Element Object)
$(a^1 \neq \emptyset) \subseteq (a^1 \subseteq \{\supset\} * (\{<\} * a^1 + \{>\} * a^1))$	(T_{S030})(Active/Passive Theorem)(Eqn1)
Let $a^1 = \emptyset$	(Eqn2)
Let $a^1 \not\subseteq \{\supset\} * (\{<\} * a^1 + \{>\} * a^1)$	(Eqn3)
$\emptyset \not\subseteq \{\supset\} * (\{<\} * \emptyset + \{>\} * \emptyset)$	(A_{MX19})(Substitution of Eqn2 into Eqn3)
$\emptyset \not\subseteq \{\supset\} * (\emptyset + \emptyset)$	(T_{S187})(Null Object Theorem)
$\emptyset \not\subseteq \{\supset\} * \emptyset$	(A_{MX01})(Disjunction Tautology)
$\emptyset \not\subseteq \sim \emptyset$	(T_{S187})(Null Object Theorem)

$\emptyset \notin E$	(T_{M054})(Negative of Null Set)
$\sim E$	(T_{M001})(Universality of (T_{M670}) Null Set Theorem)
$\emptyset$	(D_{M02})(Definition of $\emptyset$)
Thus $(a^1 = \emptyset)(a^1 \notin \{\supset\} * (\{\<\} * a^1 + \{\>\} * a^1)) = \emptyset$	(A_{MX02})(Conditional Introduction)
$(a^1 = \emptyset) \subseteq (a^1 \subseteq \{\supset\} * (\{\<\} * a^1 + \{\>\} * a^1))$	(T_{M735})($\subseteq$ Existential Form)(Eqn4)
$(a^1 \neq \emptyset) + (a^1 = \emptyset) \subseteq (a^1 \subseteq \{\supset\} * (\{\<\} * a^1 + \{\>\} * a^1)) + (a^1 \subseteq \{\supset\} * (\{\<\} * a^1 + \{\>\} * a^1))$	(A_{MX21})(Combination of Eqns1&4)
$(a^1 \neq \emptyset) + (a^1 = \emptyset) \subseteq (a^1 \subseteq \{\supset\} * (\{\<\} * a^1 + \{\>\} * a^1))$	(A_{MX01})(Disjunction Tautology)
$E \subseteq (a^1 \subseteq \{\supset\} * (\{\<\} * a^1 + \{\>\} * a^1))$	(T_{M671})(Tertium non Datur)
$E = (a^1 \subseteq \{\supset\} * (\{\<\} * a^1 + \{\>\} * a^1))$	(T_{M275})(Supersets of the Universal Set)
$a^1 \subseteq \{\supset\} * (\{\<\} * a^1 + \{\>\} * a^1)$	(A_{MX16})(Order Axiom)

$$a^1 \subseteq (\{\<\} * a^1).(\{\>\} * a^1) \quad (T_{T113})$$

Proof of (T_{T113})

$\{\>\} + \{\<\} \subseteq \{\neq\}\{\sim\supset\}\{\sim\subseteq\}$	(T_{S644})
$\{\>\} + \{\<\} \subseteq \{\neq\}$	(T_{M073})(Single Premise Syllogism)
$(\{\>\} + \{\<\}) * a^1 \subseteq \{\neq\} * a^1$	(T_{R196})(Relational Deduction)
$\{\>\} * a^1 + \{\<\} * a^1 \subseteq \{\neq\} * a^1$	(T_{R002c})(Distribution of $+$ over $*$)
$(a^1 \neq \emptyset) \subseteq (\{\<\} * a^1 + \{\>\} * a^1 \subseteq \{\neq\} * a^1)$	(T_{R245})(Negative Rule for Single Element Object)
$(a^1 \neq \emptyset) \subseteq (\{\<\} * a^1 + \{\>\} * a^1 \subseteq \sim a^1)$	(T_{A080})&(T_{M008})(Inverse of an Equality)
$(a^1 \neq \emptyset) \subseteq (a^1 \subseteq \sim(\{\<\} * a^1 + \{\>\} * a^1))$	(T_{M012})(Transposition)
$(a^1 \neq \emptyset) \subseteq (a^1 \subseteq (\{\<\} * a^1).(\{\>\} * a^1))$	(T_{M026})(DeMorgan's Law)(Eqn1)
Let $a^1 = \emptyset$	(Eqn2)
Let $a^1 \notin (\{\<\} * a^1).(\{\>\} * a^1)$	(Eqn3)
$\emptyset \notin (\{\<\} * \emptyset).(\{\>\} * \emptyset)$	(A_{MX19})(Substitution of Eqn2 into Eqn3)
$\emptyset \notin \sim \emptyset. \sim \emptyset$	(T_{S181})(Null Object Theorem)
$\emptyset \notin \sim \emptyset$	(T_{M667})(Conjunction Tautology)
$\emptyset \notin E$	(T_{M054})(Negative of Null Set)
$\sim E$	(T_{M001})(Universality of (T_{M670}) Null Set Theorem)
$\emptyset$	(D_{M02})(Definition of $\emptyset$)
Thus $(a^1 = \emptyset)(a^1 \notin (\{\<\} * a^1).(\{\>\} * a^1)) = \emptyset$	(A_{MX02})(Conditional Introduction)
$(a^1 = \emptyset) \subseteq (a^1 \subseteq (\{\<\} * a^1).(\{\>\} * a^1))$	(T_{M735})($\subseteq$ Existential Form)(Eqn4)
$(a^1 = \emptyset) + (a^1 \neq \emptyset) \subseteq (a^1 \subseteq (\{\<\} * a^1).(\{\>\} * a^1)) + (a^1 \subseteq (\{\<\} * a^1).(\{\>\} * a^1))$	(A_{MX21})(Combination of Eqns1&4)
$(a^1 = \emptyset) + (a^1 \neq \emptyset) \subseteq (a^1 \subseteq (\{\<\} * a^1).(\{\>\} * a^1))$	(A_{MX01})(Disjunction Tautology)
$E \subseteq (a^1 \subseteq (\{\<\} * a^1).(\{\>\} * a^1))$	(T_{M671})(Tertium non Datur)
$E = (a^1 \subseteq (\{\<\} * a^1).(\{\>\} * a^1))$	(T_{M275})(Supersets of the Universal Set)
$a^1 \subseteq (\{\<\} * a^1).(\{\>\} * a^1)$	(A_{MX16})(Order Axiom)

$$(a^1 \subseteq \{\<\} * b^1) = (b^1 \subseteq \{\>\} * a^1) \quad (T_{T776})$$

Proof of (T_{T776})

$(a^1 \subseteq \{\<\} * b^1) = (b^1 \subseteq \vee \{\<\} * a^1)$	(T_{R519})(Active/Passive Theorem)
$(a^1 \subseteq \{\<\} * b^1) = (b^1 \subseteq \{\>\} * a^1)$	(D_{A143})(Symmetrical Opposite of an Asymmetrical Relationship Set)

$$(a^1 \subseteq S_M S_O + S_P S_O) \subseteq (\{\<\} * \{\<\} * a = \{\<\} * a) \quad (T_{S925})$$

$$(a^1 \subseteq S_M S_O + S_P S_O) \subseteq (\{\>\} * \{\>\} * a = \{\>\} * a) \quad (T_{S926})$$

Example Proof of (T_{S925})

$(o = \{\subseteq, \{\supset\}, \{\<\}, \{\>\})(a^1 = S_M S_O + S_P S_O) \subseteq (o * a \subseteq o * o * a)$	(A_{R87})(Axiom of Reverse Transitivity)
$(a^1 \subseteq S_M S_O + S_P S_O) \subseteq (\{\<\} * a \subseteq \{\<\} * \{\<\} * a)$	Extraction (Eqn1)
$(\{\<\}^1 \subseteq R_T) \subseteq (\{\<\} * \{\<\} * a \subseteq \{\<\} * a)$	(T_{S177})(Definition of a Transitive Relationship Set)(Eqn2)
$(a^1 \subseteq S_M S_O + S_P S_O)(\{\<\}^1 \subseteq R_T) \subseteq (\{\<\} * a \subseteq \{\<\} * \{\<\} * a).(\{\<\} * \{\<\} * a \subseteq \{\<\} * a)$	

$$\begin{aligned}
& (a^1 \subseteq S_M S_O + S_P S_O) (\{<\}^1 \subseteq R_T) \subseteq (\{<\} * \{<\} * a = \{<\} * a) & (T_{M669}) (\text{Combination of Eqns1\&2}) \\
& \{<\}^1 \subseteq R_T R_A & (T_{M430}) (\text{Biconditional Introduction}) (\text{Eqn3}) \\
& \{<\}^1 \subseteq R_T & (T_{T921}) \\
& (\{<\}^1 \subseteq R_T) = E & (T_{M073}) (\text{Single Premise Syllogism}) \\
& (a^1 \subseteq S_M S_O + S_P S_O) E \subseteq (\{<\} * \{<\} * a = \{<\} * a) & (A_{MX16}) (\text{Order Axiom}) (\text{Eqn4}) \\
& (a^1 \subseteq S_M S_O + S_P S_O) \subseteq (\{<\} * \{<\} * a = \{<\} * a) & (A_{MX19}) (\text{Substitution of Eqn4 into Eqn3}) \\
& & (A_{MX11}) (\text{Universal Set Conjunction Axiom})
\end{aligned}$$

C. Equations using Ordering Relationships

$$((\{<\}^1 \subseteq R_L) = E) \subseteq ((a \trianglelefteq b) = (a^1 \trianglelefteq b^1)) \quad (T_{T889})$$

Proof of (T_{T889})

$$\begin{aligned}
& ((o^1 \subseteq R_L) = E) \subseteq ((a \langle o \rangle b) = (a^1 \langle o \rangle b^1)) & (A_{S93}) (\text{Relative Axiom}) \\
& ((\{<\}^1 \subseteq R_L) = E) \subseteq ((a \triangleleft b) = (a^1 \triangleleft b^1)) & \text{Resymbolising} \\
& ((\{<\}^1 \subseteq R_L) = E) \subseteq ((a \triangleleft b) = (a^1 \triangleleft b^1)).E & (A_{MX11}) (\text{Universal Set Conjunction Axiom}) (\text{Eqn1}) \\
& (a^1 = b^1) = (a = b) & (T_{M479}) \\
& E = ((a^1 = b^1) = (a = b)) & (A_{MX16}) (\text{Order Axiom}) (\text{Eqn2}) \\
& ((\{<\}^1 \subseteq R_L) = E) \subseteq ((a \triangleleft b) = (a^1 \triangleleft b^1)).((a^1 = b^1) = (a = b)) & (A_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\
& ((\{<\}^1 \subseteq R_L) = E) \subseteq ((a \triangleleft b) + (a = b) = (a^1 \triangleleft b^1) + (a^1 = b^1)) & (A_{MX21}) (\text{Combination}) \\
& ((\{<\}^1 \subseteq R_L) = E) \subseteq ((a \trianglelefteq b) = (a^1 \trianglelefteq b^1)) & (D_{M18}) (\text{Definition of } \trianglelefteq)
\end{aligned}$$

$$((\{<\}^1 \subseteq R_L) = E) \subseteq ((a^1 \trianglelefteq b^1) = (a^1 \triangleleft b^1) + (a^1 = b^1)) \quad (T_{S435}) \text{ Definition of } \trianglelefteq$$

Proof of (T_{S435})

$$\begin{aligned}
& (a \trianglelefteq b) = (a \triangleleft b) + (a = b) & (D_{M18}) (\text{Definition of } \trianglelefteq) \\
& \text{Let } \{<\}^1 \subseteq R_L \\
& (a^1 \trianglelefteq b^1) = (a^1 \triangleleft b^1) + (a^1 = b^1) & (A_{S93}) (\text{Relative Axiom}) (T_{M479}) \& (T_{T889}) \\
& \text{Thus } ((\{<\}^1 \subseteq R_L) = E) \subseteq ((a^1 \trianglelefteq b^1) = (a^1 \triangleleft b^1) + (a^1 = b^1)) & (A_{MX02}) (\text{Conditional Introduction})
\end{aligned}$$

$$(a^1 \leq b^1) = (a^1 < b^1) + (a^1 = b^1) \quad (T_{T887}) \text{ Definition of } \leq$$

$$(a^1 \geq b^1) = (a^1 > b^1) + (a^1 = b^1) \quad (T_{T888}) \text{ Definition of } \geq$$

Example Proof of (T_{T887})

$$\begin{aligned}
& ((\{<\}^1 \subseteq R_L) = E) \subseteq ((a^1 \trianglelefteq b^1) = (a^1 \triangleleft b^1) + (a^1 = b^1)) & (T_{S435}) (\text{Definition of } \trianglelefteq) (\text{Eqn1}) \\
& \text{Let } \{<\} = \{<\} & (\text{Eqn2}) \\
& ((\{<\}^1 \subseteq R_L) = E) \subseteq ((a^1 \leq b^1) = (a^1 < b^1) + (a^1 = b^1)) & (A_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) (\text{Eqn3}) \\
& (o = \{=, \{<\}, \{>\}, \{\leq\}, \{\geq\}) = (o^1 \subseteq R_L) & (D_{R96}) (\text{Definition of an Ordering Relationship Set}) \\
& \{<\}^1 \subseteq R_L & \text{Extraction} \\
& (\{<\}^1 \subseteq R_L) = E & (A_{MX16}) (\text{Order Axiom}) (\text{Eqn4}) \\
& (E = E) \subseteq ((a^1 \leq b^1) = (a^1 < b^1) + (a^1 = b^1)) & (A_{MX19}) (\text{Substitution of Eqn4 into Eqn3}) \\
& E \subseteq ((a^1 \leq b^1) = (a^1 < b^1) + (a^1 = b^1)) & (T_{M342}) \\
& E = ((a^1 \leq b^1) = (a^1 < b^1) + (a^1 = b^1)) & (T_{M275}) (\text{Supersets of the Universal Set}) \\
& (a^1 \leq b^1) = (a^1 < b^1) + (a^1 = b^1) & (A_{MX16}) (\text{Order Axiom})
\end{aligned}$$

$$((\{<\}^1 \subseteq R_L) = E) \subseteq ((a^1 \triangleleft b^1) = (b^1 \triangleright a^1)) \quad (T_{S436}) \text{ Definition of } \triangleleft$$

Proof of (T_{S436})

$$\begin{aligned}
& (a \triangleleft b) = (b \triangleright a) & (D_{S23}) (\text{Definition of } \triangleleft) (\text{Eqn1}) \\
& ((\{<\}^1 \subseteq R_L) = E) \subseteq ((a^1 \triangleleft b^1) = (b^1 \triangleright a^1)) & (A_{S93}) (\text{Relative Axiom})
\end{aligned}$$

$$(a^1 < b^1) = (b^1 > a^1) \quad (T_{T886})$$

Proof of (T_{T886})

$((\{<\}^1 \subseteq R_L) = E) \subseteq ((a^1 < b^1) = (b^1 > a^1))$	(T_{S436})(Definition of $<$)(Eqn1)
$\{<\} = \{<, \{>\}, \{<\}, \{>\}$	(D_{S08})(Definition of $<$)
$\{<\} = \{<\}$	Extraction (Eqn2)
$(a^1 < b^1) = (b^1 > a^1)$	(A_{MX19})(Substitution of Eqn2 into Eqn1)

11.3 Proofs of Theorems of Unordered and Non-unordered Sets

$$(a^1 \subseteq S_u) \subseteq (\forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa \subseteq \{#\}\{\#\}) \quad (T_{S654})$$

$$(a^1 \subseteq S_u) \subseteq (\forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa \subseteq \{#\}\{\#\}) \quad (T_{S654A})$$

Example Proof of (T_{S654})

$(a^1 \subseteq S_u) = (\forall Sa. \forall Qa \subseteq \{#\}\{\#\})$	(D_{R84})(Definition of an Unordered Set)(Eqn1)
$\forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa \subseteq \forall Sa. \forall Qa$	(T_{M083})(Conjunction Elimination)(Eqn2)
$(a^1 \subseteq S_u) \subseteq (\forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa \subseteq \{#\}\{\#\})$	(T_{M139})(Hypothetical Syllogism Eqns2&1)

$$(a^1 \subseteq S_u) \subseteq (\forall Sa. \forall Qa \subseteq \{=\} + \{>\} + \{<\} + \{\|\} + \{#\}\{\sim\}\{\sim\}\{\#\}) \quad (T_{S653})$$

$$(a^1 \subseteq S_u) \subseteq (\forall Sa. \forall Qa \subseteq \{=\} + \{>\} + \{<\} + \{\|\} + \{#\}\{\sim\}\{\sim\}\{\#\}) \quad (T_{S653A})$$

Example Proof of (T_{S653})

$\forall Sa. \forall Qa \subseteq E$	(T_{M744})(Universal Set Theorem)(Eqn1)
$c + c = E$	(T_{M671})(Tertium non Datur)(Eqn2)
$\forall Sa. \forall Qa \subseteq c + c$	(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)
Let $c = \{>\} + \{<\} + \{=\} + \{>\} + \{<\} + \{\ \}$	(Eqn4)
$c = \sim(\{>\} + \{<\} + \{=\} + \{>\} + \{<\} + \{\ \})$	(T_{M008})(Inverse of an Equality)
$c = \{#\}\{\#\}\{\sim\}\{\sim\}\{\#\}$	(T_{M188})(DeMorgan's Law for Series)(Eqn5)
$\forall Sa. \forall Qa \subseteq \{>\} + \{<\} + \{=\} + \{>\} + \{<\} + \{\ \} + \{#\}\{\#\}\{\sim\}\{\sim\}\{\#\}$	(A_{MX19})(Substitution of Eqns4&5 into Eqn3)
$\forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa \subseteq \{>\} + \{<\} + \{#\}\{\#\}\{\sim\}\{\sim\}\{\#\}$	(T_{M468})(Eqn6)
$(a^1 \subseteq S_u) \subseteq (\forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa \subseteq \{#\}\{\#\})$	(T_{S654})(Eqn7)
$(a^1 \subseteq S_u) \subseteq (\forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa. \forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa \subseteq \{#\}\{\#\}. (\{>\} + \{<\} + \{#\}\{\#\}\{\sim\}\{\sim\}\{\#\}))$	(T_{M669})(Combination of Eqns6&7)
$(a^1 \subseteq S_u) \subseteq (\forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa. \subseteq \{#\}\{\#\}. (\{>\} + \{<\} + \{#\}\{\#\}\{\sim\}\{\sim\}\{\#\}))$	(T_{M667})(Conjunction Tautology)
$(a^1 \subseteq S_u) \subseteq (\forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa. \subseteq (\{#\}\{\#\}. \{>\} + \{#\}\{\#\}. \{<\} + \{#\}\{\#\}\{\sim\}\{\sim\}\{\#\}))$	(T_{M407})(Distribution of . over +)
$(a^1 \subseteq S_u) \subseteq (\forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa. \subseteq (\emptyset. \{#\} + \emptyset. \{#\} + \{#\}\{\#\}\{\sim\}\{\sim\}\{\#\}))$	(T_{M621})(Complement Conjunction)
$(a^1 \subseteq S_u) \subseteq (\forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa. \subseteq (\emptyset + \emptyset + \{#\}\{\#\}\{\sim\}\{\sim\}\{\#\}))$	(T_{M050})(Null Set Conjunction)
$(a^1 \subseteq S_u) \subseteq (\forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa. \subseteq \{#\}\{\#\}\{\sim\}\{\sim\}\{\#\})$	(T_{M668})(Null Set Disjunction)
$(a^1 \subseteq S_u) \subseteq (\forall Sa. \{#\}\{\sim\}\{\sim\}\{\#\}. \forall Qa. \subseteq \{#\}\{\sim\}\{\sim\}\{\#\})$	(T_{M073})(Single Premise Syllogism)
$(a^1 \subseteq S_u) \subseteq (\forall Sa. \forall Qa. \subseteq \{=\} + \{>\} + \{<\} + \{\ \} + \{#\}\{\sim\}\{\sim\}\{\#\})$	(T_{M468})

$$(a^1 \subseteq S_u) \subseteq (a \subseteq (\{>\} * a) (\{<\} * a)) \quad (T_{S655})$$

$$(a^1 \subseteq S_u) \subseteq (a \subseteq (\{>\} * a) (\{<\} * a)) \quad (T_{S655A})$$

Example Proof of (T_{S655})

$(a^1 \subseteq S_u) \subseteq (\forall Sa. \forall Qa \subseteq \{#\}\{\#\})$	(D_{R84})(Definition of an Unordered Set)
$(a^1 \subseteq S_u) \subseteq (\forall Sa. \sim(\{#\}\{\#\}). \forall Qa = \emptyset)$	(T_{M735})($\subseteq$ Existential Form)
$(a^1 \subseteq S_u) \subseteq (\forall Sa. (\{<\} + \{>\}). \forall Qa = \emptyset)$	(T_{M025})(DeMorgan's Law)
$(a^1 \subseteq S_u) \subseteq (\forall Sa. \{>\}. \forall Qa + \forall Sa. \{<\}. \forall Qa = \emptyset)$	(T_{M407})(Distribution of . over +)

$$\begin{aligned}
(a^1 \subseteq S_u) &\subseteq (\forall Sa.\{>\}.\forall Qa = \emptyset).(\forall Sa.\{<\}.\forall Qa = \emptyset) & (T_{M071})(\text{Probabilistic Conjunction}) \\
(a^1 \subseteq S_u) &\subseteq (a \subseteq \{>\} * a)(a \subseteq \{<\} * a) & (T_{S082}) \\
(a^1 \subseteq S_u) &\subseteq (a \subseteq (\{>\} * a)(\{<\} * a)) & (T_{M132})(\text{Composition})
\end{aligned}$$

$$(a^1 \not\subseteq S_u) = (a.\{>\} * a \neq \emptyset)(a.\{<\} * a \neq \emptyset) \quad (T_{S648})$$

$$(a^1 \not\subseteq S_u) = (a.\{>\} * a \neq \emptyset)(a.\{<\} * a \neq \emptyset) \quad (T_{S648_A})$$

Example Proof of (T_{S648})

$$\begin{aligned}
(a^1 \subseteq S_u) &= (a \subseteq (\{>\} * a)(\{<\} * a)) & (T_{S655}) \\
(a^1 \subseteq S_u) &= (a.\sim((\{>\} * a)(\{<\} * a)) = \emptyset) & (T_{M735})(\subseteq \text{Existential Form}) \\
(a^1 \subseteq S_u) &= (a.((\{>\} * a) + (\{<\} * a)) = \emptyset) & (T_{M026})(\text{DeMorgan's Law}) \\
(a^1 \subseteq S_u) &= (a.(\{>\} * a) + a.(\{<\} * a) = \emptyset) & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
(a^1 \subseteq S_u) &= (a.(\{>\} * a) = \emptyset).(a.(\{<\} * a) = \emptyset) & (T_{M071})(\text{Probabilistic Conjunction}) \\
(a^1 \not\subseteq S_u) &= \sim((a.(\{>\} * a) = \emptyset).(a.(\{<\} * a) = \emptyset)) & (T_{M008})(\text{Inverse of an Equality}) \\
(a^1 \not\subseteq S_u) &= ((a.(\{>\} * a) \neq \emptyset) + (a.(\{<\} * a) \neq \emptyset)) & (T_{M025})(\text{DeMorgan's Law})(\text{Eqn1}) \\
(a.(\{>\} * a) \neq \emptyset) &= (a.(\{<\} * a) \neq \emptyset) & (T_{S647}) \\
(a.(\{>\} * a) \neq \emptyset) &= (a.(\{>\} * a) \neq \emptyset)(a.(\{<\} * a) \neq \emptyset) & (T_{M521})(\subseteq \text{Equality Form})(\text{Eqn2}) \\
(a.(\{<\} * a) \neq \emptyset) &= (a.(\{<\} * a) \neq \emptyset)(a.(\{>\} * a) \neq \emptyset) & (T_{M521})(\subseteq \text{Equality Form})(\text{Eqn3}) \\
(a^1 \not\subseteq S_u) &= (a.(\{>\} * a) \neq \emptyset)(a.(\{<\} * a) \neq \emptyset) + (a.(\{<\} * a) \neq \emptyset)(a.(\{>\} * a) \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\
(a^1 \not\subseteq S_u) &= (a.(\{>\} * a) \neq \emptyset)(a.(\{<\} * a) \neq \emptyset) & (A_{MX01})(\text{Disjunction Tautology})
\end{aligned}$$

$$(a^1 \not\subseteq S_u) \subseteq (\forall Sa.(\{>\} + \{<\}).\forall Qa \neq \emptyset) \quad (T_{S664})$$

$$(a^1 \not\subseteq S_u) \subseteq (\forall Sa.(\{>\} + \{<\}).\forall Qa \neq \emptyset) \quad (T_{S664_A})$$

Example Proof of (T_{S664})

$$\begin{aligned}
(a^1 \not\subseteq S_u) &\subseteq (a.\{>\} * a \neq \emptyset)(a.\{<\} * a \neq \emptyset) & (T_{S648}) \\
(a^1 \not\subseteq S_u) &\subseteq (\forall Sa.\{>\}.\forall Qa \neq \emptyset)(\forall Sa.\{<\}.\forall Qa \neq \emptyset) & (T_{R227})(\text{Eqn1}) \\
(\forall Sa.\{>\}.\forall Qa \neq \emptyset)(\forall Sa.\{<\}.\forall Qa \neq \emptyset) &\subseteq (\forall Sa.\{>\}.\forall Qa + \forall Sa.\{<\}.\forall Qa \neq \emptyset) & (T_{M312})(\text{Probabilistic Conjunction})(\text{Eqn2}) \\
(a^1 \not\subseteq S_u) &\subseteq (\forall Sa.\{>\}.\forall Qa + \forall Sa.\{<\}.\forall Qa \neq \emptyset) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\
(a^1 \not\subseteq S_u) &\subseteq (\forall Sa.(\{>\} + \{<\}).\forall Qa \neq \emptyset) & (T_{M407})(\text{Distribution of } . \text{ over } +)
\end{aligned}$$

$$(a^1 \not\subseteq S_u) \subseteq (a.\{ \neq \} * a \neq \emptyset) \quad (T_{S639})$$

$$(a^1 \not\subseteq S_u) \subseteq (a.\{ \neq \} * a \neq \emptyset) \quad (T_{S639_A})$$

Example Proof of (T_{S639})

$$\begin{aligned}
(a^1 \not\subseteq S_u) &\subseteq (a.\{>\} * a \neq \emptyset)(a.\{<\} * a \neq \emptyset) & (T_{S648})(\text{Eqn1}) \\
\{>\} &\subseteq \{ \neq \} & (T_{S642}) \\
\{>\} * a &\subseteq \{ \neq \} * a & (T_{R197})(\text{Relational Deduction}) \\
a.\{>\} * a &\subseteq a.\{ \neq \} * a & (T_{M029})(\text{Deduction}) \\
(a.\{>\} * a \subseteq a.\{ \neq \} * a) &= E & (A_{MX16})(\text{Order Axiom}) \\
(a.\{>\} * a \neq \emptyset) &\subseteq (a.\{ \neq \} * a \neq \emptyset) & (T_{M096})(\text{Probabilistic Rule})(\text{Eqn2}) \\
\{<\} &\subseteq \{ \neq \} & (T_{S643}) \\
\{<\} * a &\subseteq \{ \neq \} * a & (T_{R197})(\text{Relational Deduction}) \\
a.\{<\} * a &\subseteq a.\{ \neq \} * a & (T_{M029})(\text{Deduction}) \\
(a.\{<\} * a \subseteq a.\{ \neq \} * a) &= E & (A_{MX16})(\text{Order Axiom}) \\
(a.\{<\} * a \neq \emptyset) &\subseteq (a.\{ \neq \} * a \neq \emptyset) & (T_{M096})(\text{Probabilistic Rule})(\text{Eqn3}) \\
(a.\{>\} * a \neq \emptyset)(a.\{<\} * a \neq \emptyset) &\subseteq (a.\{ \neq \} * a \neq \emptyset)(a.\{ \neq \} * a \neq \emptyset) & (T_{M669})(\text{Combination of Eqns2\&3}) \\
(a.\{>\} * a \neq \emptyset)(a.\{<\} * a \neq \emptyset) &\subseteq (a.\{ \neq \} * a \neq \emptyset) & (T_{M667})(\text{Conjunction Tautology})(\text{Eqn4}) \\
(a^1 \not\subseteq S_u) &\subseteq (a.\{ \neq \} * a \neq \emptyset) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&4})
\end{aligned}$$

$$(a^1 \subseteq S_c.S_u) \subseteq (\forall Sa.\forall Qa \subseteq \{=\} + \{>\} + \{<\} + \{\parallel\} + \{\neq\}\{\sim\}\{\sim\}\{\parallel\}) \quad (T_{R723})$$

$$(a^1 \subseteq S_c.S_u) \subseteq (\forall Sa.\forall Qa \subseteq \{=\} + \{>\} + \{<\} + \{\parallel\} + \{\neq\}\{\sim\}\{\sim\}\{\parallel\}) \quad (T_{R723_A})$$

Example Proof of (T_{R723})

$$\begin{aligned}
 (a^1 \subseteq S_u) &\subseteq (\forall Sa. \forall Qa \subseteq \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\} \{\sim\} \{\neg\} \{\times\}) & (T_{S653})(\text{Eqn1}) \\
 \text{Let } a^1 &\subseteq S_c.S_u & (\text{Eqn2}) \\
 S_c.S_u &\subseteq S_u & (T_{M083})(\text{Conjunction Elimination})(\text{Eqn3}) \\
 a^1 &\subseteq S_u & (T_{M139})(\text{Hypothetical Syllogism Eqns2\&3}) \\
 \text{Thus } (a^1 &\subseteq S_c.S_u) \subseteq (a^1 \subseteq S_u) & (A_{MX02})(\text{Conditional Introduction})(\text{Eqn4}) \\
 (a^1 \subseteq S_c.S_u) &\subseteq (\forall Sa. \forall Qa \subseteq \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\} \{\sim\} \{\neg\} \{\times\}) & (T_{M139})(\text{Hypothetical Syllogism Eqns4\&1})
 \end{aligned}$$

$$(a^1 \subseteq S_d.S_u) \subseteq (\forall Sa. \forall Qa \subseteq \{=\} + \{\neq\}) \quad (T_{R721})$$

$$(a^1 \subseteq S_d.S_u) \subseteq (\forall Sa. \forall Qa \subseteq \{=\} + \{\neq\}) \quad (T_{R721A})$$

Example Proof of (T_{R721})

$$\begin{aligned}
 (a^1 \subseteq S_u) &\subseteq (\forall Sa. \forall Qa \subseteq \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\} \{\sim\} \{\neg\} \{\times\}) & (T_{S653})(\text{Eqn1}) \\
 (a^1 \subseteq S_d) &\subseteq (\forall Sa. \forall Qa \subseteq \{\neg\} \{\sim\} \{\times\}) & (T_{S656})(\text{Eqn2}) \\
 (a^1 \subseteq S_d)(a^1 \subseteq S_u) &\subseteq (\forall Sa. \forall Qa \subseteq \{\neg\} \{\sim\} \{\times\})(\forall Sa. \forall Qa \subseteq \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\} \{\sim\} \{\neg\} \{\times\}) & (T_{M669})(\text{Combination of Eqns1\&2}) \\
 (a^1 \subseteq S_d.S_u) &\subseteq (\forall Sa. \forall Qa. \subseteq (\{\neg\} \{\sim\} \{\times\}).(\{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\} \{\sim\} \{\neg\} \{\times\})) & (T_{M132})(\text{Composition}) \\
 (a^1 \subseteq S_d.S_u) &\subseteq (\forall Sa. \forall Qa \subseteq \{=\} \{\neg\} \{\sim\} \{\times\} + \{\supset\} \{\neg\} \{\sim\} \{\times\} + \{\subset\} \{\neg\} \{\sim\} \{\times\} + \{\parallel\} \{\neg\} \{\sim\} \{\times\} + \{\neq\} \{\sim\} \{\neg\} \{\times\} \{\neg\} \{\sim\} \{\times\}) & (T_{M407})(\text{Distribution of } \cdot \text{ over } +) \\
 (a^1 \subseteq S_d.S_u) &\subseteq (\forall Sa. \forall Qa \subseteq \{=\} \{\neg\} \{\sim\} \{\times\} + \{\supset\} \{\neg\} \{\sim\} \{\times\} + \{\subset\} \{\neg\} \{\sim\} \{\times\} + \{\parallel\} \{\neg\} \{\sim\} \{\times\} + \{\neq\} \{\sim\} \{\neg\} \{\times\}) & (T_{M667})(\text{Conjunction Tautology}) \\
 (a^1 \subseteq S_d.S_u) &\subseteq (\forall Sa. \forall Qa \subseteq \{=\} \{\neg\} \{\sim\} \{\times\} + \emptyset \{\neg\} \{\sim\} \{\times\} + \emptyset \{\sim\} \{\neg\} \{\times\} + \emptyset \{\neg\} \{\sim\} \{\times\} + \{\neq\} \{\sim\} \{\neg\} \{\times\}) & (T_{M621})(\text{Complement Conjunction}) \\
 (a^1 \subseteq S_d.S_u) &\subseteq (\forall Sa. \forall Qa \subseteq \{=\} \{\neg\} \{\sim\} \{\times\} + \emptyset + \emptyset + \emptyset + \{\neq\} \{\sim\} \{\neg\} \{\times\}) & (T_{M050})(\text{Null Set Conjunction}) \\
 (a^1 \subseteq S_d.S_u) &\subseteq (\forall Sa. \forall Qa \subseteq \{=\} \{\neg\} \{\sim\} \{\times\} + \{\neq\} \{\sim\} \{\neg\} \{\times\}) & (T_{M668})(\text{Null Set Disjunction}) \\
 (a^1 \subseteq S_d.S_u) &\subseteq (\forall Sa. \forall Qa \subseteq (\{\neg\} \{\sim\} \{\times\}).(\{=\} + \{\neq\})) & (T_{M407})(\text{Distribution of } \cdot \text{ over } +) \\
 (a^1 \subseteq S_d.S_u) &\subseteq (\forall Sa. \forall Qa \subseteq \{=\} + \{\neq\}) & (T_{M073})(\text{Single Premise Syllogism})
 \end{aligned}$$

11.4 Proofs of Theorems of Ordered Sets

A. Relationships

$$\forall Sa. \forall Qa \subseteq \{\supset\} + \{\subset\} + \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\} \{\sim\} \{\neg\} \{\times\} \quad (T_{S651})$$

Proof of (T_{S651})

$$\begin{aligned}
 \forall Sa. \forall Qa &\subseteq E & (T_{M744})(\text{Universal Set Theorem})(\text{Eqn1}) \\
 c + c &= E & (T_{M671})(\text{Tertium non Datur})(\text{Eqn2}) \\
 \forall Sa. \forall Qa &\subseteq c + c & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\
 \text{Let } c &= \{\supset\} + \{\subset\} + \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} & (\text{Eqn4}) \\
 c &= \sim(\{\supset\} + \{\subset\} + \{=\} + \{\supset\} + \{\subset\} + \{\parallel\}) & (T_{M008})(\text{Inverse of an Equality}) \\
 c &= \{\neq\} \{\sim\} \{\neg\} \{\times\} & (T_{M188})(\text{DeMorgan's Law for Series})(\text{Eqn5}) \\
 \forall Sa. \forall Qa &\subseteq \{\supset\} + \{\subset\} + \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\} \{\sim\} \{\neg\} \{\times\} & (A_{MX19})(\text{Substitution of Eqns4\&5 into Eqn3})
 \end{aligned}$$

$$\forall Sa. \forall Qa = \forall Sa. (\{\supset\} + \{\subset\} + \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\} \{\sim\} \{\neg\} \{\times\}). \forall Qa \quad (T_{S657})$$

Proof of (T_{S657})

$$\begin{aligned}
 \forall Sa. \forall Qa &\subseteq \{\supset\} + \{\subset\} + \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\} \{\sim\} \{\neg\} \{\times\} & (T_{S651}) \\
 \forall Sa. \forall Qa &= \forall Sa. (\{\supset\} + \{\subset\} + \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\} \{\sim\} \{\neg\} \{\times\}). \forall Qa & (A_{MX44})(\subseteq \text{Equality Form})
 \end{aligned}$$

$${}^v\text{Sa}.(\{>\} + \{<\} + \{\supset\} + \{\subset\} + \{\parallel\}).{}^v\text{Qa} \subseteq {}^v\text{Sa}. \{#\}.{}^v\text{Qa} \quad (T_{S660})$$

Proof of (T_{S660})

$$\begin{aligned} {}^v\text{Sa}.{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \\ &\quad (T_{S657}) \\ {}^v\text{Sa}. \{#\}.{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}). \{#\}.{}^v\text{Qa} \\ &\quad (T_{M233})(\text{Deduction}) \\ {}^v\text{Sa}. \{#\}.{}^v\text{Qa} &= \\ {}^v\text{Sa}.(\{>\}. \{#\} + \{<\}. \{#\} + \{=\}. \{#\} + \{\supset\}. \{#\} + \{\subset\}. \{#\} + \{\parallel\}. \{#\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \\ &\quad (T_{M407})(\text{Distribution of } . \text{ over } +) \\ {}^v\text{Sa}. \{#\}.{}^v\text{Qa} &= \\ {}^v\text{Sa}.(\{>\}. \{#\} + \{<\}. \{#\} + \{=\}. \{#\} + \{\supset\}. \{#\} + \{\subset\}. \{#\} + \{\parallel\}. \{#\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \\ &\quad (T_{M667})(\text{Conjunction Tautology}) \\ {}^v\text{Sa}. \{#\}.{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \{=\}. \{#\} + \{\supset\}. \{#\} + \{\subset\}. \{#\} + \{\parallel\}. \{#\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \\ &\quad (T_{S689}) \& (T_{S685}) \\ {}^v\text{Sa}. \{#\}.{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \emptyset + \{\supset\}. \{#\} + \{\subset\}. \{#\} + \{\parallel\}. \{#\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \\ &\quad (T_{M621})(\text{Complement Conjunction}) \\ {}^v\text{Sa}. \{#\}.{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \{\supset\}. \{#\} + \{\subset\}. \{#\} + \{\parallel\}. \{#\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \\ &\quad (T_{M668})(\text{Null Set Disjunction}) \\ {}^v\text{Sa}. \{#\}.{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \\ &\quad (T_{A104}), (T_{A103}), (T_{A105}) \& (A_{MX44})(\subseteq \text{Equality Form}) \\ {}^v\text{Sa}. \{#\}.{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \{\supset\} + \{\subset\} + \{\parallel\}).{}^v\text{Qa} + {}^v\text{Sa}.(\{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \\ &\quad (T_{M407})(\text{Distribution of } . \text{ over } +) \\ {}^v\text{Sa}.(\{>\} + \{<\} + \{\supset\} + \{\subset\} + \{\parallel\}).{}^v\text{Qa} &\subseteq {}^v\text{Sa}. \{#\}.{}^v\text{Qa} \\ &\quad (T_{M072b})(\text{Single Premise Syllogism}) \end{aligned}$$

$${}^v\text{Sa}.(\{>\} + \{<\} + \{\parallel\}).{}^v\text{Qa} \subseteq {}^v\text{Sa}.(\{#\}\{\sim\}\{\sim\}).{}^v\text{Qa} \quad (T_{S659})$$

Proof of (T_{S659})

$$\begin{aligned} {}^v\text{Sa}.{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \\ &\quad (T_{S657}) \\ {}^v\text{Sa}.(\{#\}\{\sim\}\{\sim\}).{}^v\text{Qa} &= \\ {}^v\text{Sa}.(\{>\} + \{<\} + \{=\} + \{\supset\} + \{\subset\} + \{\parallel\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).(\{#\}\{\sim\}\{\sim\}).{}^v\text{Qa} \\ &\quad (T_{M233})(\text{Deduction}) \\ {}^v\text{Sa}.(\{#\}\{\sim\}\{\sim\}).{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} \{#\}\{\sim\}\{\sim\} + \{<\} \{#\}\{\sim\}\{\sim\} + \{=\} \{#\}\{\sim\}\{\sim\} + \{\supset\} \{#\}\{\sim\}\{\sim\} + \\ &\quad \{\subset\} \{#\}\{\sim\}\{\sim\} + \{\parallel\} \{#\}\{\sim\}\{\sim\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}\{\sim\}\{\sim\}).{}^v\text{Qa} \\ &\quad (T_{M407})(\text{Distribution of } . \text{ over } +) \\ {}^v\text{Sa}.(\{#\}\{\sim\}\{\sim\}).{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} \{#\}\{\sim\}\{\sim\} + \{<\} \{#\}\{\sim\}\{\sim\} + \{=\} \{#\}\{\sim\}\{\sim\} + \{\supset\} \{#\}\{\sim\}\{\sim\} + \\ &\quad \{\subset\} \{#\}\{\sim\}\{\sim\} + \{\parallel\} \{#\}\{\sim\}\{\sim\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}\{\sim\}\{\sim\}).{}^v\text{Qa} \\ &\quad (T_{M667})(\text{Conjunction Tautology}) \\ {}^v\text{Sa}.(\{#\}\{\sim\}\{\sim\}).{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \{=\} \{#\}\{\sim\}\{\sim\} + \{\supset\} \{#\}\{\sim\}\{\sim\} + \{\subset\} \{#\}\{\sim\}\{\sim\} + \\ &\quad \{\parallel\} \{#\}\{\sim\}\{\sim\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}\{\sim\}\{\sim\}).{}^v\text{Qa} \quad (T_{S644}) \& (A_{MX44})(\subseteq \text{Equality Form}) \\ {}^v\text{Sa}.(\{#\}\{\sim\}\{\sim\}).{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \emptyset \{\sim\}\{\sim\} + \emptyset \{#\}\{\sim\} + \emptyset \{#\}\{\sim\} + \{\parallel\} \{#\}\{\sim\}\{\sim\} + \\ &\quad \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \quad (T_{M621})(\text{Complement Conjunction}) \\ {}^v\text{Sa}.(\{#\}\{\sim\}\{\sim\}).{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \emptyset + \emptyset + \emptyset + \{\parallel\} \{#\}\{\sim\}\{\sim\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \\ &\quad (T_{M050})(\text{Null Set Conjunction}) \\ {}^v\text{Sa}.(\{#\}\{\sim\}\{\sim\}).{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \{\parallel\} \{#\}\{\sim\}\{\sim\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \\ &\quad (T_{M668})(\text{Null Set Disjunction}) \\ {}^v\text{Sa}.(\{#\}\{\sim\}\{\sim\}).{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \{\parallel\} + \{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \\ &\quad (T_{A105}), (T_{A107}), (T_{A108}) \& (A_{MX44})(\subseteq \text{Equality Form}) \\ {}^v\text{Sa}.(\{#\}\{\sim\}\{\sim\}).{}^v\text{Qa} &= {}^v\text{Sa}.(\{>\} + \{<\} + \{\parallel\}).{}^v\text{Qa} + {}^v\text{Sa}.(\{\neq\}\{\neq\}\{\sim\}\{\sim\}\{\neq\}\{\neq\}).{}^v\text{Qa} \\ &\quad (T_{M407})(\text{Distribution of } . \text{ over } +) \\ {}^v\text{Sa}.(\{>\} + \{<\} + \{\parallel\}).{}^v\text{Qa} &\subseteq {}^v\text{Sa}.(\{#\}\{\sim\}\{\sim\}).{}^v\text{Qa} \\ &\quad (T_{M072b})(\text{Single Premise Syllogism}) \end{aligned}$$

$${}^v\text{Sa}.(\{>\} + \{<\}).{}^v\text{Qa} \subseteq {}^v\text{Sa}.(\{#\}\{\sim\}\{\sim\}).{}^v\text{Qa} \quad (T_{S658})$$

Proof of (T_{S658})

$$\begin{aligned} \{>\} + \{<\} &\subseteq \{\neq\}\{\sim\supset\}\{\sim\subset\} && (T_{S644}) \\ \forall Sa.(\{>\} + \{<\}).\forall Qa \subseteq \forall Sa.(\{\neq\}\{\sim\supset\}\{\sim\subset\}).\forall Qa && (T_{M233})(\text{Deduction}) \end{aligned}$$

$$\forall Sa.(\{>\} + \{<\})\{\neq\}.\forall Qa \subseteq \forall Sa.(\{\neq\}\{\sim\supset\}\{\sim\subset\}\{\neq\}).\forall Qa \quad (T_{S676})$$

Proof of (T_{S676})

$$\begin{aligned} \{>\} + \{<\} &\subseteq \{\neq\}\{\sim\supset\}\{\sim\subset\} && (T_{S644}) \\ \forall Sa.(\{>\} + \{<\})\{\neq\}.\forall Qa \subseteq \forall Sa.(\{\neq\}\{\sim\supset\}\{\sim\subset\}\{\neq\}).\forall Qa && (T_{M233})(\text{Deduction}) \end{aligned}$$

B. General Ordered Sets

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (a = \{\supset\} * a^1) \quad (T_{760})$$

Proof of (T₇₆₀)

$$\begin{aligned} (a^1 \subseteq S_A) &\subseteq (a = \{\supset\} * a^1) && (D_{S28})(\text{Definition of Reverse Abstraction for Members of } S_A) \\ (a^1 \subseteq S_M + S_D) &\subseteq (a = \{\supset\} * a^1) && (D_{S35})(\text{Definition of } S_A)(\text{Eqn1}) \\ (S_M + S_D) S_O &\subseteq S_M + S_D && (T_{M083})(\text{Conjunction Elimination}) \\ S_M S_O + S_D S_O &\subseteq S_M + S_D && (T_{M407})(\text{Distribution of } . \text{ over } +) \\ (S_M S_O + S_D S_O \subseteq S_M + S_D) &= E && (A_{MX16})(\text{Order Axiom}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (a^1 \subseteq S_M + S_D) && (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn2}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (a = \{\supset\} * a^1) && (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$(a^1 \subseteq S_O) \subseteq (a \sim a^1 \neq \emptyset) \quad (T_{S701})$$

Proof of (T_{S701})

$$\begin{aligned} (a^1 \subseteq S_O) &\subseteq (\forall S(a \sim a^1).(\{>\} + \{<\}).\{\neq\}.\forall Q(a \sim a^1) \neq \emptyset) && (A_{R33})(\text{Axiom of Ordered Sets})(\text{Eqn1}) \\ \forall S(a \sim a^1).(\{>\} + \{<\}).\{\neq\}.\forall Q(a \sim a^1) &\subseteq \forall S(a \sim a^1) && (T_{M083})(\text{Conjunction Elimination}) \\ (\forall S(a \sim a^1).(\{>\} + \{<\}).\{\neq\}.\forall Q(a \sim a^1) \subseteq \forall S(a \sim a^1)) &= E && (A_{MX16})(\text{Order Axiom}) \\ (\forall S(a \sim a^1).(\{>\} + \{<\}).\{\neq\}.\forall Q(a \sim a^1) \neq \emptyset) &\subseteq (\forall S(a \sim a^1) \neq \emptyset) && (T_{M096})(\text{Probabilistic Rule})(\text{Eqn2}) \\ (a^1 \subseteq S_O) &\subseteq (\forall S(a \sim a^1) \neq \emptyset) && (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\ (a^1 \subseteq S_O) &\subseteq (a \sim a^1 \neq \emptyset) && (T_{S200})(\text{Matching Theorem}) \end{aligned}$$

$$(a^1 \subseteq S_O) \subseteq (a \neq \emptyset) \quad (T_{S702})$$

Proof of (T_{S702})

$$\begin{aligned} (a^1 \subseteq S_O) &\subseteq (a \sim a^1 \neq \emptyset) && (T_{S701})(\text{Eqn1}) \\ a \sim a^1 &\subseteq a && (T_{M083})(\text{Conjunction Elimination}) \\ (a \sim a^1 \subseteq a) &= E && (A_{MX16})(\text{Order Axiom}) \\ (a \sim a^1 \neq \emptyset) &\subseteq (a \neq \emptyset) && (T_{M096})(\text{Probabilistic Rule})(\text{Eqn2}) \\ (a^1 \subseteq S_O) &\subseteq (a \neq \emptyset) && (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

Alternative Proof of (T_{S702})

$$\begin{aligned} (a^1 \subseteq S_O) &\subseteq (a \sim a^1 \neq \emptyset) && (T_{S701})(\text{Eqn1}) \\ (a \sim a^1 \neq \emptyset) &\subseteq (a \neq \emptyset)(\sim a^1 \neq \emptyset) && (T_{M242}) \\ (a \sim a^1 \neq \emptyset) &\subseteq (a \neq \emptyset) && (T_{M073})(\text{Single Premise Syllogism})(\text{Eqn2}) \\ (a^1 \subseteq S_O) &\subseteq (a \neq \emptyset) && (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$(a^1 \subseteq S_O) \subseteq (a^1 \neq \emptyset) \quad (T_{S638})$$

Proof of (T_{S638})

$$(a^1 \subseteq S_O) \subseteq (a \neq \emptyset) \quad (T_{S702})(\text{Eqn1})$$

$$\begin{aligned} (a \neq \emptyset) &\subseteq (a^1 \neq \emptyset) & (T_{M503})(\text{Eqn2}) \\ (a^1 \subseteq S_0) &\subseteq (a^1 \neq \emptyset) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$(a^1 \subseteq S_0) \subseteq (a \sim a^1 \subseteq \{\neq\} * a \sim a^1) \quad (T_{S700})$$

Proof of (T_{S700})

$$\begin{aligned} (a^1 \subseteq S_0) &= (a \sim a^1 \subseteq (\{>\} * a \sim a^1) + (\{<\} * a \sim a^1)) & (D_{R60})(\text{Definition of an Ordered Set})(\text{Eqn1}) \\ \{>\} + \{<\} &\subseteq \{\neq\} \{\sim\} \{\neg\} & (T_{S644}) \\ \{>\} + \{<\} &\subseteq \{\neq\} & (T_{M073})(\text{Single Premise Syllogism}) \\ (\{>\} + \{<\}) * a \sim a^1 &\subseteq \{\neq\} * a \sim a^1 & (T_{R197})(\text{Relational Deduction}) \\ \{>\} * a \sim a^1 + \{<\} * a \sim a^1 &\subseteq \{\neq\} * a \sim a^1 & (T_{R002c})(\text{Distribution of } + \text{ over } *) (\text{Eqn2}) \\ (a^1 \subseteq S_0) &\subseteq (a \sim a^1 \subseteq \{\neq\} * a \sim a^1) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$(a^1 \subseteq S_0) \subseteq (a \sim a^1 \subseteq \{\neq\} * a) \quad (T_{S673})$$

Proof of (T_{S673})

$$\begin{aligned} (a^1 \subseteq S_0) &\subseteq (a \sim a^1 \subseteq \{\neq\} * a \sim a^1) & (T_{S700})(\text{Eqn1}) \\ a \sim a^1 &\subseteq a & (T_{M083})(\text{Conjunction Elimination}) \\ \{\neq\} * a \sim a^1 &\subseteq \{\neq\} * a & (T_{R196})(\text{Relational Deduction})(\text{Eqn2}) \text{ since} \\ (a^1 \subseteq S_0) &\subseteq (a \sim a^1 \subseteq \{\neq\} * a) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$(a^1 \subseteq S_0) \subseteq (a \cdot \{\neq\} * a \neq \emptyset) \quad (T_{S703})$$

Proof of (T_{S703})

$$\begin{aligned} (a^1 \subseteq S_0) &\subseteq (a \sim a^1 \subseteq \{\neq\} * a) & (T_{S673})(\text{Eqn1}) \\ ((a \sim a^1 \neq \emptyset) = E) &\subseteq ((a \sim a^1 \subseteq \{\neq\} * a) \subseteq (a \sim a^1 \not\subseteq \{\neq\} * a)) & (T_{M175})(\text{Negation Introduction})(\text{Eqn2}) \\ ((a \sim a^1 \neq \emptyset) = E) &\subseteq ((a^1 \subseteq S_0) \subseteq (a \sim a^1 \not\subseteq \{\neq\} * a)) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\ ((a \sim a^1 \neq \emptyset) = E) &\subseteq ((a^1 \subseteq S_0) \subseteq (a \sim a^1 \cdot \{\neq\} * a \neq \emptyset)) & (T_{M735})(\subseteq \text{Existential Form}) \& (T_{M008})(\text{Inverse of an Equality}) \\ (a^1 \subseteq S_0)(a \sim a^1 \neq \emptyset) &\subseteq (a \sim a^1 \cdot \{\neq\} * a \neq \emptyset) & (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets}(\text{Eqn3}) \\ (a^1 \subseteq S_0) &\subseteq (a \sim a^1 \neq \emptyset) & (T_{S701}) \\ (a^1 \subseteq S_0) &= (a^1 \subseteq S_0)(a \sim a^1 \neq \emptyset) & (T_{M521})(\subseteq \text{Equality Form})(\text{Eqn4}) \\ (a^1 \subseteq S_0) &\subseteq (a \sim a^1 \cdot \{\neq\} * a \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})(\text{Eqn5}) \\ a \sim a^1 \cdot \{\neq\} * a &\subseteq a \cdot \{\neq\} * a & (T_{M083})(\text{Conjunction Elimination}) \\ (a \sim a^1 \cdot \{\neq\} * a \subseteq a \cdot \{\neq\} * a) &= E & (A_{MX16})(\text{Order Axiom}) \\ (a \sim a^1 \cdot \{\neq\} * a \neq \emptyset) &\subseteq (a \cdot \{\neq\} * a \neq \emptyset) & (T_{M096})(\text{Probabilistic Rule})(\text{Eqn6}) \\ (a^1 \subseteq S_0) &\subseteq (a \cdot \{\neq\} * a \neq \emptyset) & (T_{M139})(\text{Hypothetical Syllogism Eqns5\&6}) \end{aligned}$$

$$(a^1 \subseteq S_0) \subseteq (a \neq a^1)(a \neq \emptyset) \quad (T_{S641})$$

Proof of (T_{S641})

$$\begin{aligned} (a^1 \subseteq S_0) &\subseteq (a \cdot \{\neq\} * a \neq \emptyset) & (T_{S703}) \\ (a^1 \subseteq S_0) &\subseteq (a \neq a^1)(a \neq \emptyset) & (A_{R37})(\text{Axiom of Plural Sets}) \end{aligned}$$

$$(a^1 \subseteq S_0) \subseteq (a \neq \emptyset^1) \quad (T_{T691})$$

Proof of (T_{T691})

$$\begin{aligned} (a^1 \subseteq S_0) &\subseteq (a \neq a^1)(a \neq \emptyset) & (T_{S641})(\text{Eqn1}) \\ \text{Let } a &= \emptyset^1 & (\text{Eqn2}) \\ a^1 &= \emptyset^1 & (T_{M499})(\text{Eqn3}) \\ (a^1 \subseteq S_0) &\subseteq (\emptyset^1 \neq \emptyset^1)(\emptyset^1 \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})(\text{Eqn4}) \end{aligned}$$

$\emptyset^1 = \emptyset^1$	(D_{M00})(Definition of =)
$(\emptyset^1 = \emptyset^1) = E$	($AMX16$)(Order Axiom)
$(\emptyset^1 \neq \emptyset^1) = \sim E$	(T_{M008})(Inverse of an Equality)
$(\emptyset^1 \neq \emptyset^1) = \emptyset$	(D_{M02})(Definition of $\emptyset$)(Eqn5)
$(a^1 \subseteq S_0) \subseteq \emptyset.(\emptyset^1 \neq \emptyset)$	($AMX19$)(Substitution of Eqn5 into Eqn4)
$(a^1 \subseteq S_0) \subseteq \emptyset$	(T_{M050})(Null Set Conjunction)
$(a^1 \subseteq S_0) = \emptyset$	(T_{M052})(Subsets of the Null Set)
$(a^1 \not\subseteq S_0) = \sim \emptyset$	(T_{M008})(Inverse of an Equality)
$(a^1 \not\subseteq S_0) = E$	(T_{M054})(Negative of Null Set)
$a^1 \not\subseteq S_0$	($AMX16$)(Order Axiom)
Thus $(a = \emptyset^1) \subseteq (a^1 \not\subseteq S_0)$	($AMX02$)(Conditional Introduction)
$(a^1 \subseteq S_0) \subseteq (a \neq \emptyset^1)$	(T_{M012})(Transposition)

$$(a^1 \subseteq S_0) \subseteq (a^1 \neq \emptyset^1) \quad (T_{T692})$$

Proof of (T_{T692})

$(a^1 \subseteq S_0) \subseteq (a \neq a^1)(a \neq \emptyset)$	(T_{S641})(Eqn1)
Let $a^1 = \emptyset^1$	(Eqn2)
$(a^1 \subseteq S_0) \subseteq (a \neq \emptyset^1)(a \neq \emptyset)$	($AMX19$)(Substitution of Eqn2 into Eqn1)
$(a^1 \subseteq S_0) \subseteq (a^1 \neq \emptyset^1)(a^1 \neq \emptyset)$	(T_{M618})(Eqn3)
$(a^1 \subseteq S_0) \subseteq (\emptyset^1 \neq \emptyset^1)(\emptyset^1 \neq \emptyset)$	($AMX19$)(Substitution of Eqn2 into Eqn3)(Eqn4)
$\emptyset^1 = \emptyset^1$	(D_{M00})(Definition of =)
$(\emptyset^1 = \emptyset^1) = E$	($AMX16$)(Order Axiom)
$(\emptyset^1 \neq \emptyset^1) = \sim E$	(T_{M008})(Inverse of an Equality)
$(\emptyset^1 \neq \emptyset^1) = \emptyset$	(D_{M02})(Definition of $\emptyset$)(Eqn5)
$(a^1 \subseteq S_0) \subseteq \emptyset(\emptyset^1 \neq \emptyset)$	($AMX19$)(Substitution of Eqn5 into Eqn4)
$(a^1 \subseteq S_0) \subseteq \emptyset$	(T_{M050})(Null Set Conjunction)
$(a^1 \subseteq S_0) = \emptyset$	(T_{M052})(Subsets of the Null Set)
$(a^1 \not\subseteq S_0) = \sim \emptyset$	(T_{M008})(Inverse of an Equality)
$(a^1 \not\subseteq S_0) = E$	(T_{M054})(Negative of Null Set)
$a^1 \not\subseteq S_0$	($AMX16$)(Order Axiom)
Thus $(a^1 = \emptyset^1) \subseteq (a^1 \not\subseteq S_0)$	($AMX02$)(Conditional Introduction)
$(a^1 \subseteq S_0) \subseteq (a^1 \neq \emptyset^1)$	(T_{M012})(Transposition)

$$(a^1 \subseteq S_0) \subseteq (\forall Sa.\{ \neq \} \{ \sim \} \{ \sim \} \{ \parallel \} . \forall Qa \subseteq \{ > \} + \{ < \}) \quad (T_{S652})$$

Proof of (T_{S652})

$(a^1 \subseteq S_0) = (\forall Sa.\forall Qa \subseteq \{ > \} + \{ < \} + \{ = \} + \{ \supset \} + \{ \subset \} + \{ \parallel \})$	(D_{R99})(Definition of an Ordered Set Part 2)
$(a^1 \subseteq S_0) \subseteq (\forall Sa.\{ \neq \} \{ \sim \} \{ \sim \} \{ \parallel \} . \forall Qa \subseteq \{ > \} + \{ < \})$	(T_{M468})

$$(a^1 \subseteq S_0) \subseteq (a \subseteq \{ > \} * a + \{ < \} * a + \{ = \} * a + \{ \supset \} * a + \{ \subset \} * a + \{ \parallel \} * a) \quad (T_{S650})$$

Proof of (T_{S650})

$(a^1 \subseteq S_0) \subseteq (\forall Sa.\forall Qa \subseteq \{ > \} + \{ < \} + \{ = \} + \{ \supset \} + \{ \subset \} + \{ \parallel \})$	(D_{R99})(Definition of an Ordered Set Part 2)(Eqn1)
$(\forall Sa.\forall Qa \subseteq \{ > \} + \{ < \} + \{ = \} + \{ \supset \} + \{ \subset \} + \{ \parallel \}) \subseteq$	
$(\forall Sa.\forall Qa.\forall Qa \subseteq (\{ > \} + \{ < \} + \{ = \} + \{ \supset \} + \{ \subset \} + \{ \parallel \}).\forall Qa)$	(T_{M029})(Deduction)
$(\forall Sa.\forall Qa \subseteq \{ > \} + \{ < \} + \{ = \} + \{ \supset \} + \{ \subset \} + \{ \parallel \}) \subseteq$	
$(\forall Sa.\forall Qa \subseteq (\{ > \} + \{ < \} + \{ = \} + \{ \supset \} + \{ \subset \} + \{ \parallel \}).\forall Qa)$	(T_{M667})(Conjunction Tautology)
$(\forall Sa.\forall Qa \subseteq \{ > \} + \{ < \} + \{ = \} + \{ \supset \} + \{ \subset \} + \{ \parallel \}) \subseteq$	
$(\forall Sa.\forall Qa \subseteq \{ > \}.\forall Qa + \{ < \}.\forall Qa + \{ = \}.\forall Qa + \{ \supset \}.\forall Qa + \{ \subset \}.\forall Qa + \{ \parallel \}.\forall Qa)$	(T_{M407})(Distribution of . over +)(Eqn2)
$(a^1 \subseteq S_0) \subseteq (\forall Sa.\forall Qa \subseteq \{ > \}.\forall Qa + \{ < \}.\forall Qa + \{ = \}.\forall Qa + \{ \supset \}.\forall Qa + \{ \subset \}.\forall Qa + \{ \parallel \}.\forall Qa)$	(T_{M139})(Hypothetical Syllogism Eqns1&2)(Eqn3)

$$\begin{aligned}
& (\forall Sa. \forall Qa \subseteq \{>\}. \forall Qa + \{<\}. \forall Qa + \{=\}. \forall Qa + \{\supset\}. \forall Qa + \{\subset\}. \forall Qa + \{\|\}. \forall Qa) \subseteq \\
& (S(\forall Sa. \forall Qa) \subseteq S(\{>\}. \forall Qa + \{<\}. \forall Qa + \{=\}. \forall Qa + \{\supset\}. \forall Qa + \{\subset\}. \forall Qa + \{\|\}. \forall Qa)) \\
& \quad (T_{R196})(\text{Relational Deduction})(\text{Eqn4}) \\
& (a^1 \subseteq S_0) \subseteq (S(\forall Sa. \forall Qa) \subseteq S(\{>\}. \forall Qa + \{<\}. \forall Qa + \{=\}. \forall Qa + \{\supset\}. \forall Qa + \{\subset\}. \forall Qa + \{\|\}. \forall Qa)) \\
& \quad (T_{M139})(\text{Hypothetical Syllogism Eqns3\&4}) \\
& (a^1 \subseteq S_0) \subseteq \\
& (S(\forall Sa. \forall Qa) \subseteq S(\{>\}. \forall Qa) + S(\{<\}. \forall Qa) + S(\{=\}. \forall Qa) + S(\{\supset\}. \forall Qa) + S(\{\subset\}. \forall Qa) + S(\{\|\}. \forall Qa)) \\
& \quad (T_{S291})(\text{Distribution of } S^* \text{ over } +)(\text{Eqn5}) \\
& (a \neq \emptyset) + (a = \emptyset) = (S(\forall Sa \forall Qa) = a) \\
& \quad (T_{S248}) \\
& E = (S(\forall Sa \forall Qa) = a) \\
& \quad (T_{M671})(\text{Tertium non Datur}) \\
& S(\forall Sa \forall Qa) = a \\
& \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn6}) \\
& (a^1 \subseteq S_0) \subseteq (a \subseteq S(\{>\}. \forall Qa) + S(\{<\}. \forall Qa) + S(\{=\}. \forall Qa) + S(\{\supset\}. \forall Qa) + S(\{\subset\}. \forall Qa) + S(\{\|\}. \forall Qa)) \\
& \quad (A_{MX19})(\text{Substitution of Eqn6 into Eqn5}) \\
& (a^1 \subseteq S_0) \subseteq (a \subseteq \{>\} * a + \{<\} * a + \{=\} * a + \{\supset\} * a + \{\subset\} * a + \{\|\} * a) \\
& \quad (A_{R27})(\text{Function Axiom})
\end{aligned}$$

$$(a^1 \subseteq S_0) \subseteq (\forall Sa. (\{>\} + \{<\}). \{\|\}. \forall Qa \neq \emptyset) \quad (T_{S704})$$

Proof of (T_{S704})

$$\begin{aligned}
& (a^1 \subseteq S_0) \subseteq (\forall S(a. \sim a^1). (\{>\} + \{<\}). \{\|\}. \forall Q(a. \sim a^1) \neq \emptyset) \quad (A_{R33})(\text{Axiom of Ordered Sets})(\text{Eqn1}) \\
& a. \sim a^1 \subseteq a \quad (T_{M083})(\text{Conjunction Elimination}) \\
& \forall S(a. \sim a^1) \subseteq \forall Sa \quad (T_{R196})(\text{Relational Deduction})(\text{Eqn2}) \\
& \forall Q(a. \sim a^1) \subseteq \forall Qa \quad (T_{R196})(\text{Relational Deduction})(\text{Eqn3}) \\
& \forall S(a. \sim a^1). \forall Q(a. \sim a^1) \subseteq \forall Sa. \forall Qa \quad (T_{M669})(\text{Combination of Eqns2\&3}) \\
& \forall S(a. \sim a^1). (\{>\} + \{<\}). \{\|\}. \forall Q(a. \sim a^1) \subseteq \forall Sa. (\{>\} + \{<\}). \{\|\}. \forall Qa \quad (T_{M029})(\text{Deduction}) \\
& (\forall S(a. \sim a^1). (\{>\} + \{<\}). \{\|\}. \forall Q(a. \sim a^1) \subseteq \forall Sa. (\{>\} + \{<\}). \{\|\}. \forall Qa) = E \\
& \quad (A_{MX16})(\text{Order Axiom}) \\
& (\forall S(a. \sim a^1). (\{>\} + \{<\}). \{\|\}. \forall Q(a. \sim a^1) \neq \emptyset) \subseteq (\forall Sa. (\{>\} + \{<\}). \{\|\}. \forall Qa \neq \emptyset) \\
& \quad (T_{M096})(\text{Probabilistic Rule})(\text{Eqn4}) \\
& (a^1 \subseteq S_0) \subseteq (\forall Sa. (\{>\} + \{<\}). \{\|\}. \forall Qa \neq \emptyset) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&4})
\end{aligned}$$

$$(a^1 \subseteq S_0) \subseteq (\forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa \neq \emptyset) \quad (T_{S674})$$

Proof of (T_{S674})

$$\begin{aligned}
& (a^1 \subseteq S_0) \subseteq (\forall Sa. (\{>\} + \{<\}). \{\|\}. \forall Qa \neq \emptyset) \quad (T_{S704})(\text{Eqn1}) \\
& \forall Sa. (\{>\} + \{<\}). \{\|\}. \forall Qa \subseteq \forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa \quad (T_{S676}) \\
& (\forall Sa. (\{>\} + \{<\}). \{\|\}. \forall Qa \subseteq \forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa) = E \quad (A_{MX16})(\text{Order Axiom}) \\
& (\forall Sa. (\{>\} + \{<\}). \{\|\}. \forall Qa \neq \emptyset) \subseteq (\forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa \neq \emptyset) \quad (T_{M096})(\text{Probabilistic Rule})(\text{Eqn2}) \\
& (a^1 \subseteq S_0) \subseteq (\forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa \neq \emptyset) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})
\end{aligned}$$

$$S_0 \subseteq \sim S_u \quad (T_{R706})$$

Proof of (T_{R706})

$$\begin{aligned}
& (a^1 \subseteq S_u) \subseteq (\forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa \subseteq \{>\} \{<\}) \quad (T_{S654}) \\
& (\forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa \not\subseteq \{>\} \{<\}) \subseteq (a^1 \not\subseteq S_u) \quad (T_{M012})(\text{Transposition})(\text{Eqn3}) \\
& ((\forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa \neq \emptyset) = E) \subseteq \\
& ((\forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa \subseteq \{>\} + \{<\}) \subseteq (\forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa \not\subseteq \{>\} \{<\})) \\
& \quad (T_{M175})(\text{Negation Introduction}) \\
& ((\forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa \neq \emptyset) = E) \subseteq \\
& ((a^1 \subseteq S_0) \subseteq (\forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa \not\subseteq \{>\} \{<\})) \quad (T_{S652}) \\
& (\forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa \neq \emptyset) (a^1 \subseteq S_0) \subseteq (\forall Sa. (\{ \neq \} \{ \sim \} \{ \sim \} \{ \subset \} \{ \|\} \}. \forall Qa \not\subseteq \{>\} \{<\})) \\
& \quad (A_{MX35})(\text{Exportation/Importation}) \\
& \quad \text{for truth sets (Eqn4)}
\end{aligned}$$

$(a^1 \subseteq S_0) \subseteq (\forall Sa. \{ \neq \} \{ \sim \} \{ \neg \} \{ \} \}. \forall Qa \neq \emptyset)$	(T_{S674})
$(a^1 \subseteq S_0) = (a^1 \subseteq S_0) (\forall Sa. \{ \neq \} \{ \sim \} \{ \neg \} \{ \} \}. \forall Qa \neq \emptyset) (A_{MX44}) (\subseteq \text{Equality Form}) (\text{Eqn5})$	
$(a^1 \subseteq S_0) \subseteq (\forall Sa. \{ \neq \} \{ \sim \} \{ \neg \} \{ \} \}. \forall Qa \subseteq \{ \neq \} \{ \neq \})$	$(A_{MX19}) (\text{Substitution of Eqn5 into Eqn4}) (\text{Eqn6})$
$(a^1 \subseteq S_0) \subseteq (a^1 \not\subseteq S_u)$	$(T_{M139}) (\text{Hypothetical Syllogism Eqns6\&3}) (\text{Eqn7})$
$(a^1 \not\subseteq S_u) \subseteq (a^1 \subseteq \sim S_u)$	$(T_{M604}) (\text{Eqn8})$
$(a^1 \subseteq S_0) \subseteq (a^1 \subseteq \sim S_u)$	$(T_{M139}) (\text{Hypothetical Syllogism Eqns7\&8}) (\text{Eqn9})$
Let $a_1^1 \text{ } \text{+++} \text{ } a_n^1 = E$	(Eqn10)
For $a_r^1 = a_1^1$ to a_n^1 Let $a^1 = a_r^1$	(Eqn11)
$((a_1^1 \subseteq S_0) \subseteq (a_1^1 \subseteq \sim S_u)) \dots ((a_n^1 \subseteq S_0) \subseteq (a_n^1 \subseteq \sim S_u))$	$(A_{MX19}) (\text{Substitution of Eqn11 into Eqn9})$
$((a_1^1 \subseteq S_0) \dots (a_n^1 \subseteq S_0) \subseteq (a_1^1 \subseteq \sim S_u) \dots (a_n^1 \subseteq \sim S_u))$	$(T_{M669}) (\text{Combination Axiom})$
$(a_1^1 \text{+++} a_n^1 \subseteq S_0) \subseteq (a_1^1 \text{+++} a_n^1 \subseteq \sim S_u)$	$(T_{M192}) (\text{Conjunction of Sufficiencies for a Series}) (\text{Eqn12})$
$(E \subseteq S_0) \subseteq (E \subseteq \sim S_u)$	$(A_{MX19}) (\text{Substitution of Eqn10 into Eqn12})$
$(E = S_0) \subseteq (E = \sim S_u)$	$(T_{M275}) (\text{Supersets of the Universal Set})$
$(S_0 \subseteq \sim S_u) = E$	$(T_{M094}) (\text{Probabilistic Rule})$
$S_0 \subseteq \sim S_u$	$(A_{MX16}) (\text{Order Axiom})$

C. Continuum Ordered Sets

$$(a^1 \subseteq S_c.S_0) \subseteq (\forall Sa. \forall Qa \subseteq \{=\} + \{<\} + \{>\} + \{||\} + \{<\} + \{>\}) \quad (T_{R724})$$

Proof of (T_{R724})

$(a^1 \subseteq S_0) \subseteq (\forall Sa. \forall Qa \subseteq \{>\} + \{<\} + \{=\} + \{>\} + \{<\} + \{ \})$	$(D_{R99}) (\text{Definition of an Ordered Set Part 2}) (\text{Eqn1})$
Let $a^1 \subseteq S_c.S_0$	(Eqn2)
$S_c.S_0 \subseteq S_0$	$(T_{M083}) (\text{Conjunction Elimination}) (\text{Eqn3})$
$a^1 \subseteq S_0$	$(T_{M139}) (\text{Hypothetical Syllogism Eqns2\&3})$
Thus $(a^1 \subseteq S_c.S_0) \subseteq (a^1 \subseteq S_0)$	$(A_{MX02}) (\text{Conditional Introduction}) (\text{Eqn4})$
$(a^1 \subseteq S_c.S_0) \subseteq (\forall Sa. \forall Qa \subseteq \{=\} + \{<\} + \{>\} + \{ \} + \{<\} + \{>\})$	$(T_{M139}) (\text{Hypothetical Syllogism Eqns4\&1})$

$$(a^1 \subseteq S_c.S_0) \subseteq (a \neq \emptyset) \quad (T_{S896})$$

$$(a^1 \subseteq S_c.S_0) \subseteq (a^1 \neq \emptyset) \quad (T_{S898})$$

Proof of (T_{S896}) & (T_{S898})

$(a^1 \subseteq S_c.S_0) \subseteq (\forall Sa. (\{>\} + \{<\}). \{ \}. \forall Qa \neq \emptyset)$	$(A_{R40}) (\text{Axiom of Continuum Ordered Sets})$
$(a^1 \subseteq S_c.S_0) \subseteq (a. (\{>\} + \{<\}). \{ \}) * a \neq \emptyset$	(T_{R227})
$(a^1 \subseteq S_c.S_0) \subseteq (a \neq \emptyset)$	$(T_{M350}) \& (T_{M012}) (\text{Transposition}) (\text{Eqn1})$
$(a \neq \emptyset) \subseteq (a^1 \neq \emptyset)$	$(A_{MX50}) (\text{Components of a Null Entity}) \& (T_{M012}) (\text{Transposition}) (\text{Eqn2})$
$(a^1 \subseteq S_c.S_0) \subseteq (a^1 \neq \emptyset)$	$(T_{M139}) (\text{Hypothetical Syllogism Eqns1\&2})$

$$(a^1 \subseteq S_c.S_0) (\bar{b} \subseteq a) (\bar{b} \neq \emptyset) (\bar{b} \neq \bar{b}^1) \subseteq (\bar{b}^1 \subseteq S_c.S_0) \quad (T_{T991})$$

Proof of (T_{T991})

$(a^1 \subseteq S_c) (\bar{b} \subseteq a) (\bar{b} \neq \emptyset) (\bar{b} \neq \bar{b}^1) \subseteq (\bar{b}^1 \subseteq S_c)$	(T_{T990})
$(a^1 \subseteq S_c) (\bar{b} \subseteq a) (\bar{b} \neq \emptyset) (\bar{b} \neq \bar{b}^1) \subseteq (\bar{b}^1 \subseteq S_c)$	Resymbolising (Eqn1)
$((\bar{b} \subseteq a) (\bar{b} \neq \emptyset) (\bar{b} \neq \bar{b}^1) = E) \subseteq ((a^1 \subseteq S_0) \subseteq (\bar{b}^1 \subseteq S_0))$	(T_{T971})
$(a^1 \subseteq S_0) (\bar{b} \subseteq a) (\bar{b} \neq \emptyset) (\bar{b} \neq \bar{b}^1) \subseteq (\bar{b}^1 \subseteq S_0)$	$(A_{MX35}) (\text{Exportation/Importation}) \text{ for truth sets } (\text{Eqn2})$
$(a^1 \subseteq S_c) (\bar{b} \subseteq a) (\bar{b} \neq \emptyset) (\bar{b} \neq \bar{b}^1) (a^1 \subseteq S_0) (\bar{b} \subseteq a) (\bar{b}^1 \neq \emptyset) (\bar{b} \neq \bar{b}^1) \subseteq (\bar{b}^1 \subseteq S_c) (\bar{b}^1 \subseteq S_0)$	$(T_{M669}) (\text{Combination of Eqns1\&2})$

$$(a^1 \subseteq S_c)(a^1 \subseteq S_o)(b \subseteq a)(b \neq \emptyset)(b \neq b^1) \subseteq (b^1 \subseteq S_c)(b^1 \subseteq S_o) \quad (T_{M667})(\text{Conjunction Tautology})$$

$$(a^1 \subseteq S_c.S_o)(b \subseteq a)(b \neq \emptyset)(b \neq b^1) \subseteq (b^1 \subseteq S_c.S_o) \quad (T_{M132})(\text{Composition})$$

D. Discrete Ordered Sets

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\forall S\alpha. \forall Q\alpha \subseteq \{=\} + \{<\} + \{>\}) \quad (T_{R722})$$

Proof of (T_{R722})

$$(\alpha^1 \subseteq S_o) \subseteq (\forall S\alpha. \forall Q\alpha \subseteq \{>\} + \{<\} + \{=\} + \{>\} + \{<\} + \{\parallel\}) \quad (D_{R99})(\text{Definition of an Ordered Set Part 2})$$

$$(\alpha^1 \subseteq S_o) \subseteq (\forall S\alpha. \forall Q\alpha \subseteq \{\sim\} \{>\} \{\parallel\}) \quad (Eqn1)$$

$$(\alpha^1 \subseteq S_o)(\alpha^1 \subseteq S_o) \subseteq (\forall S\alpha. \forall Q\alpha \subseteq \{\sim\} \{>\} \{\parallel\}). (\forall S\alpha. \forall Q\alpha \subseteq \{>\} + \{<\} + \{=\} + \{>\} + \{<\} + \{\parallel\}) \quad (T_{S656})(Eqn2)$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\forall S\alpha. \forall Q\alpha \subseteq (\{\sim\} \{>\} \{\parallel\}). (\{>\} + \{<\} + \{=\} + \{>\} + \{<\} + \{\parallel\})) \quad (T_{M669})(\text{Combination of Eqns1\&2})$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\forall S\alpha. \forall Q\alpha \subseteq (\{\sim\} \{>\} \{\parallel\}). (\{>\} + \{<\} + \{=\} + \{>\} + \{<\} + \{\parallel\})) \quad (T_{M132})(\text{Composition})$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\forall S\alpha. \forall Q\alpha \subseteq \{>\} \{\sim\} \{>\} \{\parallel\} + \{<\} \{\sim\} \{>\} \{\parallel\} + \{=\} \{\sim\} \{>\} \{\parallel\} + \{>\} \{\sim\} \{>\} \{\parallel\} + \{<\} \{\sim\} \{>\} \{\parallel\} + \{\parallel\} \{\sim\} \{>\} \{\parallel\}) \quad (T_{M407})(\text{Distribution of } \cdot \text{ over } +)$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\forall S\alpha. \forall Q\alpha \subseteq \{>\} \{\sim\} \{>\} \{\parallel\} + \{<\} \{\sim\} \{>\} \{\parallel\} + \{=\} \{\sim\} \{>\} \{\parallel\} + \emptyset \{\sim\} \{>\} \{\parallel\} + \emptyset \{\sim\} \{>\} \{\parallel\} + \emptyset \{\sim\} \{>\} \{\parallel\}) \quad (T_{M621})(\text{Complement Conjunction})$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\forall S\alpha. \forall Q\alpha \subseteq \{>\} \{\sim\} \{>\} \{\parallel\} + \{<\} \{\sim\} \{>\} \{\parallel\} + \{=\} \{\sim\} \{>\} \{\parallel\} + \emptyset \{\sim\} \{>\} \{\parallel\} + \emptyset \{\sim\} \{>\} \{\parallel\} + \emptyset \{\sim\} \{>\} \{\parallel\}) \quad (T_{M050})(\text{Null Set Conjunction})$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\forall S\alpha. \forall Q\alpha \subseteq \{>\} \{\sim\} \{>\} \{\parallel\} + \{<\} \{\sim\} \{>\} \{\parallel\} + \{=\} \{\sim\} \{>\} \{\parallel\}) \quad (T_{M668})(\text{Null Set Disjunction})$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\forall S\alpha. \forall Q\alpha \subseteq (\{\sim\} \{>\} \{\parallel\}). (\{>\} + \{<\} + \{=\})) \quad (T_{M407})(\text{Distribution of } \cdot \text{ over } +)$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\forall S\alpha. \forall Q\alpha \subseteq \{>\} + \{<\} + \{=\}) \quad (T_{M073})(\text{Single Premise Syllogism})$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\alpha. \{>\} * \alpha = \alpha. \{<\} \diamond \alpha) \quad (T_{S826})$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\alpha. \{<\} * \alpha = \alpha. \{>\} \diamond \alpha) \quad (T_{S827})$$

Example Proof of (T_{S826})

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\forall S\alpha. \forall Q\alpha \subseteq \{=\} + \{<\} + \{>\}) \quad (T_{R722})(Eqn1)$$

$$(\alpha^1 \subseteq S_o) \subseteq (\alpha \neq \emptyset) \quad (T_{S702}) \text{ Resymbolised}$$

$$\text{Let } \alpha = \alpha_1^1 + \dots + \alpha_n^1 \quad (Eqn2)$$

$$\text{Where } (\alpha_1^1 \neq \emptyset) \dots (\alpha_n^1 \neq \emptyset)$$

$$(\alpha_1^1 \neq \emptyset) \dots (\alpha_n^1 \neq \emptyset) = E \quad (A_{MX16})(\text{Order Axiom})(Eqn3)$$

$$\text{For } r=1 \text{ to } n$$

$$\alpha_r^1 \subseteq \alpha \quad (T_{M072b})(\text{Single Premise Syllogism})$$

$$\forall Q\alpha_r^1 \subseteq \forall Q\alpha \quad (T_{R196})(\text{Relational Deduction})$$

$$\forall S\alpha. \forall Q\alpha_r^1 \subseteq \forall S\alpha. \forall Q\alpha \quad (T_{M029})(\text{Deduction})(Eqn4)$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\forall S\alpha. \forall Q\alpha_r^1 \subseteq \{=\} + \{<\} + \{>\}) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&4})$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq ((\forall S\alpha. \forall Q\alpha_r^1) * \alpha_r^1 \subseteq (\{=\} + \{<\} + \{>\}) * \alpha_r^1) \quad (T_{R197})(\text{Relational Deduction})$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\alpha \subseteq (\{=\} + \{<\} + \{>\}) * \alpha_r^1) \quad (T_{S826}) \text{ since } \alpha_r^1 \neq \emptyset$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\alpha \subseteq (\{<\} + \{>\}) * \alpha_r^1) \quad (D_{A141})(\text{Disjunction with } \{=\})$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\alpha \subseteq \{<\} * \alpha_r^1 + \{>\} * \alpha_r^1) \quad (T_{R002c})(\text{Distribution of } + \text{ over } *)$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\alpha. \{>\} * \alpha_r^1 \subseteq \{<\} * \alpha_r^1) \quad (T_{M468})$$

$$(\alpha^1 \subseteq S_o.S_o) \subseteq (\alpha. \{>\} * \alpha_r^1 = \alpha. \{>\} * \alpha_r^1. \{<\} * \alpha_r^1) \quad (A_{MX44})(\subseteq \text{ Equality Form})$$

$$(\alpha^1 \subseteq S_o.S_o)(\alpha_r^1 \neq \emptyset) \subseteq (\alpha. \{>\} * \alpha_r^1 = \alpha. \{>\} * \alpha_r^1. \{<\} * \alpha_r^1) \quad (T_{R245})(\text{Negative Rule for Single Element Object})$$

$$\begin{aligned}
(\alpha^1 \subseteq S_D.S_O)(\alpha_r^1 \neq \emptyset) &\subseteq (\alpha.\{>\}_* \alpha_r^1 = \alpha.\{\{>\}_*\} * \alpha_r^1) \quad (T_{R239})(\text{Distribution of } . \text{ over } * \text{ for Entities})(\text{Eqn5}) \\
\{\leq\} &\subseteq \{>\} \quad (T_{S829}) \\
\{\leq\} &= \{\leq\}\{>\} \quad (A_{MX44})(\subseteq \text{Equality Form}) (\text{Eqn6}) \\
(\alpha^1 \subseteq S_D.S_O)(\alpha_r^1 \neq \emptyset) &\subseteq (\alpha.\{>\}_* \alpha_r^1 = \alpha.\{\leq\}_* \alpha_r^1) \quad (A_{MX19})(\text{Substitution of Eqn6 into Eqn5}) \\
(\alpha^1 \subseteq S_D.S_O)((\alpha_1^1 \neq \emptyset) \dots (\alpha_n^1 \neq \emptyset)) &\subseteq (\alpha.\{>\}_* \alpha_1^1 = \alpha.\{\leq\}_* \alpha_1^1) \dots (\alpha.\{>\}_* \alpha_n^1 = \alpha.\{\leq\}_* \alpha_n^1) \\
&\quad (T_{M669})(\text{Combination for } r=1 \text{ to } n) \\
(\alpha^1 \subseteq S_D.S_O)((\alpha_1^1 \neq \emptyset) \dots (\alpha_n^1 \neq \emptyset)) &\subseteq (\alpha.\{>\}_* \alpha_1^1 \dots \alpha.\{>\}_* \alpha_n^1 = \alpha.\{\leq\}_* \alpha_1^1 \dots \alpha.\{\leq\}_* \alpha_n^1) \\
&\quad (T_{M231})(\text{Combination}) \\
(\alpha^1 \subseteq S_D.S_O)((\alpha_1^1 \neq \emptyset) \dots (\alpha_n^1 \neq \emptyset)) &\subseteq (\alpha.\{>\}_* \alpha_1^1 \dots \{>\}_* \alpha_n^1 = \alpha.\{\leq\}_* \alpha_1^1 \dots \{\leq\}_* \alpha_n^1) \\
&\quad (T_{M667})(\text{Conjunction Tautology})(\text{Eqn7}) \\
(\alpha^1 \subseteq S_D.S_O)E &\subseteq (\alpha.\{>\}_* \alpha_1^1 \dots \{>\}_* \alpha_n^1 = \alpha.\{\leq\}_* \alpha_1^1 \dots \{\leq\}_* \alpha_n^1) \\
&\quad (A_{MX19})(\text{Substitution of Eqn3 into Eqn7}) \\
(\alpha^1 \subseteq S_D.S_O) &\subseteq (\alpha.\{>\}_* \alpha_1^1 \dots \{>\}_* \alpha_n^1 = \alpha.\{\leq\}_* \alpha_1^1 \dots \{\leq\}_* \alpha_n^1) \\
&\quad (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\
(\alpha^1 \subseteq S_D.S_O) &\subseteq (\alpha.\{>\}_* \alpha_1^1 \dots \{>\}_* \alpha_n^1 = \alpha.\{\leq\} \diamond (\alpha_1^1 + \dots + \alpha_n^1)) \\
&\quad (D_{S15})(\text{Definition of the Intersection Function Operator}) \\
(\alpha^1 \subseteq S_D.S_O) &\subseteq (\alpha.\sim(\{>\}_* \alpha_1^1 + \dots + \{>\}_* \alpha_n^1) = \alpha.\{\leq\} \diamond (\alpha_1^1 + \dots + \alpha_n^1)) \\
&\quad (T_{M188})(\text{DeMorgan's Law for Series}) \\
(\alpha^1 \subseteq S_D.S_O) &\subseteq (\alpha.\{>\}_* (\alpha_1^1 + \dots + \alpha_n^1) = \alpha.\{\leq\} \diamond (\alpha_1^1 + \dots + \alpha_n^1)) \\
&\quad (A_{S64})(\text{Distribution of } * \text{ over } +)(\text{Eqn8}) \\
(\alpha^1 \subseteq S_D.S_O) &\subseteq (\alpha.\{>\}_* \alpha = \alpha.\{\leq\} \diamond \alpha) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn8})
\end{aligned}$$

$$(\alpha^1 \subseteq S_D)(\beta^1 \subseteq \alpha) \subseteq (\beta^1 = \{\supset\}_* \beta^1) \quad (T_{T128})$$

$$(\alpha^1 \subseteq S_D)(\beta^1 \subseteq \alpha) \subseteq (\beta^1 = \{\subseteq\}_* \beta^1) \quad (T_{T140})$$

Example Proof of (T_{T128})

$$\begin{aligned}
\text{Let } \alpha^1 &\subseteq S_D \\
(\alpha^1 \subseteq S_D) &= E \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn1}) \\
(\alpha^1 \subseteq S_D) &= (\alpha \subseteq S_E)(\alpha \neq \alpha^1) \quad (A_{S53A})(\text{Axiom of Discrete Sets and Entities}) \\
(\alpha^1 \subseteq S_D) &\subseteq (\alpha \subseteq S_E) \quad (T_{M073b})(\text{Single Premise Syllogism})(\text{Eqn2}) \\
E &\subseteq (\alpha \subseteq S_E) \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\
E &= (\alpha \subseteq S_E) \quad (T_{M275})(\text{Supersets of the Universal Set}) \\
\alpha &\subseteq S_E \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\
\text{Let } \beta^1 &\subseteq \alpha \quad (\text{Eqn4}) \\
\beta^1 &\subseteq S_E \quad (T_{M139})(\text{Hypothetical Syllogism Eqns4\&3}) \\
\beta^1 &= \beta \quad (D_{R03A})(\text{Eqn5}) \\
\{=\}_* \beta^1 &= \beta \quad (D_{A144})(\text{Definition of the Reflexive Relationship Set}) \\
\emptyset + \{=\}_* \beta^1 &= \beta \quad (T_{M050})(\text{Null Set Conjunction}) \\
\{\supset\}_* \beta^1 + \{=\}_* \beta^1 &= \beta \quad (D_{R06A}) \text{ i.e. } (\alpha^1 \subseteq S_E) = (\{\supset\}_* \alpha^1 = \emptyset) \\
(\{\supset\}_* + \{=\}_*) \beta^1 &= \beta \quad (T_{R002c})(\text{Distribution of } + \text{ over } *) \\
\{\supset\}_* \beta^1 &= \beta \quad (D_{A141})(\text{Disjunction of } \{\supset\} \text{ with } \{=\})(\text{Eqn6}) \\
\beta^1 &= \{\supset\}_* \beta^1 \quad (A_{MX19})(\text{Substitution of Eqn6 into Eqn5}) \\
\text{Thus } (\alpha^1 \subseteq S_D)(\beta^1 \subseteq \alpha) &\subseteq (\beta^1 = \{\supset\}_* \beta^1) \quad (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

E. Mixed Ordered Sets

$$(\alpha^1 \subseteq S_M.S_O) \subseteq (\alpha = \{\supset\}_* \alpha^1) \quad (T_{T930})$$

Proof of (T_{T930})

$$\begin{aligned}
(\alpha^1 \subseteq S_M.S_O + S_D.S_O) &\subseteq (\alpha = \{\supset\}_* \alpha^1) \quad (T_{T760})(\text{Eqn1}) \\
S_M.S_O &\subseteq S_M.S_O + S_D.S_O \quad (T_{M082})(\text{Disjunction Introduction}) \\
(S_M.S_O \subseteq S_M.S_O + S_D.S_O) &= E \quad (A_{MX16})(\text{Order Axiom})
\end{aligned}$$

$$(a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq S_M S_O + S_D S_O) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn2})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a = \{\supset\} * a^1) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns2\&1})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq a) \quad (T_{T072})$$

Proof of (T_{T072})

$$(a^1 \subseteq S_M S_O) \subseteq (a = \{\supset\} * a^1) \quad (T_{T930})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a = (\{\supset\} + \{=\}) * a^1) \quad (D_{A141})(\text{Disjunction with } \{=\})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a = \{\supset\} * a^1 + \{=\} * a^1) \quad (T_{R002c})(\text{Distribution of } + \text{ over } *)$$

$$(a^1 \subseteq S_M S_O) \subseteq (a = \{\supset\} * a^1 + a^1) \quad (D_{A144})(\text{Definition of the Reflexive Relationship Set})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq a) \quad (T_{M072b})(\text{Single Premise Syllogism})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a \neq \emptyset) \quad (T_{T049})$$

Proof of (T_{T049})

$$S_M S_O \subseteq S_O \quad (T_{M083})(\text{Conjunction Elimination})$$

$$(S_M S_O \subseteq S_O) = E \quad (A_{MX16})(\text{Order Axiom})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq S_O) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn1})$$

$$(a^1 \subseteq S_O) \subseteq (a \neq \emptyset) \quad (T_{S702})(\text{Eqn2})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a \neq \emptyset) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a^1 \neq \emptyset) \quad (T_{S904})$$

Proof of (T_{S904})

$$S_M S_O \subseteq S_O \quad (T_{M083})(\text{Conjunction Elimination})$$

$$(S_M S_O \subseteq S_O) = E \quad (A_{MX16})(\text{Order Axiom})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq S_O) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn1})$$

$$(a^1 \subseteq S_O) \subseteq (a^1 \neq \emptyset) \quad (T_{S638})(\text{Eqn2})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a^1 \neq \emptyset) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$$

F. Sets of Points

$$(a^1 \subseteq S_P S_O) \subseteq (\alpha = a) \quad (T_{T116})$$

Proof of (T_{T116})

$$(a^1 \subseteq S_D S_O) \subseteq (\alpha = a) \quad (D_{S31})(\text{Definition of } \alpha \text{ Symbolism})(\text{Eqn1})$$

$$S_P \subseteq S_D \quad (A_{R53})(\text{Axiom of Points})$$

$$S_P S_O \subseteq S_D S_O \quad (T_{M029})(\text{Deduction})$$

$$(a^1 \subseteq S_P S_O) \subseteq (a^1 \subseteq S_D S_O) \quad (T_{M272})(\text{Deductive Rule})(\text{Eqn2})$$

$$(a^1 \subseteq S_P S_O) \subseteq (\alpha = a) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$$

$$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha.(\{\subset\} * \alpha) \neq \emptyset) \quad (T_{S995})$$

Proof of (T_{S995})

$$(a^1 \subseteq S_D) \subseteq (a.(\{\subset\} * a) \neq \emptyset) \quad (T_{S994A})$$

$$(\alpha^1 \subseteq S_D) \subseteq (\alpha.(\{\subset\} * \alpha) \neq \emptyset) \quad \text{Resymbolising (Eqn1)}$$

$$S_P \subseteq S_D \quad (A_{R53})(\text{Axiom of Points})$$

$$S_P S_O \subseteq S_D S_O \quad (T_{M029})(\text{Deduction})$$

$$S_P S_O \subseteq S_D \quad (T_{M073})(\text{Single Premise Syllogism})$$

$$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha^1 \subseteq S_D) \quad (T_{M272})(\text{Deductive Rule})(\text{Eqn2})$$

$$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha.(\{\subset\} * \alpha) \neq \emptyset) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns2\&1})$$

$$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \neq \emptyset)$$

(T_{7050})

Proof of (T_{7050})

$(\alpha^1 \subseteq S_O) = (\alpha \neq \emptyset)$	(T_{S702})(Eqn1)
$S_P \subseteq S_D$	(A_{R53})(Axiom of Points)
$S_P S_O \subseteq S_D S_O$	(T_{M029})(Deduction)
$S_P S_O \subseteq S_O$	(T_{M073})(Single Premise Syllogism)
$(S_P S_O \subseteq S_O) = E$	(A_{MX16})(Order Axiom)
$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha^1 \subseteq S_O)$	(T_{M272})(Deductive Rule)(Eqn2)
$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \neq \emptyset)$	(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha^1 \neq \emptyset)$$

(T_{7051})

Proof of (T_{7051})

$(\alpha^1 \subseteq S_O) = (\alpha^1 \neq \emptyset)$	(T_{S638})(Eqn1)
$S_P \subseteq S_D$	(A_{R53})(Axiom of Points)
$S_P S_O \subseteq S_D S_O$	(T_{M029})(Deduction)
$S_P S_O \subseteq S_O$	(T_{M073})(Single Premise Syllogism)
$(S_P S_O \subseteq S_O) = E$	(A_{MX16})(Order Axiom)
$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha^1 \subseteq S_O)$	(T_{M272})(Deductive Rule)(Eqn2)
$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha^1 \neq \emptyset)$	(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha = \alpha \cdot \{\subseteq\} * \alpha)$$

(T_{7312})

Proof of (T_{7312})

$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \cdot \{\subseteq\} * \alpha) \neq \emptyset$	(T_{S995})(Eqn1)
$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha^1 \neq \emptyset)$	(T_{7051})
$(\alpha^1 \subseteq S_P S_O) = (\alpha^1 \subseteq S_P S_O)(\alpha^1 \neq \emptyset)$	(A_{MX44})($\subseteq$ Equality Form)(Eqn2)
$(\alpha^1 \subseteq S_P S_O)(\alpha \neq \emptyset) \subseteq (\alpha \cdot \{\subseteq\} * \alpha) \neq \emptyset$	(A_{MX19})(Substitution of Eqn2 into Eqn1)
$((\alpha^1 \subseteq S_P S_O) = E) \subseteq ((\alpha \neq \emptyset) \subseteq (\alpha \cdot \{\subseteq\} * \alpha) \neq \emptyset)$	(A_{MX35})(Exportation/Importation) for truth sets
$((\alpha^1 \subseteq S_P S_O) = E) \subseteq ((\alpha \subseteq \alpha \cdot \{\subseteq\} * \alpha) = E)$	(T_{M096})(Probabilistic Rule)
$((\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \alpha \cdot \{\subseteq\} * \alpha)) = E$	(T_{M094})(Probabilistic Rule)
$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \alpha \cdot \{\subseteq\} * \alpha)$	(A_{MX16})(Order Axiom)
$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha = \alpha \cdot \alpha \cdot \{\subseteq\} * \alpha)$	(A_{MX44})($\subseteq$ Equality Form)
$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha = \alpha \cdot \{\subseteq\} * \alpha)$	(T_{M667})(Conjunction Tautology)

G. Mixed Ordered Sets and Sets of Points

$$((a^1 \subseteq S_M S_O + S_P S_O)(\beta \subseteq a)(\chi \subseteq a) = E) \subseteq (((\beta + \chi)^1 \subseteq S_P S_O) = (\beta^1 \subseteq S_P S_O)(\chi^1 \subseteq S_P S_O)) \quad (T_{S929})$$

Proof of (T_{S929})

$((a^1 \subseteq S_M S_O + S_P S_O)(\beta \subseteq a)(\chi \subseteq a) = E) \subseteq (((\beta + \chi)^1 \subseteq S_P S_O) = (\beta^1 + \chi^1 \subseteq S_P S_O))$	(A_{R88})(Disjunctions of Sets of Points)(Eqn1)
$(\beta^1 \subseteq S_P S_O)(\chi^1 \subseteq S_P S_O) = (\beta^1 + \chi^1 \subseteq S_P S_O)$	(T_{M131b})(Conjunction of Sufficiencies)(Eqn2)
$((a^1 \subseteq S_M S_O + S_P S_O)(\beta \subseteq a)(\chi \subseteq a) = E) \subseteq (((\beta + \chi)^1 \subseteq S_P S_O) = (\beta^1 \subseteq S_P S_O)(\chi^1 \subseteq S_P S_O))$	(A_{MX19})(Substitution of Eqn2 into Eqn1)

11.5 Proofs of Theorems of Bounds of Extent

A. Bounds of Extent in General

$$\mathcal{B}^1 \subseteq \mathbf{R}_0 \quad (\mathcal{T}_{S728})$$

$$\mathcal{C}^1 \subseteq \mathbf{R}_0 \quad (\mathcal{T}_{S731})$$

Example Proof of ($\mathcal{T}_{S728}$)

$$\begin{aligned} (\mathbf{X} = \mathcal{B}, \mathcal{C}) \subseteq (\mathbf{X}^1 \subseteq \mathbf{R}_0) & \quad (\mathcal{A}_{R41}) \text{(Axiom of Bounds of Extent)} \\ \mathcal{B}^1 \subseteq \mathbf{R}_0 & \quad \text{Extraction} \end{aligned}$$

$$\mathcal{B} \subseteq \{\supset\} \quad (\mathcal{T}_{T001})$$

$$\mathcal{C} \subseteq \{\supset\} \quad (\mathcal{T}_{T002})$$

Example Proof of ($\mathcal{T}_{T001}$)

$$\begin{aligned} (\mathbf{X} = \mathcal{B}, \mathcal{C}) \subseteq (\mathbf{X} \subseteq \{\supset\}) & \quad (\mathcal{A}_{R99}) \text{(Axiom of Bounds of Extent)} \\ \mathcal{B} \subseteq \{\supset\} & \quad \text{Extraction} \end{aligned}$$

$$\forall \mathcal{B} \subseteq \{\subseteq\} \quad (\mathcal{T}_{T003})$$

$$\forall \mathcal{C} \subseteq \{\subseteq\} \quad (\mathcal{T}_{T004})$$

Example Proof of ($\mathcal{T}_{T003}$)

$$\begin{aligned} \mathcal{B} \subseteq \{\supset\} & \quad (\mathcal{T}_{T001}) \\ \forall \mathcal{B} \subseteq \forall \{\supset\} & \quad (\mathcal{T}_{R196}) \text{(Relational Deduction)} \\ \forall \mathcal{B} \subseteq \{\subseteq\} & \quad (\mathcal{T}_{T741}) \text{(Definition of } \forall \{\supset\}) \end{aligned}$$

$$\mathcal{B}\mathbf{a} \subseteq \{\supset\} * \mathbf{a} \quad (\mathcal{T}_{T006})$$

$$\mathcal{C}\mathbf{a} \subseteq \{\supset\} * \mathbf{a} \quad (\mathcal{T}_{T007})$$

Example Proof of ($\mathcal{T}_{T006}$)

$$\begin{aligned} \mathcal{B} \subseteq \{\supset\} & \quad (\mathcal{T}_{T001}) \\ \mathcal{B}\mathbf{a} \subseteq \{\supset\} * \mathbf{a} & \quad (\mathcal{T}_{R197}) \text{(Relational Deduction)} \end{aligned}$$

$$\mathcal{B}\mathbf{a} + \mathcal{C}\mathbf{a} \subseteq \{\supset\} * \mathbf{a} \quad (\mathcal{T}_{T008})$$

Proof of ($\mathcal{T}_{T008}$)

$$\begin{aligned} \mathcal{B}\mathbf{a} \subseteq \{\supset\} * \mathbf{a} & \quad (\mathcal{T}_{T006}) \text{(Eqn1)} \\ \mathcal{C}\mathbf{a} \subseteq \{\supset\} * \mathbf{a} & \quad (\mathcal{T}_{T007}) \text{(Eqn2)} \\ \mathcal{B}\mathbf{a} + \mathcal{C}\mathbf{a} \subseteq \{\supset\} * \mathbf{a} + \{\supset\} * \mathbf{a} & \quad (\mathcal{A}_{MX21}) \text{(Combination of Eqns1&2)} \\ \mathcal{B}\mathbf{a} + \mathcal{C}\mathbf{a} \subseteq \{\supset\} * \mathbf{a} & \quad (\mathcal{A}_{MX01}) \text{(Disjunction Tautology)} \end{aligned}$$

$$\forall \mathcal{B}\mathbf{a} \subseteq \{\subseteq\} * \mathbf{a} \quad (\mathcal{T}_{T009})$$

$$\forall \mathcal{C}\mathbf{a} \subseteq \{\subseteq\} * \mathbf{a} \quad (\mathcal{T}_{T010})$$

Example Proof of ($\mathcal{T}_{T009}$)

$$\begin{aligned} \forall \mathcal{B} \subseteq \{\subseteq\} & \quad (\mathcal{T}_{T003}) \\ \forall \mathcal{B}\mathbf{a} \subseteq \{\subseteq\} * \mathbf{a} & \quad (\mathcal{T}_{R197}) \text{(Relational Deduction)} \end{aligned}$$

$$\forall \mathcal{B}\mathbf{a}. \forall \mathcal{C}\mathbf{a} \subseteq \{\subseteq\} * \mathbf{a} \quad (\mathcal{T}_{T011})$$

Proof of ($\mathcal{T}_{T011}$)

$$\forall \mathcal{B}\mathbf{a} \subseteq \{\subseteq\} * \mathbf{a} \quad (\mathcal{T}_{T009})$$

$$\begin{array}{ll}
\forall Ca \subseteq \{\subseteq\} * a & (T_{7010}) \\
\forall Ba. \forall Ca \subseteq \{\subseteq\} * a. \{\subseteq\} * a & (T_{669})(\text{Combination of Eqns1\&2}) \\
\forall Ba. \forall Ca \subseteq \{\subseteq\} * a & (T_{667})(\text{Conjunction Tautology})
\end{array}$$

$$Ba^1 \subseteq \{\supseteq\} * a^1 \quad (T_{7012})$$

$$Ca^1 \subseteq \{\supseteq\} * a^1 \quad (T_{7013})$$

Example Proof of (T₇₀₁₂)

$$\begin{array}{ll}
Ba \subseteq \{\supseteq\} * a & (T_{7006})(\text{Eqn1}) \\
\text{Let } a = a^1 & (\text{Eqn2}) \\
Ba^1 \subseteq \{\supseteq\} * a^1 & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
\end{array}$$

$$(Ba^1 \neq \emptyset) \subseteq (a^1 \subseteq QB) \quad (T_{5710})$$

$$(Ca^1 \neq \emptyset) \subseteq (a^1 \subseteq QC) \quad (T_{5711})$$

Example Proof of (T₅₇₁₀)

$$\begin{array}{ll}
(c * b^1 \neq \emptyset) \subseteq (b^1 \subseteq Qc) & (T_{5410b}) \\
(Ba^1 \neq \emptyset) \subseteq (a^1 \subseteq QB) & \text{Resymbolising}
\end{array}$$

$$(a^1 \subseteq QB)(a^1 \neq \emptyset) \subseteq (Ba^1 = (Ba^1)^1) \quad (T_{5729})$$

$$(a^1 \subseteq QC)(a^1 \neq \emptyset) \subseteq (Ca^1 = (Ca^1)^1) \quad (T_{5732})$$

Example Proof of (T₅₇₂₉)

$$\begin{array}{ll}
(B^1 \subseteq Ro)(a^1 \subseteq QB)(a^1 \neq \emptyset) \subseteq (Ba^1 = (Ba^1)^1) & (T_{533})(\text{Definition of a One-to-One Relationship Set}) \\
E.(a^1 \subseteq QB)(a^1 \neq \emptyset) \subseteq (Ba^1 = (Ba^1)^1) & (T_{001})(\text{Universality of (T}_{5728}\text{)}) \\
(a^1 \subseteq QB)(a^1 \neq \emptyset) \subseteq (Ba^1 = (Ba^1)^1) & (A_{MX11})(\text{Universal Set Conjunction Axiom})
\end{array}$$

$$(Ba^1 \neq \emptyset) \subseteq (Ba^1 = (Ba^1)^1) \quad (T_{047})$$

$$(Ca^1 \neq \emptyset) \subseteq (Ca^1 = (Ca^1)^1) \quad (T_{048})$$

Example Proof of (T₀₄₇)

$$\begin{array}{ll}
(a^1 \subseteq QB)(a^1 \neq \emptyset) \subseteq (Ba^1 = (Ba^1)^1) & (T_{5729})(\text{Eqn1}) \\
(Ba^1 \neq \emptyset) \subseteq (a^1 \subseteq QB) & (T_{5710}) \\
(Ba^1 \neq \emptyset)(a^1 \neq \emptyset) \subseteq (a^1 \subseteq QB)(a^1 \neq \emptyset) & (T_{029})(\text{Deduction})(\text{Eqn2}) \\
(Ba^1 \neq \emptyset) \subseteq (a^1 \neq \emptyset) & (T_{171b}) \\
(Ba^1 \neq \emptyset) = (Ba^1 \neq \emptyset)(a^1 \neq \emptyset) & (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn3}) \\
(Ba^1 \neq \emptyset) \subseteq (a^1 \subseteq QB)(a^1 \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqn3 into Eqn2})(\text{Eqn4}) \\
(Ba^1 \neq \emptyset) \subseteq (Ba^1 = (Ba^1)^1) & (T_{139})(\text{Hypothetical Syllogism Eqns4\&1})
\end{array}$$

$$(Ba^1 \neq \emptyset) \subseteq (a^1 \subseteq \{\subseteq\} * Ba^1) \quad (T_{014})$$

$$(Ca^1 \neq \emptyset) \subseteq (a^1 \subseteq \{\subseteq\} * Ca^1) \quad (T_{015})$$

Example Proof of (T₀₁₄)

$$\begin{array}{ll}
Ba^1 \subseteq \{\supseteq\} * a^1 & (T_{012}) \\
(Ba^1 \neq \emptyset) \subseteq ((Ba^1)^1 \subseteq \{\supseteq\} * a^1) & (T_{047}) \\
(Ba^1 \neq \emptyset) \subseteq (a^1 \subseteq \forall \{\supseteq\} * (Ba^1)^1) & (T_{519})(\text{Active/Passive Theorem}) \\
(Ba^1 \neq \emptyset) \subseteq (a^1 \subseteq \forall \{\supseteq\} * Ba^1) & (T_{047}) \\
(Ba^1 \neq \emptyset) \subseteq (a^1 \subseteq \{\subseteq\} * Ba^1) & (T_{741})(\text{Definition of } \forall \{\supseteq\})
\end{array}$$

B. Bounds of Extent in General Ordered Sets

$$(a^1 \subseteq S_0) \subseteq ((Ba)^1 \subseteq S_0 S_0) \quad (T_{S968})$$

$$(a^1 \subseteq S_0) \subseteq ((Ca)^1 \subseteq S_0 S_0) \quad (T_{S969})$$

$$(a^1 \subseteq S_0) \subseteq ((Ba)^1 \subseteq S_0) \quad (T_{S885})$$

$$(a^1 \subseteq S_0) \subseteq ((Ca)^1 \subseteq S_0) \quad (T_{S886})$$

$$(a^1 \subseteq S_0) \subseteq ((Ba)^1 \subseteq S_0) \quad (T_{S970})$$

$$(a^1 \subseteq S_0) \subseteq ((Ca)^1 \subseteq S_0) \quad (T_{S971})$$

Example Proof of (T_{S968}) and (T_{S885})

$$((X = B, C) = E) \subseteq ((a^1 \subseteq S_0) \subseteq ((Xa)^1 \subseteq S_0 S_0)) \quad (A_{R50})(\text{Beginnings and Ends in an Ordered Set})$$

$$(a^1 \subseteq S_0) \subseteq ((Ba)^1 \subseteq S_0 S_0) \quad \text{Extraction (Eqn1)}$$

$$S_0 S_0 \subseteq S_0 \quad (T_{M083})(\text{Conjunction Elimination})$$

$$((Ba)^1 \subseteq S_0 S_0) \subseteq ((Ba)^1 \subseteq S_0) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn2})$$

$$(a^1 \subseteq S_0) \subseteq ((Ba)^1 \subseteq S_0) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$$

$$(a^1 \subseteq S_0) \subseteq (Ba \neq \emptyset) \quad (T_{S887})$$

$$(a^1 \subseteq S_0) \subseteq (Ca \neq \emptyset) \quad (T_{S888})$$

Example Proof of (T_{S887})

$$(a^1 \subseteq S_0) \subseteq ((Ba)^1 \subseteq S_0) \quad (T_{S885})(\text{Eqn1})$$

$$((Ba)^1 \subseteq S_0) \subseteq (Ba \neq \emptyset) \quad (T_{S702})(\text{Eqn2})$$

$$(a^1 \subseteq S_0) \subseteq (Ba \neq \emptyset) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$$

$$(Ba = \emptyset) \subseteq (a^1 \subseteq \sim S_0) \quad (T_{S951})$$

$$(Ca = \emptyset) \subseteq (a^1 \subseteq \sim S_0) \quad (T_{S952})$$

Example Proof of (T_{S951})

$$(a^1 \subseteq S_0) \subseteq (Ba \neq \emptyset) \quad (T_{S887})$$

$$(Ba = \emptyset) \subseteq (a^1 \not\subseteq S_0) \quad (T_{M012})(\text{Transposition})$$

$$(Ba = \emptyset) \subseteq (a^1 \not\subseteq S_0) \quad \text{Resymbolising (Eqn1)}$$

$$(a^1 \not\subseteq S_0) \subseteq (a^1 \subseteq \sim S_0) \quad (T_{M604})(\text{Eqn2})$$

$$(Ba = \emptyset) \subseteq (a^1 \subseteq \sim S_0) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$$

$$(Ba^1 \neq \emptyset) \subseteq (a^1 \subseteq S_0) \quad (T_{S713})$$

$$(Ca^1 \neq \emptyset) \subseteq (a^1 \subseteq S_0) \quad (T_{S714})$$

Example Proof of (T_{S713})

$$(a^1 \subseteq QB) \subseteq (a^1 \subseteq S_0) \quad (T_{S717})(\text{Eqn1})$$

$$(Ba^1 \neq \emptyset) \subseteq (a^1 \subseteq QB) \quad (T_{S710})(\text{Eqn2})$$

$$(Ba^1 \neq \emptyset) \subseteq (a^1 \subseteq S_0) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns2\&1})$$

$$(a^1 \subseteq S_0)(a_n \subseteq a)(Ba_n^1 \neq \emptyset) \subseteq (Ba_n^1 = (Ba_n^1)^1) \quad (T_{T955})$$

$$(a^1 \subseteq S_0)(a_n \subseteq a)(Ca_n^1 \neq \emptyset) \subseteq (Ca_n^1 = (Ca_n^1)^1) \quad (T_{T956})$$

Example Proof of (T_{T955})

$$(Ba_n^1 \neq \emptyset) \subseteq (Ba_n^1 = (Ba_n^1)^1) \quad (T_{T047})$$

$$(Ba_n^1 \neq \emptyset)(a_n \subseteq a) \subseteq (Ba_n^1 = (Ba_n^1)^1)(a_n \subseteq a) \quad (T_{M029})(\text{Deduction})(\text{Eqn1})$$

$$(Ba_n^1 \neq \emptyset)(a_n \subseteq a)(a^1 \subseteq S_0) \subseteq (Ba_n^1 \neq \emptyset)(a_n \subseteq a) \quad (T_{M083})(\text{Conjunction Elimination})(\text{Eqn2})$$

$$(Ba_n^1 \neq \emptyset)(a_n \subseteq a)(a^1 \subseteq S_0) \subseteq (Ba_n^1 = (Ba_n^1)^1)(a_n \subseteq a) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns2\&1})$$

$$(\mathcal{B}a_n^1 \neq \emptyset)(a_n \subseteq a)(a^1 \subseteq S_o) \subseteq (\mathcal{B}a_n^1 = (\mathcal{B}a_n^1)^1)$$

(T_{M073})(Single Premise Syllogism)

$$(a \subseteq QB) \subseteq (a^1 \subseteq S_o)$$

(T_{S974})

$$(a \subseteq QC) \subseteq (a^1 \subseteq S_o)$$

(T_{S975})

Example Proof of (T_{S974})

$$((X = \{<\}, \{>\}, \mathcal{B}, C) = E) \subseteq ((a^1 \subseteq \sim S_o) \subseteq (a^1 \not\subseteq QX)(a \not\subseteq QX))$$

(A_{R05})(Axiom of Non-ordered Sets)

$$((X = \{<\}, \{>\}, \mathcal{B}, C) = E) \subseteq ((a^1 \subseteq \sim S_o) \subseteq (a \not\subseteq QX))$$

(T_{M073})(Single Premise Syllogism)

$$(a^1 \subseteq \sim S_o) \subseteq (a \not\subseteq QB)$$

Extraction

$$(a \subseteq QB) \subseteq (a^1 \not\subseteq \sim S_o)$$

(T_{M012})(Transposition)(Eqn1)

$$(a^1 \not\subseteq \sim S_o) \subseteq (a^1 \subseteq S_o)$$

(T_{M604})(Eqn2)

$$(a \subseteq QB) \subseteq (a^1 \subseteq S_o)$$

(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$(a \subseteq QB) \subseteq (a^1 \subseteq S_o)$$

Resymbolising

$$(a \subseteq QB) + (a \subseteq QC) \subseteq (a^1 \subseteq S_o)$$

(T_{S981})

Proof of (T_{S981})

$$(a \subseteq QB) \subseteq (a^1 \subseteq S_o)$$

(T_{S974})(Eqn1)

$$(a \subseteq QC) \subseteq (a^1 \subseteq S_o)$$

(T_{S975})(Eqn2)

$$(a \subseteq QB) + (a \subseteq QC) \subseteq (a^1 \subseteq S_o) + (a^1 \subseteq S_o)$$

(A_{MX27})(Combination of Eqns1&2)

$$(a \subseteq QB) + (a \subseteq QC) \subseteq (a^1 \subseteq S_o)$$

(A_{MX07})(Disjunction Tautology)

$$((X = \mathcal{B}, C) = E) \subseteq ((a^1 \subseteq S_M S_o + S_D S_o) \subseteq (Xa \subseteq a.(\{<\} * a)))$$

(T_{T723})

$$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\mathcal{B}a \subseteq a.(\{<\} * a))$$

(T_{T726})

$$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\mathcal{C}a \subseteq a.(\{<\} * a))$$

(T_{T727})

Proof of (T_{T723}) (T_{T726}) & (T_{T727})

$$(a^1 \subseteq S_M S_o + S_D S_o)(a_n \subseteq a) \subseteq (\mathcal{B}a_n^1 = a_n.(\{>\} * a_n).(\{<\} * a_n))$$

(D_{R53})(Definition of the Beginning Function)

$$(a^1 \subseteq S_M S_o + S_D S_o)(a_n \subseteq a) \subseteq (\mathcal{B}a_n^1 \subseteq a_n.(\{<\} * a_n))$$

(T_{M073})(Single Premise Syllogism)(Eqn1)

$$(a^1 \subseteq S_M S_o + S_D S_o)(a_n \subseteq a) \subseteq (\mathcal{C}a_n^1 = a_n.(\{<\} * a_n).(\{<\} * a_n))$$

(D_{R54})(Definition of the End Function)

$$(a^1 \subseteq S_M S_o + S_D S_o)(a_n \subseteq a) \subseteq (\mathcal{C}a_n^1 \subseteq a_n.(\{<\} * a_n))$$

(T_{M073})(Single Premise Syllogism)(Eqn2)

$$(a^1 \subseteq S_M S_o + S_D S_o)(a_n \subseteq a) \subseteq (\mathcal{B}a_n^1 + \mathcal{C}a_n^1 \subseteq a_n.(\{<\} * a_n) + a_n.(\{<\} * a_n))$$

(A_{MX27})(Combination of Eqns1&2)

$$(a^1 \subseteq S_M S_o + S_D S_o)(a_n \subseteq a) \subseteq (\mathcal{B}a_n^1 + \mathcal{C}a_n^1 \subseteq a_n.(\{<\} * a_n))$$

(A_{MX07})(Disjunction Tautology)(Eqn3)

Let $a_n = a$

(Eqn4)

$$(a^1 \subseteq S_M S_o + S_D S_o)(a \subseteq a) \subseteq (\mathcal{B}a + \mathcal{C}a \subseteq a.(\{<\} * a))$$

(A_{MX19})(Substitution of Eqn4 into Eqn3)

$$(a^1 \subseteq S_M S_o + S_D S_o)((a \subset a) + (a = a)) \subseteq (\mathcal{B}a + \mathcal{C}a \subseteq a.(\{<\} * a))$$

(T_{M266})(Definition of $\subseteq$)

$$(a^1 \subseteq S_M S_o + S_D S_o)((a \subset a) + E) \subseteq (\mathcal{B}a + \mathcal{C}a \subseteq a.(\{<\} * a))$$

(T_{M001})(Universality of (D_{M00}) Definition of $=$)

$$(a^1 \subseteq S_M S_o + S_D S_o)E \subseteq (\mathcal{B}a + \mathcal{C}a \subseteq a.(\{<\} * a))$$

(T_{M055})(Universal Set Disjunction)

$$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\mathcal{B}a + \mathcal{C}a \subseteq a.(\{<\} * a))$$

(A_{MX11})(Universal Set Conjunction)

$$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\mathcal{B}a \subseteq a.(\{<\} * a))$$

(T_{M072})(Single Premise Syllogism)(Eqn4)

$$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\mathcal{C}a \subseteq a.(\{<\} * a))$$

(T_{M072})(Single Premise Syllogism)(Eqn5)

$$((X = \mathcal{B}, C) = E) \subseteq ((a^1 \subseteq S_M S_o + S_D S_o) \subseteq (Xa \subseteq a.(\{<\} * a)))$$

(Reverse Extraction Eqns4&5)

$$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\mathcal{B}a + \mathcal{C}a \subseteq a.(\{<\} * a))$$

(T_{T728})

Proof of (T728)

$$\begin{aligned}
 (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba \subseteq a.\{\subseteq\}^* a)) & (T726)(Eqn1) \\
 (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ca \subseteq a.\{\subseteq\}^* a)) & (T727)(Eqn2) \\
 (a^1 \subseteq S_M S_O + S_D S_O)(a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba \subseteq a.\{\subseteq\}^* a))(Ca \subseteq a.\{\subseteq\}^* a)) & (T669)(Combination of Eqns1&2) \\
 (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba \subseteq a.\{\subseteq\}^* a))(Ca \subseteq a.\{\subseteq\}^* a)) & (T667)(Conjunction Tautology)(Eqn3) \\
 (Ba \subseteq a.\{\subseteq\}^* a))(Ca \subseteq a.\{\subseteq\}^* a)) &\subseteq (Ba + Ca \subseteq a.\{\subseteq\}^* a) + a.\{\subseteq\}^* a)) & (AMX21)(Combination Axiom) \\
 (Ba \subseteq a.\{\subseteq\}^* a))(Ca \subseteq a.\{\subseteq\}^* a)) &\subseteq (Ba + Ca \subseteq a.\{\subseteq\}^* a)) & (AMX01)(Disjunction Tautology)(Eqn4) \\
 (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba + Ca \subseteq a.\{\subseteq\}^* a)) & (T139)(Hypothetical Syllogism Eqns4&3)
 \end{aligned}$$

$$((X = B, C) = E) \subseteq ((a^1 \subseteq S_M S_O + S_D S_O) \subseteq ((Xa)^1 \subseteq S_O)) \quad (T713)$$

Proof of (T713)

$$\begin{aligned}
 ((X = B, C) = E) &\subseteq ((a^1 \subseteq S_O) \subseteq ((Xa)^1 \subseteq S_D S_O)) & (AR50)(Beginnings and Ends in an Ordered Set)(Eqn1) \\
 ((Xa)^1 \subseteq S_D S_O) &\subseteq ((Xa)^1 \subseteq S_O) & (T073)(Single Premise Syllogism)(Eqn2) \\
 ((X = B, C) = E) &\subseteq ((a^1 \subseteq S_O) \subseteq ((Xa)^1 \subseteq S_O)) & (T139)(Hypothetical Syllogism Eqns1&2)(Eqn3) \\
 (a^1 \subseteq (S_M + S_D) S_O) &\subseteq (a^1 \subseteq S_O) & (T089)(Single Premise Syllogism) \\
 (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (a^1 \subseteq S_O) & (T407)(Distribution of . over +)(Eqn4) \\
 ((X = B, C) = E) &\subseteq ((a^1 \subseteq S_M S_O + S_D S_O) \subseteq ((Xa)^1 \subseteq S_O)) & (T139)(Hypothetical Syllogism Eqns4&3)
 \end{aligned}$$

$$((X = B, C) = E) \subseteq ((a^1 \subseteq S_M S_O + S_P S_O) \subseteq ((Xa)^1 \subseteq S_P S_O)) \quad (T712)$$

Proof of (T712)

$$\begin{aligned}
 ((X = B, C) = E) &\subseteq ((a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Xa \subseteq a.\{\subseteq\}^* a)) & (T723)(Eqn1) \\
 S_P \subseteq S_D & & (AR53)(Axiom of Points) \\
 S_P S_O \subseteq S_D S_O & & (T029)(Deduction) \\
 S_M S_O + S_P S_O \subseteq S_M S_O + S_D S_O & & (T030)(Deduction) \\
 (S_M S_O + S_P S_O \subseteq S_M S_O + S_D S_O) = E & & (AMX16)(Order Axiom) \\
 (a^1 \subseteq S_M S_O + S_P S_O) &\subseteq (a^1 \subseteq S_M S_O + S_D S_O) & (T099)(Entity Mode / Set Mode Rule)(Eqn2) \\
 ((X = B, C) = E) &\subseteq ((a^1 \subseteq S_M S_O + S_P S_O) \subseteq (Xa \subseteq a.\{\subseteq\}^* a)) & (T139)(Hypothetical Syllogism Eqns1&2) \\
 ((X = B, C) = E) &\subseteq ((a^1 \subseteq S_M S_O + S_P S_O) \subseteq ((Xa)^1 \subseteq \{\supseteq\}(a.\{\subseteq\}^* a)^1)) & (T422)(Eqn3) \\
 (a^1 \subseteq S_M + S_P) &\subseteq ((a.\{\subseteq\}^* a)^1 \subseteq S_P) & (AR76A)(Axiom of Abstract Point Sets) \\
 (S_M + S_P) S_O \subseteq S_M + S_P & & (T083)(Conjunction Elimination) \\
 S_M S_O + S_P S_O \subseteq S_M + S_P & & (T407)(Distribution of . over +) \\
 (S_M S_O + S_P S_O \subseteq S_M + S_P) = E & & (AMX16)(Order Axiom) \\
 (a^1 \subseteq S_M S_O + S_P S_O) &\subseteq (a^1 \subseteq S_M + S_P) & (T099)(Entity Mode / Set Mode Rule)(Eqn4) \\
 (a^1 \subseteq S_M S_O + S_P S_O) &\subseteq ((a.\{\subseteq\}^* a)^1 \subseteq S_P) & (T139)(Hypothetical Syllogism Eqns3&4) \\
 (a^1 \subseteq S_M S_O + S_P S_O) &\subseteq (\{\supseteq\}(a.\{\subseteq\}^* a)^1 \subseteq \{\supseteq\} S_P) & (TR196)(Relational Deduction)(Eqn5) \\
 ((X = B, C) = E) &\subseteq ((a^1 \subseteq S_M S_O + S_P S_O) \subseteq ((Xa)^1 \subseteq \{\supseteq\} S_P)) & (T139)(Hypothetical Syllogism Eqns3&5) \\
 ((X = B, C) = E) &\subseteq ((a^1 \subseteq S_M S_O + S_P S_O) \subseteq ((Xa)^1 \subseteq S_P)) & (T716A)(Eqn6) \\
 ((X = B, C) = E) &\subseteq ((a^1 \subseteq S_M S_O + S_P S_O) \subseteq ((Xa)^1 \subseteq S_O)) & (T713)(Eqn7) \\
 ((X = B, C) = E) &\subseteq ((a^1 \subseteq S_M S_O + S_P S_O) \subseteq ((Xa)^1 \subseteq S_P S_O)) & (T132)(Composition of Eqns6&7)
 \end{aligned}$$

$$((X = B, C) = E) \subseteq ((a^1 \subseteq S_M S_O) \subseteq ((Xa)^1 \subseteq S_P S_O)) \quad (T778)$$

Proof of (T778)

$$((X = B, C) = E) \subseteq ((a^1 \subseteq S_M S_O + S_P S_O) \subseteq ((Xa)^1 \subseteq S_P S_O)) \quad (T712)(Bounds of Extent are Points)(Eqn1)$$

$S_M S_O \subseteq S_M S_O + S_P S_O$	(T_{M082}) (Disjunction Introduction)
$(S_M S_O \subseteq S_M S_O + S_P S_O) = E$	(A_{MX16}) (Order Axiom)
$(a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq S_M S_O + S_P S_O)$	(T_{M099}) (Entity Mode / Set Mode Rule)(Eqn2)
$((X = B, C) = E) \subseteq ((a^1 \subseteq S_M S_O) \subseteq ((Xa)^1 \subseteq S_P S_O))$	(T_{M139}) (Hypothetical Syllogism Eqns1&2)

C. Bounds of Extent of Discrete Ordered Sets

$$(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (B\alpha_n^1 = \alpha_n.(\{>\}_* \alpha_n)) \quad (T_{S818b})$$

$$(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (C\alpha_n^1 = \alpha_n.(\{<\}_* \alpha_n)) \quad (T_{S819b})$$

Example Proof of (T_{S818b})

$(a^1 \subseteq S_M S_O + S_D S_O)(a_n \subseteq a) \subseteq (Ba_n^1 = a_n.(\{>\}_* a_n).(\{<\}_* a_n))$	(D_{R53}) (Definition of the Beginning Function)(Eqn1)
$S_D S_O \subseteq S_M S_O + S_D S_O$	(T_{M082}) (Disjunction Introduction)
$(S_D S_O \subseteq S_M S_O + S_D S_O) = E$	(A_{MX16}) (Order Axiom)
$(a^1 \subseteq S_D S_O) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)$	(T_{M099}) (Entity Mode / Set Mode Rule)
$(a^1 \subseteq S_D S_O)(a_n \subseteq a) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)(a_n \subseteq a)$	(Eqn2)
$(a^1 \subseteq S_D S_O)(a_n \subseteq a) \subseteq (Ba_n^1 = a_n.(\{>\}_* a_n).(\{<\}_* a_n))$	(T_{M139}) (Hypothetical Syllogism Eqns2&1)(Eqn3)
Let $a^1 \subseteq S_D S_O$	
$a \subseteq \{<\}_* a$	(T_{R719A})
$a = a.\{<\}_* a$	(A_{MX44}) ($\subseteq$ Equality Form)(Eqn4)
$(a^1 \subseteq S_D S_O)(a_n \subseteq a) \subseteq (Ba_n^1 = a_n.(\{>\}_* a_n))$	(A_{MX19}) (Substitution of Eqn4 into Eqn3)
$(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (B\alpha_n^1 = \alpha_n.(\{>\}_* \alpha_n))$	Resymbolising

$$(\alpha^1 \subseteq S_D S_O) \subseteq (B\alpha^1 = \alpha.(\{>\}_* \alpha)) \quad (T_{S818})$$

$$(\alpha^1 \subseteq S_D S_O) \subseteq (C\alpha^1 = \alpha.(\{<\}_* \alpha)) \quad (T_{S819})$$

Example Proof of (T_{S818})

$(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (B\alpha_n^1 = \alpha_n.(\{>\}_* \alpha_n))$	(T_{S818b}) (Eqn1)
Let $\alpha_n = \alpha$	(Eqn2)
$(\alpha^1 \subseteq S_D S_O)(\alpha \subseteq \alpha) \subseteq (B\alpha^1 = \alpha.(\{>\}_* \alpha))$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)
$(\alpha^1 \subseteq S_D S_O)((\alpha \subset \alpha) + (\alpha = \alpha)) \subseteq (B\alpha^1 = \alpha.(\{>\}_* \alpha))$	(T_{M266}) (Definition of $\subseteq$)
$(\alpha^1 \subseteq S_D S_O)((\alpha \subset \alpha) + E) \subseteq (B\alpha^1 = \alpha.(\{>\}_* \alpha))$	(T_{M007}) (Universality of (D_{M00}) Definition of $=$)
$(\alpha^1 \subseteq S_D S_O)E \subseteq (B\alpha^1 = \alpha.(\{>\}_* \alpha))$	(T_{M055}) (Universal Set Disjunction)
$(\alpha^1 \subseteq S_D S_O) \subseteq (B\alpha^1 = \alpha.(\{>\}_* \alpha))$	(A_{MX11}) (Universal Set Conjunction Axiom)(Eqn3)

$$(\alpha^1 \subseteq S_D S_O)(\alpha_n^1 \subseteq \alpha) \subseteq (B\alpha_n^1 = \alpha_n^1) \quad (T_{T978})$$

$$(\alpha^1 \subseteq S_D S_O)(\alpha_n^1 \subseteq \alpha) \subseteq (C\alpha_n^1 = \alpha_n^1) \quad (T_{T979})$$

Example Proof of (T_{T978})

$((X = B, C) = E) \subseteq ((\alpha^1 \subseteq S_D S_O)(\alpha_n^1 \subseteq \alpha) \subseteq (\alpha_n^1 = X\alpha_n^1)(\alpha_n^1 \subseteq {}^V X\alpha_n^1))$	(A_{S69}) (Axiom of Discrete Ordered Sets)
$(\alpha^1 \subseteq S_D S_O)(\alpha_n^1 \subseteq \alpha) \subseteq (\alpha_n^1 = B\alpha_n^1)(\alpha_n^1 \subseteq {}^V B\alpha_n^1)$	Extraction
$(\alpha^1 \subseteq S_D S_O)(\alpha_n^1 \subseteq \alpha) \subseteq (\alpha_n^1 = B\alpha_n^1)$	(T_{M073}) (Single Premise Syllogism)

$$((X = B, C) = E) \subseteq ((\alpha^1 \subseteq S_D S_O) \subseteq (\alpha = X\alpha).(\alpha \subseteq {}^V X\alpha)) \quad (T_{T251})$$

Proof of (T_{T251})

Let $X = B, C$

Let $\alpha = \alpha_1^1 + \alpha_n^1$

(Eqn1)

$(\alpha^1 \subseteq S_D S_O)(\alpha_r^1 \subseteq \alpha) \subseteq (\alpha_r^1 = X\alpha_r^1)(\alpha_r^1 \subseteq \forall X\alpha_r^1)$	(A_{S69})(Axiom of Discrete Ordered Sets)(Eqn2)
$(\alpha^1 \subseteq S_D S_O)(\alpha_1^1 \subseteq \alpha) \dots (\alpha_n^1 \subseteq S_D S_O)(\alpha_n^1 \subseteq \alpha) \subseteq (\alpha_1^1 = X\alpha_1^1)(\alpha_1^1 \subseteq \forall X\alpha_1^1) \dots (\alpha_n^1 = X\alpha_n^1)(\alpha_n^1 \subseteq \forall X\alpha_n^1)$	(T_{M669})(Combination of Eqn2 for $r = 1$ to n)
$(\alpha^1 \subseteq S_D S_O)((\alpha_1^1 \subseteq \alpha) \dots (\alpha_n^1 \subseteq \alpha)) \subseteq (\alpha_1^1 = X\alpha_1^1)(\alpha_1^1 \subseteq \forall X\alpha_1^1) \dots (\alpha_n^1 = X\alpha_n^1)(\alpha_n^1 \subseteq \forall X\alpha_n^1)$	(T_{M667})(Conjunction Tautology)
$(\alpha^1 \subseteq S_D S_O)((\alpha_1^1 \subseteq \alpha) \dots (\alpha_n^1 \subseteq \alpha)) \subseteq ((\alpha_1^1 = X\alpha_1^1) \dots (\alpha_n^1 = X\alpha_n^1)).((\alpha_1^1 \subseteq \forall X\alpha_1^1) \dots (\alpha_n^1 \subseteq \forall X\alpha_n^1))$	(T_{M404})(Commutative .)
$(\alpha^1 \subseteq S_D S_O)(\alpha_1^1 + \dots + \alpha_n^1 \subseteq \alpha) \subseteq ((\alpha_1^1 = X\alpha_1^1) \dots (\alpha_n^1 = X\alpha_n^1)).((\alpha_1^1 \subseteq \forall X\alpha_1^1) \dots (\alpha_n^1 \subseteq \forall X\alpha_n^1))$	(T_{M131b})(Conjunction of Sufficiencies)
$(\alpha^1 \subseteq S_D S_O)(\alpha_1^1 + \dots + \alpha_n^1 \subseteq \alpha) \subseteq (\alpha_1^1 + \dots + \alpha_n^1 = X\alpha_1^1 + \dots + X\alpha_n^1).((\alpha_1^1 \subseteq \forall X\alpha_1^1) \dots (\alpha_n^1 \subseteq \forall X\alpha_n^1))$	(T_{M231})(Combination)
$(\alpha^1 \subseteq S_D S_O)(\alpha_1^1 + \dots + \alpha_n^1 \subseteq \alpha) \subseteq (\alpha_1^1 + \dots + \alpha_n^1 = X(\alpha_1^1 + \dots + \alpha_n^1)).((\alpha_1^1 \subseteq \forall X\alpha_1^1) \dots (\alpha_n^1 \subseteq \forall X\alpha_n^1))$	(T_{M407})(Distribution of . over +)
$(\alpha^1 \subseteq S_D S_O)(\alpha_1^1 + \dots + \alpha_n^1 \subseteq \alpha) \subseteq (\alpha_1^1 + \dots + \alpha_n^1 = X(\alpha_1^1 + \dots + \alpha_n^1)).((\alpha_1^1 + \dots + \alpha_n^1 \subseteq \forall X\alpha_1^1 + \dots + \forall X\alpha_n^1))$	(A_{MX21})(Combination)
$(\alpha^1 \subseteq S_D S_O)(\alpha_1^1 + \dots + \alpha_n^1 \subseteq \alpha) \subseteq (\alpha_1^1 + \dots + \alpha_n^1 = X(\alpha_1^1 + \dots + \alpha_n^1)).((\alpha_1^1 + \dots + \alpha_n^1 \subseteq \forall X(\alpha_1^1 + \dots + \alpha_n^1)))$	(T_{M407})(Distribution of . over +)(Eqn3)
$(\alpha^1 \subseteq S_D S_O)(\alpha \subseteq \alpha) \subseteq (\alpha = X\alpha).(\alpha \subseteq \forall X\alpha)$	(A_{MX19})(Substitution of Eqn1 into Eqn3)
$(\alpha^1 \subseteq S_D S_O)((\alpha \subseteq \alpha) + (\alpha = \alpha)) \subseteq (\alpha = X\alpha).(\alpha \subseteq \forall X\alpha)$	(T_{M266})(Definition of $\subseteq$)
$(\alpha^1 \subseteq S_D S_O)((\alpha \subseteq \alpha) + E) \subseteq (\alpha = X\alpha).(\alpha \subseteq \forall X\alpha)$	(T_{M001})(Universality of (D_{M00}) Definition of $=$)
$(\alpha^1 \subseteq S_D S_O)E \subseteq (\alpha = X\alpha).(\alpha \subseteq \forall X\alpha)$	(T_{M055})(Universal Set Disjunction)
$(\alpha^1 \subseteq S_D S_O) \subseteq (\alpha = X\alpha).(\alpha \subseteq \forall X\alpha)$	(A_{MX17})(Universal Set Conjunction Axiom)
Thus $((X = B, C) = E) \subseteq ((\alpha^1 \subseteq S_D S_O) \subseteq (\alpha = X\alpha).(\alpha \subseteq \forall X\alpha))$	(A_{MX02})(Conditional Introduction)

$$(\alpha^1 \subseteq S_D S_O) \subseteq (B\alpha = \alpha) \quad (T_{S893})$$

$$(\alpha^1 \subseteq S_D S_O) \subseteq (C\alpha = \alpha) \quad (T_{S894})$$

$$(\alpha^1 \subseteq S_D S_O) \subseteq (\alpha \subseteq \forall B\alpha) \quad (T_{T252})$$

$$(\alpha^1 \subseteq S_D S_O) \subseteq (\alpha \subseteq \forall C\alpha) \quad (T_{T253})$$

Example Proof of (T_{S893})

$((X = B, C) = E) \subseteq ((\alpha^1 \subseteq S_D S_O) \subseteq (\alpha = X\alpha).(\alpha \subseteq \forall X\alpha))$	(T_{T251})
$(\alpha^1 \subseteq S_D S_O) \subseteq (\alpha = B\alpha).(\alpha \subseteq \forall B\alpha)$	Extraction
$(\alpha^1 \subseteq S_D S_O) \subseteq (\alpha = B\alpha)$	(T_{M073})(Single Premise Syllogism)

$$(\alpha^1 \subseteq S_D S_O) \subseteq (B\alpha = C\alpha) \quad (T_{T932})$$

Proof of (T_{T932})

$(\alpha^1 \subseteq S_D S_O) \subseteq (C\alpha = \alpha)$	(T_{S894})(Eqn1)
$(\alpha^1 \subseteq S_D S_O) \subseteq (B\alpha = \alpha)$	(T_{S893})(Eqn2)
$(\alpha^1 \subseteq S_D S_O) \subseteq (B\alpha = C\alpha)$	(A_{MX19})(Substitution of Eqn1 into Eqn2)

$$(\alpha^1 \subseteq S_D S_O) \subseteq (\alpha = B\alpha + C\alpha) \quad (T_{T931})$$

Proof of (T_{T931})

$(\alpha^1 \subseteq S_D S_O) \subseteq (C\alpha = \alpha)$	(T_{S894})(Eqn1)
$(\alpha^1 \subseteq S_D S_O) \subseteq (B\alpha = \alpha)$	(T_{S893})(Eqn2)
$(\alpha^1 \subseteq S_D S_O)(\alpha^1 \subseteq S_D S_O) \subseteq (B\alpha = \alpha)(C\alpha = \alpha)$	(T_{M669})(Combination of Eqns1&2)
$(\alpha^1 \subseteq S_D S_O) \subseteq (B\alpha = \alpha)(C\alpha = \alpha)$	(T_{M667})(Conjunction Tautology)(Eqn3)
$(\alpha = B\alpha).(\alpha = C\alpha) \subseteq (\alpha + \alpha = B\alpha + C\alpha)$	(T_{M230})(Combination)
$(\alpha = B\alpha).(\alpha = C\alpha) \subseteq (\alpha = B\alpha + C\alpha)$	(A_{MX01})(Disjunction Tautology)(Eqn4)
$(\alpha^1 \subseteq S_D S_O) \subseteq (\alpha = B\alpha + C\alpha)$	(T_{M139})(Hypothetical Syllogism Eqns3&4)

$$(\alpha^1 \subseteq S_0)(\alpha_n^1 \subseteq \alpha) \subseteq (\alpha_n^1 \neq \emptyset^1) \quad (T_{S600})$$

$$(\alpha^1 \subseteq S_0)(\alpha_n^1 \subseteq \alpha) \subseteq (\alpha_n^1 \neq E^1) \quad (T_{S605})$$

Example Proof of (T_{S600})

$$(\alpha^1 \subseteq S_0)(\alpha_n^1 \subseteq \alpha) \subseteq (\alpha_n^1 \neq \emptyset^1)(\alpha_n^1 \neq E^1) \quad (A_{A01})(\emptyset^1 \text{ and } E^1 \text{ are not ordered})$$

$$(\alpha^1 \subseteq S_0)(\alpha_n^1 \subseteq \alpha) \subseteq (\alpha_n^1 \neq \emptyset^1) \quad (T_{M073})(\text{Single Premise Syllogism})$$

$$(\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n.(\{\leq\} \diamond \alpha_n)) \quad (T_{S730b})$$

$$(\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{C}\alpha_n^1 = \alpha_n.(\{\geq\} \diamond \alpha_n)) \quad (T_{S733b})$$

Example Proof of (T_{S730b})

$$(\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n.(\{\>\} \ast \alpha_n)) \quad (T_{S818b})(\text{Eqn1})$$

$$((\alpha_n \subseteq \alpha)(\alpha_n \neq \emptyset)(\alpha_n \neq \alpha_n^1) = E) \subseteq ((\alpha^1 \subseteq S_D S_0) \subseteq (\alpha_n^1 \subseteq S_D S_0))$$

(T_{T975})

$$((\alpha_n \neq \emptyset)(\alpha_n \neq \alpha_n^1) = E) \subseteq ((\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \subseteq (\alpha_n^1 \subseteq S_D S_0))$$

(A_{MX35})(Exportation/Importation) for truth sets (Eqn2)

$$(\alpha_n^1 \subseteq S_D S_0) \subseteq (\alpha_n.(\{\>\} \ast \alpha_n = \alpha_n.(\{\leq\} \diamond \alpha_n)) \quad (T_{S826})(\text{Eqn3})$$

$$((\alpha_n \neq \emptyset)(\alpha_n \neq \alpha_n^1) = E) \subseteq ((\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \subseteq (\alpha_n.(\{\>\} \ast \alpha_n = \alpha_n.(\{\leq\} \diamond \alpha_n)))$$

(T_{M139})(Hypothetical Syllogism Eqns2&3)(Eqn4)

$$((\alpha_n \neq \emptyset)(\alpha_n \neq \alpha_n^1) = E) \subseteq$$

$$((\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha)(\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n.(\{\>\} \ast \alpha_n)(\alpha_n.(\{\>\} \ast \alpha_n = \alpha_n.(\{\leq\} \diamond \alpha_n)))$$

(T_{M669})(Combination of Eqns1&4)

$$((\alpha_n \neq \emptyset)(\alpha_n \neq \alpha_n^1) = E) \subseteq ((\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n.(\{\>\} \ast \alpha_n)(\alpha_n.(\{\>\} \ast \alpha_n = \alpha_n.(\{\leq\} \diamond \alpha_n)))$$

(T_{M667})(Conjunction Tautology)

$$((\alpha_n \neq \emptyset)(\alpha_n \neq \alpha_n^1) = E) \subseteq ((\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n.(\{\leq\} \diamond \alpha_n))$$

(A_{MX19})(Substitution)

$$(\alpha_n \neq \emptyset)(\alpha_n \neq \alpha_n^1)(\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n.(\{\leq\} \diamond \alpha_n))$$

(A_{MX35})(Exportation/Importation) for truth sets (Eqn5)

Let $\alpha_n = \emptyset$

$$(\alpha_n^1 = \emptyset) + (\alpha_n^1 = \emptyset^1)$$

(T_{M496})

$$(\alpha_n^1 = \emptyset) + \emptyset$$

(T_{M024})(Non existence of anti-(T_{S600}))

$$\alpha_n^1 = \emptyset$$

(T_{M668})(Null Set Disjunction)(Eqn6)

$$\text{Let } (\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \alpha_n.(\{\leq\} \diamond \alpha_n))$$

(Eqn7)

$$(\emptyset \subseteq S_D S_0)(\emptyset \subseteq \alpha) \not\subseteq (\mathcal{B}\emptyset = \emptyset.(\{\leq\} \diamond \emptyset))$$

(A_{MX19})(Substitution of Eqn6 into Eqn7)

$$EE \not\subseteq (\mathcal{B}\emptyset = \emptyset.(\{\leq\} \diamond \emptyset))$$

(T_{M001})(Universality of (T_{M670}) Null Set Theorem)

$$E \not\subseteq (\mathcal{B}\emptyset = \emptyset.(\{\leq\} \diamond \emptyset))$$

(T_{M667})(Conjunction Tautology)

$$E \neq (\mathcal{B}\emptyset = \emptyset.(\{\leq\} \diamond \emptyset))$$

(T_{M275})(Supersets of the Universal Set) &

(T_{M008})(Inverse of an Equality)

$$E \neq (\emptyset = \emptyset.(\{\leq\} \diamond \emptyset))$$

(T_{S181})(Null Object Theorem)

$$E \neq (\emptyset = \emptyset)$$

(T_{M050})(Null Set Conjunction)

$$E \neq E$$

(T_{M343})

$$\emptyset$$

(T_{M345})

$$\text{Thus } ((\alpha_n = \emptyset) = E)((\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \alpha_n.(\{\leq\} \diamond \alpha_n)) = \emptyset$$

(A_{MX02})(Conditional Introduction)

$$((\alpha_n = \emptyset) = E) \subseteq ((\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n.(\{\leq\} \diamond \alpha_n))$$

(T_{M735})($\subseteq$ Existential Form)

$$(\alpha_n = \emptyset)(\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n.(\{\leq\} \diamond \alpha_n))$$

(A_{MX35})(Exportation/Importation) for truth sets (Eqn8)

Let $\alpha_n = \alpha_n^1$

(Eqn9)

$$\text{Let } (\alpha^1 \subseteq S_D S_0)(\alpha_n \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \alpha_n.(\{\leq\} \diamond \alpha_n))$$

(Eqn10)

$$(\alpha^1 \subseteq S_D S_0)(\alpha_n^1 \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \alpha_n^1.(\{\leq\} \diamond \alpha_n^1))$$

(A_{MX19})(Substitution of Eqn9 into Eqn10)

$$(\alpha^1 \subseteq S_D S_0)(\alpha_n^1 \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \alpha_n^1.(\{\leq\} \ast \alpha_n^1))$$

(T_{R232})(Interchangeability of $\ast$ and $\diamond$ for Entities)

$$(\alpha^1 \subseteq S_D S_0)(\alpha_n^1 \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \alpha_n^1.(\{\<\} + \{=\}) \ast \alpha_n^1) \quad (D_{A141})(\text{Disjunction with } \{=\})$$

$$(\alpha^1 \subseteq S_D S_0)(\alpha_n^1 \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \alpha_n^1.(\{\<\} \ast \alpha_n^1 + \{=\} \ast \alpha_n^1))$$

$(\alpha^1 \subseteq S_D S_O)(\alpha_n^1 \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \alpha_n^1 \cdot \{\leq\} * \alpha_n^1 + \alpha_n^1)$	(T_{R002c}) (Distribution of $+$ over $*$)
$(\alpha^1 \subseteq S_D S_O)(\alpha_n^1 \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \alpha_n^1 \cdot \{\leq\} * \alpha_n^1 + \alpha_n^1 \cdot \alpha_n^1)$	(D_{A144}) (Definition of the Reflexive Relationship Set)
$\alpha_n^1 \subseteq \{\leq\} * \alpha_n^1$	(T_{M407}) (Distribution of $.$ over $+$)(Eqn11)
$\alpha_n^1 \cdot \{\leq\} * \alpha_n^1 = \emptyset$	(T_{T119})
$(\alpha^1 \subseteq S_D S_O)(\alpha_n^1 \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \emptyset + \alpha_n^1 \cdot \alpha_n^1)$	(T_{M735}) ($\subseteq$ Existential Form)(Eqn12)
$(\alpha^1 \subseteq S_D S_O)(\alpha_n^1 \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \alpha_n^1 \cdot \alpha_n^1)$	(A_{MX19}) (Substitution of Eqn12 into Eqn11)
$(\alpha^1 \subseteq S_D S_O)(\alpha_n^1 \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \alpha_n^1)$	(T_{M668}) (Null Set Disjunction)
$(\alpha^1 \subseteq S_D S_O)(\alpha_n^1 \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \alpha_n^1)$	(T_{M667}) (Conjunction Tautology)
$\emptyset$	(T_{M024}) (Non-existence of Anti- (T_{T978}))
Thus $((\alpha_n^1 = \alpha_n^1) = E)((\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \not\subseteq (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n)) = \emptyset$	(A_{MX02}) (Conditional Introduction)
$((\alpha_n^1 = \alpha_n^1) = E) \subseteq ((\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n))$	(T_{M735}) ($\subseteq$ Existential Form)
$(\alpha_n^1 = \alpha_n^1)(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n)$	(A_{MX35}) (Exportation/Importation) for truth sets(Eqn13)
$(\alpha_n = \emptyset)(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) + (\alpha_n^1 = \alpha_n^1)(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n) + (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n)$	(A_{MX21}) (Combination of Eqns8&13)
$(\alpha_n = \emptyset)(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) + (\alpha_n^1 = \alpha_n^1)(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n)$	(A_{MX01}) (Disjunction Tautology)
$((\alpha_n = \emptyset) + (\alpha_n^1 = \alpha_n^1))(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n)$	(T_{M407}) (Distribution of $.$ over $+$)
$\sim((\alpha_n \neq \emptyset)(\alpha_n^1 \neq \alpha_n^1))(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n)$	(T_{M025}) (DeMorgan's Law)(Eqn14)
$(\alpha_n \neq \emptyset)(\alpha_n \neq \alpha_n^1)(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) + \sim((\alpha_n \neq \emptyset)(\alpha_n^1 \neq \alpha_n^1))(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n) + (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n)$	(A_{MX21}) (Combination of Eqns5&14)
$(\alpha_n \neq \emptyset)(\alpha_n \neq \alpha_n^1)(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) + \sim((\alpha_n \neq \emptyset)(\alpha_n^1 \neq \alpha_n^1))(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n)$	(A_{MX01}) (Disjunction Tautology)
$((\alpha_n \neq \emptyset)(\alpha_n \neq \alpha_n^1) + \sim((\alpha_n \neq \emptyset)(\alpha_n^1 \neq \alpha_n^1)))(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n)$	(T_{M407}) (Distribution of $.$ over $+$)
$E(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n)$	(T_{M671}) (Tertium non Datur)
$(\alpha^1 \subseteq S_D S_O)(\alpha_n \subseteq \alpha) \subseteq (\mathcal{B}\alpha_n^1 = \alpha_n \cdot \{\leq\} \diamond \alpha_n)$	(A_{MX11}) (Universal Set Conjunction Axiom)

$$(\alpha^1 \subseteq S_D S_O) \subseteq (\mathcal{B}\alpha^1 = \alpha \cdot \{\leq\} \diamond \alpha) \quad (T_{S730})$$

$$(\alpha^1 \subseteq S_D S_O) \subseteq (C\alpha^1 = \alpha \cdot \{\geq\} \diamond \alpha) \quad (T_{S733})$$

Example Proof of (T_{S730})

$(\alpha^1 \subseteq S_D S_O) \subseteq (\mathcal{B}\alpha^1 = \alpha \cdot \{\geq\} * \alpha)$	(T_{S818}) (Eqn1)
$(\alpha^1 \subseteq S_D S_O) \subseteq (\alpha \cdot \{\geq\} * \alpha = \alpha \cdot \{\leq\} \diamond \alpha)$	(T_{S826}) (Eqn2)
$(\alpha^1 \subseteq S_D S_O) \subseteq (\mathcal{B}\alpha^1 = \alpha \cdot \{\leq\} \diamond \alpha)$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)

D. Bounds of Extent in Sets of Points

$$(\alpha^1 \subseteq S_P S_O) \subseteq (\mathcal{B}\alpha \neq \emptyset) \quad (T_{T036})$$

$$(\alpha^1 \subseteq S_P S_O) \subseteq (C\alpha \neq \emptyset) \quad (T_{T037})$$

Example Proof of (T_{T036})

$(\alpha^1 \subseteq S_O) \subseteq (\mathcal{B}\alpha \neq \emptyset)$	(T_{S887}) (Eqn1)
$S_P \subseteq S_D$	(A_{R53}) (Axiom of Points)
$S_P S_O \subseteq S_D S_O$	(T_{M029}) (Deduction)
$S_P S_O \subseteq S_O$	(T_{M073}) (Single Premise Syllogism)
$(S_P S_O \subseteq S_O) = E$	(A_{MX16}) (Order Axiom)

$$\begin{array}{ll}
(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha^1 \subseteq S_O) & (T_{M272})(\text{Deductive Rule})(\text{Eqn2}) \\
(\alpha^1 \subseteq S_P S_O) \subseteq (B\alpha \neq \emptyset) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})
\end{array}$$

$$((X = B, C) = E) \subseteq ((\alpha^1 \subseteq S_P S_O) \subseteq (\alpha = X\alpha).(\alpha \subseteq \forall X\alpha)) \quad (T_{T250})$$

Proof of (T_{T250})

$$\begin{array}{ll}
((X = B, C) = E) \subseteq ((\alpha^1 \subseteq S_O S_O) \subseteq (\alpha = X\alpha).(\alpha \subseteq \forall X\alpha)) & (T_{T251}) \\
(X = B, C)(\alpha^1 \subseteq S_O S_O) \subseteq (\alpha = X\alpha).(\alpha \subseteq \forall X\alpha) & (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\
& (\text{Eqn1}) \\
S_P \subseteq S_O & (A_{R53})(\text{Axiom of Points}) \\
S_P S_O \subseteq S_O S_O & (T_{M029})(\text{Deduction}) \\
(S_P S_O \subseteq S_O S_O) = E & (A_{MX16})(\text{Order Axiom}) \\
(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha^1 \subseteq S_O S_O) & (T_{M272})(\text{Deductive Rule}) \\
(X = B, C)(\alpha^1 \subseteq S_P S_O) \subseteq (X = B, C)(\alpha^1 \subseteq S_O S_O) & (T_{M029})(\text{Deduction})(\text{Eqn2}) \\
(X = B, C)(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha = X\alpha).(\alpha \subseteq \forall X\alpha) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\
((X = B, C) = E) \subseteq ((\alpha^1 \subseteq S_P S_O) \subseteq (\alpha = X\alpha).(\alpha \subseteq \forall X\alpha)) & (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets}
\end{array}$$

$$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \forall B\alpha) \quad (T_{T226})$$

$$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \forall C\alpha) \quad (T_{T227})$$

Example Proof of (T_{T226})

$$\begin{array}{ll}
((X = B, C) = E) \subseteq ((\alpha^1 \subseteq S_P S_O) \subseteq (\alpha = X\alpha).(\alpha \subseteq \forall X\alpha)) & (T_{T250}) \\
((X = B, C) = E) \subseteq ((\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \forall X\alpha)) & (T_{M073})(\text{Single Premise Syllogism}) \\
(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \forall B\alpha) & \text{Extraction}
\end{array}$$

$$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \forall B\alpha. \forall C\alpha) \quad (T_{T228})$$

Proof of (T_{T228})

$$\begin{array}{ll}
(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \forall B\alpha) & (T_{T226})(\text{Eqn1}) \\
(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \forall C\alpha) & (T_{T227})(\text{Eqn2}) \\
(\alpha^1 \subseteq S_P S_O)(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \forall B\alpha)(\alpha \subseteq \forall C\alpha) & (T_{M669})(\text{Combination of Eqns1\&2}) \\
(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \forall B\alpha)(\alpha \subseteq \forall C\alpha) & (T_{M667})(\text{Conjunction Tautology}) \\
(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \forall B\alpha. \forall C\alpha) & (T_{M132})(\text{Composition})
\end{array}$$

$$(\alpha^1 \subseteq S_P S_O) \subseteq (\forall B\alpha = \forall C\alpha) \quad (T_{T229})$$

Proof of (T_{T229})

$$\begin{array}{ll}
(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \forall B\alpha) & (T_{T226})(\text{Eqn1}) \\
(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha \subseteq \forall C\alpha) & (T_{T227})(\text{Eqn2}) \\
(\alpha^1 \subseteq S_P S_O) \subseteq (\forall B\alpha = \forall C\alpha) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})
\end{array}$$

E. Bounds of Extent in Mixed Ordered Sets

$$(a^1 \subseteq S_M S_O) \subseteq (\alpha = B\alpha) \quad (T_{S916})$$

$$(a^1 \subseteq S_M S_O) \subseteq (\alpha = C\alpha) \quad (T_{S917})$$

Example Proof of (T_{S916})

$$\begin{array}{ll}
((X = B, C) = E) \subseteq ((a^1 \subseteq S_M S_O) \subseteq (\alpha = X\alpha)(\forall X\alpha \subseteq a)) & (A_{R64})(\text{Axiom of Mixed Ordered Sets}) \\
(a^1 \subseteq S_M S_O) \subseteq (\alpha = B\alpha)(a \subseteq \forall B\alpha) & \text{Extraction} \\
(a^1 \subseteq S_M S_O) \subseteq (\alpha = B\alpha) & (T_{M073})(\text{Single Premise Syllogism})
\end{array}$$

$$(a^1 \subseteq S_M S_0) \subseteq (Ba = Ca) \quad (T_{7094})$$

Proof of (T_{7094})

$$\begin{aligned} (a^1 \subseteq S_M S_0) &\subseteq (\alpha = Ba) && (T_{S916})(\text{Eqn1}) \\ (a^1 \subseteq S_M S_0) &\subseteq (\alpha = Ca) && (T_{S917})(\text{Eqn2}) \\ (a^1 \subseteq S_M S_0)(a^1 \subseteq S_M S_0) &\subseteq (\alpha = Ba)(\alpha = Ca) && (T_{M669})(\text{Combination of Eqns1\&2}) \\ (a^1 \subseteq S_M S_0) &\subseteq (\alpha = Ba)(\alpha = Ca) && (T_{M667})(\text{Conjunction Tautology}) \\ (a^1 \subseteq S_M S_0) &\subseteq (Ba = Ca) && (A_{MX19})(\text{Substitution}) \end{aligned}$$

$$(a^1 \subseteq S_M S_0 + S_D S_0) \subseteq (Ba = Ca) \quad (T_{7944})$$

$$(a^1 \subseteq S_M S_0 + S_D S_0) \subseteq (a.\sim(Ba) = a.\sim(Ca)) \quad (T_{7950})$$

Proof of (T_{7944}) & (T_{7950})

$$\begin{aligned} (a^1 \subseteq S_M S_0) &\subseteq (Ba = Ca) && (T_{7094})(\text{Eqn1}) \\ (\alpha^1 \subseteq S_D S_0) &\subseteq (Ba = Ca) && (T_{7932}) \\ (a^1 \subseteq S_D S_0) &\subseteq (Ba = Ca) && \text{Resymbolising (Eqn2)} \\ (a^1 \subseteq S_M S_0) + (a^1 \subseteq S_D S_0) &\subseteq (Ba = Ca) + (Ba = Ca) && (A_{MX21})(\text{Combination of Eqns1\&2}) \\ (a^1 \subseteq S_M S_0) + (a^1 \subseteq S_D S_0) &\subseteq (Ba = Ca) && (A_{MX01})(\text{Disjunction Tautology}) \\ (a^1 \subseteq S_M S_0 + S_D S_0) &\subseteq (Ba = Ca) && (T_{M526}) \\ (a^1 \subseteq S_M S_0 + S_D S_0) &\subseteq (\sim(Ba) = \sim(Ca)) && (T_{M008})(\text{Inverse of an Equality}) \\ (a^1 \subseteq S_M S_0 + S_D S_0) &\subseteq (a.\sim(Ba) = a.\sim(Ca)) && (T_{M029})(\text{Deduction}) \end{aligned}$$

$$(a^1 \subseteq S_M S_0) \subseteq (\alpha = Ba + Ca) \quad (T_{7980})$$

Proof of (T_{7980})

$$\begin{aligned} (a^1 \subseteq S_M S_0) &\subseteq (\alpha = Ba) && (T_{S916})(\text{Eqn1}) \\ (a^1 \subseteq S_M S_0) &\subseteq (\alpha = Ca) && (T_{S917})(\text{Eqn2}) \\ (a^1 \subseteq S_M S_0)(a^1 \subseteq S_M S_0) &\subseteq (\alpha = Ba)(\alpha = Ca) && (T_{M669})(\text{Combination of Eqns1\&2}) \\ (a^1 \subseteq S_M S_0) &\subseteq (\alpha = Ba)(\alpha = Ca) && (T_{M667})(\text{Conjunction Tautology}) \\ (a^1 \subseteq S_M S_0) &\subseteq (\alpha + \alpha = Ba + Ca) && (A_{MX21})(\text{Combination}) \\ (a^1 \subseteq S_M S_0) &\subseteq (\alpha = Ba + Ca) && (A_{MX01})(\text{Disjunction Tautology}) \end{aligned}$$

$$(a^1 \subseteq S_M S_0) \subseteq (\alpha^1 \subseteq S_P S_0) \quad (T_{7777})$$

Proof of (T_{7777})

$$\begin{aligned} ((X = B, C) = E) &\subseteq ((a^1 \subseteq S_M S_0) \subseteq ((Xa)^1 \subseteq S_P S_0)) && (T_{7778}) \\ (a^1 \subseteq S_M S_0) &\subseteq ((Ba)^1 \subseteq S_P S_0) && \text{Extraction (Eqn1)} \\ (a^1 \subseteq S_M S_0) &\subseteq ((Ca)^1 \subseteq S_P S_0) && \text{Extraction (Eqn2)} \\ (a^1 \subseteq S_M S_0)(a^1 \subseteq S_M S_0) &\subseteq ((Ba)^1 \subseteq S_P S_0)((Ca)^1 \subseteq S_P S_0) && (T_{M669})(\text{Combination of Eqns1\&2}) \\ (a^1 \subseteq S_M S_0) &\subseteq ((Ba)^1 \subseteq S_P S_0)((Ca)^1 \subseteq S_P S_0) && (T_{M667})(\text{Conjunction Tautology}) \\ (a^1 \subseteq S_M S_0) &\subseteq ((Ba)^1 + (Ca)^1 \subseteq S_P S_0) && (T_{M131b})(\text{Conjunction of Sufficiencies}) \\ (a^1 \subseteq S_M S_0) &\subseteq ((Ba + Ca)^1 \subseteq S_P S_0) && (A_{R88})(\text{Disjunctions of Sets of Points}) \\ (a^1 \subseteq S_M S_0) &\subseteq (\alpha^1 \subseteq S_P S_0) && (T_{7980}) \end{aligned}$$

$$(a^1 \subseteq S_M S_0) \subseteq ((Ba + Ca)^1 \subseteq S_P S_0) \quad (T_{7143}) \text{ Beginnings and Ends in a Mixed Ordered Set}$$

Proof of (T_{7143})

$$\begin{aligned} (a^1 \subseteq S_M S_0) &\subseteq (\alpha^1 \subseteq S_P S_0) && (T_{7777})(\text{Eqn1}) \\ (a^1 \subseteq S_M S_0) &\subseteq (\alpha = Ba + Ca) && (T_{7980}) \\ (a^1 \subseteq S_M S_0) &\subseteq (\alpha^1 = (Ba + Ca)^1) && (T_{M479})(\text{Eqn2}) \\ (a^1 \subseteq S_M S_0) &\subseteq ((Ba + Ca)^1 \subseteq S_P S_0) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(a^1 \subseteq S_M S_O) \subseteq ((Ba)^1 \subseteq S_P S_O) \quad (T_{S927})$$

$$(a^1 \subseteq S_M S_O) \subseteq ((Ca)^1 \subseteq S_P S_O) \quad (T_{S928})$$

Example Proof of (T_{S927})

$$(a^1 \subseteq S_M S_O) \subseteq ((Ba + Ca)^1 \subseteq S_P S_O) \quad (T_{143})(\text{Beginnings and Ends in a Mixed Ordered Set})(\text{Eqn1})$$

$$((Ba + Ca)^1 \subseteq S_P S_O) = ((Ba)^1 \subseteq S_P S_O)((Ca)^1 \subseteq S_P S_O) \quad (T_{S929})(\text{Eqn2})$$

$$(a^1 \subseteq S_M S_O) \subseteq ((Ba)^1 \subseteq S_P S_O)((Ca)^1 \subseteq S_P S_O) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(a^1 \subseteq S_M S_O) \subseteq ((Ba)^1 \subseteq S_P S_O) \quad (T_{M073})(\text{Single Premise Syllogism})$$

$$(a^1 \subseteq S_M S_O) \subseteq ((a \sim (Ba + Ca))^1 \subseteq S_C S_O) \quad (T_{936})$$

Proof of (T_{936})

$$((X = B, C) = E) \subseteq ((a^1 \subseteq S_M S_O) = ((a \sim (Xa))^1 \subseteq S_C S_O)) \quad (A_{S75})(\text{Spans are Continua})$$

$$(a^1 \subseteq S_M S_O) = ((a \sim (Ba))^1 \subseteq S_C S_O) \quad \text{Extraction (Eqn1)}$$

$$(a^1 \subseteq S_M S_O) = ((a \sim (Ca))^1 \subseteq S_C S_O) \quad \text{Extraction (Eqn2)}$$

$$(a^1 \subseteq S_M S_O)(a^1 \subseteq S_M S_O) = ((a \sim (Ba))^1 \subseteq S_C S_O)((a \sim (Ca))^1 \subseteq S_C S_O) \quad (T_{M669})(\text{Combination of Eqns1\&2})$$

$$(a^1 \subseteq S_M S_O) = ((a \sim (Ba))^1 \subseteq S_C S_O)((a \sim (Ca))^1 \subseteq S_C S_O) \quad (T_{M667})(\text{Conjunction Tautology})$$

$$(a^1 \subseteq S_M S_O) = ((a \sim (Ba))^1 + (a \sim (Ca))^1 \subseteq S_C S_O) \quad (T_{M131b})(\text{Conjunction of Sufficiencies})$$

$$(a^1 \subseteq S_M S_O) = (a \sim (Ba) + a \sim (Ca) \subseteq S_C S_O) \quad (A_{R01A})(\text{Axiom of Continuum Sets})$$

$$(a^1 \subseteq S_M S_O) = (a \sim (Ba) + \sim(Ca)) \subseteq S_C S_O \quad (T_{M407})(\text{Distribution of } \sim \text{ over } +)(\text{Eqn3})$$

$$\sim(Ba) \sim(Ca) \subseteq \sim(Ba) + \sim(Ca) \quad (T_{M146})(\text{Conjunction a Subset of Disjunction})$$

$$a \sim (Ba) \sim(Ca) \subseteq a \sim (Ba) + \sim(Ca) \quad (T_{M029})(\text{Deduction})(\text{Eqn4})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a \sim (Ba) \sim(Ca)) \subseteq S_C S_O \quad (T_{M139})(\text{Hypothetical Syllogism Eqns4\&3})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a \sim (Ba + Ca)) \subseteq S_C S_O \quad (T_{M026})(\text{DeMorgan's Law})$$

$$(a^1 \subseteq S_M S_O) \subseteq ((a \sim (Ba + Ca))^1 \subseteq S_C S_O) \quad (A_{R01A})(\text{Axiom of Continuum Sets})$$

$$(a^1 \subseteq S_M S_O) \subseteq ((\alpha \sim a)^1 \subseteq S_C S_O) \quad (T_{S946})$$

Proof of (T_{S946})

$$(a^1 \subseteq S_M S_O) \subseteq ((a \sim (Ba + Ca))^1 \subseteq S_C S_O) \quad (T_{936})(\text{Eqn1})$$

$$(a^1 \subseteq S_M S_O) \subseteq (\alpha = Ba + Ca) \quad (T_{980})(\text{Eqn2})$$

$$(a^1 \subseteq S_M S_O) \subseteq ((a \sim \alpha)^1 \subseteq S_C S_O) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(a^1 \subseteq S_M S_O) \subseteq (\forall Ba \subseteq a) \quad (T_{S918})$$

$$(a^1 \subseteq S_M S_O) \subseteq (\forall Ca \subseteq a) \quad (T_{S919})$$

Example Proof of (T_{S918})

$$((X = B, C) = E) \subseteq ((a^1 \subseteq S_M S_O) \subseteq (\alpha = Xa)(\forall Xa \subseteq a)) \quad (A_{R64})(\text{Axiom of Mixed Ordered Sets})$$

$$(a^1 \subseteq S_M S_O) \subseteq (\alpha = Ba)(\forall Ba \subseteq a) \quad \text{Extraction}$$

$$(a^1 \subseteq S_M S_O) \subseteq (\forall Ba \subseteq a) \quad (T_{M073})(\text{Single Premise Syllogism})$$

$$(a^1 \subseteq S_M S_O) \subseteq (\forall B Ba \subseteq a) \quad (T_{S866})$$

$$(a^1 \subseteq S_M S_O) \subseteq (\forall B Ca \subseteq a) \quad (T_{S867})$$

$$(a^1 \subseteq S_M S_O) \subseteq (\forall C Ba \subseteq a) \quad (T_{S868})$$

$$(a^1 \subseteq S_M S_O) \subseteq (\forall C Ca \subseteq a) \quad (T_{S869})$$

Example Proof of (T_{S866})

$$(a^1 \subseteq S_M S_O) \subseteq (\forall Ba \subseteq a) \quad (T_{S918})(\text{Eqn1})$$

$$(a^1 \subseteq S_M S_O) \subseteq (\alpha = Ba) \quad (T_{S918})(\text{Eqn2})$$

$$(a^1 \subseteq S_M S_O) \subseteq (\forall B B a \subseteq a)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(a^1 \subseteq S_M S_O) \subseteq (\forall B a. \forall C a \subseteq a)$$

(T_{S871})

Proof of (T_{S871})

$$(a^1 \subseteq S_M S_O) \subseteq (\forall B a \subseteq a)$$

(T_{S918})(Eqn1)

$$(a^1 \subseteq S_M S_O) \subseteq (\forall C a \subseteq a)$$

(T_{S919})(Eqn2)

$$(a^1 \subseteq S_M S_O)(a^1 \subseteq S_M S_O) \subseteq (\forall B a \subseteq a)(\forall C a \subseteq a)$$

(T_{M669})(Combination of Eqns1&2)

$$(a^1 \subseteq S_M S_O) \subseteq (\forall B a \subseteq a)(\forall C a \subseteq a)$$

(T_{M667})(Conjunction Tautology)

$$(a^1 \subseteq S_M S_O) \subseteq (\forall B a. \forall C a \subseteq a.a)$$

(T_{M669})(Combination)

$$(a^1 \subseteq S_M S_O) \subseteq (\forall B a. \forall C a \subseteq a)$$

(T_{M667})(Conjunction Tautology)

$$(a^1 \subseteq S_M S_O) \subseteq ((\underline{a}. \forall B a. \forall C a)^1 \subseteq S_C S_O)$$

(T_{S914})

Proof of (T_{S914})

$$(a^1 \subseteq S_M S_O) \subseteq ((\underline{a}. a)^1 \subseteq S_C S_O)$$

(T_{S946})(Eqn1)

$$(a^1 \subseteq S_M S_O) \subseteq (\forall B a. \forall C a \subseteq a)$$

(T_{S871})

$$(a^1 \subseteq S_M S_O) \subseteq (\underline{a}. \forall B a. \forall C a \subseteq \underline{a}. a)$$

(T_{M029})(Deduction)(Eqn2)

$$(a^1 \subseteq S_M S_O) \subseteq (\underline{a}. \forall B a. \forall C a \neq \emptyset)(\underline{a}. \forall B a. \forall C a \neq (\underline{a}. \forall B a. \forall C a)^1)$$

(A_{S54})(Axiom of Finite Spans)(Eqn3)

$$(a^1 \subseteq S_M S_O)(a^1 \subseteq S_M S_O)(a^1 \subseteq S_M S_O) \subseteq$$

$$((\underline{a}. a)^1 \subseteq S_C S_O)(\underline{a}. \forall B a. \forall C a \subseteq \underline{a}. a)(\underline{a}. \forall B a. \forall C a \neq \emptyset)(\underline{a}. \forall B a. \forall C a \neq (\underline{a}. \forall B a. \forall C a)^1)$$

(T_{M669})(Combination of Eqns1,2&3)

$$(a^1 \subseteq S_M S_O) \subseteq$$

$$((\underline{a}. a)^1 \subseteq S_C S_O)(\underline{a}. \forall B a. \forall C a \subseteq \underline{a}. a)(\underline{a}. \forall B a. \forall C a \neq \emptyset)(\underline{a}. \forall B a. \forall C a \neq (\underline{a}. \forall B a. \forall C a)^1)$$

(T_{M667})(Conjunction Tautology)

$$(a^1 \subseteq S_M S_O) \subseteq ((\underline{a}. \forall B a. \forall C a)^1 \subseteq S_C S_O)$$

(T_{T991})

$$(a^1 \subseteq S_M S_O)(\beta^1 \subseteq a) \subseteq (\forall B \beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1)).(\forall C \beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1))$$

(T_{T114})

Proof of (T_{T114})

$$(a^1 \subseteq S_M S_O + S_D S_O)(\beta^1 \subseteq a) \subseteq (\forall B \beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1)).(\forall C \beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1))$$

(D_{R50})(Definition of $\forall B$ & $\forall C$)(Eqn1)

$$S_M S_O \subseteq S_M S_O + S_D S_O$$

(T_{M082})(Disjunction Introduction)

$$(a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)$$

(T_{M272})(Deductive Rule)

$$(a^1 \subseteq S_M S_O)(\beta^1 \subseteq a) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)(\beta^1 \subseteq a)$$

(T_{M029})(Deduction)(Eqn2)

$$(a^1 \subseteq S_M S_O)(\beta^1 \subseteq a) \subseteq (\forall B \beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1)).(\forall C \beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1))$$

(T_{M139})(Hypothetical Syllogism Eqns2&1)

F. Bounds of Extent in Mixed and Discrete Ordered Sets

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (B a^1 = B a^1)$$

(T_{T077})

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (C a^1 = C a^1)$$

(T_{T078})

Example Proof of (T_{T077})

$$(X = B, C)(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (X a^1 = X a^1)$$

(A_{S55})(Bounds of Extent of Sets)

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (B a^1 = B a^1)$$

Extraction

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (B a \neq \emptyset)$$

(T_{S889})

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (C a \neq \emptyset)$$

(T_{S890})

Example Proof of (T_{S889})

$(a^1 \subseteq S_o) = (Ba \neq \emptyset)$	(T _{S887})(Eqn1)
$S_M S_o \subseteq S_o$	(T _{M083})(Conjunction Elimination)(Eqn2)
$S_D S_o \subseteq S_o$	(T _{M083})(Conjunction Elimination)(Eqn3)
$S_M S_o + S_D S_o \subseteq S_o + S_o$	(A _{MX21})(Combination of Eqns2&3)
$S_M S_o + S_D S_o \subseteq S_o$	(A _{MX01})(Disjunction Tautology)
$(S_M S_o + S_D S_o \subseteq S_o) = E$	(A _{MX16})(Order Axiom)
$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (a^1 \subseteq S_o)$	(T _{M272})(Deductive Rule)(Eqn4)
$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (Ba \neq \emptyset)$	(A _{MX19})(Substitution of Eqn1 into Eqn4)

$$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (Ba + Ca \neq \emptyset) \quad (T_{S891})$$

Proof of (T_{S891})

$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (Ba \neq \emptyset)$	(T _{S889})(Eqn1)
$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (Ca \neq \emptyset)$	(T _{S890})(Eqn2)
$(a^1 \subseteq S_M S_o + S_D S_o). (a^1 \subseteq S_M S_o + S_D S_o) \subseteq (Ba \neq \emptyset). (Ca \neq \emptyset)$	(T _{M669})(Combination of Eqns1&2)
$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (Ba \neq \emptyset). (Ca \neq \emptyset)$	(T _{M667})(Conjunction Tautology)(Eqn3)
$(Ba \neq \emptyset). (Ca \neq \emptyset) \subseteq (Ba + Ca \neq \emptyset)$	(T _{M312})(Probabilistic Conjunction)(Eqn4)
$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (Ba + Ca \neq \emptyset)$	(T _{M139})(Hypothetical Syllogism Eqns3&4)

$$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (Ba = Ca) \quad (T_{T932c})$$

Proof of (T_{T932c})

$(a^1 \subseteq S_M S_o) \subseteq (Ba = Ca)$	(T _{T094})(Eqn1)
$(a^1 \subseteq S_D S_o) \subseteq (Ba = Ca)$	(T _{T932})
$(a^1 \subseteq S_D S_o) \subseteq (Ba = Ca)$	Resymbolising (Eqn2)
$(a^1 \subseteq S_M S_o) + (a^1 \subseteq S_D S_o) \subseteq (Ba = Ca) + (Ba = Ca)$	(A _{MX21})(Combination of Eqns1&2)
$(a^1 \subseteq S_M S_o) + (a^1 \subseteq S_D S_o) \subseteq (Ba = Ca)$	(A _{MX01})(Disjunction Tautology)
$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (Ba = Ca)$	(T _{M526})

$$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\alpha = Ba + Ca) \quad (T_{T931b}) \text{ Definition of } \alpha \text{ from } a$$

Proof of (T_{T931b})

$(a^1 \subseteq S_M S_o) \subseteq (\alpha = Ba + Ca)$	(T _{T980})(Eqn1)
$(a^1 \subseteq S_D S_o) \subseteq (\alpha = Ba + Ca)$	(T _{T931})
$(a^1 \subseteq S_D S_o) \subseteq (\alpha = Ba + Ca)$	Resymbolising (Eqn2)
$(a^1 \subseteq S_M S_o) + (a^1 \subseteq S_D S_o) \subseteq (\alpha = Ba + Ca) + (\alpha = Ba + Ca)$	(A _{MX21})(Combination of Eqns1&2)
$(a^1 \subseteq S_M S_o) + (a^1 \subseteq S_D S_o) \subseteq (\alpha = Ba + Ca)$	(A _{MX01})(Disjunction Tautology)
$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\alpha = Ba + Ca)$	(T _{M526})

$$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\alpha \neq \emptyset) \quad (T_{S882})$$

$$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\alpha^1 \neq \emptyset) \quad (T_{S883})$$

Proof of (T_{S882}) & (T_{S883})

$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (Ba + Ca \neq \emptyset)$	(T _{S891})(Eqn1)
$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\alpha = Ba + Ca)$	(T _{T931b})(Definition of α from a)(Eqn2)
$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\alpha \neq \emptyset)$	(A _{MX19})(Substitution of Eqn1 into Eqn2)(Eqn3)
$(\alpha \neq \emptyset) \subseteq (\alpha^1 \neq \emptyset)$	(T _{M503})(Eqn4)
$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\alpha^1 \neq \emptyset)$	(T _{M139})(Hypothetical Syllogism Eqns3&4)

$$(a^1 \subseteq S_M S_o + S_D S_o) \subseteq (\alpha. (\{ \subseteq \} \alpha) \neq \emptyset) \quad (T_{T017})$$

Proof of (T₇₀₁₇)

$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\alpha \neq \emptyset)$	(T _{S882})(Eqn1)
$(a^1 \subseteq S_D) \subseteq (a \subseteq \{\subseteq\}^* a)$	(T _{R719A})(Eqn2)
$S_D S_O \subseteq S_D$	(T _{M083})(Conjunction Elimination)
$(a^1 \subseteq S_D S_O) \subseteq (a^1 \subseteq S_D)$	(T _{M099})(Entity Mode / Set Mode Rule)(Eqn3)
$(a^1 \subseteq S_D S_O) \subseteq (a \subseteq \{\subseteq\}^* a)$	(T _{M139})(Hypothetical Syllogism Eqns3&2)
$(a^1 \subseteq S_D S_O) \subseteq (\alpha \subseteq \{\subseteq\}^* \alpha)$	(D _{S37})(Definition of α Symbolism)(Eqn4)
$(a^1 \subseteq S_D S_O) = (\alpha = a)$	(D _{S37})(Definition of α Symbolism)
$\alpha^1 \subseteq S_D S_O$	(A _{MX19})(Substitution)
$(\alpha^1 \subseteq S_D S_O) = E$	(A _{MX16})(Order Axiom)(Eqn5)
$E \subseteq (\alpha \subseteq \{\subseteq\}^* \alpha)$	(A _{MX19})(Substitution of Eqn5 into Eqn4)
$E = (\alpha \subseteq \{\subseteq\}^* \alpha)$	(T _{M275})(Supersets of the Universal Set)
$\alpha \subseteq \{\subseteq\}^* \alpha$	(A _{MX16})(Order Axiom)
$\alpha = \alpha.(\{\subseteq\}^* \alpha)$	(A _{MX44})($\subseteq$ Equality Form)(Eqn6)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\alpha.(\{\subseteq\}^* \alpha) \neq \emptyset)$	(A _{MX19})(Substitution of Eqn6 into Eqn1)

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\alpha^1 = (Ba + Ca)^1) \quad (T_{S895})$$

Proof of (T_{S895})

$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\alpha = Ba + Ca)$	(T _{T931b})(Definition of α from a)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\alpha^1 = (Ba + Ca)^1)$	(T _{M479})

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba + Ca = a.(\{\subseteq\}^* a)) \quad (T_{7059})$$

Proof of (T₇₀₅₉)

$(a^1 \subseteq S_M S_O + S_D S_O)(X = B, C) \subseteq (Xa = a.(\{\subseteq\}^* a))$	(A _{S74})(4 th Axiom of Bounds of Extent)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba = a.(\{\subseteq\}^* a))$	Extraction (Eqn1)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ca = a.(\{\subseteq\}^* a))$	Extraction (Eqn2)
$(a^1 \subseteq S_M S_O + S_D S_O). (a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba = a.(\{\subseteq\}^* a)). (Ca = a.(\{\subseteq\}^* a))$	(T _{M669})(Combination of Eqns1&2)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba = a.(\{\subseteq\}^* a)). (Ca = a.(\{\subseteq\}^* a))$	(T _{M667})(Conjunction Tautology)(Eqn3)
$(Ba = a.(\{\subseteq\}^* a)). (Ca = a.(\{\subseteq\}^* a)) \subseteq (Ba + Ca = a.(\{\subseteq\}^* a) + a.(\{\subseteq\}^* a))$	(T _{M230})(Combination)
$(Ba = a.(\{\subseteq\}^* a)). (Ca = a.(\{\subseteq\}^* a)) \subseteq (Ba + Ca = a.(\{\subseteq\}^* a))$	(A _{MX01})(Disjunction Tautology)(Eqn4)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba + Ca = a.(\{\subseteq\}^* a))$	(T _{M139})(Hypothetical Syllogism Eqns3&4)

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba^1 = a.(\{\>\}^* a).(\{\subseteq\}^* a)) \quad (T_{7986})$$

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ca^1 = a.(\{\<\}^* a).(\{\subseteq\}^* a)) \quad (T_{7987})$$

Example Proof of (T₇₉₈₆)

$(a^1 \subseteq S_M S_O + S_D S_O)(a_n \subseteq a) \subseteq (Ba_n^1 = a_n.(\{\>\}^* a_n).(\{\subseteq\}^* a_n))$	(D _{R53})(Definition of the Beginning Function)(Eqn1)
Let $a_n = a$	(Eqn2)
$(a^1 \subseteq S_M S_O + S_D S_O)(a \subseteq a) \subseteq (Ba^1 = a.(\{\>\}^* a).(\{\subseteq\}^* a))$	(A _{MX19})(Substitution of Eqn2 into Eqn1)
$(a^1 \subseteq S_M S_O + S_D S_O)((a \subset a) + (a = a)) \subseteq (Ba^1 = a.(\{\>\}^* a).(\{\subseteq\}^* a))$	(T _{M266})(Definition of $\subseteq$)
$(a^1 \subseteq S_M S_O + S_D S_O)((a \subset a) + E) \subseteq (Ba^1 = a.(\{\>\}^* a).(\{\subseteq\}^* a))$	(T _{M001})(Universality of (D _{M00}) Definition of $=$)
$(a^1 \subseteq S_M S_O + S_D S_O)E \subseteq (Ba^1 = a.(\{\>\}^* a).(\{\subseteq\}^* a))$	(T _{M053})(Universal Set Disjunction)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba^1 = a.(\{\>\}^* a).(\{\subseteq\}^* a))$	(A _{MX11})(Universal Set Conjunction Axiom)

$$((a^1 \subseteq S_M S_O + S_D S_O) = E) \subseteq ((a.(\{>\}_* a).(\{<\}_* a) \neq \emptyset) \subseteq (a^1 \subseteq Q B)) \quad (T_{R710})$$

$$((a^1 \subseteq S_M S_O + S_D S_O) = E) \subseteq ((a.(\{<\}_* a).(\{>\}_* a) \neq \emptyset) \subseteq (a^1 \subseteq Q C)) \quad (T_{R708})$$

Example Proof of (T_{R710})

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (B a^1 = a.(\{>\}_* a).(\{<\}_* a)) \quad (T_{986})$$

$$((a^1 \subseteq S_M S_O + S_D S_O) \subseteq (B a^1 = a.(\{>\}_* a).(\{<\}_* a))) = E \quad (A_{MX16})(\text{Order Axiom})$$

$$((a^1 \subseteq S_M S_O + S_D S_O) = E) \subseteq ((B a^1 = a.(\{>\}_* a).(\{<\}_* a)) = E)$$

(T_{M094})(Probabilistic Rule)

$$((a^1 \subseteq S_M S_O + S_D S_O) = E) \subseteq ((B a^1 \neq \emptyset) = (a.(\{>\}_* a).(\{<\}_* a) \neq \emptyset))$$

(T_{M523})(Probabilistic Rule)(Eqn1)

$$(B a^1 \neq \emptyset) \subseteq (a^1 \subseteq Q B)$$

(T_{S710})(Eqn2)

$$((a^1 \subseteq S_M S_O + S_D S_O) = E) \subseteq ((a.(\{>\}_* a).(\{<\}_* a) \neq \emptyset) \subseteq (a^1 \subseteq Q B))$$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

$$((a^1 \subseteq S_M S_O + S_D S_O) = E) \subseteq ((a.(\{<\}_* a) = \emptyset) \subseteq (B a^1 = \emptyset)) \quad (T_{S990})$$

$$((a^1 \subseteq S_M S_O + S_D S_O) = E) \subseteq ((a.(\{<\}_* a) = \emptyset) \subseteq (C a^1 = \emptyset)) \quad (T_{S991})$$

$$((a^1 \subseteq S_M S_O + S_D S_O) = E) \subseteq ((B a^1 \neq \emptyset) \subseteq (a.(\{<\}_* a) \neq \emptyset)) \quad (T_{S992})$$

$$((a^1 \subseteq S_M S_O + S_D S_O) = E) \subseteq ((C a^1 \neq \emptyset) \subseteq (a.(\{<\}_* a) \neq \emptyset)) \quad (T_{S993})$$

Example Proof of (T_{S990}) & (T_{S992})

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (B a^1 = a.(\{>\}_* a).(\{<\}_* a)) \quad (T_{986})$$

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (B a^1 \subseteq a.(\{<\}_* a)) \quad (T_{M073b})(\text{Single Premise Syllogism})$$

$$((a^1 \subseteq S_M S_O + S_D S_O) \subseteq (B a^1 \subseteq a.(\{<\}_* a))) = E$$

(A_{MX16})(Order Axiom)

$$((a^1 \subseteq S_M S_O + S_D S_O) = E) \subseteq ((B a^1 \subseteq a.(\{<\}_* a)) = E)$$

(T_{M094})(Probabilistic Rule)

$$((a^1 \subseteq S_M S_O + S_D S_O) = E) \subseteq ((B a^1 \neq \emptyset) \subseteq (a.(\{<\}_* a) \neq \emptyset))$$

(T_{M096})(Probabilistic Rule)

$$((a^1 \subseteq S_M S_O + S_D S_O) = E) \subseteq ((a.(\{<\}_* a) = \emptyset) \subseteq (B a^1 = \emptyset))$$

(T_{M012})(Transposition)

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (a.(\{<\}_* a) \neq \emptyset) \quad (T_{989})$$

Proof of (T_{989})

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (B a \neq \emptyset) \quad (T_{S889})$$

$$(a^1 \subseteq S_M S_O + S_D S_O) = (a^1 \subseteq S_M S_O + S_D S_O)(B a \neq \emptyset) \quad (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn1})$$

$$((a^1 \subseteq S_M S_O + S_D S_O) = E) \subseteq ((B a^1 \neq \emptyset) \subseteq (a.(\{<\}_* a) \neq \emptyset))$$

(T_{S992})

$$(a^1 \subseteq S_M S_O + S_D S_O)(B a^1 \neq \emptyset) \subseteq (a.(\{<\}_* a) \neq \emptyset)$$

(A_{MX35})(Exportation/Importation) for truth sets (Eqn2)

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (a.(\{<\}_* a) \neq \emptyset)$$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

$$(a^1 \subseteq S_M S_O + S_D S_O)(a_n \subseteq a)(B a_n^1 \neq \emptyset)(C a_n^1 \neq \emptyset) \subseteq (B a_n^1 \subseteq \{<\}_* C a_n^1) \quad (T_{S956})$$

Proof of (T_{S956})

$$(a^1 \subseteq S_M S_O + S_D S_O)(a_n \subseteq a)(B a_n^1 \neq \emptyset)(C a_n^1 \neq \emptyset) \subseteq (C a_n^1 \subseteq \{>\}_* B a_n^1) \quad (A_{R98})(\text{Order of Bounds of Extent 1})$$

$$(a^1 \subseteq S_M S_O + S_D S_O)(a_n \subseteq a)(B a_n^1 \neq \emptyset)(C a_n^1 \neq \emptyset) \subseteq (B a_n^1 \subseteq \{<\}_* C a_n^1) \quad (T_{R519})(\text{Active/Passive Theorem})$$

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (B a^1 = (\{<\}_* a^1).(\{>\}_* \{<\}_* a^1).(\{>\}_* \{<\}_* a^1)) \quad (T_{068})$$

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (C a^1 = (\{<\}_* a^1).(\{>\}_* \{<\}_* a^1).(\{<\}_* \{<\}_* a^1)) \quad (T_{069})$$

Example Proof of (T_{068})

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (B a^1 = a.(\{>\}_* a).(\{<\}_* a))$$

(T_{986})(Eqn1)

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (a = \{>\}_* a^1) \quad (T_{760})(\text{Eqn2})$$

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (B a^1 = \{<\}_* a^1.(\{>\}_* \{<\}_* a^1).(\{>\}_* \{<\}_* a^1))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\alpha = a.\{\subseteq\}^* \alpha) \quad (T_{7070})$$

Proof of (T₇₀₇₀)

$$\begin{aligned} (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba + Ca = a.\{\subseteq\}^* \alpha) & (T_{7059})(\text{Eqn1}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha = Ba + Ca) & (T_{7931b})(\text{Definition of } \alpha \text{ from } a)(\text{Eqn2}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha = a.\{\subseteq\}^* \alpha) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(\beta^1 \subseteq S_D S_O)(\alpha \subseteq \beta) \subseteq (\alpha = \alpha.\{\subseteq\}^* \alpha) \quad (T_{7737})$$

Proof of (T₇₇₃₇)

$$\begin{aligned} (\beta^1 \subseteq S_D S_O)(\alpha \subseteq \beta)(\alpha \neq \emptyset)(\alpha \neq \alpha^1) &\subseteq (\alpha^1 \subseteq S_D S_O) & (T_{7957})(\text{Eqn1}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha = a.\{\subseteq\}^* \alpha) & (T_{7070})(\text{Eqn2}) \\ S_D S_O &\subseteq S_M S_O + S_D S_O & (T_{M082})(\text{Disjunction Introduction}) \\ (S_D S_O \subseteq S_M S_O + S_D S_O) &= E & (A_{MX16})(\text{Order Axiom}) \\ (a^1 \subseteq S_D S_O) &\subseteq (a^1 \subseteq S_M S_O + S_D S_O) & (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn3}) \\ (a^1 \subseteq S_D S_O) &\subseteq (\alpha = a.\{\subseteq\}^* \alpha) & (T_{M139})(\text{Hypothetical Syllogism Eqns3\&2})(\text{Eqn4}) \\ (a^1 \subseteq S_D S_O) &\subseteq (\alpha = \alpha) & (D_{S31})(\text{Definition of } \alpha \text{ Symbolism})(\text{Eqn5}) \\ (a^1 \subseteq S_D S_O) &\subseteq (\alpha = \alpha.\{\subseteq\}^* \alpha) & (A_{MX19})(\text{Substitution of Eqn5 into Eqn4})(\text{Eqn6}) \\ (\beta^1 \subseteq S_D S_O)(\alpha \subseteq \beta)(\alpha \neq \emptyset)(\alpha \neq \alpha^1) &\subseteq (\alpha = \alpha.\{\subseteq\}^* \alpha) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&6})(\text{Eqn7}) \end{aligned}$$

$$\begin{aligned} \text{Let } \alpha &= \emptyset & (\text{Eqn8}) \\ \text{Let } \alpha &\neq \alpha.\{\subseteq\}^* \alpha & (\text{Eqn9}) \\ \emptyset &\neq \emptyset.\{\subseteq\}^* \emptyset & (A_{MX19})(\text{Substitution of Eqn9 into Eqn8}) \\ \emptyset &\neq \emptyset & (T_{M050})(\text{Null Set Conjunction}) \\ \emptyset & & (T_{M346}) \\ \text{Thus } (\alpha = \emptyset)(\alpha &\neq \alpha.\{\subseteq\}^* \alpha) &= \emptyset & (A_{MX02})(\text{Conditional Introduction}) \\ (\alpha = \emptyset) &\subseteq (\alpha = \alpha.\{\subseteq\}^* \alpha) & (T_{M735})(\subseteq \text{ Existential Form})(\text{Eqn10}) \end{aligned}$$

$$\begin{aligned} \text{Let } \alpha &= \alpha^1 & (\text{Eqn11}) \\ \text{Let } \alpha &\neq \alpha.\{\subseteq\}^* \alpha & (\text{Eqn12}) \\ \alpha^1 &\neq \alpha^1.\{\subseteq\}^* \alpha^1 & (A_{MX19})(\text{Substitution of Eqn11 into Eqn12})(\text{Eqn13}) \\ \{\subseteq\}^* \alpha^1 &\subseteq \{\subseteq\}^* \alpha^1 & (T_{S212}) \\ \alpha^1 &\subseteq \{\subseteq\}^* \alpha^1 & (D_{A144})(\text{Definition of the Reflexive Relationship Set}) \\ \alpha^1 &= \alpha^1.\{\subseteq\}^* \alpha^1 & (T_{M735})(\subseteq \text{ Existential Form})(\text{Eqn14}) \\ \alpha^1 &\neq \alpha^1 & (A_{MX19})(\text{Substitution of Eqn14 into Eqn13}) \\ \emptyset & & (T_{M024})(\text{Non existence of anti-}(D_{M00}) \text{ Definition of } =) \\ \text{Thus } (\alpha = \alpha^1)(\alpha &\neq \alpha.\{\subseteq\}^* \alpha) &= \emptyset & (A_{MX02})(\text{Conditional Introduction}) \\ (\alpha = \alpha^1) &\subseteq (\alpha = \alpha.\{\subseteq\}^* \alpha) & (T_{M735})(\subseteq \text{ Existential Form})(\text{Eqn15}) \end{aligned}$$

$$\begin{aligned} (\alpha = \emptyset) + (\alpha = \alpha^1) &\subseteq (\alpha = \alpha.\{\subseteq\}^* \alpha) + (\alpha = \alpha.\{\subseteq\}^* \alpha) & (A_{MX21})(\text{Combination of Eqns10\&15}) \\ (\alpha = \emptyset) + (\alpha = \alpha^1) &\subseteq (\alpha = \alpha.\{\subseteq\}^* \alpha) & (A_{MX01})(\text{Disjunction Tautology}) \\ \sim((\alpha \neq \emptyset)(\alpha \neq \alpha^1)) &\subseteq (\alpha = \alpha.\{\subseteq\}^* \alpha) & (T_{M025})(\text{DeMorgan's Law}) \\ (\beta^1 \subseteq S_D S_O)(\alpha \subseteq \beta). \sim((\alpha \neq \emptyset)(\alpha \neq \alpha^1)) &\subseteq (\alpha = \alpha.\{\subseteq\}^* \alpha) & (T_{M089})(\text{Single Premise Syllogism})(\text{Eqn16}) \\ (\beta^1 \subseteq S_D S_O)(\alpha \subseteq \beta)(\alpha \neq \emptyset)(\alpha \neq \alpha^1) + (\beta^1 \subseteq S_D S_O)(\alpha \subseteq \beta). \sim((\alpha \neq \emptyset)(\alpha \neq \alpha^1)) &\subseteq \\ (\alpha = \alpha.\{\subseteq\}^* \alpha) + (\alpha = \alpha.\{\subseteq\}^* \alpha) & & (A_{MX21})(\text{Combination of Eqns7\&16}) \\ (\beta^1 \subseteq S_D S_O)(\alpha \subseteq \beta)(\alpha \neq \emptyset)(\alpha \neq \alpha^1) + (\beta^1 \subseteq S_D S_O)(\alpha \subseteq \beta). \sim((\alpha \neq \emptyset)(\alpha \neq \alpha^1)) &\subseteq (\alpha = \alpha.\{\subseteq\}^* \alpha) & (A_{MX01})(\text{Disjunction Tautology}) \\ (\beta^1 \subseteq S_D S_O)(\alpha \subseteq \beta)((\alpha \neq \emptyset)(\alpha \neq \alpha^1) + \sim((\alpha \neq \emptyset)(\alpha \neq \alpha^1))) &\subseteq (\alpha = \alpha.\{\subseteq\}^* \alpha) & (T_{M407})(\text{Distribution of } . \text{ over } +) \\ (\beta^1 \subseteq S_D S_O)(\alpha \subseteq \beta)E &\subseteq (\alpha = \alpha.\{\subseteq\}^* \alpha) & (T_{M671})(\text{Tertium non Datur}) \\ (\beta^1 \subseteq S_D S_O)(\alpha \subseteq \beta) &\subseteq (\alpha = \alpha.\{\subseteq\}^* \alpha) & (A_{MX11})(\text{Universal Set Conjunction Axiom}) \end{aligned}$$

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba + Ca \subseteq a) \quad (T_{S986})$$

Proof of (T_{S986})

$$\begin{aligned} (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba + Ca = a.(\{-\}_* a)) & (T_{T059})(\text{Eqn1}) \\ (Ba + Ca = a.(\{-\}_* a)) &\subseteq (Ba + Ca \subseteq a) & (T_{M073b})(\text{Single Premise Syllogism})(\text{Eqn2}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba + Ca \subseteq a) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\alpha \subseteq a) \quad (T_{S987})$$

Proof of (T_{S987})

$$\begin{aligned} (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba + Ca \subseteq a) & (T_{S986}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha \subseteq a) & (T_{T931b})(\text{Definition of } \alpha \text{ from } a) \end{aligned}$$

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba \subseteq a) \quad (T_{S921})$$

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ca \subseteq a) \quad (T_{S922})$$

Example Proof of (T_{S921})

$$\begin{aligned} (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba + Ca \subseteq a) & (T_{S986})(\text{Eqn1}) \\ (Ba + Ca \subseteq a) &\subseteq (Ba \subseteq a) & (T_{M072})(\text{Single Premise Syllogism})(\text{Eqn2}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba \subseteq a) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$((a^1 \subseteq S_M S_O + S_P S_O) = E) \subseteq (((Ba + Ca)^1 \subseteq S_P S_O) = ((Ba)^1 + (Ca)^1 \subseteq S_P S_O)) \quad (T_{T935})$$

Proof of (T_{T935})

$$\begin{aligned} ((a^1 \subseteq S_M S_O + S_P S_O)(\beta \subseteq a)(\chi \subseteq a) = E) &\subseteq (((\beta + \chi)^1 \subseteq S_P S_O) = (\beta^1 + \chi^1 \subseteq S_P S_O)) & (A_{R88})(\text{Disjunctions of Sets of Points})(\text{Eqn1}) \\ \text{Let } \beta &= Ba & (\text{Eqn2}) \\ \text{Let } \chi &= Ca & (\text{Eqn3}) \\ ((a^1 \subseteq S_M S_O + S_P S_O)(Ba \subseteq a)(Ca \subseteq a) = E) &\subseteq (((Ba + Ca)^1 \subseteq S_P S_O) = ((Ba)^1 + (Ca)^1 \subseteq S_P S_O)) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})(\text{Eqn4}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba \subseteq a) & (T_{S921}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &= (a^1 \subseteq S_M S_O + S_D S_O)(Ba \subseteq a) & (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn5}) \\ ((a^1 \subseteq S_M S_O + S_P S_O)(Ca \subseteq a) = E) &\subseteq (((Ba + Ca)^1 \subseteq S_P S_O) = ((Ba)^1 + (Ca)^1 \subseteq S_P S_O)) & (A_{MX19})(\text{Substitution of Eqn5 into Eqn4})(\text{Eqn6}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ca \subseteq a) & (T_{S922}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &= (a^1 \subseteq S_M S_O + S_D S_O)(Ca \subseteq a) & (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn7}) \\ ((a^1 \subseteq S_M S_O + S_P S_O) = E) &\subseteq (((Ba + Ca)^1 \subseteq S_P S_O) = ((Ba)^1 + (Ca)^1 \subseteq S_P S_O)) & (A_{MX19})(\text{Substitution of Eqn7 into Eqn6}) \end{aligned}$$

$$(a^1 \subseteq S_M S_O) \subseteq ((a. \sim (Ba + Ca))^1 \subseteq S_C S_O).((a. (Ba + Ca))^1 \subseteq S_P S_O).(a^1 \subseteq S_O).(Ba + Ca = a.(\{-\}_* a))$$

(T_{T822}) Definition of a Mixed Continuum Ordered Set

Proof of (T_{T822})

$$\begin{aligned} ((X = B, C) = E) &\subseteq ((a^1 \subseteq S_M S_O) = ((a. \sim (Xa))^1 \subseteq S_C S_O)) & (A_{S75})(\text{Spans are Continua}) \\ (a^1 \subseteq S_M S_O) &= ((a. \sim (Ba))^1 \subseteq S_C S_O) & \text{Extraction} \\ (a^1 \subseteq S_M S_O) &= ((a. \sim (Ba).a. \sim (Ba))^1 \subseteq S_C S_O) & (T_{M667})(\text{Conjunction Tautology}) \\ (a^1 \subseteq S_M S_O) &= ((a. \sim (Ba).a. \sim (Ca))^1 \subseteq S_C S_O) & (T_{T950}) \\ (a^1 \subseteq S_M S_O) &= ((a. \sim (Ba). \sim (Ca))^1 \subseteq S_C S_O) & (T_{M667})(\text{Conjunction Tautology}) \\ (a^1 \subseteq S_M S_O) &= ((a. \sim (Ba + Ca))^1 \subseteq S_C S_O) & (T_{M026})(\text{DeMorgan's Law})(\text{Eqn1}) \\ ((X = B, C) = E) &\subseteq ((a^1 \subseteq S_M S_O) \subseteq ((Xa)^1 \subseteq S_P S_O)) & (T_{T778})(\text{Bounds of Extent are Points}) \\ (a^1 \subseteq S_M S_O) &\subseteq ((Ba)^1 \subseteq S_P S_O) & \text{Extraction (Eqn2)} \\ (a^1 \subseteq S_M S_O) &\subseteq ((Ca)^1 \subseteq S_P S_O) & \text{Extraction (Eqn3)} \end{aligned}$$

$(a^1 \subseteq S_M S_O) + (a^1 \subseteq S_M S_O) \subseteq ((Ba)^1 \subseteq S_P S_O) + ((Ca)^1 \subseteq S_P S_O)$	(A_{MX21}) (Combination of Eqns2&3)
$(a^1 \subseteq S_M S_O) \subseteq ((Ba)^1 \subseteq S_P S_O) + ((Ca)^1 \subseteq S_P S_O)$	(A_{MX01}) (Disjunction Tautology)
$(a^1 \subseteq S_M S_O) \subseteq ((Ba)^1 + (Ca)^1 \subseteq S_P S_O)$	(T_{M526})
$(a^1 \subseteq S_M S_O) \subseteq ((Ba + Ca)^1 \subseteq S_P S_O)$	(T_{T935}) (Eqn4)
$S_M S_O \subseteq S_O$	(T_{M083}) (Conjunction Elimination)
$(S_M S_O \subseteq S_O) = E$	(A_{MX16}) (Order Axiom)
$(a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq S_O)$	(T_{M099}) (Entity Mode / Set Mode Rule)(Eqn5)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba + Ca = a.(\{\subseteq\}^* a))$	(T_{T059}) (Eqn6)
$S_M S_O \subseteq S_M S_O + S_D S_O$	(T_{M082}) (Disjunction Introduction)
$(S_M S_O \subseteq S_M S_O + S_D S_O) = E$	(A_{MX16}) (Order Axiom)
$(a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)$	(T_{M099}) (Entity Mode / Set Mode Rule)(Eqn7)
$(a^1 \subseteq S_M S_O) \subseteq (Ba + Ca = a.(\{\subseteq\}^* a))$	(T_{M139}) (Hypothetical Syllogism Eqns6&7)(Eqn8)
$(a^1 \subseteq S_M S_O)(a^1 \subseteq S_M S_O)(a^1 \subseteq S_M S_O)(a^1 \subseteq S_M S_O) \subseteq$ $((a. \sim(Ba + Ca))^1 \subseteq S_C S_O).((a.(Ba + Ca))^1 \subseteq S_P S_O).(a^1 \subseteq S_O).(Ba + Ca = a.(\{\subseteq\}^* a))$	(T_{M669}) (Combination of Eqns1,4,5&8)
$(a^1 \subseteq S_M S_O) \subseteq ((a. \sim(Ba + Ca))^1 \subseteq S_C S_O).((a.(Ba + Ca))^1 \subseteq S_P S_O).(a^1 \subseteq S_O).(Ba + Ca = a.(\{\subseteq\}^* a))$	(T_{M667}) (Conjunction Tautology)

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall B(a\underline{a}) = \emptyset) \quad (T_{S923})$$

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall C(a\underline{a}) = \emptyset) \quad (T_{S924})$$

Example Proof of (T_{S923})

$(a^1 \subseteq S_M S_O + S_D S_O)(X = B, C) \subseteq (\forall X(a\underline{a}) = \emptyset)$	(A_{R67}) (Axiom of Mixed and Discrete Ordered Sets)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall B(a\underline{a}) = \emptyset)$	Extraction

$$(a^1 \subseteq S_M S_O + S_D S_O)(b \subseteq a) \subseteq (\forall B(\underline{a}b) = \emptyset) \quad (T_{T097})$$

$$(a^1 \subseteq S_M S_O + S_D S_O)(b \subseteq a) \subseteq (\forall C(\underline{a}b) = \emptyset) \quad (T_{T098})$$

Example Proof of (T_{T097})

Let $b \subseteq a$	
$\underline{a}b \subseteq \underline{a}a$	(T_{M029}) (Deduction)
$\forall B(\underline{a}b) \subseteq \forall B(\underline{a}a)$	(T_{R196}) (Relational Deduction)
$(\forall B(\underline{a}b) \subseteq \forall B(\underline{a}a)) = E$	(A_{MX16}) (Order Axiom)
$(\forall B(\underline{a}b) \neq \emptyset) \subseteq (\forall B(\underline{a}a) \neq \emptyset)$	(T_{M096}) (Probabilistic Rule)
$(\forall B(\underline{a}a) = \emptyset) \subseteq (\forall B(\underline{a}b) = \emptyset)$	(T_{M008}) (Inverse of an Equality)(Eqn1)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall B(\underline{a}a) = \emptyset)$	(T_{S923}) (Eqn2)
$(a^1 \subseteq S_M S_O) \subseteq (\forall B(\underline{a}b) = \emptyset)$	(T_{M139}) (Hypothetical Syllogism Eqns1&2)
Thus $((b \subseteq a) = E) \subseteq ((a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall B(\underline{a}b) = \emptyset))$	(A_{MX02}) (Conditional Introduction)
$(a^1 \subseteq S_M S_O + S_D S_O)(b \subseteq a) \subseteq (\forall B(\underline{a}b) = \emptyset)$	(A_{MX35}) (Exportation/Importation) for truth sets

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall B a = \forall Ba) \quad (T_{S902})$$

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall C a = \forall Ca) \quad (T_{S907})$$

Example Proof of (T_{S902})

$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq E$	(T_{M744}) (Universal Set Theorem)(Eqn1)
$\forall Ba = \forall Ba$	(D_{M00}) (Definition of =)
$(\forall Ba = \forall Ba) = E$	(A_{MX16}) (Order Axiom)(Eqn2)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall Ba = \forall Ba)$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall Ba = \forall B(E.a))$	(A_{MX11}) (Universal Set Conjunction Axiom)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall Ba = \forall B((\alpha + \underline{a}).a))$	(T_{M671}) (Tertium non Datur)

$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall B a = \forall B(\alpha a + \underline{\alpha} a))$	(T_{M407})(Distribution of . over +)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall B a = \forall B(\alpha a) + \forall B(\underline{\alpha} a))$	(T_{R002b})(Distribution of * over +)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall B a = \forall B(\alpha a) + \emptyset)$	(T_{S923})
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall B a = \forall B(\alpha a))$	(T_{M668})(Null Set Disjunction)(Eqn3)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\alpha \subseteq a)$	(T_{S987})
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\alpha = \alpha a)$	(A_{MX44})($\subseteq$ Equality Form)(Eqn4)
$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall B a = \forall B \alpha)$	(A_{MX19})(Substitution of Eqn4 into Eqn3)

$$(\alpha^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1 = \beta^1).(\forall C \beta^1 = \beta^1) \quad (T_{T115})$$

Proof of (T_{T115})

$(a^1 \subseteq S_M S_O + S_D S_O)(\beta^1 \subseteq a) \subseteq (\forall B \beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1)).(\forall C \beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1))$	(D_{R50})(Definition of $\forall B$ & $\forall C$)(Eqn1)
$S_D S_O \subseteq S_M S_O + S_D S_O$	(T_{M082})(Disjunction Introduction)
$(a^1 \subseteq S_D S_O) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)$	(T_{M272})(Deductive Rule)
$(a^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)(\beta^1 \subseteq \alpha)$	(T_{M029})(Deduction)(Eqn2)
$(a^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1)).(\forall C \beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1))$	(T_{M139})(Hypothetical Syllogism Eqns2&1)
$(\alpha^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1)).(\forall C \beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1))$	(D_{S31})(Definition of α Symbolism)(Eqn3)
$(\alpha^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\beta^1 = \{\subseteq\} * \beta^1)$	(T_{T140})(Eqn4)
$S_D S_O \subseteq S_D$	(T_{M083})(Conjunction Elimination)
$(S_D S_O \subseteq S_D) = E$	(A_{MX16})(Order Axiom)
$(\alpha^1 \subseteq S_D S_O) \subseteq (\alpha^1 \subseteq S_D)$	(T_{M099})(Entity Mode / Set Mode Rule)
$(\alpha^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\alpha^1 \subseteq S_D)(\beta^1 \subseteq \alpha)$	(T_{M029})(Deduction)(Eqn5)
$(\alpha^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\beta^1 = \{\subseteq\} * \beta^1)$	(T_{M139})(Hypothetical Syllogism Eqns4&5)(Eqn6)
$(\alpha^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1 = \beta^1.(\{\subseteq\} * \{\<\} * \beta^1)).(\forall C \beta^1 = \beta^1.(\{\subseteq\} * \{\>\} * \beta^1))$	(A_{MX19})(Substitution of Eqn6 into Eqn3)(Eqn7)
$\beta^1 = \beta^1.(\{\subseteq\} * \{\<\} * \beta^1)$	(T_{T124})(Eqn8)
$\beta^1 = \beta^1.(\{\subseteq\} * \{\>\} * \beta^1)$	(T_{T125})(Eqn9)
$(\alpha^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1 = \beta^1).(\forall C \beta^1 = \beta^1)$	(A_{MX19})(Substitution of Eqns8&9 into Eqn7)

$$(a^1 \subseteq S_M S_O + S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1 \forall C \beta^1 = \beta^1) \quad (T_{T780})$$

Proof of (T_{T780})

$(a^1 \subseteq S_M S_O) \subseteq (\alpha^1 \subseteq S_P S_O)$	(T_{T777})(Eqn1)
$S_P \subseteq S_D$	(A_{R53})(Axiom of Points)
$S_P S_O \subseteq S_D S_O$	(T_{M029})(Deduction)
$(\alpha^1 \subseteq S_P S_O) \subseteq (\alpha^1 \subseteq S_D S_O)$	(T_{M272})(Deductive Rule)(Eqn2)
$(a^1 \subseteq S_M S_O) \subseteq (\alpha^1 \subseteq S_D S_O)$	(T_{M139})(Hypothetical Syllogism Eqns1&2)
$(a^1 \subseteq S_M S_O)(\beta^1 \subseteq \alpha) \subseteq (\alpha^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha)$	(T_{M029})(Deduction)(Eqn3)
$(\alpha^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1 = \beta^1).(\forall C \beta^1 = \beta^1)$	(T_{T115})(Eqn4)
$(a^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1 = \beta^1).(\forall C \beta^1 = \beta^1)$	(D_{S31})(Definition of α Symbolism)(Eqn5)
$(a^1 \subseteq S_M S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1 = \beta^1).(\forall C \beta^1 = \beta^1)$	(T_{M139})(Hypothetical Syllogism Eqns3&4)(Eqn6)
$(a^1 \subseteq S_M S_O)(\beta^1 \subseteq \alpha) + (a^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1 = \beta^1).(\forall C \beta^1 = \beta^1) + (\forall B \beta^1 = \beta^1).(\forall C \beta^1 = \beta^1)$	(A_{MX21})(Combination of Eqns5&6)
$(a^1 \subseteq S_M S_O)(\beta^1 \subseteq \alpha) + (a^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1 = \beta^1).(\forall C \beta^1 = \beta^1)$	(A_{MX01})(Disjunction Tautology)
$((a^1 \subseteq S_M S_O) + (a^1 \subseteq S_D S_O))(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1 = \beta^1).(\forall C \beta^1 = \beta^1)$	(T_{M407})(Distribution of . over +)
$(a^1 \subseteq S_M S_O + S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1 = \beta^1).(\forall C \beta^1 = \beta^1)$	(T_{M526})
$(a^1 \subseteq S_M S_O + S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall B \beta^1. \forall C \beta^1 = \beta^1 \beta^1)$	(T_{M231})(Combination)

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq \alpha) \subseteq (\vee \mathcal{B}\beta^1. \vee \mathcal{C}\beta^1 = \beta^1) \quad (T_{M667})(\text{Conjunction Tautology})$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1)) \quad (T_{T087})$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{C}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1)) \quad (T_{T088})$$

Example Proof of (T_{T087})

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1)).(\vee \mathcal{C}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1)) \quad (D_{R50})(\text{Definition of } \vee \mathcal{B} \text{ \& } \vee \mathcal{C})$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1)) \quad (T_{M073})(\text{Single Premise Syllogism})$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1. \vee \mathcal{C}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1)) \quad (T_{T089})$$

Proof of (T_{T089})

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1)) \quad (T_{T087})(\text{Eqn1})$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{C}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1)) \quad (T_{T088})(\text{Eqn2})$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a)(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1)).(\vee \mathcal{C}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1)) \quad (T_{M669})(\text{Combination of Eqns1\&2})$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1)).(\vee \mathcal{C}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1)) \quad (T_{M667})(\text{Conjunction Tautology})$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1. \vee \mathcal{C}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1)) \quad (T_{M669})(\text{Combination})$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1. \vee \mathcal{C}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1)) \quad (T_{M667})(\text{Conjunction Tautology})$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\beta^1 \subseteq \vee \mathcal{B}\beta^1. \vee \mathcal{C}\beta^1) \quad (T_{T091})$$

Proof of (T_{T091})

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1. \vee \mathcal{C}\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1)) \quad (T_{T089})$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1. \vee \mathcal{C}\beta^1 = (\{\subseteq\} * \beta^1). \sim (\{\subseteq\} * \{\<\} * \beta^1) + (\{\subseteq\} * \{\>\} * \beta^1)) \quad (T_{M026})(\text{DeMorgan's Law})$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1. \vee \mathcal{C}\beta^1 = (\{\subseteq\} * \beta^1). \sim (\{\subseteq\} * (\{\<\} * \beta^1 + \{\>\} * \beta^1)) \quad (T_{R002b})(\text{Distribution of } * \text{ over } +)$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1. \vee \mathcal{C}\beta^1 = (\{\subseteq\} * \beta^1). \{\subseteq\} * (\{\<\} * \beta^1 + \{\>\} * \beta^1)) \quad (D_{R04})(\text{Definition of } *) (\text{Eqn1})$$

$$\beta^1 \subseteq \{\subseteq\} * (\{\<\} * \beta^1 + \{\>\} * \beta^1) \quad (T_{T112})$$

$$(\{\subseteq\} * \beta^1). \beta^1 \subseteq (\{\subseteq\} * \beta^1). \{\subseteq\} * (\{\<\} * \beta^1 + \{\>\} * \beta^1) \quad (T_{M029})(\text{Deduction})(\text{Eqn2})$$

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq ((\{\subseteq\} * \beta^1). \beta^1 \subseteq \vee \mathcal{B}\beta^1. \vee \mathcal{C}\beta^1)$$

(A_{MX19})(Substitution of Eqn1 into Eqn2)(Eqn3)

(D_{M00})(Definition of =)

(D_{A144})(Definition of the Reflexive Relationship Set)

(T_{R239})(Distribution of . over * for Entities)

(D_{A141})(Disjunction with {=})

(T_{M407})(Distribution of . over +)

(T_{A109})

(T_{M668})(Null Set Disjunction)

(T_{M667})(Conjunction Tautology)

(D_{A144})(Definition of the Reflexive Relationship Set)(Eqn4)

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\beta^1 \subseteq \vee \mathcal{B}\beta^1. \vee \mathcal{C}\beta^1)$$

(A_{MX19})(Substitution of Eqn4 into Eqn3)

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\beta^1 \subseteq (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\<\} * \beta^1).(\{\subseteq\} * \{\>\} * \beta^1)) \quad (T_{T090})$$

Proof of (T_{T090})

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\vee \mathcal{B}\beta^1. \vee C\beta^1 = (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\subseteq\} * \beta^1).(\{\subseteq\} * \{\subseteq\} * \beta^1))$$

(T₁₀₈₉)(Eqn1)

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\beta^1 \subseteq \vee \mathcal{B}\beta^1. \vee C\beta^1)$$

(T₁₀₉₁)(Eqn2)

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\beta^1 \subseteq (\{\subseteq\} * \beta^1).(\{\subseteq\} * \{\subseteq\} * \beta^1).(\{\subseteq\} * \{\subseteq\} * \beta^1))$$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\beta^1 \subseteq \vee \mathcal{B}\beta^1)$$

(T₁₁₀₅)

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\beta^1 \subseteq \vee C\beta^1)$$

(T₁₁₀₆)Example Proof of (T₁₁₀₅)

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\beta^1 \subseteq \vee \mathcal{B}\beta^1. \vee C\beta^1)$$

(T₁₀₉₁)

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\beta^1 \subseteq \vee \mathcal{B}\beta^1)$$

(T₁₀₇₃)(Single Premise Syllogism)

$$(a^1 \subseteq S_m S_o + S_o S_o) \subseteq (a\beta^1 \subseteq \vee \mathcal{B}(a\beta^1))$$

(T₁₁₀₇)

$$(a^1 \subseteq S_m S_o + S_o S_o) \subseteq (a\beta^1 \subseteq \vee C(a\beta^1))$$

(T₁₁₀₈)Example Proof of (T₁₁₀₇)

$$(a^1 \subseteq S_m S_o + S_o S_o)(\beta^1 \subseteq a) \subseteq (\beta^1 \subseteq \vee \mathcal{B}\beta^1)$$

(T₁₁₀₅)(Eqn1)

$$\text{Let } \beta^1 = a\beta^1$$

(Eqn2)

$$(a^1 \subseteq S_m S_o + S_o S_o)(a\beta^1 \subseteq a) \subseteq (a\beta^1 \subseteq \vee \mathcal{B}(a\beta^1))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(a^1 \subseteq S_m S_o + S_o S_o)E \subseteq (a\beta^1 \subseteq \vee \mathcal{B}(a\beta^1))$$

(T₁₀₀₁)(Universality of (T₁₀₈₃))

$$(a^1 \subseteq S_m S_o + S_o S_o) \subseteq (a\beta^1 \subseteq \vee \mathcal{B}(a\beta^1))$$

(A_{MX11})(Universal Set Conjunction Axiom)

$$(a^1 \subseteq S_m S_o + S_o S_o) \subseteq (a\beta \subseteq \vee \mathcal{B}(a\beta))$$

(T₁₁₀₁)

$$(a^1 \subseteq S_m S_o + S_o S_o) \subseteq (a\beta \subseteq \vee C(a\beta))$$

(T₁₁₀₂)Example Proof of (T₁₁₀₁)

$$(a^1 \subseteq S_m S_o + S_o S_o) \subseteq (a\beta_r^1 \subseteq \vee \mathcal{B}(a\beta_r^1))$$

(T₁₁₀₇)(Eqn1)

$$\text{Let } \beta = \beta_1^1 + \dots + \beta_n^1$$

$$\alpha\beta = \alpha(\beta_1^1 + \dots + \beta_n^1)$$

(T₁₀₂₉)(Deduction)

$$\alpha\beta = \alpha\beta_1^1 + \dots + \alpha\beta_n^1$$

(T₁₄₀₇)(Distribution of . over +)(Eqn2)

$$(a^1 \subseteq S_m S_o + S_o S_o) \dots (a^1 \subseteq S_m S_o + S_o S_o) \subseteq (a\beta_1^1 \subseteq \vee \mathcal{B}(a\beta_1^1)) \dots (a\beta_n^1 \subseteq \vee \mathcal{B}(a\beta_n^1))$$

(T₁₆₆₉)(Combination of Eqn1 for r = 1 to n)

$$(a^1 \subseteq S_m S_o + S_o S_o) \subseteq (a\beta_1^1 \subseteq \vee \mathcal{B}(a\beta_1^1)) \dots (a\beta_n^1 \subseteq \vee \mathcal{B}(a\beta_n^1))$$

(T₁₆₆₇)(Conjunction Tautology)

$$(a^1 \subseteq S_m S_o + S_o S_o) \subseteq (a\beta_1^1 + \dots + a\beta_n^1 \subseteq \vee \mathcal{B}(a\beta_1^1)) + \dots + \vee \mathcal{B}(a\beta_n^1)$$

(T₁₆₆₉)(Combination)

$$(a^1 \subseteq S_m S_o + S_o S_o) \subseteq (a\beta_1^1 + \dots + a\beta_n^1 \subseteq \vee \mathcal{B}(a\beta_1^1 + \dots + a\beta_n^1))$$

(A₅₆₄)(Distribution of * over +)(Eqn3)

$$(a^1 \subseteq S_m S_o + S_o S_o) \subseteq (a\beta \subseteq \vee \mathcal{B}(a\beta))$$

(A_{MX19})(Substitution of Eqn2 into Eqn3)

$$((a^1 \subseteq S_m S_o + S_o S_o) = E) \subseteq ((\vee \mathcal{B}(\alpha\beta) \neq \emptyset) = (a\beta \neq \emptyset))$$

(T₁₀₉₉)

$$((a^1 \subseteq S_m S_o + S_o S_o) = E) \subseteq ((\vee C(\alpha\beta) \neq \emptyset) = (a\beta \neq \emptyset))$$

(T₁₁₀₀)Example Proof of (T₁₀₉₉)

$$\text{Let } a^1 \subseteq S_m S_o + S_o S_o$$

$$a\beta \subseteq \vee \mathcal{B}(a\beta)$$

(T₁₁₀₁)

$$(a\beta \subseteq \vee \mathcal{B}(a\beta)) = E$$

(A_{MX16})(Order Axiom)

$$(a\beta \neq \emptyset) \subseteq (\vee \mathcal{B}(a\beta) \neq \emptyset)$$

(T₁₀₉₆)(Probabilistic Rule)(Eqn1)

$$(\vee \mathcal{B}(a\beta) \neq \emptyset) \subseteq a\beta \neq \emptyset$$

(T₁₁₇₀)(Eqn2)

$$(\vee \mathcal{B}(a\beta) \neq \emptyset) = (a\beta \neq \emptyset)$$

(T₁₄₃₀)(Biconditional Introduction Eqns 1&2)

$$\text{Thus } ((a^1 \subseteq S_m S_o + S_o S_o) = E) \subseteq ((\vee \mathcal{B}(\alpha\beta) \neq \emptyset) = (a\beta \neq \emptyset))$$

(A_{MX02})(Conditional Introduction)

$$((a^1 \subseteq S_M S_O + S_D S_O)(\beta \subseteq a) = E) \subseteq ((\forall \mathcal{B}\beta \neq \emptyset) = (a\beta \neq \emptyset)) \quad (T_{109})$$

$$((a^1 \subseteq S_M S_O + S_D S_O)(\beta \subseteq a) = E) \subseteq ((\forall C\beta \neq \emptyset) = (a\beta \neq \emptyset)) \quad (T_{110})$$

Example Proof of (T_{109})

Let $a^1 \subseteq S_M S_O + S_D S_O$

Let $\beta \subseteq a$

$\beta = E\beta$

(A_{MX11})(Universal Set Conjunction Axiom)

$\beta = (a + \underline{a})\beta$

(T_{M671})(Tertium non Datur)

$\beta = a\beta + \underline{a}\beta$

(T_{M407})(Distribution of . over +)

$\forall \mathcal{B}\beta = \forall \mathcal{B}(a\beta + \underline{a}\beta)$

(T_{R522})(Relational Deduction)

$\forall \mathcal{B}\beta = \forall \mathcal{B}(a\beta) + \forall \mathcal{B}(\underline{a}\beta)$

(T_{R002b})(Distribution of * over +)

$\forall \mathcal{B}\beta = \forall \mathcal{B}(a\beta) + \emptyset$

(T_{1097})

$\forall \mathcal{B}\beta = \forall \mathcal{B}(a\beta)$

(T_{M668})(Null Set Disjunction)

$(\forall \mathcal{B}\beta = \forall \mathcal{B}(a\beta)) = E$

(A_{MX16})(Order Axiom)

$(\forall \mathcal{B}\beta \neq \emptyset) = (\forall \mathcal{B}(a\beta) \neq \emptyset)$

(T_{M096})(Probabilistic Rule)(Eqn1)

$(\forall \mathcal{B}(a\beta) \neq \emptyset) = (a\beta \neq \emptyset)$

(T_{1099})(Eqn2)

$(\forall \mathcal{B}\beta \neq \emptyset) = (a\beta \neq \emptyset)$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

Thus $((a^1 \subseteq S_M S_O + S_D S_O)(\beta \subseteq a) = E) \subseteq ((\forall \mathcal{B}\beta \neq \emptyset) = (a\beta \neq \emptyset))$

(A_{MX02})(Conditional Introduction)

$$((a^1 \subseteq S_M S_O + S_D S_O)(\beta \subseteq a) = E) \subseteq ((\forall \mathcal{B}\beta \neq \emptyset) = (\forall C\beta \neq \emptyset)) \quad (T_{1082})$$

$$((a^1 \subseteq S_M S_O + S_D S_O)(\beta^1 \subseteq a) = E) \subseteq ((\forall \mathcal{B}\beta^1 \neq \emptyset) = (\forall C\beta^1 \neq \emptyset)) \quad (T_{1083})$$

Proof of (T_{1082}) & (T_{1083})

$((a^1 \subseteq S_M S_O + S_D S_O)(\beta \subseteq a) = E) \subseteq ((\forall \mathcal{B}\beta \neq \emptyset) = (a\beta \neq \emptyset))$

(T_{109})(Eqn1)

$((a^1 \subseteq S_M S_O + S_D S_O)(\beta \subseteq a) = E) \subseteq ((\forall C\beta \neq \emptyset) = (a\beta \neq \emptyset))$

(T_{110})(Eqn2)

$((a^1 \subseteq S_M S_O + S_D S_O)(\beta \subseteq a) = E) \subseteq ((\forall \mathcal{B}\beta \neq \emptyset) = (\forall C\beta \neq \emptyset))$

(A_{MX19})(Substitution of Eqn2 into Eqn1)
(Eqn3)

Let $\beta = \beta^1$

(Eqn4)

$((a^1 \subseteq S_M S_O + S_D S_O)(\beta^1 \subseteq a) = E) \subseteq ((\forall \mathcal{B}\beta^1 \neq \emptyset) = (\forall C\beta^1 \neq \emptyset))$

(A_{MX19})(Substitution of Eqn4 into Eqn3)

11.6 Further Theorems of Bounds of Extent

A. Further Theorems of Bounds of Extent in General

$$Ba = \mathcal{B}Ba \quad (T_{S947})$$

$$Ba = CBa \quad (T_{S948})$$

$$Ca = \mathcal{B}Ca \quad (T_{S949})$$

$$Ca = CCa \quad (T_{S950})$$

Example Proof of (T_{S947})

$(\alpha^1 \subseteq S_D S_O) \subseteq (\mathcal{B}\alpha = \alpha)$

(T_{S893})(Eqn1)

Let $\alpha = \mathcal{B}a$

(Eqn2)

$((\mathcal{B}a)^1 \subseteq S_D S_O) \subseteq (\mathcal{B}Ba = \mathcal{B}a)$

(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)

$(a^1 \subseteq S_O) \subseteq ((\mathcal{B}a)^1 \subseteq S_D S_O)$

(T_{S968})(Eqn4)

$(a^1 \subseteq S_O) \subseteq (\mathcal{B}Ba = \mathcal{B}a)$

(T_{M139})(Hypothetical Syllogism Eqns4&3)(Eqn5)

$((a\mathcal{B})^1 \subseteq S_O) \subseteq (\mathcal{B}Ba\mathcal{B} = \mathcal{B}a\mathcal{B})$

Resymbolising (Eqn6)

Let $a^1 \subseteq \sim S_O$

$(a^1 \subseteq \sim S_O) = (\mathcal{B} \neq \emptyset)(\mathcal{B} \neq \mathcal{B}^1)(\mathcal{B} \subset a)((a\mathcal{B})^1 \subseteq S_O)((a\mathcal{B})^1 \subseteq S_U)$

$(a^1 \subseteq \sim So) = (\underline{b} \neq \emptyset)(\underline{b} \neq \underline{b}^1)(\underline{b} \subset a)((a\underline{b})^1 \subseteq So)((a\underline{b})^1 \subseteq Su)$	(D_{S29})(Definition of a Non-Ordered Set)
$(a^1 \subseteq \sim So) \subseteq ((a\underline{b})^1 \subseteq So)$	Resymbolising
$(a^1 \subseteq \sim So) \subseteq (B\bar{B}a\underline{b} = Ba\underline{b})$	(T_{M073b})(Single Premise Syllogism)(Eqn7)
$(a^1 \subseteq \sim So) \subseteq ((a\underline{b})^1 \subseteq Su)$	(T_{M139})(Hypothetical Syllogism Eqns7&6)(Eqn8)
$((a\underline{b})^1 \subseteq Su) \subseteq (Bab = \emptyset)$	(T_{M073b})(Single Premise Syllogism)(Eqn9)
$(a^1 \subseteq \sim So) \subseteq (Bab = \emptyset)$	(T_{S978A})(Eqn10)
$(Bab = \emptyset) \subseteq (B\bar{B}ab = \emptyset)$	(T_{M139})(Hypothetical Syllogism Eqns9&10)(Eqn11)
$(Bab = \emptyset) = (Bab = \emptyset)(B\bar{B}ab = \emptyset)$	(T_{S303})
$(Bab = \emptyset) \subseteq (B\bar{B}ab = Ba\underline{b})$	($AMX44$)($\subseteq$ Equality Form)
$(a^1 \subseteq \sim So) \subseteq (B\bar{B}ab = Ba\underline{b})$	($AMX19$)(Substitution)(Eqn12)
$(a^1 \subseteq \sim So)(a^1 \subseteq \sim So) \subseteq (B\bar{B}ab = Ba\underline{b})(B\bar{B}a\underline{b} = Ba\underline{b})$	(T_{M139})(Hypothetical Syllogism Eqns11&12)(Eqn13)
$(a^1 \subseteq \sim So) \subseteq (B\bar{B}ab = Ba\underline{b})(B\bar{B}a\underline{b} = Ba\underline{b})$	(T_{M669})(Combination of Eqns8&13)
$(a^1 \subseteq \sim So) \subseteq (B\bar{B}ab + B\bar{B}a\underline{b} = Ba\underline{b} + Ba\underline{b})$	(T_{M667})(Conjunction Tautology)
$(a^1 \subseteq \sim So) \subseteq (B\bar{B}(ab + a\underline{b}) = B(ab + a\underline{b}))$	(T_{M230})(Combination)
$(a^1 \subseteq \sim So) \subseteq (B\bar{B}a(b + \underline{b}) = Ba(b + \underline{b}))$	(T_{R002b})(Distribution of * over +)
$(a^1 \subseteq \sim So) \subseteq (B\bar{B}aE = BaE)$	(T_{M407})(Distribution of . over +)
$(a^1 \subseteq \sim So) \subseteq (B\bar{B}a = Ba)$	(T_{M671})(Tertium non Datur)
$(a^1 \not\subseteq So) \subseteq (a^1 \subseteq \sim So)$	($AMX11$)(Universal Set Conjunction Axiom)(Eqn14)
$(a^1 \not\subseteq So) \subseteq (B\bar{B}a = Ba)$	(T_{M604})(Eqn15)
$(a^1 \subseteq So) + (a^1 \not\subseteq So) \subseteq (B\bar{B}a = Ba) + (B\bar{B}a = Ba)$	(T_{M139})(Hypothetical Syllogism Eqns15&14)(Eqn16)
$(a^1 \subseteq So) + (a^1 \not\subseteq So) \subseteq (B\bar{B}a = Ba)$	($AMX21$)(Combination of Eqns5&16)
$E \subseteq (B\bar{B}a = Ba)$	($AMX01$)(Disjunction Tautology)
$E = (B\bar{B}a = Ba)$	(T_{M671})(Tertium non Datur)
$B\bar{B}a = Ba$	(T_{M275})(Supersets of the Universal Set)
	($AMX16$)(Order Axiom)

$$(b^1 \subseteq \{<\} * a^1) =$$

$$(Ba^1 \neq \emptyset).((Bb^1 \neq \emptyset).(Bb^1 \subseteq \{<\} * Ba^1) + (Bb^1 = \emptyset)).(Cb^1 \neq \emptyset).((Ca^1 \neq \emptyset).(Cb^1 \subseteq \{<\} * Ca^1) + (Ca^1 = \emptyset))$$

(T_{981}) Definition of $\{<\}$

Proof of (T_{981})

$$(b^1 \subseteq \{>\} * a^1) =$$

$$(Bb^1 \neq \emptyset).((Ba^1 \neq \emptyset).(Bb^1 \subseteq \{>\} * Ba^1) + (Ba^1 = \emptyset)).(Ca^1 \neq \emptyset).((Cb^1 \neq \emptyset).(Cb^1 \subseteq \{>\} * Ca^1) + (Cb^1 = \emptyset))$$

(D_{R63})(Definition of $\{>\}$)

$$(a^1 \subseteq \{<\} * b^1) =$$

$$(Bb^1 \neq \emptyset).((Ba^1 \neq \emptyset).(Ba^1 \subseteq \{<\} * Bb^1) + (Ba^1 = \emptyset)).(Ca^1 \neq \emptyset).((Cb^1 \neq \emptyset).(Ca^1 \subseteq \{<\} * Cb^1) + (Cb^1 = \emptyset))$$

(T_{R519})(Active/Passive Theorem)

$$(b^1 \subseteq \{<\} * a^1) =$$

$$(Ba^1 \neq \emptyset).((Bb^1 \neq \emptyset).(Bb^1 \subseteq \{<\} * Ba^1) + (Bb^1 = \emptyset)).(Cb^1 \neq \emptyset).((Ca^1 \neq \emptyset).(Cb^1 \subseteq \{<\} * Ca^1) + (Ca^1 = \emptyset))$$

Resymbolising

$$(b^1 \subseteq \{>\} * a^1) \subseteq (Bb^1 \neq \emptyset)(Ca^1 \neq \emptyset) \quad (T_{982})$$

$$(b^1 \subseteq \{<\} * a^1) \subseteq (Ba^1 \neq \emptyset)(Cb^1 \neq \emptyset) \quad (T_{983})$$

Example Proof of (T_{982})

$$(b^1 \subseteq \{>\} * a^1) =$$

$$(Bb^1 \neq \emptyset).((Ba^1 \neq \emptyset).(Bb^1 \subseteq \{>\} * Ba^1) + (Ba^1 = \emptyset)).(Ca^1 \neq \emptyset).((Cb^1 \neq \emptyset).(Cb^1 \subseteq \{>\} * Ca^1) + (Cb^1 = \emptyset))$$

(D_{R63})(Definition of $\{>\}$)

$$(b^1 \subseteq \{>\} * a^1) \subseteq (Bb^1 \neq \emptyset)(Ca^1 \neq \emptyset)$$

(T_{M073b})(Single Premise Syllogism)

$$(Ba^1 \neq \emptyset) \subseteq (\{>\} * a^1 \subseteq \{>\} * Ba^1) \quad (T_{S961})$$

$$(Ca^1 \neq \emptyset) \subseteq (\{<\} * a^1 \subseteq \{<\} * Ca^1) \quad (T_{S962})$$

Example Proof of (T_{S961})

$$\begin{aligned} &(\hat{b}^1 \subseteq \{>\} * a^1) = \\ &(\mathcal{B}\hat{b}^1 \neq \emptyset).((Ba^1 \neq \emptyset).(\mathcal{B}\hat{b}^1 \subseteq \{>\} * Ba^1) + (Ba^1 = \emptyset)).(Ca^1 \neq \emptyset).((C\hat{b}^1 \neq \emptyset).(C\hat{b}^1 \subseteq \{>\} * Ca^1) + (C\hat{b}^1 = \emptyset)) \\ &\quad (D_{R63})(\text{Definition of } \{>\}) \\ &(\hat{b}^1 \subseteq \{>\} * a^1) \subseteq (Ba^1 \neq \emptyset).(\mathcal{B}\hat{b}^1 \subseteq \{>\} * Ba^1) + (Ba^1 = \emptyset) \quad (T_{M073b})(\text{Single Premise Syllogism}) \\ &(Ba^1 \neq \emptyset).(\hat{b}^1 \subseteq \{>\} * a^1) \subseteq (Ba^1 \neq \emptyset).(\mathcal{B}\hat{b}^1 \subseteq \{>\} * Ba^1) \quad (T_{M468}) \\ &(Ba^1 \neq \emptyset).(\hat{b}^1 \subseteq \{>\} * a^1) \subseteq (\mathcal{B}\hat{b}^1 \subseteq \{>\} * Ba^1) \quad (T_{M256})(\text{Conditional Subsets Rule}) \\ &((Ba^1 \neq \emptyset) = E) \subseteq ((\hat{b}^1 \subseteq \{>\} * a^1) \subseteq (\mathcal{B}\hat{b}^1 \subseteq \{>\} * Ba^1)) \quad (A_{MX35})(\text{Exportation/Importation})(Eqn1) \\ &\text{Let } \hat{b}^1 = \mathcal{B}\hat{b}^1 \quad (Eqn2) \\ &((Ba^1 \neq \emptyset) = E) \subseteq ((\mathcal{B}\hat{b}^1 \subseteq \{>\} * a^1) \subseteq (\mathcal{B}\mathcal{B}\hat{b}^1 \subseteq \{>\} * Ba^1)) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ &((Ba^1 \neq \emptyset) = E) \subseteq ((\mathcal{B}\hat{b}^1 \subseteq \{>\} * a^1) \subseteq (\mathcal{B}\hat{b}^1 \subseteq \{>\} * Ba^1)) \quad (T_{S947}) \\ &((Ba^1 \neq \emptyset) = E) \subseteq ((\{>\} * a^1 \subseteq \{>\} * Ba^1) = E) \quad (T_{M748})(\text{Entity Mode / Set Mode Rule}) \\ &\quad \text{Since } \mathcal{B}\hat{b}^1 \neq \emptyset \\ &((Ba^1 \neq \emptyset) \subseteq (\{>\} * a^1 \subseteq \{>\} * Ba^1)) = E \quad (T_{M094})(\text{Probabilistic Rule}) \\ &(Ba^1 \neq \emptyset) \subseteq (\{>\} * a^1 \subseteq \{>\} * Ba^1) \quad (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$(Ba^1 \neq \emptyset)(\hat{b}^1 \subseteq a)(\hat{b}^1 \neq \emptyset) \subseteq (\mathcal{B}\hat{b}^1 \neq \emptyset) \quad (T_{T984})$$

$$(Ca^1 \neq \emptyset)(\hat{b}^1 \subseteq a)(\hat{b}^1 \neq \emptyset) \subseteq (C\hat{b}^1 \neq \emptyset) \quad (T_{T985})$$

Example Proof of (T_{T984})

$$\begin{aligned} &((X = B, C) = E) \subseteq ((Xa^1 \neq \emptyset)(\hat{b}^1 \subseteq a)(\hat{b}^1 \neq \emptyset) \subseteq (X\hat{b}^1 \neq \emptyset)) \quad (A_{S67})(\text{Bounds of Extent of Set Members}) \\ &(Ba^1 \neq \emptyset)(\hat{b}^1 \subseteq a)(\hat{b}^1 \neq \emptyset) \subseteq (\mathcal{B}\hat{b}^1 \neq \emptyset) \quad \text{Extraction} \end{aligned}$$

$$(Ba^1 \neq \emptyset) \subseteq (\{>\} * a \subseteq \{>\} * Ba) \quad (T_{S963})$$

$$(Ca^1 \neq \emptyset) \subseteq (\{<\} * a \subseteq \{<\} * Ca) \quad (T_{S964})$$

Example Proof of (T_{S963})

$$\begin{aligned} &\text{Let } a = a_1^1 + \dots + a_n^1 \quad (Eqn1) \\ &\text{Let } (a_1^1 \neq \emptyset) \dots (a_n^1 \neq \emptyset) \\ &(a_1^1 \neq \emptyset) \dots (a_n^1 \neq \emptyset) = E \quad (A_{MX16})(\text{Order Axiom})(Eqn2) \\ &Ba = \mathcal{B}(a_1^1 + \dots + a_n^1) \quad (T_{R196})(\text{Relational Deduction}) \\ &Ba = \mathcal{B}a_1^1 + \dots + \mathcal{B}a_n^1 \quad (A_{S64})(\text{Distribution of } * \text{ over } +)(Eqn3) \\ &(Ba^1 \neq \emptyset) \subseteq (\{>\} * a_r^1 \subseteq \{>\} * \mathcal{B}a_r^1) \quad (T_{S961})(Eqn4) \\ &(Ba_1^1 \neq \emptyset) \dots (Ba_n^1 \neq \emptyset) \subseteq (\{>\} * a_1^1 \subseteq \{>\} * \mathcal{B}a_1^1) \dots (\{>\} * a_n^1 \subseteq \{>\} * \mathcal{B}a_n^1) \\ &\quad (T_{M669})(\text{Combination of Eqn4 for } r = 1 \text{ to } n) \\ &(Ba_1^1 \neq \emptyset) \dots (Ba_n^1 \neq \emptyset) \subseteq (\{>\} * a_1^1 + \dots + \{>\} * a_n^1 \subseteq \{>\} * \mathcal{B}a_1^1 + \dots + \{>\} * \mathcal{B}a_n^1) \\ &\quad (A_{MX27})(\text{Combination}) \\ &(Ba_1^1 \neq \emptyset) \dots (Ba_n^1 \neq \emptyset) \subseteq (\{>\} * (a_1^1 + \dots + a_n^1) \subseteq \{>\} * (\mathcal{B}a_1^1 + \dots + \mathcal{B}a_n^1)) \\ &\quad (A_{S64})(\text{Distribution of } * \text{ over } +)(Eqn5) \\ &(Ba_1^1 \neq \emptyset) \dots (Ba_n^1 \neq \emptyset) \subseteq (\{>\} * a \subseteq \{>\} * Ba) \quad (A_{MX19})(\text{Substitution of Eqns1\&3 into Eqn5})(Eqn6) \\ &(Ba^1 \neq \emptyset)(a_1^1 \subseteq a)(a_1^1 \neq \emptyset) \subseteq (\mathcal{B}a_1^1 \neq \emptyset) \quad (T_{T984})(Eqn7) \\ &(Ba^1 \neq \emptyset)(a_1^1 \subseteq a)(a_1^1 \neq \emptyset) \dots (Ba^1 \neq \emptyset)(a_n^1 \subseteq a)(a_n^1 \neq \emptyset) \subseteq (Ba_1^1 \neq \emptyset) \dots (Ba_n^1 \neq \emptyset) \\ &\quad (T_{M669})(\text{Combination of Eqn7 for } r = 1 \text{ to } n) \\ &(Ba^1 \neq \emptyset)((a_1^1 \subseteq a)(a_1^1 \neq \emptyset) \dots (a_n^1 \subseteq a)(a_n^1 \neq \emptyset)) \subseteq (Ba_1^1 \neq \emptyset) \dots (Ba_n^1 \neq \emptyset) \\ &\quad (T_{M667})(\text{Conjunction Tautology}) \\ &(Ba^1 \neq \emptyset)((a_1^1 \subseteq a) \dots (a_n^1 \subseteq a)((a_1^1 \neq \emptyset) \dots (a_n^1 \neq \emptyset))) \subseteq (Ba_1^1 \neq \emptyset) \dots (Ba_n^1 \neq \emptyset) \\ &\quad (T_{M404})(\text{Commutative.})(Eqn8) \\ &(Ba^1 \neq \emptyset)((a_1^1 \subseteq a) \dots (a_n^1 \subseteq a))E \subseteq (Ba_1^1 \neq \emptyset) \dots (Ba_n^1 \neq \emptyset) \\ &\quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn8}) \end{aligned}$$

$$(\mathcal{B}a^1 \neq \emptyset)((a_1^1 \subseteq a) \dots (a_n^1 \subseteq a)) \subseteq (\mathcal{B}a_1^1 \neq \emptyset) \dots (\mathcal{B}a_n^1 \neq \emptyset)$$

($\mathbf{AMX11}$)(Universal Set Conjunction Axiom)

$$(\mathcal{B}a^1 \neq \emptyset)((a_1^1 \vdash \vdash a_n^1 \subseteq a) \subseteq (\mathcal{B}a_1^1 \neq \emptyset) \dots (\mathcal{B}a_n^1 \neq \emptyset))$$

($\mathbf{TM131b}$)(Conjunction of Sufficiencies)(Eqn9)

$$(\mathcal{B}a^1 \neq \emptyset)((a \subseteq a) \subseteq (\mathcal{B}a_1^1 \neq \emptyset) \dots (\mathcal{B}a_n^1 \neq \emptyset))$$

($\mathbf{AMX19}$)(Substitution of Eqn1 into Eqn9)

$$(\mathcal{B}a^1 \neq \emptyset)((a \subseteq a) + (a = a)) \subseteq (\mathcal{B}a_1^1 \neq \emptyset) \dots (\mathcal{B}a_n^1 \neq \emptyset)$$

($\mathbf{TM266}$)(Definition of $\subseteq$)

$$(\mathcal{B}a^1 \neq \emptyset)((a \subseteq a) + \mathbf{E}) \subseteq (\mathcal{B}a_1^1 \neq \emptyset) \dots (\mathcal{B}a_n^1 \neq \emptyset)$$

($\mathbf{TM007}$)(Universality of ($\mathbf{DM00}$) Definition of $=$)

$$(\mathcal{B}a^1 \neq \emptyset)\mathbf{E} \subseteq (\mathcal{B}a_1^1 \neq \emptyset) \dots (\mathcal{B}a_n^1 \neq \emptyset)$$

($\mathbf{TM055}$)(Universal Set Disjunction)

$$(\mathcal{B}a^1 \neq \emptyset) \subseteq (\mathcal{B}a_1^1 \neq \emptyset) \dots (\mathcal{B}a_n^1 \neq \emptyset)$$

($\mathbf{AMX11}$)(Universal Set Conjunction Axiom)(Eqn10)

$$(\mathcal{B}a^1 \neq \emptyset) \subseteq (\{>\} * a \subseteq \{>\} * \mathcal{B}a)$$

($\mathbf{TM139}$)(Hypothetical Syllogism Eqns10&6)

B. Further Theorems of Mixed Ordered Sets

$$(a^1 \subseteq S_M S_O) \subseteq (\forall \mathcal{B}a. \forall Ca \subseteq a)$$

($\mathbf{TS901}$)

Proof of ($\mathbf{TS901}$)

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall \mathcal{B}a = \forall \mathcal{B}a)$$

($\mathbf{TS902}$)(Eqn1)

$$S_M S_O \subseteq S_M S_O + S_D S_O$$

($\mathbf{TM082}$)(Disjunction Introduction)

$$(a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)$$

($\mathbf{TM099}$)(Entity Mode / Set Mode Rule)(Eqn2)

$$(a^1 \subseteq S_M S_O) \subseteq (\forall \mathcal{B}a = \forall \mathcal{B}a)$$

($\mathbf{TM139}$)(Hypothetical Syllogism Eqns2&1)(Eqn3)

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\forall Ca = \forall Ca)$$

($\mathbf{TS907}$)(Eqn4)

$$(a^1 \subseteq S_M S_O) \subseteq (\forall Ca = \forall Ca)$$

($\mathbf{TM139}$)(Hypothetical Syllogism Eqns2&4)(Eqn5)

$$(a^1 \subseteq S_M S_O) \subseteq (\forall \mathcal{B}a. \forall Ca \subseteq a)$$

($\mathbf{TS871}$)(Eqn6)

$$(a^1 \subseteq S_M S_O) \subseteq (\forall \mathcal{B}a. \forall Ca \subseteq a)$$

($\mathbf{AMX19}$)(Substitution of Eqns3&5 into Eqn6)

$$(a^1 \subseteq S_M S_O) \subseteq (\alpha. \{\subseteq\} * \alpha \neq \emptyset)$$

($\mathbf{TT018}$)

Proof of ($\mathbf{TT018}$)

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\alpha. \{\subseteq\} * \alpha \neq \emptyset)$$

($\mathbf{TT017}$)(Eqn1)

$$S_M S_O \subseteq S_M S_O + S_D S_O$$

($\mathbf{TM082}$)(Disjunction Introduction)

$$(S_M S_O \subseteq S_M S_O + S_D S_O) = \mathbf{E}$$

($\mathbf{AMX16}$)(Order Axiom)

$$(a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)$$

($\mathbf{TM099}$)(Entity Mode / Set Mode Rule)(Eqn2)

$$(a^1 \subseteq S_M S_O) \subseteq (\alpha. \{\subseteq\} * \alpha \neq \emptyset)$$

($\mathbf{TM139}$)(Hypothetical Syllogism Eqns2&1)

$$(a^1 \subseteq S_M S_O) \subseteq (a. \{\subseteq\} * a \neq \emptyset)$$

($\mathbf{TT016}$)

Proof of ($\mathbf{TT016}$)

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (a. \{\subseteq\} * a \neq \emptyset)$$

($\mathbf{TT989}$)(Eqn1)

$$S_M S_O \subseteq S_M S_O + S_D S_O$$

($\mathbf{TM082}$)(Disjunction Introduction)

$$(S_M S_O \subseteq S_M S_O + S_D S_O) = \mathbf{E}$$

($\mathbf{AMX16}$)(Order Axiom)

$$(a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)$$

($\mathbf{TM099}$)(Entity Mode / Set Mode Rule)(Eqn2)

$$(a^1 \subseteq S_M S_O) \subseteq (a. \{\subseteq\} * a \neq \emptyset)$$

($\mathbf{TM139}$)(Hypothetical Syllogism Eqns2&1)

$$(a^1 \subseteq S_M S_O) \subseteq ((\sim(\mathcal{B}a + Ca). \forall \mathcal{B}(\mathcal{B}a + Ca). \forall C(\mathcal{B}a + Ca))^1 \subseteq S_C S_O)$$

($\mathbf{TS906}$)

Proof of ($\mathbf{TS906}$)

$$(a^1 \subseteq S_M S_O) \subseteq ((\alpha. \forall \mathcal{B}a. \forall Ca)^1 \subseteq S_C S_O)$$

($\mathbf{TS914}$)

$$(a^1 \subseteq S_M S_O) \subseteq ((\sim(\mathcal{B}a + Ca). \forall \mathcal{B}(\mathcal{B}a + Ca). \forall C(\mathcal{B}a + Ca))^1 \subseteq S_C S_O)$$

($\mathbf{TT931b}$)(Definition of α from a)

C. Further Theorems of Mixed and Discrete Ordered Sets

$$(a^1 \subseteq S_M S_O + S_D S_O). (Ba^1 \neq \emptyset) \subseteq (\{>\} * \alpha = \{>\} * a) \quad (T_{7064})$$

$$(a^1 \subseteq S_M S_O + S_D S_O). (Ca^1 \neq \emptyset) \subseteq (\{<\} * \alpha = \{<\} * a) \quad (T_{7065})$$

$$(a^1 \subseteq S_M S_O + S_D S_O). (Ba^1 \neq \emptyset) \subseteq (\{>\} * a = \{>\} * \alpha) \quad (T_{7066})$$

$$(a^1 \subseteq S_M S_O + S_D S_O). (Ca^1 \neq \emptyset) \subseteq (\{<\} * a = \{<\} * \alpha) \quad (T_{7067})$$

Example Proof of (T₇₀₆₄) & (T₇₀₆₆)

$$\begin{aligned} (Ba^1 \neq \emptyset) &\subseteq (\{>\} * a \subseteq \{>\} * Ba) && (T_{5963}) \\ (Ba^1 \neq \emptyset) &\subseteq (\{>\} * a \subseteq \{>\} * Ba + \{>\} * Ca) && (T_{M150}) \text{(Single Premise Syllogism)} \\ (Ba^1 \neq \emptyset) &\subseteq (\{>\} * a \subseteq \{>\} * (Ba + Ca)) && (T_{R002b}) \text{(Distribution of } * \text{ over } +) \\ (Ba^1 \neq \emptyset) &\subseteq (\{>\} * a \subseteq \{>\} * \alpha) && (T_{7931b}) \text{(Definition of } \alpha \text{ from } a) \text{(Eqn1)} \\ (a^1 \subseteq S_M S_O + S_D S_O) & (Ba^1 \neq \emptyset) \subseteq (Ba^1 \neq \emptyset) && (T_{M083}) \text{(Conjunction Elimination) (Eqn2)} \\ (a^1 \subseteq S_M S_O + S_D S_O) & (Ba^1 \neq \emptyset) \subseteq (\{>\} * a \subseteq \{>\} * \alpha) && (T_{M139}) \text{(Hypothetical Syllogism Eqns1\&2) (Eqn3)} \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha \subseteq a) && (T_{5987}) \text{(Eqn4)} \\ (\alpha \subseteq a) &\subseteq (\{>\} * \alpha \subseteq \{>\} * a) && (T_{R196}) \text{(Relational Deduction) (Eqn5)} \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\{>\} * \alpha \subseteq \{>\} * a) && (T_{M139}) \text{(Hypothetical Syllogism Eqns4\&5) (Eqn6)} \\ (a^1 \subseteq S_M S_O + S_D S_O). (a^1 \subseteq S_M S_O + S_D S_O) & (Ba^1 \neq \emptyset) \subseteq (\{>\} * \alpha \subseteq \{>\} * a). (\{>\} * a \subseteq \{>\} * \alpha) && (T_{M669}) \text{(Combination of Eqns3\&6)} \\ (a^1 \subseteq S_M S_O + S_D S_O). (Ba^1 \neq \emptyset) &\subseteq (\{>\} * \alpha \subseteq \{>\} * a). (\{>\} * a \subseteq \{>\} * \alpha) && (T_{M667}) \text{(Conjunction Tautology)} \\ (a^1 \subseteq S_M S_O + S_D S_O). (Ba^1 \neq \emptyset) &\subseteq (\{>\} * \alpha = \{>\} * a) && (T_{M430}) \text{(Biconditional Introduction)} \\ (a^1 \subseteq S_M S_O + S_D S_O). (Ba^1 \neq \emptyset) &\subseteq (\{>\} * a = \{>\} * \alpha) && (T_{M008}) \text{(Inverse of an Equality)} \end{aligned}$$

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\{\leq\} \diamond \alpha = \{\leq\} \diamond a) \quad (T_{7075})$$

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\{\geq\} \diamond \alpha = \{\geq\} \diamond a) \quad (T_{7076})$$

Example Proof of (T₇₀₇₅)

$$\begin{aligned} (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha \subseteq a) && (T_{5987}) \\ (a^1 \subseteq S_M S_O + S_D S_O) & (\alpha \neq \emptyset) \subseteq (\{\leq\} \diamond a \subseteq \{\leq\} \diamond \alpha) && (T_{R262}) \text{(Relational Deduction) (Eqn1)} \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba + Ca \neq \emptyset) && (T_{5891}) \text{(Eqn2)} \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha = Ba + Ca) && (T_{7931b}) \text{(Definition of } \alpha \text{ from } a) \text{(Eqn3)} \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha \neq \emptyset) && (A_{MX19}) \text{(Substitution of Eqn3 into Eqn2)} \\ (a^1 \subseteq S_M S_O + S_D S_O) &= (a^1 \subseteq S_M S_O + S_D S_O). (\alpha \neq \emptyset) && (A_{MX44}) \text{(Equality Form) (Eqn4)} \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\{\leq\} \diamond a \subseteq \{\leq\} \diamond \alpha) && (A_{MX19}) \text{(Substitution of Eqn4 into Eqn1) (Eqn5)} \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha = Ba + Ca) && (T_{7931b}) \text{(Definition of } \alpha \text{ from } a) \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\{\leq\} \diamond \alpha = \{\leq\} \diamond (Ba + Ca)) && (T_{R522}) \text{(Relational Deduction)} \\ (Ba \neq \emptyset) (Ca \neq \emptyset) &\subseteq (\{\leq\} \diamond \alpha = \{\leq\} \diamond Ba). (\{\leq\} \diamond Ca) && (T_{R235}) \text{(Non Distribution of } \diamond \text{ over } +) \\ (Ba \neq \emptyset) (Ca \neq \emptyset) &\subseteq (\{\leq\} \diamond \alpha \subseteq \{\leq\} \diamond Ba) && (T_{M073b}) \text{(Single Premise Syllogism) (Eqn6)} \\ (a^1 \subseteq S_M S_O + S_D S_O) & (a_n \subseteq a) \subseteq (\{\leq\} \diamond Ba_n \subseteq \{\leq\} \diamond a_n) && (A_{502}) \text{(Order of Bounds of Extent 2)} \\ (a^1 \subseteq S_M S_O + S_D S_O) & (a \subseteq a) \subseteq (\{\leq\} \diamond Ba \subseteq \{\leq\} \diamond a) && \text{Resymbolising} \\ (a^1 \subseteq S_M S_O + S_D S_O) & ((a \subset a) + (a = a)) \subseteq (\{\leq\} \diamond Ba \subseteq \{\leq\} \diamond a) && (T_{M266}) \text{(Definition of } \subseteq) \\ (a^1 \subseteq S_M S_O + S_D S_O) & ((a \subset a) + E) \subseteq (\{\leq\} \diamond Ba \subseteq \{\leq\} \diamond a) && (T_{M001}) \text{(Universality of } (D_{M00}) \text{ Definition of } =) \\ (a^1 \subseteq S_M S_O + S_D S_O) & E \subseteq (\{\leq\} \diamond Ba \subseteq \{\leq\} \diamond a) && (T_{M055}) \text{(Universal Set Disjunction)} \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\{\leq\} \diamond Ba \subseteq \{\leq\} \diamond a) && (A_{MX17}) \text{(Universal Set Conjunction Axiom) (Eqn7)} \\ (a^1 \subseteq S_M S_O + S_D S_O) & (Ba \neq \emptyset) (Ca \neq \emptyset) \subseteq (\{\leq\} \diamond \alpha \subseteq \{\leq\} \diamond a) && (T_{M139}) \text{(Hypothetical Syllogism Eqns6\&7) (Eqn8)} \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba \neq \emptyset) && (T_{5889}) \text{(Eqn9)} \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ca \neq \emptyset) && (T_{5890}) \text{(Eqn10)} \\ (a^1 \subseteq S_M S_O + S_D S_O). (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba \neq \emptyset) (Ca \neq \emptyset) && (T_{M669}) \text{(Combination of Eqns9\&10)} \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba \neq \emptyset) (Ca \neq \emptyset) && (T_{M667}) \text{(Conjunction Tautology) (Eqn11)} \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\{\leq\} \diamond \alpha \subseteq \{\leq\} \diamond a) && (T_{M139}) \text{(Hypothetical Syllogism Eqns8\&11)} \end{aligned}$$

$$\begin{aligned}
& (a^1 \subseteq S_M S_O + S_D S_O) \cdot (a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\{\leq\} \diamond a \subseteq \{\leq\} \diamond a) \cdot (\{\leq\} \diamond a \subseteq \{\leq\} \diamond a) & (\text{Eqn12}) \\
& (a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\{\leq\} \diamond a \subseteq \{\leq\} \diamond a) \cdot (\{\leq\} \diamond a \subseteq \{\leq\} \diamond a) & (T_{M669})(\text{Combination of Eqns5\&12}) \\
& (a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\{\leq\} \diamond a \subseteq \{\leq\} \diamond a) & (T_{M667})(\text{Conjunction Tautology}) \\
& (a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\{\leq\} \diamond a = \{\leq\} \diamond a) & (T_{M430})(\text{Biconditional Introduction})
\end{aligned}$$

$$(a^1 \subseteq S_M S_O + S_D S_O)(Ba^1 \neq \emptyset) \subseteq (Ba^1 = a \cdot (\{\leq\} \diamond a) \cdot (\{\subset\} * a)) \quad (T_{S957})$$

$$(a^1 \subseteq S_M S_O + S_D S_O)(Ca^1 \neq \emptyset) \subseteq (Ca^1 = a \cdot (\{\geq\} \diamond a) \cdot (\{\subset\} * a)) \quad (T_{S958})$$

Example Proof of (T_{S957})

$$\begin{aligned}
& (a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba^1 = a \cdot (\{\geq\} * a) \cdot (\{\subset\} * a)) & (T_{T986}) \\
& (a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba^1 = a \cdot (\{\geq\} * a)) & (T_{T070})(\text{Eqn1}) \\
& (a^1 \subseteq S_M S_O + S_D S_O)(Ba^1 \neq \emptyset) \subseteq (\{\geq\} * a = \{\geq\} * a) & (T_{T066})(\text{Eqn2}) \\
& (a^1 \subseteq S_M S_O + S_D S_O)(Ba^1 \neq \emptyset) \subseteq (Ba^1 = a \cdot (\{\geq\} * a)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\
& (a^1 \subseteq S_D \cdot S_O) \subseteq (a \cdot \{\geq\} * a = a \cdot \{\leq\} \diamond a) & (T_{S826})(\text{Eqn4}) \\
& (a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq S_P S_O) & (T_{T777})(\text{Eqn5}) \\
& S_P \subseteq S_D & (A_{R53})(\text{Axiom of Points}) \\
& S_P S_O \subseteq S_D S_O & (T_{M029})(\text{Deduction}) \\
& (S_P S_O \subseteq S_D \cdot S_O) = E & (A_{MX16})(\text{Order Axiom}) \\
& (a^1 \subseteq S_P S_O) \subseteq (a^1 \subseteq S_D \cdot S_O) & (T_{M272})(\text{Deductive Rule})(\text{Eqn6}) \\
& (a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq S_D \cdot S_O) & (T_{M139})(\text{Hypothetical Syllogism Eqns5\&6})(\text{Eqn7}) \\
& (a^1 \subseteq S_D S_O) = (a^1 \subseteq S_D \cdot S_O) & \text{Resymbolising} \\
& (a^1 \subseteq S_D S_O) \subseteq (a^1 \subseteq S_D \cdot S_O) & (T_{M425})(\text{Eqn8}) \\
& (a^1 \subseteq S_M S_O) + (a^1 \subseteq S_D S_O) \subseteq (a^1 \subseteq S_D \cdot S_O) + (a^1 \subseteq S_D \cdot S_O) & (A_{MX21})(\text{Combination of Eqns7\&8}) \\
& (a^1 \subseteq S_M S_O) + (a^1 \subseteq S_D S_O) \subseteq (a^1 \subseteq S_D \cdot S_O) & (A_{MX01})(\text{Disjunction Tautology}) \\
& (a^1 \subseteq S_M S_O + S_D S_O) \subseteq (a^1 \subseteq S_D \cdot S_O) & (T_{M526})(\text{Eqn9}) \\
& (a^1 \subseteq S_M S_O + S_D S_O) \subseteq (a \cdot \{\geq\} * a = a \cdot \{\leq\} \diamond a) & (T_{M139})(\text{Hypothetical Syllogism Eqns4\&9}) \\
& & (\text{Eqn10}) \\
& (a^1 \subseteq S_M S_O + S_D S_O)(Ba^1 \neq \emptyset) \subseteq (Ba^1 = a \cdot (\{\leq\} \diamond a)) & (A_{MX19})(\text{Substitution of Eqn10 into Eqn3}) \\
& (a^1 \subseteq S_M S_O + S_D S_O)(Ba^1 \neq \emptyset) \subseteq (Ba^1 = a \cdot (\{\leq\} \diamond a) \cdot (\{\subset\} * a)) & (T_{T070})(\text{Eqn11}) \\
& (a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\{\leq\} \diamond a \subseteq \{\leq\} \diamond a) & (T_{T075})(\text{Eqn12}) \\
& (a^1 \subseteq S_M S_O + S_D S_O)(Ba^1 \neq \emptyset) \subseteq (Ba^1 = a \cdot (\{\leq\} \diamond a) \cdot (\{\subset\} * a)) & (A_{MX19})(\text{Substitution of Eqn12 into Eqn11})
\end{aligned}$$

11.7 Proofs of Theorems of Mapped Relationships

$$B_x a^1 = {}^v X B X a^1 \quad (T_{R727})$$

$$C_x a^1 = {}^v X C X a^1 \quad (T_{R728})$$

$$\{\triangleright\}_x a^1 = {}^v X \{\triangleright\} X a^1 \quad (T_{R729})$$

Example Proof of (T_{R727})

$$\begin{aligned}
& o_x * a = {}^v X * o * X * a & (T_{R571})(\text{Eqn1}) \\
& \text{Let } a = a^1 & (\text{Eqn2}) \\
& \text{Let } o = B & (\text{Eqn3}) \\
& B_x * a^1 = {}^v X * B * X * a^1 & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})
\end{aligned}$$

$$\{=\}_x a^1 = {}^v X X a^1 \quad (T_{T909})$$

Proof of (T_{T909})

$$\begin{aligned}
& o_x * a = {}^v X * o * X * a & (T_{R571})(\text{Eqn1}) \\
& \text{Let } a = a^1 & (\text{Eqn2})
\end{aligned}$$

Let $\mathbf{o} = \{=\}$ (Eqn3)
 $\{=\}_{\mathcal{X}} \mathbf{a}^1 = {}^v\mathcal{X}\{=\} \mathcal{X} \mathbf{a}^1$ ($\mathbf{A}_{MX19}$)(Substitution of Eqns2&3 into Eqn1)
 $\{=\}_{\mathcal{X}} \mathbf{a}^1 = {}^v\mathcal{X} \mathcal{X} \mathbf{a}^1$ ($\mathbf{D}_{A144}$)(Definition of the Reflexive Relationship Set)

$$((\mathcal{X}\mathbf{a})^1 = \mathcal{X}\mathbf{a}^1) \subseteq (\mathcal{B}_x \mathbf{a}^1 = {}^v\mathcal{X}\mathcal{B}(\mathcal{X}\mathbf{a})^1) \quad (\mathbf{T}_{R698})$$

$$((\mathcal{X}\mathbf{a})^1 = \mathcal{X}\mathbf{a}^1) \subseteq (\mathcal{C}_x \mathbf{a}^1 = {}^v\mathcal{X}\mathcal{C}(\mathcal{X}\mathbf{a})^1) \quad (\mathbf{T}_{R701})$$

N.B. These theorems can be simplified for the time and space dimensions where $(\mathcal{T}\mathbf{a})^1 = \mathcal{T}\mathbf{a}^1$ ($\mathbf{A}_{S05T}$) and $((\mathbf{O}\mathbf{b}^1)\mathbf{a})^1 = (\mathbf{O}\mathbf{b}^1)\mathbf{a}^1$ ($\mathbf{A}_{S05O}$)

Example Proof of ($\mathbf{T}_{R698}$)

$\mathcal{B}_x \mathbf{a}^1 = {}^v\mathcal{X}\mathcal{B}\mathcal{X}\mathbf{a}^1$ ($\mathbf{T}_{R727}$)(Eqn1)
 Let $(\mathcal{X}\mathbf{a})^1 = \mathcal{X}\mathbf{a}^1$ (Eqn2)
 $\mathcal{B}_x \mathbf{a}^1 = {}^v\mathcal{X}\mathcal{B}(\mathcal{X}\mathbf{a})^1$ ($\mathbf{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)
 Thus $((\mathcal{X}\mathbf{a})^1 = \mathcal{X}\mathbf{a}^1) \subseteq (\mathcal{B}_x \mathbf{a}^1 = {}^v\mathcal{X}\mathcal{B}(\mathcal{X}\mathbf{a})^1)$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)

$$((\mathcal{X}^1 \subseteq \mathbf{R}_0)(\mathbf{o} * \mathcal{X} * \mathbf{b}^1 \subseteq \mathbf{S}\mathcal{X}) = \mathbf{E}) \subseteq ((\mathbf{a}^1 \subseteq \mathbf{o}_x * \mathbf{b}^1) = (\mathcal{X} * \mathbf{a}^1 \subseteq \mathbf{o} * \mathcal{X} * \mathbf{b}^1)) \quad (\mathbf{T}_{S789})$$

Proof of ($\mathbf{T}_{S789}$)

Let $\mathcal{X}^1 \subseteq \mathbf{R}_0$
 Let $(\mathbf{o} * \mathcal{X} * \mathbf{b}^1 \subseteq \mathbf{S}\mathcal{X})$
 Let $\mathbf{a}^1 \subseteq \mathbf{o}_x * \mathbf{b}^1$
 $\mathbf{a}^1 \subseteq {}^v\mathcal{X} * \mathbf{o} * \mathcal{X} * \mathbf{b}^1$ ($\mathbf{T}_{R571}$)(Eqn1)
 $\mathbf{o} * \mathcal{X} * \mathbf{b}^1 \subseteq \mathbf{E}$ ($\mathbf{T}_{M744}$)(Universal Set Theorem)
 ${}^v\mathcal{X} * \mathbf{o} * \mathcal{X} * \mathbf{b}^1 \subseteq {}^v\mathcal{X}\mathbf{E}$ ($\mathbf{T}_{R196}$)(Relational Deduction)(Eqn2)
 $\mathbf{a}^1 \subseteq {}^v\mathcal{X}\mathbf{E}$ ($\mathbf{T}_{M139}$)(Hypothetical Syllogism Eqns1&2)
 $\mathbf{a}^1 \subseteq \mathbf{Q}\mathcal{X}$ ($\mathbf{T}_{R211}$)
 $\mathcal{X} * \mathbf{a}^1 \subseteq \mathcal{X} * {}^v\mathcal{X} * \mathbf{o} * \mathcal{X} * \mathbf{b}^1$ ($\mathbf{T}_{R196}$)(Relational Deduction on Eqn1)(Eqn3)
 $\mathbf{o} * \mathcal{X} * \mathbf{b}^1 = \mathcal{X} * {}^v\mathcal{X} * \mathbf{o} * \mathcal{X} * \mathbf{b}^1$ ($\mathbf{T}_{S518}$)(Eqn4) since $(\mathcal{X}^1 \subseteq \mathbf{R}_0)$ and $(\mathbf{o} * \mathcal{X} * \mathbf{b}^1 \subseteq \mathbf{S}\mathcal{X})$
 $\mathcal{X} * \mathbf{a}^1 \subseteq \mathbf{o} * \mathcal{X} * \mathbf{b}^1$ ($\mathbf{A}_{MX19}$)(Substitution of Eqn4 into Eqn3)
 Thus $(\mathbf{a}^1 \subseteq \mathbf{o}_x * \mathbf{b}^1) \subseteq (\mathcal{X} * \mathbf{a}^1 \subseteq \mathbf{o} * \mathcal{X} * \mathbf{b}^1)$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)(Eqn5)
 Let $\mathcal{X} * \mathbf{a}^1 \subseteq \mathbf{o} * \mathcal{X} * \mathbf{b}^1$
 ${}^v\mathcal{X} * \mathcal{X} * \mathbf{a}^1 \subseteq {}^v\mathcal{X} * \mathbf{o} * \mathcal{X} * \mathbf{b}^1$ ($\mathbf{T}_{R196}$)(Relational Deduction)
 ${}^v\mathcal{X} * \mathcal{X} * \mathbf{a}^1 \subseteq \mathbf{o}_x * \mathbf{b}^1$ ($\mathbf{T}_{R571}$)(Eqn6)
 $\mathbf{a}^1 \subseteq {}^v\mathcal{X} * \mathcal{X} * \mathbf{a}^1$ ($\mathbf{T}_{R527}$)(Eqn7) since $(\mathbf{a}^1 \subseteq \mathbf{Q}\mathcal{X})$
 $\mathbf{a}^1 \subseteq \mathbf{o}_x * \mathbf{b}^1$ ($\mathbf{T}_{M139}$)(Hypothetical Syllogism Eqns 6&7)
 Thus $(\mathcal{X} * \mathbf{a}^1 \subseteq \mathbf{o} * \mathcal{X} * \mathbf{b}^1) \subseteq (\mathbf{a}^1 \subseteq \mathbf{o}_x * \mathbf{b}^1)$ ($\mathbf{A}_{MX02}$)(Conditional Introduction)(Eqn8)
 $(\mathbf{a}^1 \subseteq \mathbf{o}_x * \mathbf{b}^1) = (\mathcal{X} * \mathbf{a}^1 \subseteq \mathbf{o} * \mathcal{X} * \mathbf{b}^1)$ ($\mathbf{T}_{M430}$)(Biconditional Introduction Eqns5&8)
 Thus $((\mathcal{X}^1 \subseteq \mathbf{R}_0)(\mathbf{o} * \mathcal{X} * \mathbf{b}^1 \subseteq \mathbf{S}\mathcal{X}) = \mathbf{E}) \subseteq ((\mathbf{a}^1 \subseteq \mathbf{o}_x * \mathbf{b}^1) = (\mathcal{X} * \mathbf{a}^1 \subseteq \mathbf{o} * \mathcal{X} * \mathbf{b}^1))$
 $(\mathbf{A}_{MX02}$)(Conditional Introduction)

Chapter 12 Proofs of Theorems of Dimensions

12.1 Proofs of Theorems Which Do Not Require the Axioms and Definitions of Dimensions

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\mathcal{S}\mathcal{X} = \mathcal{X}\mathcal{E}) \quad (\mathcal{T}_{7240})$$

Proof of ($\mathcal{T}_{7240}$)

$$\begin{aligned} ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) &\subseteq \mathcal{E} & (\mathcal{T}_{744})(\text{Universal Set Theorem}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) &\subseteq (\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}) & (\mathcal{T}_{7001})(\text{Universality of } (\mathcal{D}_{700}) \text{ Definition of } =) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) &\subseteq (\mathcal{S}\mathcal{X} = \mathcal{X}\mathcal{E}) & (\mathcal{T}_{7210}) \end{aligned}$$

$$(\mathcal{a}^1 \subseteq \mathcal{D}_A)(\mathcal{B}\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{a}^1 = (\mathcal{B}\mathcal{a}^1)^1) \quad (\mathcal{T}_{7044})$$

$$(\mathcal{a}^1 \subseteq \mathcal{D}_A)(\mathcal{C}\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{C}\mathcal{a}^1 = (\mathcal{C}\mathcal{a}^1)^1) \quad (\mathcal{T}_{7045})$$

Example Proof of ($\mathcal{T}_{7044}$)

$$\begin{aligned} (\mathcal{B}\mathcal{a}^1 \neq \emptyset) &\subseteq (\mathcal{B}\mathcal{a}^1 = (\mathcal{B}\mathcal{a}^1)^1) & (\mathcal{T}_{7047}) \\ \text{Let } \mathcal{a}^1 &\subseteq \mathcal{D}_A \\ \text{Thus } ((\mathcal{a}^1 \subseteq \mathcal{D}_A) = \mathcal{E}) &\subseteq ((\mathcal{B}\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{a}^1 = (\mathcal{B}\mathcal{a}^1)^1)) & (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\ ((\mathcal{a}^1 \subseteq \mathcal{D}_A)(\mathcal{B}\mathcal{a}^1 \neq \emptyset)) &\subseteq (\mathcal{B}\mathcal{a}^1 = (\mathcal{B}\mathcal{a}^1)^1) & (\mathcal{A}_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(\mathcal{B}(\mathcal{X}\mathcal{E})^1 \neq \emptyset) \subseteq (\mathcal{B}(\mathcal{X}\mathcal{E})^1 = (\mathcal{B}(\mathcal{X}\mathcal{E})^1)^1) \quad (\mathcal{T}_{7199})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(\mathcal{C}(\mathcal{X}\mathcal{E})^1 \neq \emptyset) \subseteq (\mathcal{C}(\mathcal{X}\mathcal{E})^1 = (\mathcal{C}(\mathcal{X}\mathcal{E})^1)^1) \quad (\mathcal{T}_{7200})$$

Example Proof of ($\mathcal{T}_{7199}$)

$$\begin{aligned} (\mathcal{a}^1 \subseteq \mathcal{D}_A)(\mathcal{B}\mathcal{a}^1 \neq \emptyset) &\subseteq (\mathcal{B}\mathcal{a}^1 = (\mathcal{B}\mathcal{a}^1)^1) & (\mathcal{T}_{7044})(\text{Eqn1}) \\ \text{Let } \mathcal{a}^1 &= (\mathcal{X}\mathcal{E})^1 & (\text{Eqn2}) \\ \mathcal{a} &= \mathcal{X}\mathcal{E} & (\mathcal{T}_{7479})(\text{Eqn3}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(\mathcal{B}(\mathcal{X}\mathcal{E})^1 \neq \emptyset) &\subseteq (\mathcal{B}(\mathcal{X}\mathcal{E})^1 = (\mathcal{B}(\mathcal{X}\mathcal{E})^1)^1) & (\mathcal{A}_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(\mathcal{B}\mathcal{X}\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}\mathcal{a}^1 = (\mathcal{B}\mathcal{X}\mathcal{a}^1)^1) \quad (\mathcal{T}_{7740})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(\mathcal{C}\mathcal{X}\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{C}\mathcal{X}\mathcal{a}^1 = (\mathcal{C}\mathcal{X}\mathcal{a}^1)^1) \quad (\mathcal{T}_{7030})$$

Example Proof of ($\mathcal{T}_{7740}$)

$$\begin{aligned} (\mathcal{B}\mathcal{a}^1 \neq \emptyset) &\subseteq (\mathcal{B}\mathcal{a}^1 = (\mathcal{B}\mathcal{a}^1)^1) & (\mathcal{T}_{7047})(\text{Eqn1}) \\ \text{Let } \mathcal{a}^1 &= \mathcal{X}\mathcal{a}^1 & (\text{Eqn2}) \\ \text{where } (\mathcal{X}\mathcal{E})^1 &\subseteq \mathcal{D}_A \\ (\mathcal{B}\mathcal{X}\mathcal{a}^1 \neq \emptyset) &\subseteq (\mathcal{B}\mathcal{X}\mathcal{a}^1 = (\mathcal{B}\mathcal{X}\mathcal{a}^1)^1) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \text{Thus } (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) = \mathcal{E}) &\subseteq ((\mathcal{B}\mathcal{X}\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}\mathcal{a}^1 = (\mathcal{B}\mathcal{X}\mathcal{a}^1)^1)) & (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(\mathcal{B}\mathcal{X}\mathcal{a}^1 \neq \emptyset) &\subseteq (\mathcal{B}\mathcal{X}\mathcal{a}^1 = (\mathcal{B}\mathcal{X}\mathcal{a}^1)^1) & (\mathcal{A}_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(\mathcal{X}\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}\mathcal{a}^1 = \mathcal{B}(\mathcal{X}\mathcal{a}^1)^1) \quad (\mathcal{T}_{7743})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(\mathcal{X}\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{C}\mathcal{X}\mathcal{a}^1 = \mathcal{C}(\mathcal{X}\mathcal{a}^1)^1) \quad (\mathcal{T}_{7032})$$

Example Proof of ($\mathcal{T}_{7743}$)

$$\begin{aligned} (\mathcal{a}^1 \neq \emptyset) &\subseteq (\mathcal{a}^{11} = \mathcal{a}^1) & (\mathcal{D}_{717})(\text{Definition of } \mathcal{a}^{11})(\text{Eqn1}) \\ \text{Let } \mathcal{a}^1 &= \mathcal{X}\mathcal{a}^1 & (\text{Eqn2}) \\ \text{where } (\mathcal{X}\mathcal{E})^1 &\subseteq \mathcal{D}_A \end{aligned}$$

$(\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{X}a^1 = (\mathcal{X}a^1)^1)$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn2 into Eqn1)
$(\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}a^1 = \mathcal{B}(\mathcal{X}a^1)^1)$	$(\mathbf{T}_{R522})$ (Relational Deduction)
Thus $((\mathcal{X}E)^1 \subseteq \mathcal{D}_A) = E \subseteq ((\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}a^1 = \mathcal{B}(\mathcal{X}a^1)^1))$	$(\mathbf{A}_{MX02})$ (Conditional Introduction)
$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}a^1 = \mathcal{B}(\mathcal{X}a^1)^1)$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets

$$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{B}\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}a^1 = (\mathcal{B}(\mathcal{X}a^1)^1)^1) \quad (\mathbf{T}_{S745})$$

$$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{C}\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{C}\mathcal{X}a^1 = (\mathcal{C}(\mathcal{X}a^1)^1)^1) \quad (\mathbf{T}_{T034})$$

Example Proof of $(\mathbf{T}_{S745})$

$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}a^1 = \mathcal{B}(\mathcal{X}a^1)^1)$	$(\mathbf{T}_{S743})$ (Eqn1)
$(a^1 \neq \emptyset) \subseteq (a^1 = a^{11})$	$(\mathbf{D}_{M17})$ (Definition of a^{11})(Eqn2)
Let $a^1 = \mathcal{B}(\mathcal{X}a^1)^1$	(Eqn3)
Where $(\mathcal{X}E)^1 \subseteq \mathcal{D}_A$	
$(\mathcal{B}(\mathcal{X}a^1)^1 \neq \emptyset) \subseteq (\mathcal{B}(\mathcal{X}a^1)^1 = (\mathcal{B}(\mathcal{X}a^1)^1)^1)$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn3 into Eqn2)
Thus $((\mathcal{X}E)^1 \subseteq \mathcal{D}_A) = E \subseteq ((\mathcal{B}(\mathcal{X}a^1)^1 \neq \emptyset) \subseteq (\mathcal{B}(\mathcal{X}a^1)^1 = (\mathcal{B}(\mathcal{X}a^1)^1)^1))$	$(\mathbf{A}_{MX02})$ (Conditional Introduction)
$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{B}(\mathcal{X}a^1)^1 \neq \emptyset) \subseteq (\mathcal{B}(\mathcal{X}a^1)^1 = (\mathcal{B}(\mathcal{X}a^1)^1)^1)$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets (Eqn4)
$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{X}a^1 \neq \emptyset).((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{B}(\mathcal{X}a^1)^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}a^1 = \mathcal{B}(\mathcal{X}a^1)^1).(\mathcal{B}(\mathcal{X}a^1)^1 = (\mathcal{B}(\mathcal{X}a^1)^1)^1)$	$(\mathbf{T}_{M669})$ (Combination of Eqns1&4)
$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{X}a^1 \neq \emptyset).(\mathcal{B}(\mathcal{X}a^1)^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}a^1 = \mathcal{B}(\mathcal{X}a^1)^1).(\mathcal{B}(\mathcal{X}a^1)^1 = (\mathcal{B}(\mathcal{X}a^1)^1)^1)$	$(\mathbf{T}_{M667})$ (Conjunction Tautology)
$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{X}a^1 \neq \emptyset).(\mathcal{B}(\mathcal{X}a^1)^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}a^1 = (\mathcal{B}(\mathcal{X}a^1)^1)^1)$	$(\mathbf{A}_{MX19})$ (Substitution)(Eqn5)
$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}a^1 = \mathcal{B}(\mathcal{X}a^1)^1)$	$(\mathbf{T}_{S743})$
$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}a^1 = \mathcal{B}(\mathcal{X}a^1)^1) = E$	$(\mathbf{A}_{MX16})$ (Order Axiom)
$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{X}a^1 \neq \emptyset) = E \subseteq (\mathcal{B}\mathcal{X}a^1 = \mathcal{B}(\mathcal{X}a^1)^1) = E$	$(\mathbf{T}_{M094})$ (Probabilistic Rule)
$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{X}a^1 \neq \emptyset) = E \subseteq ((\mathcal{B}\mathcal{X}a^1 \neq \emptyset) = (\mathcal{B}(\mathcal{X}a^1)^1 \neq \emptyset))$	$(\mathbf{T}_{M096})$ (Probabilistic Rule)
$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{X}a^1 \neq \emptyset)(\mathcal{B}\mathcal{X}a^1 \neq \emptyset) = ((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{X}a^1 \neq \emptyset)(\mathcal{B}(\mathcal{X}a^1)^1 \neq \emptyset)$	$(\mathbf{T}_{M444b})$ (Exportation from a Biconditional) for truth sets (Eqn6)
$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{X}a^1 \neq \emptyset).(\mathcal{B}\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}a^1 = (\mathcal{B}(\mathcal{X}a^1)^1)^1)$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn6 into Eqn5)(Eqn7)
$(\mathcal{B}\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{X}a^1 \neq \emptyset)$	$(\mathbf{T}_{R171b})$
$(\mathcal{B}\mathcal{X}a^1 \neq \emptyset) = (\mathcal{B}\mathcal{X}a^1 \neq \emptyset)(\mathcal{X}a^1 \neq \emptyset)$	$(\mathbf{A}_{MX44})$ ($\subseteq$ Equality Form)(Eqn8)
$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A)(\mathcal{B}\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{B}\mathcal{X}a^1 = (\mathcal{B}(\mathcal{X}a^1)^1)^1)$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn8 into Eqn7)

$$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A) = E \subseteq ((\mathcal{X}b^1 \subseteq \{>\} * \mathcal{X}a^1) = (\mathcal{B}\mathcal{X}b^1 \neq \emptyset).((\mathcal{B}\mathcal{X}a^1 \neq \emptyset).(\mathcal{B}\mathcal{X}b^1 \subseteq \{>\} * \mathcal{B}\mathcal{X}a^1) + (\mathcal{B}\mathcal{X}a^1 = \emptyset)).(\mathcal{C}\mathcal{X}a^1 \neq \emptyset).((\mathcal{C}\mathcal{X}b^1 \neq \emptyset).(\mathcal{C}\mathcal{X}b^1 \subseteq \{>\} * \mathcal{C}\mathcal{X}a^1) + (\mathcal{C}\mathcal{X}b^1 = \emptyset))) \quad (\mathbf{T}_{T949})$$

$$((\mathcal{X}E)^1 \subseteq \mathcal{D}_A) = E \subseteq ((\mathcal{X}b^1 \subseteq \{<\} * \mathcal{X}a^1) = (\mathcal{B}\mathcal{X}b^1 \neq \emptyset).((\mathcal{B}\mathcal{X}b^1 \subseteq \{<\} * \mathcal{B}\mathcal{X}a^1) + (\mathcal{B}\mathcal{X}b^1 = \emptyset)).(\mathcal{C}\mathcal{X}b^1 \neq \emptyset).((\mathcal{C}\mathcal{X}a^1 \neq \emptyset).(\mathcal{C}\mathcal{X}b^1 \subseteq \{<\} * \mathcal{C}\mathcal{X}a^1) + (\mathcal{C}\mathcal{X}a^1 = \emptyset))) \quad (\mathbf{T}_{T948})$$

Example Proof of $(\mathbf{T}_{T949})$

$(\mathcal{B}^1 \subseteq \{>\} * a^1) = (\mathcal{B}\mathcal{B}^1 \neq \emptyset).((\mathcal{B}a^1 \neq \emptyset).(\mathcal{B}\mathcal{B}^1 \subseteq \{>\} * \mathcal{B}a^1) + (\mathcal{B}a^1 = \emptyset)).(\mathcal{C}a^1 \neq \emptyset).((\mathcal{C}\mathcal{B}^1 \neq \emptyset).(\mathcal{C}\mathcal{B}^1 \subseteq \{>\} * \mathcal{C}a^1) + (\mathcal{C}\mathcal{B}^1 = \emptyset))$	$(\mathbf{D}_{R63})$ (Eqn1)
Let $a^1 = \mathcal{X}a^1$	(Eqn2)
Let $\mathcal{B}^1 = \mathcal{X}b^1$	(Eqn3)
Where $(\mathcal{X}E)^1 \subseteq \mathcal{D}_A$	
$(\mathcal{X}b^1 \subseteq \{>\} * \mathcal{X}a^1) = (\mathcal{B}\mathcal{X}b^1 \neq \emptyset).((\mathcal{B}\mathcal{X}a^1 \neq \emptyset).(\mathcal{B}\mathcal{X}b^1 \subseteq \{>\} * \mathcal{B}\mathcal{X}a^1) + (\mathcal{B}\mathcal{X}a^1 = \emptyset)).(\mathcal{C}\mathcal{X}a^1 \neq \emptyset).((\mathcal{C}\mathcal{X}b^1 \neq \emptyset).(\mathcal{C}\mathcal{X}b^1 \subseteq \{>\} * \mathcal{C}\mathcal{X}a^1) + (\mathcal{C}\mathcal{X}b^1 = \emptyset))$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn2&3 into Eqn1)

Thus $((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A = \mathcal{E}) \subseteq$
 $((\mathcal{X}\mathcal{b}^1 \subseteq \{>\} * \mathcal{X}\mathcal{a}^1) = (\mathcal{B}\mathcal{X}\mathcal{b}^1 \neq \emptyset).((\mathcal{B}\mathcal{X}\mathcal{a}^1 \neq \emptyset).(\mathcal{B}\mathcal{X}\mathcal{b}^1 \subseteq \{>\} * \mathcal{B}\mathcal{X}\mathcal{a}^1) +$
 $(\mathcal{B}\mathcal{X}\mathcal{a}^1 = \emptyset)).(\mathcal{C}\mathcal{X}\mathcal{a}^1 \neq \emptyset).((\mathcal{C}\mathcal{X}\mathcal{b}^1 \neq \emptyset).(\mathcal{C}\mathcal{X}\mathcal{b}^1 \subseteq \{>\} * \mathcal{C}\mathcal{X}\mathcal{a}^1) + (\mathcal{C}\mathcal{X}\mathcal{b}^1 = \emptyset)))$
 $(\mathcal{A}_{MX02})(\text{Conditional Introduction})$

$((\mathcal{Q}\mathcal{D})^1 \subseteq \mathcal{D}_M + \mathcal{P}_M)((\mathcal{S}\mathcal{D})^1 \subseteq \mathcal{D}_D)(\mathcal{a}^1 \neq \emptyset)(\mathcal{a}^1 \subseteq \mathcal{Q}\mathcal{D})(\mathcal{b}^1 \subseteq \mathcal{Q}\mathcal{D}) = \mathcal{E}) \subseteq ((\mathcal{a}^1 = \mathcal{b}^1) \subseteq (\mathcal{D}\mathcal{a}^1 = \mathcal{D}\mathcal{b}^1)) \quad (\mathcal{T}_{7404})$

Proof of $(\mathcal{T}_{7404})$

Let $(\mathcal{Q}\mathcal{D})^1 \subseteq \mathcal{D}_M + \mathcal{P}_M$

Let $(\mathcal{S}\mathcal{D})^1 \subseteq \mathcal{D}_D$

Let $\mathcal{a}^1 \neq \emptyset$

Let $\mathcal{a}^1 \subseteq \mathcal{Q}\mathcal{D}$

Let $\mathcal{b}^1 \subseteq \mathcal{Q}\mathcal{D}$

Let $\mathcal{a}^1 = \mathcal{b}^1$

$\mathcal{D}\mathcal{a}^1 = \mathcal{D}\mathcal{b}^1$

$(\mathcal{T}_{R522})(\text{Relational Deduction})$

Thus $((\mathcal{Q}\mathcal{D})^1 \subseteq \mathcal{D}_M + \mathcal{P}_M)((\mathcal{S}\mathcal{D})^1 \subseteq \mathcal{D}_D)(\mathcal{a}^1 \neq \emptyset)(\mathcal{a}^1 \subseteq \mathcal{Q}\mathcal{D})(\mathcal{b}^1 \subseteq \mathcal{Q}\mathcal{D})(\mathcal{a}^1 = \mathcal{b}^1) \subseteq (\mathcal{D}\mathcal{a}^1 = \mathcal{D}\mathcal{b}^1))$

$(\mathcal{A}_{MX02})(\text{Conditional Introduction})$

$((\mathcal{Q}\mathcal{D})^1 \subseteq \mathcal{D}_M + \mathcal{P}_M)((\mathcal{S}\mathcal{D})^1 \subseteq \mathcal{D}_D)(\mathcal{a}^1 \neq \emptyset)(\mathcal{a}^1 \subseteq \mathcal{Q}\mathcal{D})(\mathcal{b}^1 \subseteq \mathcal{Q}\mathcal{D}) = \mathcal{E}) \subseteq ((\mathcal{a}^1 = \mathcal{b}^1) \subseteq (\mathcal{D}\mathcal{a}^1 = \mathcal{D}\mathcal{b}^1))$

$(\mathcal{A}_{MX35})(\text{Exportation/Importation})$ for truth sets

12.2 Proofs of Theorems Which Require the Axioms and Definitions of Dimensions

$\mathcal{D}_A \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O \quad (\mathcal{T}_{7929})$

$(\mathcal{a}^1 \subseteq \mathcal{D}_A) \subseteq (\mathcal{a}^1 \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O) \quad (\mathcal{T}_{7792})$

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O) \quad (\mathcal{T}_{R004})$

Proof of $(\mathcal{T}_{7929})$, $(\mathcal{T}_{7792})$ & $(\mathcal{T}_{R004})$

$\mathcal{D}_A = \mathcal{D}_M + \mathcal{D}_P + \mathcal{D}_D$

$(\mathcal{D}_{S33})(\text{Definition of } \mathcal{D}_A)$

$\mathcal{D}_A \mathcal{S}_O = (\mathcal{D}_M + \mathcal{D}_P + \mathcal{D}_D) \mathcal{S}_O$

$(\mathcal{T}_{M233})(\text{Deduction})(\text{Eqn1})$

$\mathcal{D}_A \subseteq \mathcal{S}_O$

$(\mathcal{A}_{S28})(\text{Axiom of Ordered Dimensions})$

$\mathcal{D}_A = \mathcal{D}_A \mathcal{S}_O$

$(\mathcal{A}_{MX44})(\subseteq \text{Equality Form})(\text{Eqn2})$

$\mathcal{D}_A = (\mathcal{D}_M + \mathcal{D}_P + \mathcal{D}_D) \mathcal{S}_O$

$(\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3})$

$\mathcal{D}_P \subseteq \mathcal{D}_D$

$(\mathcal{A}_{R94})(\text{Point Dimensions are Discrete})$

$\mathcal{D}_D = \mathcal{D}_P + \mathcal{D}_D$

$(\mathcal{T}_{M764})(\subseteq \text{Equality Form})(\text{Eqn4})$

$\mathcal{D}_A = (\mathcal{D}_M + \mathcal{D}_D) \mathcal{S}_O$

$(\mathcal{A}_{MX19})(\text{Substitution of Eqn4 into Eqn3})(\text{Eqn5})$

$\mathcal{D}_M + \mathcal{P}_M \subseteq \mathcal{S}_M$

$(\mathcal{A}_{S12})(\text{Axiom of Mixed Continuum Dimensions})$

$\mathcal{D}_M \subseteq \mathcal{S}_M$

$(\mathcal{T}_{M072})(\text{Single Premise Syllogism})(\text{Eqn6})$

$\mathcal{D}_D + \mathcal{P}_D \subseteq \mathcal{S}_D$

$(\mathcal{A}_{R97})(\text{Axiom of Discrete Dimensions})$

$\mathcal{D}_D \subseteq \mathcal{S}_D$

$(\mathcal{T}_{M072})(\text{Single Premise Syllogism})(\text{Eqn7})$

$\mathcal{D}_M + \mathcal{D}_D \subseteq \mathcal{S}_M + \mathcal{S}_D$

$(\mathcal{A}_{MX21})(\text{Combination of Eqns6\&7})$

$(\mathcal{D}_M + \mathcal{D}_D) \mathcal{S}_O \subseteq (\mathcal{S}_M + \mathcal{S}_D) \mathcal{S}_O$

$(\mathcal{T}_{M233})(\text{Deduction})(\text{Eqn8})$

$\mathcal{D}_A \subseteq (\mathcal{S}_M + \mathcal{S}_D) \mathcal{S}_O$

$(\mathcal{A}_{MX19})(\text{Substitution of Eqn5 into Eqn8})$

$\mathcal{D}_A \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O$

$(\mathcal{T}_{M407})(\text{Distribution of } \cdot \text{ over } +)$

$(\mathcal{D}_A \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O) = \mathcal{E}$

$(\mathcal{A}_{MX16})(\text{Order Axiom})$

$(\mathcal{a}^1 \subseteq \mathcal{D}_A) \subseteq (\mathcal{a}^1 \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O)$

$(\mathcal{T}_{M099})(\text{Entity Mode / Set Mode Rule})$

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O)$

$(\mathcal{T}_{M099})(\text{Entity Mode / Set Mode Rule})$

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) \subseteq (\mathcal{B}\mathcal{X}\mathcal{E} = \mathcal{C}\mathcal{X}\mathcal{E})$

$(\mathcal{T}_{7940})$

Proof of $(\mathcal{T}_{7940})$

$$\begin{array}{ll}
(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba = Ca) & (T_{7932c})(Eqn1) \\
\text{Let } a = \mathcal{X}E & (Eqn2) \\
\text{Where } (\mathcal{X}E)^1 \subseteq D_A & \\
((\mathcal{X}E)^1 \subseteq S_M S_O + S_D S_O) \subseteq (B\mathcal{X}E = C\mathcal{X}E) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(Eqn3) \\
((\mathcal{X}E)^1 \subseteq D_A) \subseteq ((\mathcal{X}E)^1 \subseteq S_M S_O + S_D S_O) & (T_{R004})(Eqn4) \\
((\mathcal{X}E)^1 \subseteq D_A) \subseteq (B\mathcal{X}E = C\mathcal{X}E) & (T_{M139})(\text{Hypothetical Syllogism Eqns4\&3})
\end{array}$$

$$((\mathcal{X}E)^1 \subseteq D_A) \subseteq (B\mathcal{X}E + C\mathcal{X}E \subseteq \mathcal{X}E) \quad (T_{S945})$$

Proof of (T_{S945})

$$\begin{array}{ll}
(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba + Ca \subseteq a) & (T_{S986})(Eqn1) \\
\text{Let } a = \mathcal{X}E & (Eqn2) \\
\text{Where } (\mathcal{X}E)^1 \subseteq D_A & \\
((\mathcal{X}E)^1 \subseteq S_M S_O + S_D S_O) \subseteq (B\mathcal{X}E + C\mathcal{X}E \subseteq \mathcal{X}E) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(Eqn3) \\
((\mathcal{X}E)^1 \subseteq D_A) \subseteq ((\mathcal{X}E)^1 \subseteq S_M S_O + S_D S_O) & (T_{R004})(Eqn4) \\
((\mathcal{X}E)^1 \subseteq D_A) \subseteq (B\mathcal{X}E + C\mathcal{X}E \subseteq \mathcal{X}E) & (T_{M139})(\text{Hypothetical Syllogism Eqns4\&3})
\end{array}$$

$$((\mathcal{X}E)^1 \subseteq D_A)(X = B, C) \subseteq (X\mathcal{X}E = \mathcal{X}E.(\{\subset\}_* \mathcal{X}E)) \quad (T_{7942})$$

Proof of (T_{7942})

$$\begin{array}{ll}
(a^1 \subseteq S_M S_O + S_D S_O)(X = B, C) \subseteq (Xa = a.(\{\subset\}_* a)) & (A_{S74})(Eqn1) \\
\text{Let } a = \mathcal{X}E & (Eqn2) \\
\text{Where } (\mathcal{X}E)^1 \subseteq D_A & \\
((\mathcal{X}E)^1 \subseteq S_M S_O + S_D S_O)(X = B, C) \subseteq (X\mathcal{X}E = \mathcal{X}E.(\{\subset\}_* \mathcal{X}E)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(Eqn3) \\
((\mathcal{X}E)^1 \subseteq D_A) \subseteq ((\mathcal{X}E)^1 \subseteq S_M S_O + S_D S_O) & (T_{R004}) \\
((\mathcal{X}E)^1 \subseteq D_A)(X = B, C) \subseteq ((\mathcal{X}E)^1 \subseteq S_M S_O + S_D S_O)(X = B, C) & (T_{M029})(\text{Deduction})(Eqn4) \\
((\mathcal{X}E)^1 \subseteq D_A)(X = B, C) \subseteq (X\mathcal{X}E = \mathcal{X}E.(\{\subset\}_* \mathcal{X}E)) & (T_{M139})(\text{Hypothetical Syllogism Eqns4\&3})
\end{array}$$

$$((\mathcal{X}E)^1 \subseteq D_A) \subseteq (B(\mathcal{X}a)^1 = \mathcal{X}a.(\{\supset\}_* \mathcal{X}a)(\{\subset\}_* \mathcal{X}a)) \quad (T_{7209})$$

$$((\mathcal{X}E)^1 \subseteq D_A) \subseteq (C(\mathcal{X}a)^1 = \mathcal{X}a.(\{\supset\}_* \mathcal{X}a)(\{\subset\}_* \mathcal{X}a)) \quad (T_{7210})$$

Example Proof of (T_{7209})

$$\begin{array}{ll}
(a^1 \subseteq S_M S_O + S_D S_O)(a_n \subseteq a) \subseteq (Ba_n^1 = a_n.(\{\supset\}_* a_n).(\{\subset\}_* a_n)) & (D_{R53})(\text{Definition of the Beginning Function})(Eqn1) \\
D_A \subseteq S_M S_O + S_D S_O & (T_{7929}) \\
(D_A \subseteq S_M S_O + S_D S_O) = E & (A_{MX16})(\text{Order Axiom}) \\
(a^1 \subseteq D_A) \subseteq (a^1 \subseteq S_M S_O + S_D S_O) & (T_{M099})(\text{Entity Mode / Set Mode Rule}) \\
(a^1 \subseteq D_A)(a_n \subseteq a) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)(a_n \subseteq a) & (T_{M029})(\text{Deduction})(Eqn2) \\
(a^1 \subseteq D_A)(a_n \subseteq a) \subseteq (Ba_n^1 = a_n.(\{\supset\}_* a_n).(\{\subset\}_* a_n)) & (T_{M139})(\text{Hypothetical Syllogism Eqns2\&1})(Eqn3) \\
\text{Let } a_n = \mathcal{X}a & (Eqn4) \\
\text{Let } a = \mathcal{X}E & (Eqn5) \\
((\mathcal{X}E)^1 \subseteq D_A)(\mathcal{X}a \subseteq \mathcal{X}E) \subseteq (B(\mathcal{X}a)^1 = \mathcal{X}a.(\{\supset\}_* \mathcal{X}a)(\{\subset\}_* \mathcal{X}a)) & (A_{MX19})(\text{Substitution of Eqns4\&5 into Eqn3})(Eqn6) \\
a \subseteq E & (T_{M744})(\text{Universal Set Theorem}) \\
\mathcal{X}a \subseteq \mathcal{X}E & (T_{R196})(\text{Relational Deduction}) \\
(\mathcal{X}a \subseteq \mathcal{X}E) = E & (A_{MX16})(\text{Order Axiom})(Eqn7) \\
((\mathcal{X}E)^1 \subseteq D_A)E \subseteq (B(\mathcal{X}a)^1 = \mathcal{X}a.(\{\supset\}_* \mathcal{X}a)(\{\subset\}_* \mathcal{X}a)) & (A_{MX19})(\text{Substitution of Eqn7 into Eqn6}) \\
((\mathcal{X}E)^1 \subseteq D_A) \subseteq (B(\mathcal{X}a)^1 = \mathcal{X}a.(\{\supset\}_* \mathcal{X}a)(\{\subset\}_* \mathcal{X}a)) & (A_{MX11})(\text{Universal Set Conjunction Axiom})
\end{array}$$

$$((\mathcal{X}E)^1 \subseteq D_A) \subseteq (\{\leq\} \diamond B\mathcal{X}a \subseteq \{\leq\} \diamond \mathcal{X}a) \quad (T_{7938})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) \subseteq (\{\geq\} \diamond C\mathcal{X}a \subseteq \{\geq\} \diamond \mathcal{X}a) \quad (\mathcal{T}_{7939})$$

Example Proof of ($\mathcal{T}_{7938}$)

$$\begin{aligned} (a^1 \subseteq S_M S_O + S_D S_O)(a_n \subseteq a) &\subseteq (\{\leq\} \diamond B a_n \subseteq \{\leq\} \diamond a_n) & (\mathcal{A}_{502})(\text{Order of Bounds of Extent 2})(\text{Eqn1}) \\ \mathcal{D}_A &\subseteq S_M S_O + S_D S_O & (\mathcal{T}_{7929}) \\ (\mathcal{D}_A \subseteq S_M S_O + S_D S_O) &= E & (\mathcal{A}_{MX16})(\text{Order Axiom}) \\ (a^1 \subseteq \mathcal{D}_A) &\subseteq (a^1 \subseteq S_M S_O + S_D S_O) & (\mathcal{T}_{M099})(\text{Entity Mode / Set Mode Rule}) \\ (a^1 \subseteq \mathcal{D}_A)(a_n \subseteq a) &\subseteq (a^1 \subseteq S_M S_O + S_D S_O)(a_n \subseteq a) & (\mathcal{T}_{M029})(\text{Deduction})(\text{Eqn2}) \\ (a^1 \subseteq \mathcal{D}_A)(a_n \subseteq a) &\subseteq (\{\leq\} \diamond B a_n \subseteq \{\leq\} \diamond a_n) & (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns2\&1})(\text{Eqn3}) \\ \text{Let } a_n &= \mathcal{X}a & (\text{Eqn4}) \\ \text{Let } a &= \mathcal{X}\mathcal{E} & (\text{Eqn5}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) &\subseteq (\{\leq\} \diamond B \mathcal{X}a \subseteq \{\leq\} \diamond \mathcal{X}a) & (\mathcal{A}_{MX19})(\text{Substitution of Eqns4\&5 into Eqn3})(\text{Eqn6}) \\ a &\subseteq E & (\mathcal{T}_{M744})(\text{Universal Set Theorem}) \\ \mathcal{X}a &\subseteq \mathcal{X}\mathcal{E} & (\mathcal{T}_{R196})(\text{Relational Deduction}) \\ (\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) &= E & (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn7}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)E &\subseteq (\{\leq\} \diamond B \mathcal{X}a \subseteq \{\leq\} \diamond \mathcal{X}a) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn7 into Eqn6}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) &\subseteq (\{\leq\} \diamond B \mathcal{X}a \subseteq \{\leq\} \diamond \mathcal{X}a) & (\mathcal{A}_{MX11})(\text{Universal Set Conjunction Axiom}) \end{aligned}$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a \subseteq \mathcal{X}\mathcal{E}) = E) \subseteq ((a \neq \emptyset) = (\forall \mathcal{X}a \neq \emptyset)) \quad (\mathcal{T}_{7274})$$

Proof of ($\mathcal{T}_{7274}$)

$$\begin{aligned} (((\forall c)^1 \subseteq \mathcal{R}_O)(b \subseteq \mathcal{Q}c) = E) &\subseteq ((b = \forall c * a) = (a = c * b)) & (\mathcal{T}_{5528})(\text{Eqn1}) \\ \text{Let } a &= \emptyset & (\text{Eqn2}) \\ (((\forall c)^1 \subseteq \mathcal{R}_O)(b \subseteq \mathcal{Q}c) = E) &\subseteq ((b = \forall c * \emptyset) = (\emptyset = c * b)) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (((\forall c)^1 \subseteq \mathcal{R}_O)(b \subseteq \mathcal{Q}c) = E) &\subseteq ((b = \emptyset) = (\emptyset = c * b)) & (\mathcal{T}_{5181})(\text{Null Object Theorem})(\text{Eqn3}) \\ \text{Let } c &= \forall c & (\text{Eqn4}) \\ ((c^1 \subseteq \mathcal{R}_O)(b \subseteq \mathcal{Q}^V c) = E) &\subseteq ((b = \emptyset) = (\emptyset = \forall c * b)) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\ ((c^1 \subseteq \mathcal{R}_O)(b \subseteq \mathcal{S}c) = E) &\subseteq ((b = \emptyset) = (\emptyset = \forall c * b)) & (\mathcal{T}_{R015})(\text{Reversibility of Matching Entities}) \\ ((c^1 \subseteq \mathcal{R}_O)(b \subseteq c * E) = E) &\subseteq ((b = \emptyset) = (\emptyset = \forall c * b)) & (\mathcal{T}_{R210})(\text{Eqn5}) \\ \text{Let } c^1 &= \mathcal{X}^1 & (\text{Eqn6}) \\ \text{Where } (\mathcal{X}\mathcal{E})^1 &\subseteq \mathcal{D}_A + \mathcal{P}_A \\ ((\mathcal{X}^1 \subseteq \mathcal{R}_O)(b \subseteq \mathcal{X}\mathcal{E}) = E) &\subseteq ((b = \emptyset) = (\emptyset = \forall \mathcal{X}b)) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn6 into Eqn5})(\text{Eqn7}) \\ \mathcal{X}^1 &\subseteq \mathcal{R}_O & (\mathcal{A}_{545})(\text{Axiom of One-to-One Mapping}) \\ (\mathcal{X}^1 \subseteq \mathcal{R}_O) &= E & (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn8}) \\ (E(b \subseteq \mathcal{X}\mathcal{E}) = E) &\subseteq ((b = \emptyset) = (\emptyset = \forall \mathcal{X}b)) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn8 into Eqn7}) \\ ((b \subseteq \mathcal{X}\mathcal{E}) = E) &\subseteq ((b = \emptyset) = (\emptyset = \forall \mathcal{X}b)) & (\mathcal{A}_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn9}) \\ \text{Let } b &= a & (\text{Eqn10}) \\ ((a \subseteq \mathcal{X}\mathcal{E}) = E) &\subseteq ((a = \emptyset) = (\emptyset = \forall \mathcal{X}a)) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn10 into Eqn9}) \\ \text{Thus } (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(a \subseteq \mathcal{X}\mathcal{E}) = E) &\subseteq ((a = \emptyset) = (\emptyset = \forall \mathcal{X}a)) & (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a \subseteq \mathcal{X}\mathcal{E}) = E) &\subseteq ((a \neq \emptyset) = (\emptyset \neq \forall \mathcal{X}a)) & (\mathcal{T}_{M008})(\text{Inverse of an Equality}) \end{aligned}$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = E) \subseteq ((a \neq \emptyset) = (\mathcal{X}a \neq \emptyset)) \quad (\mathcal{T}_{7275})$$

Proof of ($\mathcal{T}_{7275}$)

$$\begin{aligned} (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a \subseteq \mathcal{X}\mathcal{E}) = E) &\subseteq ((a \neq \emptyset) = (\forall \mathcal{X}a \neq \emptyset)) & (\mathcal{T}_{7274})(\text{Eqn1}) \\ \text{Let } a &= \mathcal{X}a & (\text{Eqn2}) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) = E) &\subseteq ((\mathcal{X}a \neq \emptyset) = (\forall \mathcal{X}\mathcal{X}a \neq \emptyset)) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\ (a \subseteq \mathcal{Q}\mathcal{X}) &\subseteq (a \subseteq \forall \mathcal{X}\mathcal{X}a) & (\mathcal{T}_{R527}) \\ ((a \subseteq \mathcal{Q}\mathcal{X}) \subseteq (a \subseteq \forall \mathcal{X}\mathcal{X}a)) &= E & (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) = E) &\subseteq E & (\mathcal{T}_{M744})(\text{Universal Set Theorem})(\text{Eqn5}) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) = E) &\subseteq ((a \subseteq \mathcal{Q}\mathcal{X}) \subseteq (a \subseteq \forall \mathcal{X}\mathcal{X}a)) \end{aligned}$$

$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) \subseteq (Q\mathcal{X} = E)$	(A_{MX19}) (Substitution of Eqn4 into Eqn5)(Eqn6)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)$	(A_{S36}) (Axiom of Universal Mapping)(Eqn7)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq ((Q\mathcal{X} = E) = E)$	(T_{M083}) (Conjunction Elimination)(Eqn8)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq ((a \subseteq Q\mathcal{X})(Q\mathcal{X} = E) \subseteq (a \subseteq \forall \mathcal{X}\mathcal{X}a)E)$	(T_{M139}) (Hypothetical Syllogism Eqns8&7)(Eqn9)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq ((a \subseteq Q\mathcal{X})(Q\mathcal{X} = E) \subseteq (a \subseteq \forall \mathcal{X}\mathcal{X}a)E)$	(T_{M669}) (Combination of Eqns6&9)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq ((a \subseteq Q\mathcal{X})(Q\mathcal{X} = E) \subseteq (a \subseteq \forall \mathcal{X}\mathcal{X}a))$	(A_{MX11}) (Universal Set Conjunction Axiom)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq ((a \subseteq E) \subseteq (a \subseteq \forall \mathcal{X}\mathcal{X}a))$	(A_{MX19}) (Substitution)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq (E \subseteq (a \subseteq \forall \mathcal{X}\mathcal{X}a))$	(T_{M001}) (Universality of (T_{M744}) Universal Set Theorem)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq (E = (a \subseteq \forall \mathcal{X}\mathcal{X}a))$	(T_{M275}) (Supersets of the Universal Set)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq ((a \neq \emptyset) \subseteq (\forall \mathcal{X}\mathcal{X}a \neq \emptyset))$	(T_{M096}) (Probabilistic Rule)(Eqn10)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq ((a \neq \emptyset) \subseteq (\mathcal{X}a \neq \emptyset))$	(A_{MX19}) (Substitution of Eqn3 into Eqn10)(Eqn11)
$(a = \emptyset) \subseteq (\mathcal{X}a = \emptyset)$	(T_{S303})
$(\mathcal{X}a \neq \emptyset) \subseteq (a \neq \emptyset)$	(T_{M012}) (Transposition)
$((\mathcal{X}a \neq \emptyset) \subseteq (a \neq \emptyset) = E)$	(A_{MX16}) (Order Axiom)(Eqn12)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq E$	(T_{M744}) (Universal Set Theorem)(Eqn13)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq ((\mathcal{X}a \neq \emptyset) \subseteq (a \neq \emptyset))$	(A_{MX19}) (Substitution of Eqn12 into Eqn13)(Eqn14)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq ((a \neq \emptyset) = (\mathcal{X}a \neq \emptyset))$	(T_{M430}) (Biconditional Introduction Eqns11&14)(Eqn15)
$a \subseteq E$	(T_{M744}) (Universal Set Theorem)
$\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}$	(T_{R196}) (Relational Deduction)
$(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) = E$	(A_{MX16}) (Order Axiom)(Eqn16)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)E = E \subseteq ((a \neq \emptyset) = (\mathcal{X}a \neq \emptyset))$	(A_{MX19}) (Substitution of Eqn16 into Eqn15)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) = E \subseteq ((a \neq \emptyset) = (\mathcal{X}a \neq \emptyset))$	(A_{MX11}) (Universal Set Conjunction Axiom)

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{X}a^1 = (\mathcal{X}a^1)^1) \quad (T_{S746})$$

Proof of (T_{S746})

$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) \subseteq (\mathcal{X}^1 \subseteq R_0)$	(A_{S45}) (Axiom of One-to-One Mapping)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) \subseteq (\mathcal{X}^1 \subseteq R_0) = E$	(A_{MX16}) (Order Axiom)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) = E \subseteq ((\mathcal{X}^1 \subseteq R_0) = E)$	(T_{M094}) (Probabilistic Rule)(Eqn1)
$((c^1 \subseteq R_0) = E) = ((b^1 \subseteq Qc)(b^1 \neq \emptyset) \subseteq (c * b^1 = (c * b^1)^1))$	(D_{R30}) (Definition of a One-to-One Relationship Set)
$((\mathcal{X}^1 \subseteq R_0) = E) = ((a^1 \subseteq Q\mathcal{X})(a^1 \neq \emptyset) \subseteq (\mathcal{X}a^1 = (\mathcal{X}a^1)^1))$	Resymbolising (Eqn2)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) = E \subseteq ((a^1 \subseteq Q\mathcal{X})(a^1 \neq \emptyset) \subseteq (\mathcal{X}a^1 = (\mathcal{X}a^1)^1))$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) = E \subseteq ((a^1 \subseteq E)(a^1 \neq \emptyset) \subseteq (\mathcal{X}a^1 = (\mathcal{X}a^1)^1))$	(A_{S36}) (Axiom of Universal Mapping)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) = E \subseteq (E(a^1 \neq \emptyset) \subseteq (\mathcal{X}a^1 = (\mathcal{X}a^1)^1))$	(T_{M001}) (Universality of (T_{M744}) Universal Set Theorem)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) = E \subseteq ((a^1 \neq \emptyset) \subseteq (\mathcal{X}a^1 = (\mathcal{X}a^1)^1))$	(A_{MX11}) (Universal Set Conjunction Axiom)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(a^1 \neq \emptyset) \subseteq (\mathcal{X}a^1 = (\mathcal{X}a^1)^1)$	(A_{MX35}) (Exportation/Importation)
$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\mathcal{X}a^1 \neq \emptyset) \subseteq (\mathcal{X}a^1 = (\mathcal{X}a^1)^1)$	(T_{T275})

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(\mathcal{B}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{C}\mathcal{X}\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{C}\mathcal{X}\mathcal{a}^1 \subseteq \{\geq\} * \mathcal{B}\mathcal{X}\mathcal{a}^1) \quad (\mathcal{T}_{7937})$$

Proof of ($\mathcal{T}_{7937}$)

$$\begin{aligned} & (\mathcal{a}^1 \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O)(\mathcal{a}_n \subseteq \mathcal{a}).(\mathcal{B}\mathcal{a}_n^1 \neq \emptyset)(\mathcal{C}\mathcal{a}_n^1 \neq \emptyset) \subseteq (\mathcal{C}\mathcal{a}_n^1 \subseteq \{\geq\} * \mathcal{B}\mathcal{a}_n^1) & (\mathcal{A}_{R98})(\text{Order of Bounds of Extent 1})(\text{Eqn1}) \\ & \text{Let } \mathcal{a} = \mathcal{X}\mathcal{E} & (\text{Eqn2}) \\ & \text{Let } \mathcal{a}_n = \mathcal{X}\mathcal{a}^1 & (\text{Eqn3}) \\ & \text{Where } (\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A & (\text{Eqn4}) \\ & ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O)(\mathcal{X}\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}).(\mathcal{B}(\mathcal{X}\mathcal{a}^1)^1 \neq \emptyset)(\mathcal{C}(\mathcal{X}\mathcal{a}^1)^1 \neq \emptyset) \subseteq (\mathcal{C}(\mathcal{X}\mathcal{a}^1)^1 \subseteq \{\geq\} * \mathcal{B}(\mathcal{X}\mathcal{a}^1)^1) & (\mathcal{A}_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\ & ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O)(\mathcal{X}\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}).(\mathcal{B}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{C}\mathcal{X}\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{C}\mathcal{X}\mathcal{a}^1 \subseteq \{\geq\} * \mathcal{B}\mathcal{X}\mathcal{a}^1) & (\mathcal{T}_{5746})(\text{Eqn4}) \\ & \mathcal{a}^1 \subseteq \mathcal{E} & (\mathcal{T}_{M744})(\text{Universal Set Theorem}) \\ & \mathcal{X}\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E} & (\mathcal{T}_{R196})(\text{Relational Deduction}) \\ & (\mathcal{X}\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E} & (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn5}) \\ & ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O)\mathcal{E}(\mathcal{B}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{C}\mathcal{X}\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{C}\mathcal{X}\mathcal{a}^1 \subseteq \{\geq\} * \mathcal{B}\mathcal{X}\mathcal{a}^1) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn5 into Eqn4}) \\ & ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O)(\mathcal{B}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{C}\mathcal{X}\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{C}\mathcal{X}\mathcal{a}^1 \subseteq \{\geq\} * \mathcal{B}\mathcal{X}\mathcal{a}^1) & (\mathcal{A}_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn6}) \\ & ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O) & (\mathcal{T}_{R004}) \\ & ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(\mathcal{B}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{C}\mathcal{X}\mathcal{a}^1 \neq \emptyset) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_M \mathcal{S}_O + \mathcal{S}_D \mathcal{S}_O)(\mathcal{B}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{C}\mathcal{X}\mathcal{a}^1 \neq \emptyset) & (\mathcal{T}_{M029})(\text{Deduction})(\text{Eqn7}) \\ & ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(\mathcal{B}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{C}\mathcal{X}\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{C}\mathcal{X}\mathcal{a}^1 \subseteq \{\geq\} * \mathcal{B}\mathcal{X}\mathcal{a}^1) & (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns7\&6}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E} \subseteq ((\mathcal{a} = \emptyset^1) \subseteq (\mathcal{X} * \mathcal{a}^1 = \emptyset)) \quad (\mathcal{T}_{5796}) \emptyset^1 \text{ does not Map}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E} \subseteq ((\mathcal{a} = \mathcal{E}^1) \subseteq (\mathcal{X} * \mathcal{a}^1 = \emptyset)) \quad (\mathcal{T}_{5788}) \mathcal{E}^1 \text{ does not Map}$$

Example Proof of ($\mathcal{T}_{5796}$)

$$\begin{aligned} & (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{Y} = \emptyset^1, \mathcal{E}^1) = \mathcal{E}) \subseteq ((\mathcal{a} = \mathcal{Y}) \subseteq (\mathcal{X} * \mathcal{a}^1 = \emptyset)).((\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((\forall \mathcal{X}\mathcal{a}^1)^1 \neq \mathcal{Y})) & (\mathcal{A}_{R92})(\emptyset^1 \text{ and } \mathcal{E}^1 \text{ do not Map}) \\ & (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{Y} = \emptyset^1, \mathcal{E}^1) = \mathcal{E}) \subseteq ((\mathcal{a} = \mathcal{Y}) \subseteq (\mathcal{X} * \mathcal{a}^1 = \emptyset)) & (\mathcal{T}_{M073})(\text{Single Premise Syllogism}) \\ & (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\mathcal{a} = \emptyset^1) \subseteq (\mathcal{X} * \mathcal{a}^1 = \emptyset)) & \text{Extraction} \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_O) \quad (\mathcal{T}_{7933})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{P}_A) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_U) \quad (\mathcal{T}_{7922})$$

Example Proof of ($\mathcal{T}_{7933}$)

$$\begin{aligned} & \mathcal{D}_A \subseteq \mathcal{S}_O & (\mathcal{A}_{S28})(\text{Axiom of Ordered Dimensions}) \\ & (\mathcal{D}_A \subseteq \mathcal{S}_O) = \mathcal{E} & (\mathcal{A}_{MX16})(\text{Order Axiom}) \\ & ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_O) & (\mathcal{T}_{M099})(\text{Entity Mode / Set Mode Rule}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_O) \quad (\mathcal{T}_{7159})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_P) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_O) \quad (\mathcal{T}_{7998})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_D) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_O) \quad (\mathcal{T}_{7161})$$

Example Proof of ($\mathcal{T}_{7159}$)

$$\begin{aligned} & ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_O) & (\mathcal{T}_{7933}) \\ & ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M + \mathcal{D}_P + \mathcal{D}_D) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_O) & (\mathcal{D}_{S33})(\text{Definition of } \mathcal{D}_A) \\ & ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) + ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_P) + ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_D) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_O) & (\mathcal{T}_{M526}) \\ & ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_O) & (\mathcal{T}_{M072})(\text{Single Premise Syllogism}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)(\mathcal{X}\mathcal{a} \neq \emptyset)(\mathcal{X}\mathcal{a} \neq (\mathcal{X}\mathcal{a})^1) \subseteq ((\mathcal{X}\mathcal{a})^1 \subseteq \mathcal{S}_O) \quad (\mathcal{T}_{7920})$$

Proof of (T₉₂₀)

$$\begin{aligned}
 &(\hat{b}^1 \subseteq S_0)(a \subseteq \hat{b})(a \neq \emptyset)(a \neq a^1) \subseteq (a^1 \subseteq S_0) && (A_{S48})(\text{Subsets of Ordered Sets})(\text{Eqn1}) \\
 &\text{Let } \hat{b} = \chi E && (\text{Eqn2}) \\
 &\text{Let } a = \chi a && (\text{Eqn3}) \\
 &((\chi E)^1 \subseteq S_0)(\chi a \subseteq \chi E)(\chi a \neq \emptyset)(\chi a \neq (\chi a)^1) \subseteq ((\chi a)^1 \subseteq S_0) && (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})(\text{Eqn4}) \\
 &a \subseteq E && (T_{M744})(\text{Universal Set Theorem}) \\
 &\chi a \subseteq \chi E && (T_{R196})(\text{Relational Deduction}) \\
 &(\chi a \subseteq \chi E) = E && (A_{MX16})(\text{Order Axiom})(\text{Eqn5}) \\
 &((\chi E)^1 \subseteq S_0)E(\chi a \neq \emptyset)(\chi a \neq (\chi a)^1) \subseteq ((\chi a)^1 \subseteq S_0) && (A_{MX19})(\text{Substitution of Eqn5 into Eqn4}) \\
 &((\chi E)^1 \subseteq S_0)(\chi a \neq \emptyset)(\chi a \neq (\chi a)^1) \subseteq ((\chi a)^1 \subseteq S_0) && (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn6}) \\
 &((\chi E)^1 \subseteq D_A) \subseteq ((\chi E)^1 \subseteq S_0) && (T_{933}) \\
 &((\chi E)^1 \subseteq D_A)(\chi a \neq \emptyset)(\chi a \neq (\chi a)^1) \subseteq ((\chi E)^1 \subseteq S_0)((\chi a)^1 \neq \emptyset)(\chi a \neq (\chi a)^1) && (T_{M029})(\text{Deduction})(\text{Eqn7}) \\
 &((\chi E)^1 \subseteq D_A)(\chi a \neq \emptyset)(\chi a \neq (\chi a)^1) \subseteq ((\chi a)^1 \subseteq S_0) && (T_{M139})(\text{Hypothetical Syllogism Eqns7\&6})
 \end{aligned}$$

$$((\chi E)^1 \subseteq P_A) \subseteq ((\chi a)^1 \subseteq S_u) \quad (T_{919})$$

Proof of (T₉₁₉)

$$\begin{aligned}
 &(\hat{b}^1 \subseteq S_u)(a \subseteq \hat{b}) \subseteq (a^1 \subseteq S_u) && (A_{S27,a})(\text{Subsets of Unordered Sets})(\text{Eqn1}) \\
 &\text{Let } \hat{b} = \chi E && (\text{Eqn2}) \\
 &\text{Let } a = \chi a && (\text{Eqn3}) \\
 &((\chi E)^1 \subseteq S_u)(\chi a \subseteq \chi E) \subseteq ((\chi a)^1 \subseteq S_u) && (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})(\text{Eqn4}) \\
 &a \subseteq E && (T_{M744})(\text{Universal Set Theorem}) \\
 &\chi a \subseteq \chi E && (T_{R196})(\text{Relational Deduction}) \\
 &(\chi a \subseteq \chi E) = E && (A_{MX16})(\text{Order Axiom})(\text{Eqn5}) \\
 &((\chi E)^1 \subseteq S_u)E \subseteq ((\chi a)^1 \subseteq S_u) && (A_{MX19})(\text{Substitution of Eqn5 into Eqn4}) \\
 &((\chi E)^1 \subseteq S_u) \subseteq ((\chi a)^1 \subseteq S_u) && (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn6}) \\
 &((\chi E)^1 \subseteq P_A) \subseteq ((\chi E)^1 \subseteq S_u) && (T_{922})(\text{Eqn7}) \\
 &((\chi E)^1 \subseteq P_A) \subseteq ((\chi a)^1 \subseteq S_u) && (T_{M139})(\text{Hypothetical Syllogism Eqns7\&6})
 \end{aligned}$$

$$((\chi E)^1 \subseteq D_A + P_A) \subseteq (\chi E \neq (\chi E)^1) \quad (T_{934})$$

Proof of (T₉₃₄)

$$\begin{aligned}
 &(a^1 \subseteq S_0) \subseteq (a \neq a^1)(a \neq \emptyset) && (T_{S641}) \\
 &(a^1 \subseteq S_0) \subseteq (a \neq a^1) && (T_{M073})(\text{Single Premise Syllogism}) \\
 &((\chi E)^1 \subseteq S_0) \subseteq (\chi E \neq (\chi E)^1) && \text{Resymbolising (Eqn1)} \\
 &((\chi E)^1 \subseteq D_A) \subseteq ((\chi E)^1 \subseteq S_0) && (T_{933})(\text{Eqn2}) \\
 &((\chi E)^1 \subseteq D_A) \subseteq (\chi E \neq (\chi E)^1) && (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\
 &((\chi E)^1 \subseteq P_A) \subseteq (\chi E \neq (\chi E)^1) && (A_{S06})(\text{Axiom of Polydimension Plurality})(\text{Eqn4}) \\
 &((\chi E)^1 \subseteq D_A) + ((\chi E)^1 \subseteq P_A) \subseteq (\chi E \neq (\chi E)^1) + (\chi E \neq (\chi E)^1) && (A_{MX21})(\text{Combination of Eqns3\&4}) \\
 &((\chi E)^1 \subseteq D_A) + ((\chi E)^1 \subseteq P_A) \subseteq (\chi E \neq (\chi E)^1) && (A_{MX01})(\text{Disjunction Tautology}) \\
 &((\chi E)^1 \subseteq D_A + P_A) \subseteq (\chi E \neq (\chi E)^1) && (T_{M526})
 \end{aligned}$$

$$((\chi E)^1 \subseteq D_M + P_M) \subseteq ((\chi E)^1 \subseteq S_M) \quad (T_{918})$$

$$((\chi E)^1 \subseteq D_P + P_P) \subseteq ((\chi E)^1 \subseteq S_P) \quad (T_{915})$$

$$((\chi E)^1 \subseteq D_D + P_D) \subseteq ((\chi E)^1 \subseteq S_D) \quad (T_{916})$$

Example Proof of (T₉₁₈)

$$\begin{aligned}
 &D_M + P_M \subseteq S_M && (A_{S12})(\text{Axiom of Mixed Continuum Dimensions}) \\
 &(D_M + P_M \subseteq S_M) = E && (A_{MX16})(\text{Order Axiom})
 \end{aligned}$$

$$((\chi E)^1 \subseteq D_M + P_M) \subseteq ((\chi E)^1 \subseteq S_M) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})$$

$$((\chi E)^1 \subseteq D_M + P_M)(\chi a^1 \neq \emptyset)(\chi a \neq (\chi a)^1) \subseteq ((\chi a)^1 \subseteq S_M) \quad (T_{T917})$$

$$((\chi E)^1 \subseteq D_D + P_D) \subseteq ((\chi a)^1 \subseteq S_D) \quad (T_{T912})$$

$$((\chi E)^1 \subseteq D_P + P_P) \subseteq ((\chi a)^1 \subseteq S_D) \quad (T_{T913})$$

Example Proof of (T_{T917})

$$(\hat{b}^1 \subseteq S_M)(a \subseteq \hat{b})(a \neq \emptyset)(a \neq a^1) \subseteq (a^1 \subseteq S_M) \quad (A_{R96})(\text{Subsets of Mixed Continuum Sets})(\text{Eqn1})$$

$$\text{Let } \hat{b} = \chi E \quad (\text{Eqn2})$$

$$\text{Let } a = \chi a \quad (\text{Eqn3})$$

$$((\chi E)^1 \subseteq S_M)(\chi a \subseteq \chi E)(\chi a \neq \emptyset)(\chi a \neq (\chi a)^1) \subseteq ((\chi a)^1 \subseteq S_M)$$

$$(A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})(\text{Eqn4})$$

$$a \subseteq E \quad (T_{M744})(\text{Universal Set Theorem})$$

$$\chi a \subseteq \chi E \quad (T_{R196})(\text{Relational Deduction})$$

$$(\chi a \subseteq \chi E) = E \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn5})$$

$$((\chi E)^1 \subseteq S_M)E(\chi a \neq \emptyset)(\chi a \neq (\chi a)^1) \subseteq ((\chi a)^1 \subseteq S_M)$$

$$(A_{MX19})(\text{Substitution of Eqn5 into Eqn4})$$

$$((\chi E)^1 \subseteq S_M)(\chi a \neq \emptyset)(\chi a \neq (\chi a)^1) \subseteq ((\chi a)^1 \subseteq S_M)$$

$$(A_{MX17})(\text{Universal Set Conjunction Axiom})(\text{Eqn6})$$

$$((\chi E)^1 \subseteq D_M + P_M) \subseteq ((\chi E)^1 \subseteq S_M)$$

$$(T_{T918})$$

$$((\chi E)^1 \subseteq D_M + P_M)(\chi a \neq \emptyset)(\chi a \neq (\chi a)^1) \subseteq ((\chi E)^1 \subseteq S_M)((\chi a)^1 \neq \emptyset)(\chi a \neq (\chi a)^1)$$

$$(T_{M029})(\text{Deduction})(\text{Eqn7})$$

$$((\chi E)^1 \subseteq D_M + P_M)(\chi a \neq \emptyset)(\chi a \neq (\chi a)^1) \subseteq ((\chi a)^1 \subseteq S_M)$$

$$(T_{M139})(\text{Hypothetical Syllogism Eqns6\&7})$$

$$D_M \subseteq D_A \quad (T_{T911})$$

$$D_P \subseteq D_A \quad (T_{T910})$$

$$D_D \subseteq D_A \quad (T_{T885})$$

$$P_M \subseteq P_A \quad (T_{T884})$$

$$P_P \subseteq P_A \quad (T_{T851})$$

$$P_D \subseteq P_A \quad (T_{T852})$$

Example Proof of (T_{T911}) & (T_{T884})

$$D_A = D_M + D_P + D_D \quad (D_{S33})(\text{Definition of } D_A)$$

$$D_M \subseteq D_A \quad (T_{M072b})(\text{Single Premise Syllogism})$$

$$P_A = P_M + P_P + P_D \quad (D_{S34})(\text{Definition of } P_A)$$

$$P_M \subseteq P_A \quad (T_{M072b})(\text{Single Premise Syllogism})$$

$$D_M + P_M \subseteq D_A + P_A \quad (T_{T806})$$

$$D_P + P_P \subseteq D_A + P_A \quad (T_{T807})$$

$$D_D + P_D \subseteq D_A + P_A \quad (T_{T808})$$

$$((\chi E)^1 \subseteq D_M + P_M) \subseteq ((\chi E)^1 \subseteq D_A + P_A) \quad (T_{T809})$$

$$((\chi E)^1 \subseteq D_P + P_P) \subseteq ((\chi E)^1 \subseteq D_A + P_A) \quad (T_{T810})$$

$$((\chi E)^1 \subseteq D_D + P_D) \subseteq ((\chi E)^1 \subseteq D_A + P_A) \quad (T_{T811})$$

Example Proof of (T_{T806}) & (T_{T809})

$$D_M \subseteq D_A \quad (T_{T911})(\text{Eqn1})$$

$$P_M \subseteq P_A \quad (T_{T884})(\text{Eqn2})$$

$$D_M + P_M \subseteq D_A + P_A \quad (A_{MX27})(\text{Combination of Eqns1\&2})$$

$$(D_M + P_M \subseteq D_A + P_A) = E \quad (A_{MX16})(\text{Order Axiom})$$

$$((\chi E)^1 \subseteq D_M + P_M) \subseteq ((\chi E)^1 \subseteq D_A + P_A) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_M}) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_A}) \quad (\mathcal{T}_{7923})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_P}) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_A}) \quad (\mathcal{T}_{7924})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_D}) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_A}) \quad (\mathcal{T}_{7925})$$

Example Proof of ($\mathcal{T}_{7923}$)

$$\mathbf{D_M} \subseteq \mathbf{D_A} \quad (\mathcal{T}_{7911})$$

$$(\mathbf{D_M} \subseteq \mathbf{D_A}) = \mathbf{E} \quad (\mathcal{A}_{MX16})(\text{Order Axiom})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_M}) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_A}) \quad (\mathcal{T}_{M099})(\text{Entity Mode / Set Mode Rule})$$

$$\mathbf{D_M} \subseteq \mathbf{S_M.S_O} \quad (\mathcal{T}_{7926})$$

$$\mathbf{D_P} \subseteq \mathbf{S_P.S_O} \quad (\mathcal{T}_{7927})$$

$$\mathbf{D_D} \subseteq \mathbf{S_D.S_O} \quad (\mathcal{T}_{7928})$$

Example Proof of ($\mathcal{T}_{7926}$)

$$\mathbf{D_M} + \mathbf{P_M} \subseteq \mathbf{S_M} \quad (\mathcal{A}_{S12})(\text{Axiom of Mixed Continuum Dimensions})(\text{Eqn1})$$

$$\mathbf{D_M} \subseteq \mathbf{D_M} + \mathbf{P_M} \quad (\mathcal{T}_{M082})(\text{Disjunction Introduction})(\text{Eqn2})$$

$$\mathbf{D_M} \subseteq \mathbf{S_M} \quad (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns2\&1})(\text{Eqn3})$$

$$\mathbf{D_A} \subseteq \mathbf{S_O} \quad (\mathcal{A}_{S28})(\text{Axiom of Ordered Dimensions})$$

$$\mathbf{D_M} + \mathbf{D_P} + \mathbf{D_D} \subseteq \mathbf{S_O} \quad (\mathcal{D}_{S33})(\text{Definition of } \mathbf{D_A})$$

$$\mathbf{D_M} \subseteq \mathbf{S_O} \quad (\mathcal{T}_{M072})(\text{Single Premise Syllogism})(\text{Eqn4})$$

$$\mathbf{D_M} \subseteq \mathbf{S_M.S_O} \quad (\mathcal{T}_{M132})(\text{Composition Eqns3\&4})$$

$$\mathbf{P_M} \subseteq \mathbf{S_M.S_U} \quad (\mathcal{T}_{7999})$$

$$\mathbf{P_P} \subseteq \mathbf{S_P.S_U} \quad (\mathcal{T}_{7994})$$

$$\mathbf{P_D} \subseteq \mathbf{S_D.S_U} \quad (\mathcal{T}_{7993})$$

Example Proof of ($\mathcal{T}_{7999}$)

$$\mathbf{D_M} + \mathbf{P_M} \subseteq \mathbf{S_M} \quad (\mathcal{A}_{S12})(\text{Axiom of Mixed Continuum Dimensions})(\text{Eqn1})$$

$$\mathbf{P_M} \subseteq \mathbf{D_M} + \mathbf{P_M} \quad (\mathcal{T}_{M082})(\text{Disjunction Introduction})(\text{Eqn2})$$

$$\mathbf{P_M} \subseteq \mathbf{S_M} \quad (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns2\&1})(\text{Eqn3})$$

$$\mathbf{P_A} \subseteq \mathbf{S_U} \quad (\mathcal{A}_{S29})(\text{Axiom of Unordered Polydimensions})$$

$$\mathbf{P_M} + \mathbf{P_P} + \mathbf{P_D} \subseteq \mathbf{S_U} \quad (\mathcal{D}_{S34})(\text{Definition of } \mathbf{P_A})$$

$$\mathbf{P_M} \subseteq \mathbf{S_U} \quad (\mathcal{T}_{M072})(\text{Single Premise Syllogism})(\text{Eqn4})$$

$$\mathbf{P_M} \subseteq \mathbf{S_M.S_U} \quad (\mathcal{T}_{M132})(\text{Composition Eqns3\&4})$$

$$(\mathbf{a}^1 \subseteq \mathbf{D_M}) \subseteq (\mathbf{a}^1 \subseteq \mathbf{S_M.S_O}) \quad (\mathcal{T}_{S988})$$

$$(\mathbf{a}^1 \subseteq \mathbf{D_P}) \subseteq (\mathbf{a}^1 \subseteq \mathbf{S_P.S_O}) \quad (\mathcal{T}_{7817})$$

$$(\mathbf{a}^1 \subseteq \mathbf{D_D}) \subseteq (\mathbf{a}^1 \subseteq \mathbf{S_D.S_O}) \quad (\mathcal{T}_{S989})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_M}) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S_M.S_O}) \quad (\mathcal{T}_{7022})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_P}) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S_P.S_O}) \quad (\mathcal{T}_{7023})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_D}) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S_D.S_O}) \quad (\mathcal{T}_{S723})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{P_M}) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S_M.S_U}) \quad (\mathcal{T}_{7849})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{P_P}) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S_P.S_U}) \quad (\mathcal{T}_{7847})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{P_D}) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S_D.S_U}) \quad (\mathcal{T}_{7842})$$

Example Proof of (T₀₂₂) & (T₉₈₈)

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M + \mathbf{P}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_M)$	(T ₉₁₈)(Eqn1)
$\mathbf{D}_M \subseteq \mathbf{D}_M + \mathbf{P}_M$	(T ₀₈₂)(Disjunction Introduction)
$(\mathbf{D}_M \subseteq \mathbf{D}_M + \mathbf{P}_M) = \mathbf{E}$	(A _{MX16})(Order Axiom)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M + \mathbf{P}_M)$	(T ₀₉₉)(Entity Mode / Set Mode Rule)(Eqn2)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_M)$	(T ₁₃₉)(Hypothetical Syllogism Eqns2&1)(Eqn3)
$\mathbf{D}_M \subseteq \mathbf{D}_A$	(T ₉₁₁)(Eqn4)
$\mathbf{D}_A \subseteq \mathbf{S}_O$	(A _{S28})(Axiom of Ordered Dimensions)(Eqn5)
$\mathbf{D}_M \subseteq \mathbf{S}_O$	(T ₁₃₉)(Hypothetical Syllogism Eqns4&5)
$(\mathbf{D}_M \subseteq \mathbf{S}_O) = \mathbf{E}$	(A _{MX16})(Order Axiom)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_O)$	(T ₀₉₉)(Entity Mode / Set Mode Rule)(Eqn6)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M)((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_O)((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_M)$	(T ₆₆₉)(Combination of Eqns3&6)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_O)((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_M)$	(T ₆₆₇)(Conjunction Tautology)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_M \mathbf{S}_O)$	(T ₁₃₂)(Composition)(Eqn7)
Let $\mathcal{X}\mathcal{E} = a$	(Eqn8)
$(a^1 \subseteq \mathbf{D}_M) \subseteq (a^1 \subseteq \mathbf{S}_M \mathbf{S}_O)$	(A _{MX19})(Substitution of Eqn8 into Eqn7)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\sim(\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E}).\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_C \mathbf{S}_O) \quad (\text{T}_{941})$$

Proof of (T₉₄₁)

$(a^1 \subseteq \mathbf{S}_M \mathbf{S}_O) \subseteq ((a.\sim(\mathcal{B}a + \mathcal{C}a))^1 \subseteq \mathbf{S}_C \mathbf{S}_O)$	(T ₉₃₆)(Eqn1)
Let $a = \mathcal{X}\mathcal{E}$	(Eqn2)
Where $(\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M$	
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_M \mathbf{S}_O) \subseteq ((\mathcal{X}\mathcal{E}.\sim(\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E}))^1 \subseteq \mathbf{S}_C \mathbf{S}_O)$	(A _{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_M \mathbf{S}_O)$	(T ₉₈₈)(Eqn4)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E}.\sim(\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E}))^1 \subseteq \mathbf{S}_C \mathbf{S}_O)$	(T ₁₃₉)(Hypothetical Syllogism Eqns4&3)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_M \mathbf{S}_O).(\mathcal{X}^1 \subseteq \mathbf{R}_O).(\mathbf{Q}\mathcal{X} = \mathbf{E})$$

(T₀₂₀) Definition of a Mixed Continuum Dimension Mapping Relationship Set

Proof of (T₀₂₀)

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_M \mathbf{S}_O)$	(T ₀₂₂)(Eqn1)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) \subseteq (\mathcal{X}^1 \subseteq \mathbf{R}_O)$	(A _{S45})(Axiom of One-to-One Mapping)(Eqn2)
$\mathbf{D}_A \subseteq \mathbf{D}_A + \mathbf{P}_A$	(T ₀₈₂)(Disjunction Introduction)(Eqn3)
$\mathbf{D}_M \subseteq \mathbf{D}_A$	(T ₉₁₁)(Eqn4)
$\mathbf{D}_M \subseteq \mathbf{D}_A + \mathbf{P}_A$	(T ₁₃₉)(Hypothetical Syllogism Eqns3&4)
$(\mathbf{D}_M \subseteq \mathbf{D}_A + \mathbf{P}_A) = \mathbf{E}$	(A _{MX16})(Order Axiom)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A)$	(T ₀₉₉)(Entity Mode / Set Mode Rule)(Eqn5)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq (\mathcal{X}^1 \subseteq \mathbf{R}_O)$	(T ₁₃₉)(Hypothetical Syllogism Eqns2&5)(Eqn6)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) \subseteq (\mathbf{Q}\mathcal{X} = \mathbf{E})$	(A _{S36})(Axiom of Universal Mapping)(Eqn7)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq (\mathbf{Q}\mathcal{X} = \mathbf{E})$	(T ₁₃₉)(Hypothetical Syllogism Eqns5&7)(Eqn8)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M)((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M)((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_M \mathbf{S}_O)(\mathcal{X}^1 \subseteq \mathbf{R}_O)(\mathbf{Q}\mathcal{X} = \mathbf{E})$	(T ₆₆₉)(Combination of Eqns1,6&8)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_M \mathbf{S}_O).(\mathcal{X}^1 \subseteq \mathbf{R}_O).(\mathbf{Q}\mathcal{X} = \mathbf{E})$	(T ₆₆₇)(Conjunction Tautology)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E}.\sim(\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E}))^1 \subseteq \mathbf{S}_C \mathbf{S}_O).((\mathcal{X}\mathcal{E}.\sim(\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E}))^1 \subseteq \mathbf{S}_P \mathbf{S}_O).((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_O).(\mathcal{X}^1 \subseteq \mathbf{R}_O).(\mathbf{Q}\mathcal{X} = \mathbf{E}).$$

$$(\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}.\{\subset\}_\neq \mathcal{X}\mathcal{E}) \quad (\text{T}_{983})$$

Proof of (T₉₈₃)

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_M \mathbf{S}_O).(\mathcal{X}^1 \subseteq \mathbf{R}_O).(\mathbf{Q}\mathcal{X} = \mathbf{E})$	(T ₀₂₀)(Definition of a Mixed Continuum Dimension Mapping Relationship Set)(Eqn1)
$(a^1 \subseteq \mathbf{S}_M \mathbf{S}_O) \subseteq ((a.\sim(\mathcal{B}a + \mathcal{C}a))^1 \subseteq \mathbf{S}_C \mathbf{S}_O).((a.\sim(\mathcal{B}a + \mathcal{C}a))^1 \subseteq \mathbf{S}_P \mathbf{S}_O).(a^1 \subseteq \mathbf{S}_O).(\mathcal{B}a + \mathcal{C}a = a.\{\subset\}_\neq a)$	

(T_{7822})(Definition of a Mixed Continuum Ordered Set)

$$((\chi E)^1 \subseteq S_M S_O) \subseteq ((\chi E. \sim(B\chi E + C\chi E))^1 \subseteq S_C S_O).((\chi E.(B\chi E + C\chi E))^1 \subseteq S_P S_O).((\chi E)^1 \subseteq S_O).$$

$$(B\chi E + C\chi E = \chi E.(\{-\}_{\chi} \chi E))$$

Resymbolising (Eqn2)

$$((\chi E)^1 \subseteq D_M) \subseteq ((\chi E. \sim(B\chi E + C\chi E))^1 \subseteq S_C S_O).((\chi E.(B\chi E + C\chi E))^1 \subseteq S_P S_O).((\chi E)^1 \subseteq S_O).$$

$$(\chi^1 \subseteq R_O).(\mathcal{Q}\chi = E).(B\chi E + C\chi E = \chi E.(\{-\}_{\chi} \chi E))$$

(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$D_A = D_M + D_D \quad (T_{7815})$$

$$P_A = P_M + P_D \quad (T_{7814})$$

Example Proof of (T_{7815})

$D_A = D_M + D_P + D_D$	(D_{S33})(Definition of D_A)(Eqn1)
$D_P \subseteq D_D$	(A_{R94}) (Point Dimensions are Discrete)
$D_D = D_P + D_D$	(T_{M521})($\subseteq$ Equality Form)(Eqn2)
$D_A = D_M + D_D$	(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$P_A \subseteq S_M S_U + S_D S_U \quad (T_{7992})$$

Proof of (T_{7992})

$P_M \subseteq S_M S_U$	(T_{7999})(Eqn1)
$P_D \subseteq S_D S_U$	(T_{7993})(Eqn2)
$P_M + P_D \subseteq S_M S_U + S_D S_U$	(A_{MX21})(Combination of Eqns1&2)
$P_A \subseteq S_M S_U + S_D S_U$	(T_{7814})

$$(a^1 \subseteq D_A) \subseteq (\alpha \subseteq a) \quad (T_{7488})$$

Proof of (T_{7488})

$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\alpha \subseteq a)$	(T_{S987})(Eqn1)
$(a^1 \subseteq D_A) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)$	(T_{7792})(Eqn2)
$(a^1 \subseteq D_A) \subseteq (\alpha \subseteq a)$	(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$(a^1 \subseteq D_A) \subseteq (\alpha = Ba + Ca) \quad (T_{7791})$$

Proof of (T_{7791})

$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (\alpha = Ba + Ca)$	(T_{7931b})(Definition of α from a)(Eqn1)
$(a^1 \subseteq D_A) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)$	(T_{7792})(Eqn2)
$(a^1 \subseteq D_A) \subseteq (\alpha = Ba + Ca)$	(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$(a^1 \subseteq D_A) \subseteq (a \neq \emptyset) \quad (T_{7173})$$

$$((\chi E)^1 \subseteq D_A) \subseteq (\chi E \neq \emptyset) \quad (T_{7174})$$

Proof of (T_{7173}) & (T_{7174})

$(a^1 \subseteq S_M S_O) \subseteq (a \neq \emptyset)$	(T_{7049})(Eqn1)
$(a^1 \subseteq D_M) \subseteq (a^1 \subseteq S_M S_O)$	(T_{S988})(Eqn2)
$(a^1 \subseteq D_M) \subseteq (a \neq \emptyset)$	(T_{M139})(Hypothetical Syllogism Eqns1&2)(Eqn3)
$(a^1 \subseteq S_O) \subseteq (a \neq \emptyset)$	(T_{S702})(Eqn4)
$S_D S_O \subseteq S_O$	(T_{M083})(Conjunction Elimination)
$(S_D S_O \subseteq S_O) = E$	(A_{MX16})(Order Axiom)
$(a^1 \subseteq S_D S_O) \subseteq (a^1 \subseteq S_O)$	(T_{M099})(Entity Mode / Set Mode Rule)(Eqn5)
$(a^1 \subseteq S_D S_O) \subseteq (a \neq \emptyset)$	(T_{M139})(Hypothetical Syllogism Eqns4&5)(Eqn6)
$(a^1 \subseteq D_D) \subseteq (a^1 \subseteq S_D S_O)$	(T_{S989})(Eqn7)
$(a^1 \subseteq D_D) \subseteq (a \neq \emptyset)$	(T_{M139})(Hypothetical Syllogism Eqns6&7)(Eqn8)

$(a^1 \subseteq D_M) + (a^1 \subseteq D_D) \subseteq (a \neq \emptyset) + (a \neq \emptyset)$	(A_{MX21}) (Combination of Eqns3&8)
$(a^1 \subseteq D_M) + (a^1 \subseteq D_D) \subseteq (a \neq \emptyset)$	(A_{MX01}) (Disjunction Tautology)
$(a^1 \subseteq D_M + D_D) \subseteq (a \neq \emptyset)$	(T_{M526})
$(a^1 \subseteq D_A) \subseteq (a \neq \emptyset)$	(T_{T815}) (Eqn9)
Let $a^1 = (X E)^1$	(Eqn10)
$a = X E$	(T_{M479}) (Eqn11)
$((X E)^1 \subseteq D_A) \subseteq (X E \neq \emptyset)$	(A_{MX19}) (Substitution of Eqns10&11 into Eqn9)

$(a^1 \subseteq D_M) \subseteq (a \neq \emptyset)$	(T_{T052})
$((X E)^1 \subseteq D_M) \subseteq (X E \neq \emptyset)$	(T_{T176})
$(a^1 \subseteq D_P) \subseteq (a \neq \emptyset)$	(T_{T812})
$((X E)^1 \subseteq D_P) \subseteq (X E \neq \emptyset)$	(T_{T813})
$(a^1 \subseteq D_D) \subseteq (a \neq \emptyset)$	(T_{T053})
$((X E)^1 \subseteq D_D) \subseteq (X E \neq \emptyset)$	(T_{T178})

Example Proof of (T_{T052})

$(a^1 \subseteq D_A) \subseteq (a \neq \emptyset)$	(T_{T173})
$(a^1 \subseteq D_M + D_P + D_D) \subseteq (a \neq \emptyset)$	(D_{S33}) (Definition of D_A)
$(a^1 \subseteq D_M) + (a^1 \subseteq D_P) + (a^1 \subseteq D_D) \subseteq (a \neq \emptyset)$	(T_{M526})
$(a^1 \subseteq D_M) \subseteq (a \neq \emptyset)$	(T_{M072}) (Single Premise Syllogism)

$((X E)^1 \subseteq D_M)(X a \neq \emptyset)(X a \neq (X a)^1) \subseteq ((X a)^1 \subseteq S_M S_O)$	(T_{T024})
$((X E)^1 \subseteq D_P)(X a \neq \emptyset)(X a \neq (X a)^1) \subseteq ((X a)^1 \subseteq S_D S_O)$	(T_{T025})
$((X E)^1 \subseteq D_D)(X a \neq \emptyset)(X a \neq (X a)^1) \subseteq ((X a)^1 \subseteq S_D S_O)$	(T_{T025b})

Example Proof of (T_{T024})

$((X E)^1 \subseteq D_M + P_M)(X a \neq \emptyset)(X a \neq (X a)^1) \subseteq ((X a)^1 \subseteq S_M)$	(T_{T917}) (Eqn1)
$D_M \subseteq D_M + P_M$	(T_{M082}) (Disjunction Introduction)
$(D_M \subseteq D_M + P_M) = E$	(A_{MX16}) (Order Axiom)
$((X E)^1 \subseteq D_M) \subseteq ((X E)^1 \subseteq D_M + P_M)$	(T_{M099}) (Entity Mode / Set Mode Rule)
$((X E)^1 \subseteq D_M)(X a \neq \emptyset)(X a \neq (X a)^1) \subseteq ((X E)^1 \subseteq D_M + P_M)((X a)^1 \neq \emptyset)(X a \neq (X a)^1)$	(T_{M029}) (Deduction)(Eqn2)
$((X E)^1 \subseteq D_M)(X a \neq \emptyset)(X a \neq (X a)^1) \subseteq ((X a)^1 \subseteq S_M)$	(T_{M139}) (Hypothetical Syllogism Eqns1&2)(Eqn3)
$((X E)^1 \subseteq D_A)(X a \neq \emptyset)(X a \neq (X a)^1) \subseteq ((X a)^1 \subseteq S_O)$	(T_{T920}) (Eqn4)
$D_M \subseteq D_A$	(T_{T911})
$(D_M \subseteq D_A) = E$	(A_{MX16}) (Order Axiom)
$((X E)^1 \subseteq D_M) \subseteq ((X E)^1 \subseteq D_A)$	(T_{M099}) (Entity Mode / Set Mode Rule)
$((X E)^1 \subseteq D_M)(X a \neq \emptyset)(X a \neq (X a)^1) \subseteq ((X E)^1 \subseteq D_A)(X a \neq \emptyset)(X a \neq (X a)^1)$	(T_{M029}) (Deduction)(Eqn5)
$((X E)^1 \subseteq D_M)(X a \neq \emptyset)(X a \neq (X a)^1) \subseteq ((X a)^1 \subseteq S_O)$	(T_{M139}) (Hypothetical Syllogism Eqns4&5)(Eqn6)
$((X E)^1 \subseteq D_M)(X a \neq \emptyset)(X a \neq (X a)^1) \subseteq ((X a)^1 \subseteq S_M)((X a)^1 \subseteq S_O)$	(T_{M132}) (Composition Eqns3&6)
$((X E)^1 \subseteq D_M)(X a \neq \emptyset)(X a \neq (X a)^1) \subseteq ((X a)^1 \subseteq S_M S_O)$	(T_{M132}) (Composition)

$((X E)^1 \subseteq D_A) \subseteq (X^1 \subseteq R_O)$	(T_{S718})
$((X E)^1 \subseteq P_A) \subseteq (X^1 \subseteq R_O)$	(T_{S718b})

Example Proof of (T_{S718})

$((X E)^1 \subseteq D_A + P_A) \subseteq (X^1 \subseteq R_O)$	(A_{S45}) (Axiom of One-to-One Mapping)(Eqn1)
$D_A \subseteq D_A + P_A$	(T_{M082}) (Disjunction Introduction)

$$\begin{aligned} (D_A \subseteq D_A + P_A) &= E \\ ((\mathcal{X}E)^1 \subseteq D_A) &\subseteq ((\mathcal{X}E)^1 \subseteq D_A + P_A) \\ ((\mathcal{X}E)^1 \subseteq D_A) &\subseteq (\mathcal{X}^1 \subseteq R_O) \end{aligned}$$

(**A_{MX16}**)(Order Axiom)
(**T_{M099}**)(Entity Mode / Set Mode Rule)(Eqn2)
(**T_{M139}**)(Hypothetical Syllogism Eqns1&2)

$$((\mathcal{X}E)^1 \subseteq D_M) \subseteq (\mathcal{X}^1 \subseteq R_O) \quad (\mathbf{T}_{S782})$$

$$((\mathcal{X}E)^1 \subseteq D_P) \subseteq (\mathcal{X}^1 \subseteq R_O) \quad (\mathbf{T}_{T995})$$

$$((\mathcal{X}E)^1 \subseteq D_D) \subseteq (\mathcal{X}^1 \subseteq R_O) \quad (\mathbf{T}_{S719})$$

Example Proof of (**T_{S782}**)

$$\begin{aligned} ((\mathcal{X}E)^1 \subseteq D_A) &\subseteq (\mathcal{X}^1 \subseteq R_O) & (\mathbf{T}_{S718}) \\ ((\mathcal{X}E)^1 \subseteq D_M + D_P + D_D) &\subseteq (\mathcal{X}^1 \subseteq R_O) & (\mathbf{D}_{S33})(\text{Definition of } D_A) \\ ((\mathcal{X}E)^1 \subseteq D_M) + ((\mathcal{X}E)^1 \subseteq D_P) + ((\mathcal{X}E)^1 \subseteq D_D) &\subseteq (\mathcal{X}^1 \subseteq R_O) & (\mathbf{T}_{M526}) \\ ((\mathcal{X}E)^1 \subseteq D_M) &\subseteq (\mathcal{X}^1 \subseteq R_O) & (\mathbf{T}_{M072})(\text{Single Premise Syllogism}) \end{aligned}$$

$$((\mathcal{X}E)^1 \subseteq D_A) \subseteq (QX = E) \quad (\mathbf{T}_{S797})$$

$$((\mathcal{X}E)^1 \subseteq P_A) \subseteq (QX = E) \quad (\mathbf{T}_{S797b})$$

Example Proof of (**T_{S797}**)

$$\begin{aligned} ((\mathcal{X}E)^1 \subseteq D_A + P_A) &\subseteq (QX = E) & (\mathbf{A}_{S36})(\text{Axiom of Universal Mapping})(\text{Eqn1}) \\ D_A &\subseteq D_A + P_A & (\mathbf{T}_{M082})(\text{Disjunction Introduction}) \\ (D_A \subseteq D_A + P_A) &= E & (\mathbf{A}_{MX16})(\text{Order Axiom}) \\ ((\mathcal{X}E)^1 \subseteq D_A) &\subseteq ((\mathcal{X}E)^1 \subseteq D_A + P_A) & (\mathbf{T}_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn2}) \\ ((\mathcal{X}E)^1 \subseteq D_A) &\subseteq (QX = E) & (\mathbf{T}_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$((\mathcal{X}E)^1 \subseteq D_M) \subseteq (QX = E) \quad (\mathbf{T}_{S720})$$

$$((\mathcal{X}E)^1 \subseteq D_P) \subseteq (QX = E) \quad (\mathbf{T}_{T997})$$

$$((\mathcal{X}E)^1 \subseteq D_D) \subseteq (QX = E) \quad (\mathbf{T}_{S721})$$

Example Proof of (**T_{S720}**)

$$\begin{aligned} ((\mathcal{X}E)^1 \subseteq D_A) &\subseteq (QX = E) & (\mathbf{T}_{S797}) \\ ((\mathcal{X}E)^1 \subseteq D_M + D_P + D_D) &\subseteq (QX = E) & (\mathbf{D}_{S33})(\text{Definition of } D_A) \\ ((\mathcal{X}E)^1 \subseteq D_M) + ((\mathcal{X}E)^1 \subseteq D_P) + ((\mathcal{X}E)^1 \subseteq D_D) &\subseteq (QX = E) & (\mathbf{T}_{M526}) \\ ((\mathcal{X}E)^1 \subseteq D_M) &\subseteq (QX = E) & (\mathbf{T}_{M072})(\text{Single Premise Syllogism}) \end{aligned}$$

$$((\mathcal{X}E)^1 \subseteq D_A + P_A) \subseteq (\{\subseteq\}a^1 \subseteq \{\supset\}_x a^1) \quad (\mathbf{T}_{T382})$$

Proof of (**T_{T382}**)

$$\begin{aligned} (a^1 \subseteq QX) &\subseteq (a^1 \subseteq {}^V X X a^1) & (\mathbf{T}_{R527})(\text{Eqn1}) \\ ((\mathcal{X}E)^1 \subseteq D_A + P_A) &\subseteq (QX = E) & (\mathbf{A}_{S36})(\text{Axiom of Universal Mapping}) \\ (((\mathcal{X}E)^1 \subseteq D_A + P_A) \subseteq (QX = E)) &= E & (\mathbf{A}_{MX16})(\text{Order Axiom}) \\ (((\mathcal{X}E)^1 \subseteq D_A + P_A) = E) &\subseteq ((QX = E) = E) & (\mathbf{T}_{M094})(\text{Probabilistic Rule}) \\ (((\mathcal{X}E)^1 \subseteq D_A + P_A) = E) &\subseteq ((a^1 \subseteq QX) = (a^1 \subseteq E)) & (\mathbf{T}_{M099})(\text{Entity Mode / Set Mode Rule}) \\ (((\mathcal{X}E)^1 \subseteq D_A + P_A) = E) &\subseteq ((a^1 \subseteq QX) = E) & (\mathbf{T}_{M007})(\text{Universality of } (\mathbf{T}_{M744})(\text{Universal Set Theorem})) \\ (((\mathcal{X}E)^1 \subseteq D_A + P_A) \subseteq (a^1 \subseteq QX)) &= E & (\mathbf{T}_{M094})(\text{Probabilistic Rule}) \\ ((\mathcal{X}E)^1 \subseteq D_A + P_A) &\subseteq (a^1 \subseteq QX) & (\mathbf{A}_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\ ((\mathcal{X}E)^1 \subseteq D_A + P_A) &\subseteq (a^1 \subseteq {}^V X X a^1) & (\mathbf{T}_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\ ((\mathcal{X}E)^1 \subseteq D_A + P_A) &\subseteq (\{\subseteq\}a^1 \subseteq \{\subseteq\}{}^V X X a^1) & (\mathbf{T}_{R196})(\text{Relational Deduction})(\text{Eqn3}) \\ ((\mathcal{X}E)^1 \subseteq D_A + P_A) &\subseteq ({}^V X \{\supset\}a^1 = \{\subseteq\}{}^V X a^1) & (\mathbf{A}_{S07})(\text{Eqn4}) \\ \text{Let } a^1 &= X a^1 & (\text{Eqn5}) \\ ((\mathcal{X}E)^1 \subseteq D_A + P_A) &\subseteq ({}^V X \{\supset\} X a^1 = \{\subseteq\}{}^V X X a^1) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn5 into Eqn4})(\text{Eqn6}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) \subseteq (\{\subseteq\}a^1 \subseteq {}^v\mathcal{X}\{\supseteq\}\mathcal{X}a^1)$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) \subseteq (\{\subseteq\}a^1 \subseteq \{\supseteq\}_x a^1)$$

(**AMX19**)(Substitution of Eqn6 into Eqn3)
(**TR571**)(Definition of a Mapped Relationship)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A)(a \subseteq \mathcal{X}\mathcal{E}) \subseteq (a = \mathcal{X}^* {}^v \mathcal{X}^* a)$$

(**TS784**)

Proof of (**TS784**)

$$\text{Let } (\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A$$

$$\mathcal{X}^1 \subseteq \mathbf{R}_0$$

(**AS45**)(Axiom of One-to-One Mapping)

$$(\mathcal{X}^1 \subseteq \mathbf{R}_0) = \mathbf{E}$$

(**AMX16**)(Order Axiom)(Eqn1)

$$(c^1 \subseteq \mathbf{R}_0)(a \subseteq \mathbf{S}c) \subseteq (a = c^* {}^v c^* a)$$

(**TS518**)

$$(\mathcal{X}^1 \subseteq \mathbf{R}_0)(a \subseteq \mathbf{S}\mathcal{X}) \subseteq (a = \mathcal{X}^* {}^v \mathcal{X}^* a)$$

Resymbolising(Eqn2)

$$\mathbf{E}(a \subseteq \mathbf{S}\mathcal{X}) \subseteq (a = \mathcal{X}^* {}^v \mathcal{X}^* a)$$

(**AMX19**)(Substitution of Eqn1 into Eqn2)

$$(a \subseteq \mathbf{S}\mathcal{X}) \subseteq (a = \mathcal{X}^* {}^v \mathcal{X}^* a)$$

(**AMX11**)(Universal Set Conjunction Axiom)

$$\text{Thus } (((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) = \mathbf{E}) \subseteq ((a \subseteq \mathbf{S}\mathcal{X}) \subseteq (a = \mathcal{X}^* {}^v \mathcal{X}^* a))$$

(**AMX02**)(Conditional Introduction)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A)(a \subseteq \mathbf{S}\mathcal{X}) \subseteq (a = \mathcal{X}^* {}^v \mathcal{X}^* a)$$

(**AMX35**)(Exportation/Importation) for truth sets(Eqn3)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) \subseteq (\mathbf{S}\mathcal{X} = \mathcal{X}\mathcal{E})$$

(**TS240**)(Eqn4)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A)(a \subseteq \mathcal{X}\mathcal{E}) \subseteq (a = \mathcal{X}^* {}^v \mathcal{X}^* a)$$

(**AMX19**)(Substitution of Eqn4 into Eqn3)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) = \mathbf{E} \subseteq ((a \subseteq \mathcal{X}\mathcal{E}) = (\{\subset\}a \subseteq \mathcal{X}\mathcal{E}))$$

(**TS801**)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) = \mathbf{E} \subseteq ((a \subseteq \mathcal{X}\mathcal{E}) = (\{\supset\}a \subseteq \mathcal{X}\mathcal{E}))$$

(**TS802**)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) = \mathbf{E} \subseteq ((a \subseteq \mathcal{X}\mathcal{E}) = (\{\parallel\}a \subseteq \mathcal{X}\mathcal{E}))$$

(**TS803**)

Example Proof of (**TS801**)

$$((Y \subseteq \{\subset\} + \{\supset\} + \{\parallel\}). ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) = \mathbf{E}) \subseteq ((a \subseteq \mathcal{X}\mathcal{E}) = (Y a \subseteq \mathcal{X}\mathcal{E}))$$

(**AR73**)(Axiom of Membership Relationships)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) = \mathbf{E} \subseteq ((a \subseteq \mathcal{X}\mathcal{E}) = (\{\subset\}a \subseteq \mathcal{X}\mathcal{E}))$$

Extraction

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A) = \mathbf{E} \subseteq ((a \subseteq \mathcal{X}\mathcal{E}) = (\{\prec\}a \subseteq \mathcal{X}\mathcal{E}))$$

(**TS804**)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A) = \mathbf{E} \subseteq ((a \subseteq \mathcal{X}\mathcal{E}) = (\{\succ\}a \subseteq \mathcal{X}\mathcal{E}))$$

(**TS800**)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A) = \mathbf{E} \subseteq ((a \subseteq \mathcal{X}\mathcal{E}) = (\mathcal{B}a \subseteq \mathcal{X}\mathcal{E}))$$

(**TS805**)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A) = \mathbf{E} \subseteq ((a \subseteq \mathcal{X}\mathcal{E}) = (\mathcal{C}a \subseteq \mathcal{X}\mathcal{E}))$$

(**TS806**)

Example Proof of (**TS800**)

$$((Y \subseteq \{\prec\} + \{\succ\} + \{=\} + \mathcal{B} + \mathcal{C}). ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A) = \mathbf{E}) \subseteq ((a \subseteq \mathcal{X}\mathcal{E}) = (Y a \subseteq \mathcal{X}\mathcal{E}))$$

(**AS51**)(Axiom of Ordering Relationships)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A) = \mathbf{E} \subseteq ((a \subseteq \mathcal{X}\mathcal{E}) = (\{\succ\}a \subseteq \mathcal{X}\mathcal{E}))$$

Extraction

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A) = \mathbf{E} \subseteq ((a_1^1 \subseteq \{\succ\}_x a_2^1) = (\mathcal{X}a_1^1 \subseteq \{\succ\}\mathcal{X}a_2^1))$$

(**TS790**)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A) = \mathbf{E} \subseteq ((a_1^1 \subseteq \{\prec\}_x a_2^1) = (\mathcal{X}a_1^1 \subseteq \{\prec\}\mathcal{X}a_2^1))$$

(**TS791**)

Example Proof of (**TS790**)

$$((\mathcal{X}^1 \subseteq \mathbf{R}_0)(o^* \mathcal{X}^* b^1 \subseteq \mathbf{S}\mathcal{X}) = \mathbf{E}) \subseteq ((a^1 \subseteq o_{\mathcal{X}}^* b^1) = (\mathcal{X}^* a^1 \subseteq o^* \mathcal{X}^* b^1))$$

(**TS789**)(Eqn1)

$$\text{Let } ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A)$$

$$\mathcal{X}^1 \subseteq \mathbf{R}_0$$

(**TS718**)

$$(\mathcal{X}^1 \subseteq \mathbf{R}_0) = \mathbf{E}$$

(**AMX16**)(Order Axiom)(Eqn2)

$$\text{Let } a^1 = a_1^1$$

(Eqn3)

$$\text{Let } b^1 = a_2^1$$

(Eqn4)

$$\text{Let } o = \{\succ\}$$

(Eqn5)

$$(\mathbf{E}(\{\succ\}\mathcal{X}a_2^1 \subseteq \mathbf{S}\mathcal{X}) = \mathbf{E}) \subseteq ((a_1^1 \subseteq \{\succ\}_x a_2^1) = (\mathcal{X}a_1^1 \subseteq \{\succ\}\mathcal{X}a_2^1))$$

$((\{>\}Xa_2^1 \subseteq SX) = E) \subseteq ((a_1^1 \subseteq \{>\}_X a_2^1) = (Xa_1^1 \subseteq \{>\}Xa_2^1))$	(A_{MX19}) (Substitution of Eqns2,3,4&5 into Eqn1)
$Xa_2^1 = S(X \cdot^V Qa_2^1)$	(A_{MX11}) (Universal Set Conjunction Axiom)(Eqn6)
$Xa_2^1 = SX \cdot S^V Qa_2^1$	(A_{R27}) (Function Axiom)
$Xa_2^1 \subseteq SX$	(T_{R012d}) (Distribution of S^* over $\cdot$)
$Xa_2^1 \subseteq XE$	(T_{M073b}) (Single Premise Syllogism)
$\{>\}Xa_2^1 \subseteq XE$	(T_{R210})
$\{>\}Xa_2^1 \subseteq SX$	(T_{S800})
$(\{>\}Xa_2^1 \subseteq SX) = E$	(T_{R210})
$(E = E) \subseteq ((a_1^1 \subseteq \{>\}_X a_2^1) = (Xa_1^1 \subseteq \{>\}Xa_2^1))$	(A_{MX16}) (Order Axiom)(Eqn7)
$E \subseteq ((a_1^1 \subseteq \{>\}_X a_2^1) = (Xa_1^1 \subseteq \{>\}Xa_2^1))$	(A_{MX19}) (Substitution of Eqn7 into Eqn6)
$E = ((a_1^1 \subseteq \{>\}_X a_2^1) = (Xa_1^1 \subseteq \{>\}Xa_2^1))$	(T_{M342})
$(a_1^1 \subseteq \{>\}_X a_2^1) = (Xa_1^1 \subseteq \{>\}Xa_2^1)$	(T_{M275}) (Supersets of the Universal Set)
Thus $((XE)^1 \subseteq D_A) = E \subseteq ((a_1^1 \subseteq \{>\}_X a_2^1) = (Xa_1^1 \subseteq \{>\}Xa_2^1))$	(A_{MX16}) (Order Axiom)
	(A_{MX02}) (Conditional Introduction)

$$(((XE)^1 \subseteq D_A) = E) \subseteq ((Za = BXa) \subseteq (Yza = Y_{BX}a)) \quad (T_{S814})$$

$$(((XE)^1 \subseteq D_A) = E) \subseteq ((Za = CXa) \subseteq (Yza = Y_{CX}a)) \quad (T_{S815})$$

Example Proof of (T_{S814})

Let $(XE)^1 \subseteq D_A$	
Let $Za = BXa$	
$^VZa = ^VX^V Ba$	(A_{R49}) (Reverse Compound Function Axiom)(Eqn1)
$YZa = YBXa$	(T_{R196}) (Relational Deduction)
$^VZYZa = ^VZYBXa$	(T_{R196}) (Relational Deduction)(Eqn2)
$^VZYZa = ^VX^V BYBXa$	(A_{MX19}) (Substitution of Eqn1 into Eqn2)
$YZa = Y_{BX}a$	(T_{R571})
Thus $((XE)^1 \subseteq D_A) = E \subseteq ((Za = BXa) \subseteq (Yza = Y_{BX}a))$	(A_{MX02}) (Conditional Introduction)

$$((XE)^1 \subseteq D_A) = E \subseteq ((a_1^1 \subseteq \{>\}_{BX^*} a_2^1) = (BXa_1^1 \subseteq \{>\}BXa_2^1)) \quad (T_{S792})$$

$$((XE)^1 \subseteq D_A) = E \subseteq ((a_1^1 \subseteq \{<\}_{BX^*} a_2^1) = (BXa_1^1 \subseteq \{<\}BXa_2^1)) \quad (T_{S793})$$

$$((XE)^1 \subseteq D_A) = E \subseteq ((a_1^1 \subseteq \{>\}_{CX^*} a_2^1) = (CXa_1^1 \subseteq \{>\}CXa_2^1)) \quad (T_{S794})$$

$$((XE)^1 \subseteq D_A) = E \subseteq ((a_1^1 \subseteq \{<\}_{CX^*} a_2^1) = (CXa_1^1 \subseteq \{<\}CXa_2^1)) \quad (T_{S795})$$

Example Proof of (T_{S792})

Let $(XE)^1 \subseteq D_A$	
$((Z^1 \subseteq R_0)(Y * Z * b^1 \subseteq SZ) = E) \subseteq ((a^1 \subseteq Yz * b^1) = (Z * a^1 \subseteq Y * Z * b^1))$	(T_{S789}) (Eqn1)
Let $Za^1 = BXa^1$	(Eqn2)
$Yza^1 = Y_{BX}a^1$	(T_{S814}) (Eqn3)
$X^1 \subseteq R_0$	(T_{S718}) (Eqn4)
$B^1 \subseteq R_0$	(A_{R41}) (Axiom of Bounds of Extent)(Eqn5)
$Z^1 \subseteq R_0$	(A_{R48}) (One-to-One Relationships Combination Axiom)
$(Z^1 \subseteq R_0) = E$	(A_{MX16}) (Order Axiom)(Eqn6)
Let $a^1 = a_1^1$	(Eqn7)
Let $b^1 = a_2^1$	(Eqn8)
Let $Y = \{>\}$	(Eqn9)
$(E.(\{>\}Za_2^1 \subseteq SZ) = E) \subseteq ((a_1^1 \subseteq \{>\}_{Za_2^1}) = (Za_1^1 \subseteq \{>\}Za_2^1))$	(A_{MX19}) (Substitution of Eqns6,7,8&9 into Eqn1)
$((\{>\}Za_2^1 \subseteq SZ) = E) \subseteq ((a_1^1 \subseteq \{>\}_{Za_2^1}) = (Za_1^1 \subseteq \{>\}Za_2^1))$	

$Za_2^1 = S(Z.^V Qa_2^1)$	(A_{MX17})(Universal Set Conjunction Axiom)(Eqn10)
$Za_2^1 = SZ.S^V Qa_2^1$	(A_{R27})(Function Axiom)
$Za_2^1 \subseteq SZ$	(T_{R012d})(Distribution of S^* over $.$)
$\{>\}Za_2^1 \subseteq ZE$	(T_{M073b})(Single Premise Syllogism)
$\{>\}Za_2^1 \subseteq SZ$	(T_{S800})
$(\{>\}Za_2^1 \subseteq SZ) = E$	(T_{R210})
$(E = E) \subseteq ((a_1^1 \subseteq \{>\}za_2^1) = (Za_1^1 \subseteq \{>\}Za_2^1))$	(A_{MX16})(Order Axiom)(Eqn11)
$E \subseteq ((a_1^1 \subseteq \{>\}za_2^1) = (Za_1^1 \subseteq \{>\}Za_2^1))$	(A_{MX19})(Substitution of Eqn11 into Eqn10)
$E = ((a_1^1 \subseteq \{>\}za_2^1) = (Za_1^1 \subseteq \{>\}Za_2^1))$	(T_{M342})
$(a_1^1 \subseteq \{>\}za_2^1) = (Za_1^1 \subseteq \{>\}Za_2^1)$	(T_{M275})(Supersets of the Universal Set)
Thus $((\chi E)^1 \subseteq D_A) = E \subseteq ((a_1^1 \subseteq \{>\}za_2^1) = (Za_1^1 \subseteq \{>\}Za_2^1))$	(A_{MX16})(Order Axiom)
	(A_{MX02})(Conditional Introduction)(Eqn12)
$((\chi E)^1 \subseteq D_A) = E \subseteq ((a_1^1 \subseteq \{>\}_{BX} a_2^1) = (BXa_1^1 \subseteq \{>\}BXa_2^1))$	(A_{MX19})(Substitution of Eqns2,3&9 into Eqn12)

$$\begin{aligned}
& ((\chi E)^1 \subseteq D_A) = E \subseteq ((b^1 \subseteq \{>\}_x a^1) = \\
& (BXb^1 \neq \emptyset).((BXa^1 \neq \emptyset).(b^1 \subseteq \{>\}_{BX} a^1) + (BXa^1 = \emptyset)). \\
& (CXa^1 \neq \emptyset).((CXb^1 \neq \emptyset).(b^1 \subseteq \{>\}_{CX} a^1) + (CXb^1 = \emptyset))) \quad (T_{R738}) \text{ Definition of } \{>\}_x \\
& ((\chi E)^1 \subseteq D_A) = E \subseteq ((b^1 \subseteq \{<\}_x a^1) = \\
& (BXa^1 \neq \emptyset).((BXb^1 \neq \emptyset).(b^1 \subseteq \{<\}_{BX} a^1) + (BXb^1 = \emptyset)). \\
& (CXb^1 \neq \emptyset).((CXa^1 \neq \emptyset).(b^1 \subseteq \{<\}_{CX} a^1) + (CXa^1 = \emptyset))) \quad (T_{R739}) \text{ Definition of } \{<\}_x
\end{aligned}$$

Example Proof of (T_{R738})

$$\begin{aligned}
& (((\chi E)^1 \subseteq D_A) = E) \subseteq ((\chi b^1 \subseteq \{>\}_x \chi a^1) = \\
& (BXb^1 \neq \emptyset).((BXa^1 \neq \emptyset).(BXb^1 \subseteq \{>\}_x BXa^1) + (BXa^1 = \emptyset)). \\
& (CXa^1 \neq \emptyset).((CXb^1 \neq \emptyset).(CXb^1 \subseteq \{>\}_x CXa^1) + (CXb^1 = \emptyset))) \quad (T_{R749})(Eqn1) \\
& ((\chi E)^1 \subseteq D_A) = E \subseteq ((b^1 \subseteq \{>\}_x a^1) = (\chi b^1 \subseteq \{>\}_x \chi a^1)) \quad (T_{S790})(Eqn2) \\
& ((\chi E)^1 \subseteq D_A) = E \subseteq ((b^1 \subseteq \{>\}_{BX} a^1) = (BXb^1 \subseteq \{>\}_x BXa^1)) \quad (T_{S792})(Eqn3) \\
& ((\chi E)^1 \subseteq D_A) = E \subseteq ((b^1 \subseteq \{>\}_{CX} a^1) = (CXb^1 \subseteq \{>\}_x CXa^1)) \quad (T_{S794})(Eqn4) \\
& (((\chi E)^1 \subseteq D_A) = E) \subseteq ((b^1 \subseteq \{>\}_x a^1) = \\
& (BXb^1 \neq \emptyset).((BXa^1 \neq \emptyset).(b^1 \subseteq \{>\}_{BX} a^1) + (BXa^1 = \emptyset)). \\
& (CXa^1 \neq \emptyset).((CXb^1 \neq \emptyset).(b^1 \subseteq \{>\}_{CX} a^1) + (CXb^1 = \emptyset))) \quad (A_{MX19})(\text{Substitution of Eqns2,3\&4 into Eqn1})
\end{aligned}$$

$$((\chi E)^1 \subseteq D_A) \subseteq (B(\chi a)^1 = \chi E.(\{>\}_x \chi E)(\{<\}_x \chi E)) \quad (T_{R695})$$

$$((\chi E)^1 \subseteq D_A) \subseteq (C(\chi E)^1 = \chi a.(\{<\}_x \chi E)(\{>\}_x \chi E)) \quad (T_{R700})$$

Example Proof of (T_{R695})

$$\begin{aligned}
& (a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba^1 = a.(\{>\}_x a).(\{<\}_x a)) \quad (T_{R986})(Eqn1) \\
& D_M \subseteq S_M S_O \quad (T_{R926})(Eqn2) \\
& D_D \subseteq S_D S_O \quad (T_{R928})(Eqn3) \\
& D_M + D_D \subseteq S_M S_O + S_D S_O \quad (A_{MX21})(\text{Combination of Eqns2\&3}) \\
& D_A \subseteq S_M S_O + S_D S_O \quad (T_{R815}) \\
& (D_A \subseteq S_M S_O + S_D S_O) = E \quad (A_{MX16})(\text{Order Axiom}) \\
& (a^1 \subseteq D_A) \subseteq (a^1 \subseteq S_M S_O + S_D S_O) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})(Eqn4) \\
& (a^1 \subseteq D_A) \subseteq (Ba^1 = a.(\{>\}_x a).(\{<\}_x a)) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&4})(Eqn5) \\
& ((\chi E)^1 \subseteq D_A) \subseteq (B(\chi E)^1 = \chi a.(\{>\}_x \chi E)(\{<\}_x \chi E)) \quad \text{Resymbolising}
\end{aligned}$$

$$((\chi E)^1 \subseteq D_M) \subseteq (B\chi E + C\chi E = \chi E.(\{<\}_x \chi E)) \quad (T_{R166})$$

Proof of (T_{R166})

$$\begin{aligned}
((\lambda E)^1 \subseteq D_M) &\subseteq ((\lambda E. \sim(B\lambda E + C\lambda E))^1 \subseteq S_C S_O).((\lambda E.(B\lambda E + C\lambda E))^1 \subseteq S_P S_O).((\lambda E)^1 \subseteq S_O). \\
(\lambda^1 \subseteq R_O).(\mathcal{Q}\lambda = E).(\mathcal{B}\lambda E + C\lambda E = \lambda E.(\{\subset\}^* \lambda E)) &\quad (T_{S983}) \\
((\lambda E)^1 \subseteq D_M) &\subseteq (\mathcal{B}\lambda E + C\lambda E = \lambda E.(\{\subset\}^* \lambda E)) \quad (T_{M073b})(\text{Single Premise Syllogism})
\end{aligned}$$

$$((\lambda E)^1 \subseteq D_M) \subseteq ((\mathcal{B}\lambda E + C\lambda E)^1 \subseteq S_P S_O) \quad (T_{S841})$$

Proof of (T_{S841})

$$\begin{aligned}
((\lambda E)^1 \subseteq D_M) &\subseteq ((\lambda E. \sim(B\lambda E + C\lambda E))^1 \subseteq S_C S_O).((\lambda E.(B\lambda E + C\lambda E))^1 \subseteq S_P S_O).((\lambda E)^1 \subseteq S_O). \\
(\lambda^1 \subseteq R_O).(\mathcal{Q}\lambda = E).(\mathcal{B}\lambda E + C\lambda E = \lambda E.(\{\subset\}^* \lambda E)) &\quad (T_{S983}) \\
((\lambda E)^1 \subseteq D_M) &\subseteq ((\lambda E.(B\lambda E + C\lambda E))^1 \subseteq S_P S_O) \quad (T_{M073b})(\text{Single Premise Syllogism})(\text{Eqn1}) \\
(a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\mathcal{B}a + \mathcal{C}a \subseteq a) \quad (T_{S986})(\text{Eqn2}) \\
S_M S_O &\subseteq S_M S_O + S_D S_O \quad (T_{M082})(\text{Disjunction Introduction})(\text{Eqn3}) \\
D_M &\subseteq S_M S_O \quad (T_{T926})(\text{Eqn4}) \\
D_M &\subseteq S_M S_O + S_D S_O \quad (T_{M139})(\text{Hypothetical Syllogism Eqns3\&4}) \\
(D_M \subseteq S_M S_O + S_D S_O) &= E \quad (A_{MX16})(\text{Order Axiom}) \\
(a^1 \subseteq D_M) &\subseteq (a^1 \subseteq S_M S_O + S_D S_O) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn5}) \\
(a^1 \subseteq D_M) &\subseteq (\mathcal{B}a + \mathcal{C}a \subseteq a) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns2\&5}) \\
(a^1 \subseteq D_M) &\subseteq (\mathcal{B}a + \mathcal{C}a = a.(\mathcal{B}a + \mathcal{C}a)) \quad (A_{MX44})(\subseteq \text{Equality Form}) \\
((\lambda E)^1 \subseteq D_M) &\subseteq (\mathcal{B}\lambda E + C\lambda E = \lambda E.(\mathcal{B}\lambda E + C\lambda E)) \quad \text{Resymbolising (Eqn6)} \\
((\lambda E)^1 \subseteq D_M) &\subseteq ((\mathcal{B}\lambda E + C\lambda E)^1 \subseteq S_P S_O) \quad (A_{MX19})(\text{Substitution of Eqn6 into Eqn1})
\end{aligned}$$

$$(a^1 \subseteq D_M) \subseteq (\alpha \subseteq a) \quad (T_{S863})$$

Proof of (T_{S863})

$$\begin{aligned}
(a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha \subseteq a) \quad (T_{S987})(\text{Eqn1}) \\
S_M S_O &\subseteq S_M S_O + S_D S_O \quad (T_{M082})(\text{Disjunction Introduction}) \\
(S_M S_O \subseteq S_M S_O + S_D S_O) &= E \quad (A_{MX16})(\text{Order Axiom}) \\
(a^1 \subseteq S_M S_O) &\subseteq (a^1 \subseteq S_M S_O + S_D S_O) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn2}) \\
(a^1 \subseteq S_M S_O) &\subseteq (\alpha \subseteq a) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\
(a^1 \subseteq D_M) &\subseteq (a^1 \subseteq S_M S_O) \quad (T_{S988})(\text{Eqn4}) \\
(a^1 \subseteq D_M) &\subseteq (\alpha \subseteq a) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns3\&4})
\end{aligned}$$

$$(a^1 \subseteq D_M) \subseteq (\alpha = \mathcal{B}a + \mathcal{C}a) \quad (T_{T805})$$

Proof of (T_{T805})

$$\begin{aligned}
(a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha = \mathcal{B}a + \mathcal{C}a) \quad (T_{T931b})(\text{Definition of } \alpha \text{ from } a)(\text{Eqn1}) \\
S_M S_O &\subseteq S_M S_O + S_D S_O \quad (T_{M082})(\text{Disjunction Introduction}) \\
(S_M S_O \subseteq S_M S_O + S_D S_O) &= E \quad (A_{MX16})(\text{Order Axiom}) \\
(a^1 \subseteq S_M S_O) &\subseteq (a^1 \subseteq S_M S_O + S_D S_O) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn2}) \\
(a^1 \subseteq S_M S_O) &\subseteq (\alpha = \mathcal{B}a + \mathcal{C}a) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\
(a^1 \subseteq D_M) &\subseteq (a^1 \subseteq S_M S_O) \quad (T_{S988})(\text{Eqn4}) \\
(a^1 \subseteq D_M) &\subseteq (\alpha = \mathcal{B}a + \mathcal{C}a) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns4\&3})
\end{aligned}$$

$$(a^1 \subseteq D_M) \subseteq (\mathcal{B}a + \mathcal{C}a \subseteq a) \quad (T_{T167})$$

Proof of (T_{T167})

$$\begin{aligned}
(a^1 \subseteq D_M) &\subseteq (\alpha \subseteq a) \quad (T_{S863})(\text{Eqn1}) \\
(a^1 \subseteq D_M) &\subseteq (\alpha = \mathcal{B}a + \mathcal{C}a) \quad (T_{T805})(\text{Eqn2}) \\
(a^1 \subseteq D_M) &\subseteq (\mathcal{B}a + \mathcal{C}a \subseteq a) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$(a^1 \subseteq D_M) \subseteq ((\underline{a}.a)^1 \subseteq S_C S_O) \quad (T_{T221})$$

Proof of (T₂₂₁)

$$\begin{aligned} (a^1 \subseteq S_M S_O) &\subseteq ((\underline{\alpha}.a)^1 \subseteq S_C S_O) & (T_{S946})(\text{Eqn1}) \\ (a^1 \subseteq D_M) &\subseteq (a^1 \subseteq S_M S_O) & (T_{S988})(\text{Eqn2}) \\ (a^1 \subseteq D_M) &\subseteq ((\underline{\alpha}.a)^1 \subseteq S_C S_O) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$(a^1 \subseteq D_M) \subseteq (\forall B\alpha. \forall C\alpha \subseteq a) \quad (T_{222})$$

Proof of (T₂₂₂)

$$\begin{aligned} (a^1 \subseteq S_M S_O) &\subseteq (\forall B\alpha. \forall C\alpha \subseteq a) & (T_{S871})(\text{Eqn1}) \\ (a^1 \subseteq D_M) &\subseteq (a^1 \subseteq S_M S_O) & (T_{S988})(\text{Eqn2}) \\ (a^1 \subseteq D_M) &\subseteq (\forall B\alpha. \forall C\alpha \subseteq a) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$(a^1 \subseteq D_M) \subseteq ((\underline{\alpha}. \forall B\alpha. \forall C\alpha)^1 \subseteq S_C S_O) \quad (T_{169})$$

Proof of (T₁₆₉)

$$\begin{aligned} (a^1 \subseteq S_M S_O) &\subseteq ((\underline{\alpha}. \forall B\alpha. \forall C\alpha)^1 \subseteq S_C S_O) & (T_{S914})(\text{Eqn1}) \\ (a^1 \subseteq D_M) &\subseteq (a^1 \subseteq S_M S_O) & (T_{S988})(\text{Eqn2}) \\ (a^1 \subseteq D_M) &\subseteq ((\underline{\alpha}. \forall B\alpha. \forall C\alpha)^1 \subseteq S_C S_O) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$(a^1 \subseteq D_M) \subseteq ((\sim(Ba + Ca). \forall B(Ba + Ca). \forall C(Ba + Ca))^1 \subseteq S_C S_O) \quad (T_{170})$$

$$((\lambda E)^1 \subseteq D_M) \subseteq ((\sim(B\lambda E + C\lambda E). \forall B(B\lambda E + C\lambda E). \forall C(B\lambda E + C\lambda E))^1 \subseteq S_C S_O) \quad (T_{171})$$

Proof of (T₁₇₀) & (T₁₇₁)

$$\begin{aligned} (a^1 \subseteq D_M) &\subseteq ((\underline{\alpha}. \forall B\alpha. \forall C\alpha)^1 \subseteq S_C S_O) & (T_{169})(\text{Eqn1}) \\ (a^1 \subseteq D_M) &\subseteq (\alpha = Ba + Ca) & (T_{805})(\text{Eqn2}) \\ (a^1 \subseteq D_M) &\subseteq ((\sim(Ba + Ca). \forall B(Ba + Ca). \forall C(Ba + Ca))^1 \subseteq S_C S_O) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\ \text{Let } a^1 &= (\lambda E)^1 & (\text{Eqn4}) \\ a &= \lambda E & (T_{M479})(\text{Eqn5}) \\ ((\lambda E)^1 \subseteq D_C) &\subseteq ((\sim(B\lambda E + C\lambda E). \forall B(B\lambda E + C\lambda E). \forall C(B\lambda E + C\lambda E))^1 \subseteq S_C S_O) & (A_{MX19})(\text{Substitution of Eqns4\&5 into Eqn3}) \end{aligned}$$

$$(a^1 \subseteq D_A) \subseteq (\forall B\alpha. \forall C\alpha \subseteq a) \quad (T_{203})$$

$$((\lambda E)^1 \subseteq D_A) \subseteq (\forall B\lambda E. \forall C\lambda E \subseteq \lambda E) \quad (T_{204})$$

Proof of (T₂₀₃) & (T₂₀₄)

$$\begin{aligned} (a^1 \subseteq S_M S_O) &\subseteq (\forall B\alpha. \forall C\alpha \subseteq a) & (T_{S901})(\text{Eqn1}) \\ (a^1 \subseteq D_M) &\subseteq (a^1 \subseteq S_M S_O) & (T_{S988})(\text{Eqn2}) \\ (a^1 \subseteq D_M) &\subseteq (\forall B\alpha. \forall C\alpha \subseteq a) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\ (a^1 \subseteq S_P S_O) &\subseteq (\forall B\alpha. \forall C\alpha \subseteq a) & (T_{228})(\text{Eqn4}) \\ (a^1 \subseteq D_D) &\subseteq (a^1 \subseteq S_O S_O) & (T_{S989})(\text{Eqn5}) \\ (a^1 \subseteq D_D) &\subseteq (\forall B\alpha. \forall C\alpha \subseteq a) & (T_{M139})(\text{Hypothetical Syllogism Eqns4\&5})(\text{Eqn6}) \\ (a^1 \subseteq D_M) + (a^1 \subseteq D_D) &\subseteq (\forall B\alpha. \forall C\alpha \subseteq a) + (\forall B\alpha. \forall C\alpha \subseteq a) & (A_{MX21})(\text{Combination of Eqns3\&6}) \\ (a^1 \subseteq D_M) + (a^1 \subseteq D_D) &\subseteq (\forall B\alpha. \forall C\alpha \subseteq a) & (A_{MX01})(\text{Disjunction Tautology}) \\ (a^1 \subseteq D_M + D_D) &\subseteq (\forall B\alpha. \forall C\alpha \subseteq a) & (T_{M526}) \\ (a^1 \subseteq D_A) &\subseteq (\forall B\alpha. \forall C\alpha \subseteq a) & (T_{815})(\text{Eqn7}) \\ \text{Let } a^1 &= (\lambda E)^1 & (\text{Eqn8}) \\ a &= \lambda E & (T_{M479})(\text{Eqn9}) \\ ((\lambda E)^1 \subseteq D_A) &\subseteq (\forall B\lambda E. \forall C\lambda E \subseteq \lambda E) & (A_{MX19})(\text{Substitution of Eqns8\&9 into Eqn7}) \end{aligned}$$

$$(a^1 \subseteq D_M) \subseteq (\forall Ba. \forall Ca \subseteq a) \quad (T_{230})$$

$$((\lambda E)^1 \subseteq D_M) \subseteq (\forall B\lambda E. \forall C\lambda E \subseteq \lambda E) \quad (T_{231})$$

$$(a^1 \subseteq D_D) \subseteq (\forall Ba. \forall Ca \subseteq a) \quad (T_{232})$$

$$((\lambda E)^1 \subseteq D_D) \subseteq (\forall B\lambda E. \forall C\lambda E \subseteq \lambda E) \quad (T_{233})$$

Example Proof of (T_{230})

$$(a^1 \subseteq D_A) \subseteq (\forall Ba. \forall Ca \subseteq a) \quad (T_{203})$$

$$(a^1 \subseteq D_M + D_P + D_D) \subseteq (\forall Ba. \forall Ca \subseteq a) \quad (D_{533})(\text{Definition of } D_A)$$

$$(a^1 \subseteq D_M) + (a^1 \subseteq D_P) + (a^1 \subseteq D_D) \subseteq (\forall Ba. \forall Ca \subseteq a) \quad (T_{526})$$

$$(a^1 \subseteq D_M) \subseteq (\forall Ba. \forall Ca \subseteq a) \quad (T_{072})(\text{Single Premise Syllogism})$$

$$(a^1 \subseteq D_D) \subseteq (\alpha = a) \quad (T_{129})$$

Proof of (T_{129})

$$(a^1 \subseteq S_D S_O) \subseteq (\alpha = a) \quad (D_{531})(\text{Definition of } \alpha \text{ Symbolism})(\text{Eqn1})$$

$$(a^1 \subseteq D_D) \subseteq (a^1 \subseteq S_D S_O) \quad (T_{989})(\text{Eqn2})$$

$$(a^1 \subseteq D_D) \subseteq (\alpha = a) \quad (T_{139})(\text{Hypothetical Syllogism Eqns1\&2})$$

$$((\lambda E)^1 \subseteq D_M)(\lambda a^1 \neq \emptyset) \subseteq (\lambda a^1 = (\lambda a^1)^1) \quad (T_{153})$$

$$((\lambda E)^1 \subseteq D_D)(\lambda a^1 \neq \emptyset) \subseteq (\lambda a^1 = (\lambda a^1)^1) \quad (T_{155})$$

$$((\lambda E)^1 \subseteq D_P)(\lambda a^1 \neq \emptyset) \subseteq (\lambda a^1 = (\lambda a^1)^1) \quad (T_{996})$$

Example Proof of (T_{153})

$$((\lambda E)^1 \subseteq D_A + P_A)(\lambda a^1 \neq \emptyset) \subseteq (\lambda a^1 = (\lambda a^1)^1) \quad (T_{5746})(\text{Eqn1})$$

$$D_A \subseteq D_A + P_A \quad (T_{082})(\text{Disjunction Introduction})$$

$$(D_A \subseteq D_A + P_A) = E \quad (A_{MX16})(\text{Order Axiom})$$

$$((\lambda E)^1 \subseteq D_A) \subseteq ((\lambda E)^1 \subseteq D_A + P_A) \quad (T_{099})(\text{Entity Mode / Set Mode Rule})$$

$$((\lambda E)^1 \subseteq D_A)(\lambda a^1 \neq \emptyset) \subseteq ((\lambda E)^1 \subseteq D_A + P_A)(\lambda a^1 \neq \emptyset) \quad (T_{029})(\text{Deduction})(\text{Eqn2})$$

$$((\lambda E)^1 \subseteq D_A)(\lambda a^1 \neq \emptyset) \subseteq (\lambda a^1 = (\lambda a^1)^1) \quad (T_{139})(\text{Hypothetical Syllogism Eqns1\&2})$$

$$((\lambda E)^1 \subseteq D_M + D_D)(\lambda a^1 \neq \emptyset) \subseteq (\lambda a^1 = (\lambda a^1)^1) \quad (T_{815})$$

$$(((\lambda E)^1 \subseteq D_M) + ((\lambda E)^1 \subseteq D_D)).(\lambda a^1 \neq \emptyset) \subseteq (\lambda a^1 = (\lambda a^1)^1) \quad (T_{526})$$

$$((\lambda E)^1 \subseteq D_M)(\lambda a^1 \neq \emptyset) + ((\lambda E)^1 \subseteq D_D).(\lambda a^1 \neq \emptyset) \subseteq (\lambda a^1 = (\lambda a^1)^1) \quad (T_{407})(\text{Distribution of } . \text{ over } +)$$

$$((\lambda E)^1 \subseteq D_M)(\lambda a^1 \neq \emptyset) \subseteq (\lambda a^1 = (\lambda a^1)^1) \quad (T_{072})(\text{Single Premise Syllogism})$$

$$(a^1 \subseteq D_M) \subseteq (Ba \neq \emptyset) \quad (T_{038})$$

$$(a^1 \subseteq D_M) \subseteq (Ca \neq \emptyset) \quad (T_{039})$$

$$(a^1 \subseteq D_D) \subseteq (Ba \neq \emptyset) \quad (T_{040})$$

$$(a^1 \subseteq D_D) \subseteq (Ca \neq \emptyset) \quad (T_{041})$$

Example Proof of (T_{038})

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba \neq \emptyset) \quad (T_{889})(\text{Eqn1})$$

$$S_M S_O \subseteq S_M S_O + S_D S_O \quad (T_{082})(\text{Disjunction Introduction})$$

$$(S_M S_O \subseteq S_M S_O + S_D S_O) = E \quad (A_{MX16})(\text{Order Axiom})$$

$$(a^1 \subseteq S_M S_O) \subseteq (a^1 \subseteq S_M S_O + S_D S_O) \quad (T_{099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn2})$$

$$(a^1 \subseteq S_M S_O) \subseteq (Ba \neq \emptyset) \quad (T_{139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3})$$

$$(a^1 \subseteq D_M) \subseteq (a^1 \subseteq S_M S_O) \quad (T_{988})(\text{Eqn4})$$

$$(a^1 \subseteq D_M) \subseteq (Ba \neq \emptyset) \quad (T_{139})(\text{Hypothetical Syllogism Eqns3\&4})$$

$$(a^1 \subseteq D_A) \subseteq (Ba \neq \emptyset) \quad (T_{042})$$

$$(a^1 \subseteq D_A) \subseteq (Ca \neq \emptyset) \quad (T_{7043})$$

Example Proof of (T_{7042})

$$\begin{aligned} (a^1 \subseteq D_M) &\subseteq (Ba \neq \emptyset) & (T_{7038})(\text{Eqn1}) \\ (a^1 \subseteq D_D) &\subseteq (Ba \neq \emptyset) & (T_{7040})(\text{Eqn2}) \\ (a^1 \subseteq D_M) + (a^1 \subseteq D_D) &\subseteq (Ba \neq \emptyset) + (Ba \neq \emptyset) & (A_{MX21})(\text{Combination of Eqns1\&2}) \\ (a^1 \subseteq D_M) + (a^1 \subseteq D_D) &\subseteq (Ba \neq \emptyset) & (A_{MX01})(\text{Disjunction Tautology}) \\ (a^1 \subseteq D_M + D_D) &\subseteq (Ba \neq \emptyset) & (T_{M526}) \\ (a^1 \subseteq D_A) &\subseteq (Ba \neq \emptyset) & (T_{7815}) \end{aligned}$$

$$((\lambda E)^1 \subseteq D_A) \subseteq (B\lambda E \neq \emptyset) \quad (T_{7187})$$

$$((\lambda E)^1 \subseteq D_A) \subseteq (C\lambda E \neq \emptyset) \quad (T_{7188})$$

$$((\lambda E)^1 \subseteq D_M) \subseteq (B\lambda E \neq \emptyset) \quad (T_{7189})$$

$$((\lambda E)^1 \subseteq D_M) \subseteq (C\lambda E \neq \emptyset) \quad (T_{7190})$$

$$((\lambda E)^1 \subseteq D_D) \subseteq (B\lambda E \neq \emptyset) \quad (T_{7191})$$

$$((\lambda E)^1 \subseteq D_D) \subseteq (C\lambda E \neq \emptyset) \quad (T_{7192})$$

Example Proof of (T_{7187})

$$\begin{aligned} (a^1 \subseteq D_A) &\subseteq (Ba \neq \emptyset) & (T_{7042})(\text{Eqn1}) \\ \text{Let } a^1 &= (\lambda E)^1 & (\text{Eqn2}) \\ a &= \lambda E & (T_{M479})(\text{Eqn3}) \\ ((\lambda E)^1 \subseteq D_A) &\subseteq (B\lambda E \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \end{aligned}$$

$$(a^1 \subseteq D_A) \subseteq (\alpha \neq \emptyset) \quad (T_{7180})$$

$$(a^1 \subseteq D_M) \subseteq (\alpha \neq \emptyset) \quad (T_{7181})$$

$$(a^1 \subseteq D_D) \subseteq (\alpha \neq \emptyset) \quad (T_{7182})$$

Proof of (T_{7180}), (T_{7181}) & (T_{7182})

$$\begin{aligned} (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha \neq \emptyset) & (T_{S882})(\text{Eqn1}) \\ (a^1 \subseteq D_A) &\subseteq (a^1 \subseteq S_M S_O + S_D S_O) & (T_{7792})(\text{Eqn2}) \\ (a^1 \subseteq D_A) &\subseteq (\alpha \neq \emptyset) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\ (a^1 \subseteq D_M + D_D) &\subseteq (\alpha \neq \emptyset) & (T_{7815}) \\ (a^1 \subseteq D_M) + (a^1 \subseteq D_D) &\subseteq (\alpha \neq \emptyset) & (T_{M526}) \\ (a^1 \subseteq D_M) &\subseteq (\alpha \neq \emptyset) & (T_{M072})(\text{Single Premise Syllogism}) \\ (a^1 \subseteq D_D) &\subseteq (\alpha \neq \emptyset) & (T_{M072})(\text{Single Premise Syllogism}) \end{aligned}$$

$$(a^1 \subseteq D_A) \subseteq (\alpha = a.\{\subset\}_* a) \quad (T_{7311})$$

$$(a^1 \subseteq D_M) \subseteq (\alpha = a.\{\subset\}_* a) \quad (T_{7062})$$

$$(a^1 \subseteq D_D) \subseteq (\alpha = a.\{\subset\}_* a) \quad (T_{7789})$$

Example Proof of (T_{7311})

$$\begin{aligned} (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba + Ca = a.\{\subset\}_* a) & (T_{7059})(\text{Eqn1}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha = Ba + Ca) & (T_{7931b})(\text{Definition of } \alpha \text{ from } a)(\text{Eqn2}) \\ (a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha = a.\{\subset\}_* a) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\ (a^1 \subseteq D_A) &\subseteq (a^1 \subseteq S_M S_O + S_D S_O) & (T_{7792})(\text{Eqn4}) \\ (a^1 \subseteq D_A) &\subseteq (\alpha = a.\{\subset\}_* a) & (T_{M139})(\text{Hypothetical Syllogism Eqns3\&4}) \end{aligned}$$

$$(a^1 \subseteq D_A) \subseteq (a.\{\subset\}_* a \neq \emptyset) \quad (T_{S998})$$

$$(a^1 \subseteq D_M) \subseteq (a.(\{\subset\} * a) \neq \emptyset) \quad (T_{S997})$$

$$(a^1 \subseteq D_D) \subseteq (a.(\{\subset\} * a) \neq \emptyset) \quad (T_{S996})$$

$$((\mathcal{X}E)^1 \subseteq D_A).(\mathcal{X}a \neq \emptyset) \subseteq (\mathcal{X}a.(\{\subset\} * \mathcal{X}a) \neq \emptyset) \quad (T_{T058})$$

$$((\mathcal{X}E)^1 \subseteq D_M).(\mathcal{X}a \neq \emptyset) \subseteq (\mathcal{X}a.(\{\subset\} * \mathcal{X}a) \neq \emptyset) \quad (T_{T054})$$

$$((\mathcal{X}E)^1 \subseteq D_D).(\mathcal{X}a \neq \emptyset) \subseteq (\mathcal{X}a.(\{\subset\} * \mathcal{X}a) \neq \emptyset) \quad (T_{T056})$$

Example Proof of (T_{S996}) & (T_{T058})

$$(a^1 \subseteq D_A) \subseteq (\alpha \neq \emptyset) \quad (T_{T180})(\text{Eqn1})$$

$$(a^1 \subseteq D_A) \subseteq (\alpha = a.(\{\subset\} * a)) \quad (T_{T311})(\text{Eqn2})$$

$$(a^1 \subseteq D_A) \subseteq (a.(\{\subset\} * a) \neq \emptyset) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$((\mathcal{X}E)^1 \subseteq D_A).(\mathcal{X}a \neq \emptyset) \subseteq (\mathcal{X}a.(\{\subset\} * \mathcal{X}a) \neq \emptyset) \quad \text{Resymbolising}$$

$$(a^1 \subseteq D_A) \subseteq (Ba + Ca = a.(\{\subset\} * a)) \quad (T_{T061})$$

$$(a^1 \subseteq D_M) \subseteq (Ba + Ca = a.(\{\subset\} * a)) \quad (T_{T787})$$

$$(a^1 \subseteq D_D) \subseteq (Ba + Ca = a.(\{\subset\} * a)) \quad (T_{T788})$$

Example Proof of (T_{T061})

$$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba + Ca = a.(\{\subset\} * a)) \quad (T_{T059})(\text{Eqn1})$$

$$(a^1 \subseteq D_A) \subseteq (a^1 \subseteq S_M S_O + S_D S_O) \quad (T_{T792})(\text{Eqn2})$$

$$(a^1 \subseteq D_A) \subseteq (\alpha = a.(\{\subset\} * a)) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$$

$$(a^1 \subseteq D_A)(Ba^1 \neq \emptyset) \subseteq (\{>\} * \alpha = \{>\} * a) \quad (T_{T205})$$

$$(a^1 \subseteq D_A)(Ca^1 \neq \emptyset) \subseteq (\{<\} * \alpha = \{<\} * a) \quad (T_{T206})$$

$$((\mathcal{X}E)^1 \subseteq D_A)(B(\mathcal{X}E)^1 \neq \emptyset) \subseteq (\{>\} * (B\mathcal{X}E + C\mathcal{X}E) = \{>\} * \mathcal{X}E) \quad (T_{T207})$$

$$((\mathcal{X}E)^1 \subseteq D_A)(C(\mathcal{X}E)^1 \neq \emptyset) \subseteq (\{<\} * (B\mathcal{X}E + C\mathcal{X}E) = \{<\} * \mathcal{X}E) \quad (T_{T208})$$

Example Proof of (T_{T205}) & (T_{T207})

$$(a^1 \subseteq S_M S_O + S_D S_O)(Ba^1 \neq \emptyset) \subseteq (\{>\} * \alpha = \{>\} * a) \quad (T_{T064})(\text{Eqn1})$$

$$(a^1 \subseteq D_A) \subseteq (a^1 \subseteq S_M S_O + S_D S_O) \quad (T_{T792})$$

$$(a^1 \subseteq D_A)(Ba^1 \neq \emptyset) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)(Ba^1 \neq \emptyset) \quad (T_{M029})(\text{Deduction})(\text{Eqn2})$$

$$(a^1 \subseteq D_A)(Ba^1 \neq \emptyset) \subseteq (\{>\} * \alpha = \{>\} * a) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3})$$

$$(a^1 \subseteq D_A)(Ba^1 \neq \emptyset) \subseteq (\alpha = Ba + Ca) \quad (T_{T791})(\text{Eqn4})$$

$$(a^1 \subseteq D_A)(Ba^1 \neq \emptyset) \subseteq (\{>\} * Ba + Ca) = \{>\} * a \quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn1})$$

$$((\mathcal{X}E)^1 \subseteq D_A)(B(\mathcal{X}E)^1 \neq \emptyset) \subseteq (\{>\} * (B\mathcal{X}E + C\mathcal{X}E) = \{>\} * \mathcal{X}E) \quad \text{Resymbolising}$$

$$(a^1 \subseteq D_M)(Ba^1 \neq \emptyset) \subseteq (\{>\} * \alpha = \{>\} * a) \quad (T_{S910})$$

$$(a^1 \subseteq D_M)(Ca^1 \neq \emptyset) \subseteq (\{<\} * \alpha = \{<\} * a) \quad (T_{S911})$$

$$(a^1 \subseteq D_M)(Ba^1 \neq \emptyset) \subseteq (\{>\} * a = \{>\} * \alpha) \quad (T_{S912})$$

$$(a^1 \subseteq D_M)(Ca^1 \neq \emptyset) \subseteq (\{<\} * a = \{<\} * \alpha) \quad (T_{S913})$$

$$(a^1 \subseteq D_D)(Ba^1 \neq \emptyset) \subseteq (\{>\} * \alpha = \{>\} * a) \quad (T_{T236})$$

$$(a^1 \subseteq D_D)(Ca^1 \neq \emptyset) \subseteq (\{<\} * \alpha = \{<\} * a) \quad (T_{T237})$$

$$(a^1 \subseteq D_D)(Ba^1 \neq \emptyset) \subseteq (\{>\} * a = \{>\} * \alpha) \quad (T_{T238})$$

$$(a^1 \subseteq D_D)(Ca^1 \neq \emptyset) \subseteq (\{<\} * a = \{<\} * \alpha) \quad (T_{T239})$$

Example Proof of (T_{S910}) & (T_{S912})

$$(a^1 \subseteq D_A)(Ba^1 \neq \emptyset) \subseteq (\{>\} * \alpha = \{>\} * a) \quad (T_{T205})$$

$$\begin{aligned}
(a^1 \subseteq D_M + D_D)(Ba^1 \neq \emptyset) &\subseteq (\{>\} * \alpha = \{>\} * a) & (T_{7815}) \\
((a^1 \subseteq D_M) + (a^1 \subseteq D_D))(Ba^1 \neq \emptyset) &\subseteq (\{>\} * \alpha = \{>\} * a) & (T_{M526}) \\
(a^1 \subseteq D_M)(Ba^1 \neq \emptyset) + (a^1 \subseteq D_D)(Ba^1 \neq \emptyset) &\subseteq (\{>\} * \alpha = \{>\} * a) & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
(a^1 \subseteq D_M)(Ba^1 \neq \emptyset) &\subseteq (\{>\} * \alpha = \{>\} * a) & (T_{M072})(\text{Single Premise Syllogism}) \\
(a^1 \subseteq D_M)(Ba^1 \neq \emptyset) &\subseteq (\{>\} * a = \{>\} * \alpha) & (T_{M008})(\text{Inverse of an Equality})
\end{aligned}$$

$$(a^1 \subseteq D_M) \subseteq (Ba^1 = (\{\subseteq\} * a^1).(\{\supset\} * \{\subseteq\} * a^1).(\{>\} * \{\subseteq\} * a^1)) \quad (T_{S875})$$

$$(a^1 \subseteq D_M) \subseteq (Ca^1 = (\{\subseteq\} * a^1).(\{\supset\} * \{\subseteq\} * a^1).(\{\leftarrow\} * \{\subseteq\} * a^1)) \quad (T_{S876})$$

Example Proof of (T_{S875})

$$\begin{aligned}
(a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (Ba^1 = (\{\subseteq\} * a^1).(\{\supset\} * \{\subseteq\} * a^1).(\{>\} * \{\subseteq\} * a^1)) & (T_{T068})(\text{Eqn1}) \\
S_M S_O &\subseteq S_M S_O + S_D S_O & (T_{M082})(\text{Disjunction Introduction}) \\
(S_M S_O \subseteq S_M S_O + S_D S_O) &= E & (A_{MX16})(\text{Order Axiom}) \\
(a^1 \subseteq S_M S_O) &\subseteq (a^1 \subseteq S_M S_O + S_D S_O) & (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn2}) \\
(a^1 \subseteq S_M S_O) &\subseteq (Ba^1 = (\{\subseteq\} * a^1).(\{\supset\} * \{\subseteq\} * a^1).(\{>\} * \{\subseteq\} * a^1)) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\
(a^1 \subseteq D_M) &\subseteq (a^1 \subseteq S_M S_O) & (T_{S988})(\text{Eqn4}) \\
(a^1 \subseteq D_M) &\subseteq (Ba^1 = (\{\subseteq\} * a^1).(\{\supset\} * \{\subseteq\} * a^1).(\{>\} * \{\subseteq\} * a^1)) & (T_{M139})(\text{Hypothetical Syllogism Eqns3\&4})
\end{aligned}$$

$$(a^1 \subseteq D_M) \subseteq (a = \{\supset\} * a^1) \quad (T_{T071})$$

$$((\chi E)^1 \subseteq D_M) \subseteq (\chi E = \{\supset\} * (\chi E)^1) \quad (T_{T164})$$

Proof of (T_{T071}) & (T_{T164})

$$\begin{aligned}
(a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (a = \{\supset\} * a^1) & (T_{T760})(\text{Eqn1}) \\
S_M S_O &\subseteq S_M S_O + S_D S_O & (T_{M082})(\text{Disjunction Introduction}) \\
(S_M S_O \subseteq S_M S_O + S_D S_O) &= E & (A_{MX16})(\text{Order Axiom}) \\
(a^1 \subseteq S_M S_O) &\subseteq (a^1 \subseteq S_M S_O + S_D S_O) & (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn2}) \\
(a^1 \subseteq S_M S_O) &\subseteq (a = \{\supset\} * a^1) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\
(a^1 \subseteq D_M) &\subseteq (a^1 \subseteq S_M S_O) & (T_{S988})(\text{Eqn4}) \\
(a^1 \subseteq D_M) &\subseteq (a = \{\supset\} * a^1) & (T_{M139})(\text{Hypothetical Syllogism Eqns3\&4})(\text{Eqn5}) \\
\text{Let } a^1 &= (\chi E)^1 & (\text{Eqn6}) \\
a &= \chi E & (T_{M479})(\text{Eqn7}) \\
((\chi E)^1 \subseteq D_M) &\subseteq (\chi E = \{\supset\} * (\chi E)^1) & (A_{MX19})(\text{Substitution of Eqns6\&7 into Eqn5})
\end{aligned}$$

$$(a^1 \subseteq D_M) \subseteq (\alpha = (\{\supset\} * a^1).(\{\subseteq\} * \{\supset\} * a^1)) \quad (T_{T063})$$

Proof of (T_{T063})

$$\begin{aligned}
(a^1 \subseteq D_M) &\subseteq (\alpha = a.\{\subseteq\} * a) & (T_{T062})(\text{Eqn1}) \\
(a^1 \subseteq D_M) &\subseteq (a = \{\supset\} * a^1) & (T_{T071})(\text{Eqn2}) \\
(a^1 \subseteq D_M) &\subseteq (\alpha = (\{\supset\} * a^1).(\{\subseteq\} * \{\supset\} * a^1)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$(a^1 \subseteq D_D) \subseteq (a = a.\{\subseteq\} * a) \quad (T_{T172})$$

Proof of (T_{T172})

$$\begin{aligned}
(a^1 \subseteq D_D) &\subseteq (\alpha = a.\{\subseteq\} * a) & (T_{T789})(\text{Eqn1}) \\
(a^1 \subseteq S_D S_O) &\subseteq (\alpha = a) & (D_{S31})(\text{Definition of } \alpha \text{ Symbolism})(\text{Eqn2}) \\
(a^1 \subseteq D_D) &\subseteq (a^1 \subseteq S_D S_O) & (T_{S989})(\text{Eqn3}) \\
(a^1 \subseteq D_D) &\subseteq (\alpha = a) & (T_{M139})(\text{Hypothetical Syllogism Eqns3\&2})(\text{Eqn4}) \\
(a^1 \subseteq D_D) &\subseteq (a = a.\{\subseteq\} * a) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn1})
\end{aligned}$$

$$((\chi E)^1 \subseteq D_M) \subseteq ((\chi E.\{\subseteq\} * \chi E)^1 \subseteq S_P S_O) \quad (T_{T303})$$

Proof of (T₇₃₀₃)

$(a^1 \subseteq S_M S_O) \subseteq (\alpha^1 \subseteq S_P S_O)$	(T ₇₇₇)(Eqn1)
$(a^1 \subseteq D_M) \subseteq (a^1 \subseteq S_M S_O)$	(T ₉₈₈)(Eqn2)
$(a^1 \subseteq D_M) \subseteq (\alpha^1 \subseteq S_P S_O)$	(T ₁₃₉)(Hypothetical Syllogism Eqns1&2)(Eqn3)
$(a^1 \subseteq D_M) \subseteq (\alpha = a.\{\subset\}^* a)$	(T ₇₀₆₂)(Eqn4)
$(a^1 \subseteq D_M) \subseteq ((a.\{\subset\}^* a)^1 \subseteq S_P S_O)$	(A _{MX19})(Substitution of Eqn4 into Eqn1)(Eqn5)
Let $a^1 = (\chi E)^1$	(Eqn6)
$a = \chi E$	(T ₄₇₉)(Eqn7)
$((\chi E)^1 \subseteq D_M) \subseteq (\chi E.\{\subset\}^* \chi E)^1 \subseteq S_P S_O$	(A _{MX19})(Substitution of Eqns6&7 into Eqn5)

$$(a^1 \subseteq D_A) \subseteq (Ba^1 = a.\{\supset\}^* a).\{\subset\}^* a) \quad (T_{783})$$

$$(a^1 \subseteq D_A) \subseteq (Ca^1 = a.\{\subset\}^* a).\{\subset\}^* a) \quad (T_{784})$$

Example Proof of (T₇₈₃)

$(a^1 \subseteq S_M S_O + S_D S_O) \subseteq (Ba^1 = a.\{\supset\}^* a).\{\subset\}^* a)$	(T ₉₈₆)(Eqn1)
$D_M \subseteq S_M S_O$	(T ₉₂₆)(Eqn2)
$D_D \subseteq S_D S_O$	(T ₉₂₈)(Eqn3)
$D_M + D_D \subseteq S_M S_O + S_D S_O$	(A _{MX21})(Combination of Eqns2&3)
$D_A \subseteq S_M S_O + S_D S_O$	(T ₈₁₅)
$(D_A \subseteq S_M S_O + S_D S_O) = E$	(A _{MX16})(Order Axiom)
$(a^1 \subseteq D_A) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)$	(T ₀₉₉)(Entity Mode / Set Mode Rule)(Eqn4)
$(a^1 \subseteq D_A) \subseteq (Ba^1 = a.\{\supset\}^* a).\{\subset\}^* a)$	(T ₁₃₉)(Hypothetical Syllogism Eqns1&4)

$$(a^1 \subseteq D_A).(Ba^1 \neq \emptyset) \subseteq (\{\supset\}^* a = \{\supset\}^* \alpha) \quad (T_{781})$$

$$(a^1 \subseteq D_A).(Ca^1 \neq \emptyset) \subseteq (\{\subset\}^* a = \{\subset\}^* \alpha) \quad (T_{782})$$

Example Proof of (T₇₈₁)

$(a^1 \subseteq S_M S_O + S_D S_O).(Ba^1 \neq \emptyset) \subseteq (\{\supset\}^* a = \{\supset\}^* \alpha)$	(T ₀₆₆)(Eqn1)
$D_M \subseteq S_M S_O$	(T ₉₂₆)(Eqn2)
$D_D \subseteq S_D S_O$	(T ₉₂₈)(Eqn3)
$D_M + D_D \subseteq S_M S_O + S_D S_O$	(A _{MX21})(Combination of Eqns2&3)
$D_A \subseteq S_M S_O + S_D S_O$	(T ₈₁₅)
$(D_A \subseteq S_M S_O + S_D S_O) = E$	(A _{MX16})(Order Axiom)
$(a^1 \subseteq D_A) \subseteq (a^1 \subseteq S_M S_O + S_D S_O)$	(T ₀₉₉)(Entity Mode / Set Mode Rule)
$(a^1 \subseteq D_A).(Ba^1 \neq \emptyset) \subseteq (a^1 \subseteq S_M S_O + S_D S_O).(Ba^1 \neq \emptyset)$	(T ₀₂₉)(Deduction)(Eqn4)
$(a^1 \subseteq D_A).(Ba^1 \neq \emptyset) \subseteq (\{\supset\}^* a = \{\supset\}^* \alpha)$	(T ₁₃₉)(Hypothetical Syllogism Eqns1&4)

$$(a^1 \subseteq D_A) \subseteq (Ba^1 = \alpha.\{\supset\}^* a) \quad (T_{785})$$

$$(a^1 \subseteq D_A) \subseteq (Ca^1 = \alpha.\{\subset\}^* a) \quad (T_{786})$$

$$(a^1 \subseteq D_A) \subseteq (Ba^1 = \alpha.\{\supset\}^* a) \quad (T_{183})$$

$$(a^1 \subseteq D_A) \subseteq (Ca^1 = \alpha.\{\subset\}^* a) \quad (T_{184})$$

Example Proof of (T₇₈₅) & (T₁₈₃)

$(a^1 \subseteq D_A) \subseteq (Ba^1 = \alpha.\{\supset\}^* a).\{\subset\}^* a)$	(T ₇₈₃)(Eqn1)
$(a^1 \subseteq D_A) \subseteq (\alpha = a.\{\subset\}^* a)$	(T ₃₁₁)(Eqn2)
$(a^1 \subseteq D_A) \subseteq (Ba^1 = \alpha.\{\supset\}^* a)$	(A _{MX19})(Substitution of Eqn2 into Eqn1)
$(a^1 \subseteq D_A).(Ba^1 \neq \emptyset) \subseteq (Ba^1 = \alpha.\{\supset\}^* a)$	(T ₇₈₁)

$$(a^1 \subseteq D_M) \subseteq (Ba^1 = \alpha.\{\supset\}^* a) \quad (T_{873})$$

$$(a^1 \subseteq D_M) \subseteq (Ca^1 = \alpha.(\{<\}_* \alpha)) \quad (T_{S874})$$

$$(a^1 \subseteq D_0) \subseteq (Ba^1 = a.(\{>\}_* a)) \quad (T_{T185})$$

$$(a^1 \subseteq D_0) \subseteq (Ca^1 = a.(\{<\}_* a)) \quad (T_{T186})$$

Example Proof of (T_{S873})

$$(a^1 \subseteq D_A) \subseteq (Ba^1 = \alpha.(\{>\}_* \alpha)) \quad (T_{T183})$$

$$(a^1 \subseteq D_M + D_0) \subseteq (Ba^1 = \alpha.(\{>\}_* \alpha)) \quad (T_{T815})$$

$$(a^1 \subseteq D_M) + (a^1 \subseteq D_0) \subseteq (Ba^1 = \alpha.(\{>\}_* \alpha)) \quad (T_{M526})$$

$$(a^1 \subseteq D_M) \subseteq (Ba^1 = \alpha.(\{>\}_* \alpha)) \quad (T_{M072})(\text{Single Premise Syllogism})$$

$$((\mathcal{X}E)^1 \subseteq D_A)((\mathcal{X}a)^1 = \mathcal{X}a^1) \subseteq (\mathcal{B}_x a^1 = \{=\}_x a.(\{>\}_x a.(\{<\}_x a))) \quad (T_{R725}) \text{ Definition of } \mathcal{B}_x$$

$$((\mathcal{X}E)^1 \subseteq D_A)((\mathcal{X}a)^1 = \mathcal{X}a^1) \subseteq (C_x a^1 = \{=\}_x a.(\{<\}_x a.(\{<\}_x a))) \quad (T_{R726}) \text{ Definition of } C_x$$

Example Proof of (T_{R725})

$$((\mathcal{X}E)^1 \subseteq D_A) \subseteq (\mathcal{B}(\mathcal{X}a)^1 = \mathcal{X}a.(\{>\}_* \mathcal{X}a)(\{<\}_* \mathcal{X}a)) \quad (T_{T209})(\text{Eqn1})$$

$$(\mathcal{B}(\mathcal{X}a)^1 = \mathcal{X}a.(\{>\}_* \mathcal{X}a)(\{<\}_* \mathcal{X}a)) \subseteq (\forall \mathcal{X} \mathcal{B}(\mathcal{X}a)^1 = \forall \mathcal{X}(\mathcal{X}a.(\{>\}_* \mathcal{X}a)(\{<\}_* \mathcal{X}a))) \quad (T_{R522})(\text{Relational Deduction})(\text{Eqn2})$$

$$((\mathcal{X}E)^1 \subseteq D_A) \subseteq (\forall \mathcal{X} \mathcal{B}(\mathcal{X}a)^1 = \forall \mathcal{X}(\mathcal{X}a.(\{>\}_* \mathcal{X}a)(\{<\}_* \mathcal{X}a))) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})$$

$$((\mathcal{X}E)^1 \subseteq D_A) \subseteq (\forall \mathcal{X} \mathcal{B}(\mathcal{X}a)^1 = \forall \mathcal{X} \mathcal{X}a.(\forall \mathcal{X} \{>\}_* \mathcal{X}a)(\forall \mathcal{X} \{<\}_* \mathcal{X}a))$$

(A_{R35})(One-to-One Relationship Distribution Axiom)

$$((\mathcal{X}E)^1 \subseteq D_A) \subseteq (\forall \mathcal{X} \mathcal{B}(\mathcal{X}a)^1 = \forall \mathcal{X} \mathcal{X}a.(\{>\}_x a.(\{<\}_x a)) \quad (T_{R572})$$

$$((\mathcal{X}E)^1 \subseteq D_A) \subseteq (\forall \mathcal{X} \mathcal{B}(\mathcal{X}a)^1 = \forall \mathcal{X} \{=\}_x \mathcal{X}a.(\{>\}_x a.(\{<\}_x a)) \quad (T_{A080})$$

$$((\mathcal{X}E)^1 \subseteq D_A) \subseteq (\forall \mathcal{X} \mathcal{B}(\mathcal{X}a)^1 = \{=\}_x a.(\{>\}_x a.(\{<\}_x a)) \quad (T_{R571})(\text{Eqn3})$$

$$\text{Let } (\mathcal{X}a)^1 = \mathcal{X}a^1 \quad (\text{Eqn4})$$

$$((\mathcal{X}E)^1 \subseteq D_A) \subseteq (\forall \mathcal{X} \mathcal{B} \mathcal{X}a^1 = \{=\}_x a.(\{>\}_x a.(\{<\}_x a)) \quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})$$

$$((\mathcal{X}E)^1 \subseteq D_A) \subseteq (\mathcal{B}_x a^1 = \{=\}_x a.(\{>\}_x a.(\{<\}_x a)) \quad (T_{R571})$$

$$\text{Thus } (((\mathcal{X}a)^1 = \mathcal{X}a^1) = E) \subseteq (((\mathcal{X}E)^1 \subseteq D_A) \subseteq (\mathcal{B}_x a^1 = \{=\}_x a.(\{>\}_x a.(\{<\}_x a)))$$

(A_{MX02})(Conditional Introduction)

$$((\mathcal{X}E)^1 \subseteq D_A)((\mathcal{X}a)^1 = \mathcal{X}a^1) \subseteq (\mathcal{B}_x a^1 = \{=\}_x a.(\{>\}_x a.(\{<\}_x a))$$

(A_{MX35})(Exportation/Importation) for truth sets

$$((\mathcal{X}E)^1 \subseteq D_A)(\alpha^1 \subseteq \mathcal{X}E.(\{<\}_* \mathcal{X}E) \subseteq (\forall \mathcal{B} \alpha^1 = (\{<\}_* \alpha^1).(\{<\}_* \{<\}_* \alpha^1)) \quad (T_{S877})$$

$$((\mathcal{X}E)^1 \subseteq D_A)(\alpha^1 \subseteq \mathcal{X}E.(\{<\}_* \mathcal{X}E) \subseteq (\forall C \alpha^1 = (\{<\}_* \alpha^1).(\{<\}_* \{>\}_* \alpha^1)) \quad (T_{S878})$$

Example Proof of (T_{S877})

$$(a^1 \subseteq S_M S_0 + S_D S_0)(\beta^1 \subseteq a) \subseteq (\forall \mathcal{B} \beta^1 = (\{<\}_* \beta^1).(\{<\}_* \{<\}_* \beta^1)).(\forall C \beta^1 = (\{<\}_* \beta^1).(\{<\}_* \{>\}_* \beta^1)) \quad (D_{R50})(\text{Definition of } \forall \mathcal{B} \text{ \& } \forall C)(\text{Eqn1})$$

$$(a^1 \subseteq D_M) \subseteq (a^1 \subseteq S_M S_0) \quad (T_{S988})(\text{Eqn2})$$

$$(a^1 \subseteq D_0) \subseteq (a^1 \subseteq S_D S_0) \quad (T_{S989})$$

$$S_P \subseteq S_D \quad (A_{R53})(\text{Axiom of Points})$$

$$S_P S_0 \subseteq S_D S_0 \quad (T_{M029})(\text{Deduction})$$

$$(S_P S_0 \subseteq S_D S_0) = E \quad (A_{MX16})(\text{Order Axiom})$$

$$(a^1 \subseteq S_P S_0) \subseteq (a^1 \subseteq S_D S_0) \quad (T_{M272})(\text{Deductive Rule})(\text{Eqn3})$$

$$(a^1 \subseteq D_M) + (a^1 \subseteq D_0) \subseteq (a^1 \subseteq S_M S_0) + (a^1 \subseteq S_D S_0) \quad (A_{MX21})(\text{Combination of Eqns2\&3})$$

$$(a^1 \subseteq D_M + D_0) \subseteq (a^1 \subseteq S_M S_0 + S_D S_0) \quad (T_{M526})$$

$$(a^1 \subseteq D_A) \subseteq (a^1 \subseteq S_M S_0 + S_D S_0) \quad (T_{T815})$$

$$(a^1 \subseteq D_A)(\beta^1 \subseteq \alpha) \subseteq (a^1 \subseteq S_M S_0 + S_D S_0)(\beta^1 \subseteq \alpha) \quad (T_{M029})(\text{Deduction})(\text{Eqn4})$$

$$(a^1 \subseteq D_A)(\beta^1 \subseteq \alpha) \subseteq (\forall \mathcal{B} \beta^1 = (\{<\}_* \beta^1).(\{<\}_* \{<\}_* \beta^1)).(\forall C \beta^1 = (\{<\}_* \beta^1).(\{<\}_* \{>\}_* \beta^1))$$

(T_{M139})(Hypothetical Syllogism Eqns1\&4)(Eqn5)

$$\text{Let } a^1 = (\mathcal{X}E)^1 \quad (\text{Eqn6})$$

$$\text{Let } \beta^1 = \alpha^1 \quad (\text{Eqn7})$$

$$\text{Let } \alpha = \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E} \quad (\text{Eqn8})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq$$

$$(\forall \mathcal{B}\alpha^1 = (\{\subseteq\}*\alpha^1).(\{\subseteq\}*\{\prec\}*\alpha^1)).(\forall \mathcal{C}\alpha^1 = (\{\subseteq\}*\alpha^1).(\{\subseteq\}*\{\succ\}*\alpha^1))$$

($\mathcal{A}_{MX19}$)(Substitution of Eqns6,7&8 into Eqn5)

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 = (\{\subseteq\}*\alpha^1).(\{\subseteq\}*\{\prec\}*\alpha^1))$$

($\mathcal{T}_{M073}$)(Single Premise Syllogism)

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 = (\{\subseteq\}*\alpha^1).(\{\subseteq\}*\{\prec\}*\alpha^1)) \quad (\mathcal{T}_{T218})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{C}\alpha^1 = (\{\subseteq\}*\alpha^1).(\{\subseteq\}*\{\succ\}*\alpha^1)) \quad (\mathcal{T}_{T219})$$

Example Proof of ($\mathcal{T}_{T218}$)

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 = (\{\subseteq\}*\alpha^1).(\{\subseteq\}*\{\prec\}*\alpha^1)) \quad (\mathcal{T}_{S877})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_M + D_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 = (\{\subseteq\}*\alpha^1).(\{\subseteq\}*\{\prec\}*\alpha^1)) \quad (\mathcal{T}_{T815})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq D_M) + ((\mathcal{X}\mathcal{E})^1 \subseteq D_D))(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 = (\{\subseteq\}*\alpha^1).(\{\subseteq\}*\{\prec\}*\alpha^1))$$

($\mathcal{T}_{M526}$)

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) + ((\mathcal{X}\mathcal{E})^1 \subseteq D_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 = (\{\subseteq\}*\alpha^1).(\{\subseteq\}*\{\prec\}*\alpha^1))$$

($\mathcal{T}_{M407}$)(Distribution of . over +)

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 = (\{\subseteq\}*\alpha^1).(\{\subseteq\}*\{\prec\}*\alpha^1)) \quad (\mathcal{T}_{M072} \text{ (Single Premise Syllogism)})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 = \alpha^1) \quad (\mathcal{T}_{T130})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{C}\alpha^1 = \alpha^1) \quad (\mathcal{T}_{T131})$$

Example Proof of ($\mathcal{T}_{T130}$)

$$(\alpha^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall \mathcal{B}\beta^1 = \beta^1).(\forall \mathcal{C}\beta^1 = \beta^1) \quad (\mathcal{T}_{T115} \text{ (Eqn1)})$$

$$(\alpha^1 \subseteq D_D) \subseteq (\alpha^1 \subseteq S_D S_O) \quad (\mathcal{T}_{S989})$$

$$(\alpha^1 \subseteq D_D)(\beta^1 \subseteq \alpha) \subseteq (\alpha^1 \subseteq S_D S_O)(\beta^1 \subseteq \alpha) \quad (\mathcal{T}_{M029} \text{ (Deduction) (Eqn2)})$$

$$(\alpha^1 \subseteq D_D)(\beta^1 \subseteq \alpha) \subseteq (\forall \mathcal{B}\beta^1 = \beta^1).(\forall \mathcal{C}\beta^1 = \beta^1) \quad (\mathcal{T}_{M139} \text{ (Hypothetical Syllogism Eqns1&2)})$$

$$(\alpha^1 \subseteq D_D)(\beta^1 \subseteq \alpha) \subseteq (\forall \mathcal{B}\beta^1 = \beta^1) \quad (\mathcal{T}_{M073} \text{ (Single Premise Syllogism) (Eqn3)})$$

$$\text{Let } \alpha = \mathcal{X}\mathcal{E} \quad (\text{Eqn4})$$

$$\text{Let } \beta^1 = \alpha^1 \quad (\text{Eqn5})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 = \alpha^1) \quad (\mathcal{A}_{MX19} \text{ (Substitution of Eqns4&5 into Eqn3)})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 \forall \mathcal{C}\alpha^1 = \alpha^1) \quad (\mathcal{T}_{T138})$$

Proof of ($\mathcal{T}_{T138}$)

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 = \alpha^1) \quad (\mathcal{T}_{T130} \text{ (Eqn1)})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{C}\alpha^1 = \alpha^1) \quad (\mathcal{T}_{T131} \text{ (Eqn2)})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E})((\mathcal{X}\mathcal{E})^1 \subseteq D_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 = \alpha^1)(\forall \mathcal{C}\alpha^1 = \alpha^1)$$

($\mathcal{T}_{M669}$)(Combination of Eqns1&2)

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 = \alpha^1)(\forall \mathcal{C}\alpha^1 = \alpha^1)$$

($\mathcal{T}_{M667}$)(Conjunction Tautology)

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 \forall \mathcal{C}\alpha^1 = \alpha^1 \alpha^1)$$

($\mathcal{T}_{M231}$)(Combination)

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 \forall \mathcal{C}\alpha^1 = \alpha^1)$$

($\mathcal{T}_{M667}$)(Conjunction Tautology)

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{B}\alpha^1 \forall \mathcal{C}\alpha^1 = \alpha^1) \quad (\mathcal{T}_{T139})$$

Proof of ($\mathcal{T}_{T139}$)

$$(\alpha^1 \subseteq S_M S_O + S_D S_O)(\beta^1 \subseteq \alpha) \subseteq (\forall \mathcal{B}\beta^1 \forall \mathcal{C}\beta^1 = \beta^1) \quad (\mathcal{T}_{T780} \text{ (Eqn1)})$$

$$(\alpha^1 \subseteq D_A) \subseteq (\alpha^1 \subseteq S_M S_O + S_D S_O) \quad (\mathcal{T}_{T792} \text{ (Eqn2)})$$

$$(\alpha^1 \subseteq D_A)(\beta^1 \subseteq \alpha) \subseteq (\alpha^1 \subseteq S_M S_O + S_D S_O)(\beta^1 \subseteq \alpha) \quad (\mathcal{T}_{M029} \text{ (Deduction) (Eqn3)})$$

$$(\alpha^1 \subseteq D_A)(\beta^1 \subseteq \alpha) \subseteq (\forall \mathcal{B}\beta^1 \forall \mathcal{C}\beta^1 = \beta^1) \quad (\mathcal{T}_{M139} \text{ (Hypothetical Syllogism Eqns1&3) (Eqn4)})$$

$$\begin{aligned}
(a^1 \subseteq S_M S_O + S_D S_O) &\subseteq (\alpha = a.\{\subseteq\} * a) & (T_{7070})(\text{Eqn5}) \\
(a^1 \subseteq D_A) &\subseteq (\alpha = a.\{\subseteq\} * a) & (T_{M139})(\text{Hypothetical Syllogism Eqns2\&5})(\text{Eqn6}) \\
(a^1 \subseteq D_A)(\beta^1 \subseteq a.\{\subseteq\} * a) &\subseteq (\forall \beta \beta^1 \forall C \beta^1 = \beta^1) & (A_{MX19})(\text{Substitution of Eqn6 into Eqn4})(\text{Eqn7}) \\
\text{Let } a &= XE & (\text{Eqn8}) \\
\text{Let } \beta^1 &= \alpha^1 & (\text{Eqn9}) \\
((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall B \alpha^1 \forall C \alpha^1 = \alpha^1) & (A_{MX19})(\text{Substitution of Eqns8\&9 into Eqn7})
\end{aligned}$$

$$((XE)^1 \subseteq D_M)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) \subseteq (\forall B \alpha^1. \forall C \alpha^1 = \alpha^1) \quad (T_{T220})$$

Proof of (T_{T220})

$$\begin{aligned}
((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall B \alpha^1 \forall C \alpha^1 = \alpha^1) & (T_{T139}) \\
(((XE)^1 \subseteq D_M + D_D)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall B \alpha^1. \forall C \alpha^1 = \alpha^1) & (T_{T815}) \\
(((XE)^1 \subseteq D_M) + ((XE)^1 \subseteq D_D))(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall B \alpha^1. \forall C \alpha^1 = \alpha^1) & (T_{M526}) \\
((XE)^1 \subseteq D_M)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) + ((XE)^1 \subseteq D_D)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall B \alpha^1. \forall C \alpha^1 = \alpha^1) & (T_{M407})(\text{Distribution of } \cdot \text{ over } +) \\
((XE)^1 \subseteq D_M)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall B \alpha^1. \forall C \alpha^1 = \alpha^1) & (T_{M072})(\text{Single Premise Syllogism})
\end{aligned}$$

$$((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) \subseteq (\forall X' B \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{<\} \alpha^1)) \quad (T_{R896})$$

$$((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) \subseteq (\forall X' C \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{>\} \alpha^1)) \quad (T_{R897})$$

Example Proof of (T_{R896})

$$\begin{aligned}
((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall B \alpha^1 = (\{\subseteq\} * \alpha^1). (\{\subseteq\} * \{<\} * \alpha^1)) & (T_{S877}) \\
((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall X' B \alpha^1 = \forall X (\{\subseteq\} * \alpha^1). (\{\subseteq\} * \{<\} * \alpha^1)) & (T_{R522})(\text{Relational Deduction}) \\
((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall X' B \alpha^1 = \forall X (\{\subseteq\} * \alpha^1). (\forall X \{\subseteq\} * \{<\} * \alpha^1)) & (A_{R35})(\text{One-to-One Relationship Distribution Axiom})
\end{aligned}$$

$$((XE)^1 \subseteq D_M)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) \subseteq (\forall X' B \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{<\} \alpha^1)) \quad (T_{T241})$$

$$((XE)^1 \subseteq D_M)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) \subseteq (\forall X' C \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{>\} \alpha^1)) \quad (T_{T243})$$

Example Proof of (T_{T241})

$$\begin{aligned}
((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall X' B \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{<\} \alpha^1)) & (T_{R896}) \\
((XE)^1 \subseteq D_M + D_D)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall X' B \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{<\} \alpha^1)) & (T_{T815}) \\
(((XE)^1 \subseteq D_M) + ((XE)^1 \subseteq D_D))(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall X' B \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{<\} \alpha^1)) & (T_{M526}) \\
((XE)^1 \subseteq D_M)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) + ((XE)^1 \subseteq D_D)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall X' B \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{<\} \alpha^1)) & (T_{M407})(\text{Distribution of } \cdot \text{ over } +) \\
((XE)^1 \subseteq D_M)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall X' B \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{<\} \alpha^1)) & (T_{M072})(\text{Single Premise Syllogism})
\end{aligned}$$

$$((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) \subseteq (\forall X' B \alpha^1. \forall X' C \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{<\} \alpha^1). (\forall X \{\subseteq\} * \{>\} \alpha^1)) \quad (T_{R898})$$

Proof of (T_{R898})

$$\begin{aligned}
((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall X' B \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{<\} \alpha^1)) & (T_{R896})(\text{Eqn1}) \\
((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall X' C \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{>\} \alpha^1)) & (T_{R897})(\text{Eqn2}) \\
((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) \subseteq & \\
(\forall X' B \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{<\} \alpha^1)) &(\forall X' C \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{>\} \alpha^1)) & (T_{M669})(\text{Combination of Eqns1\&2}) \\
((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall X' B \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{<\} \alpha^1)) & (\forall X' C \alpha^1 = \forall X \{\subseteq\} \alpha^1. (\forall X \{\subseteq\} * \{>\} \alpha^1)) \\
&& (T_{M667})(\text{Conjunction Tautology}) \\
((XE)^1 \subseteq D_A)(\alpha^1 \subseteq XE.\{\subseteq\} * XE) &\subseteq (\forall X' B \alpha^1. \forall X' C \alpha^1 = \forall X \{\subseteq\} \alpha^1. \forall X \{\subseteq\} * \{<\} \alpha^1. \forall X \{\subseteq\} * \{>\} \alpha^1) & \\
&& (T_{M237})(\text{Combination})
\end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}_* \mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{X}' B\alpha^1. \forall \mathcal{X}' C\alpha^1 = \forall \mathcal{X}\{\subseteq\}\alpha^1. \forall \mathcal{X}\{\subseteq\}_* \{\subseteq\}\alpha^1. \forall \mathcal{X}\{\subseteq\}_* \{\supset\}\alpha^1)$$

(T_{M667})(Conjunction Tautology)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}_* \mathcal{X}\mathcal{E}) \subseteq ({}^V\mathcal{X}\alpha^1 = {}^V\mathcal{X}^V\mathcal{B}\alpha^1. {}^V\mathcal{X}^V\mathcal{C}\alpha^1) \quad (T_{S749})$$

Proof of (T_{S749})

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\_ \}^* \mathcal{X}\mathcal{E}) \subseteq ({}^\vee \mathcal{B}\alpha^1 {}^\vee \mathcal{C}\alpha^1 = \alpha^1)$	$(\mathcal{T}_{139})$
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\_ \}^* \mathcal{X}\mathcal{E}) \subseteq ({}^\vee \mathcal{X}({}^\vee \mathcal{B}\alpha^1. {}^\vee \mathcal{C}\alpha^1) = {}^\vee \mathcal{X}\alpha^1)$	$(\mathcal{R}_{522})(\text{Relational Deduction})$
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\_ \}^* \mathcal{X}\mathcal{E}) \subseteq ({}^\vee \mathcal{X}{}^\vee \mathcal{B}\alpha^1. {}^\vee \mathcal{X}{}^\vee \mathcal{C}\alpha^1 = {}^\vee \mathcal{X}\alpha^1)$	$(\mathcal{A}_{35})(\text{One-to-One Relationship Distribution Axiom})$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_M}) (\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}_* \mathcal{X}\mathcal{V}\mathcal{E}) \subseteq ({}^v\mathcal{X}^v B\alpha^1. {}^v\mathcal{X}^v C\alpha^1 = {}^v\mathcal{X}\{\subseteq\}\alpha^1. ({}^v\mathcal{X}\{\subseteq\}_* \{<\}\alpha^1). ({}^v\mathcal{X}\{\subseteq\}_* \{>\}\alpha^1)) \quad (T_{T247})$$

Example Proof of ($\mathcal{T}_{T247}$)

$$\begin{aligned}
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_A})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}_*\mathcal{X}\mathcal{E}) \subseteq (\mathcal{V}\mathcal{X}'\mathcal{B}\alpha^1.\mathcal{V}\mathcal{X}'\mathcal{C}\alpha^1 = \mathcal{V}\mathcal{X}\{\subseteq\}\alpha^1.(\mathcal{V}\mathcal{X}\{\subseteq\}_*\{\prec\}\alpha^1).(\mathcal{V}\mathcal{X}\{\subseteq\}_*\{\succ\}\alpha^1)) \\
& \quad (T_{R898}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_M} + \mathbf{D_D})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}_*\mathcal{X}\mathcal{E}) \subseteq (\mathcal{V}\mathcal{X}'\mathcal{B}\alpha^1.\mathcal{V}\mathcal{X}'\mathcal{C}\alpha^1 = \mathcal{V}\mathcal{X}\{\subseteq\}\alpha^1.(\mathcal{V}\mathcal{X}\{\subseteq\}_*\{\prec\}\alpha^1).(\mathcal{V}\mathcal{X}\{\subseteq\}_*\{\succ\}\alpha^1)) \\
& \quad (T_{T815}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_M}) + ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_D}))(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}_*\mathcal{X}\mathcal{E}) \subseteq \\
& (\mathcal{V}\mathcal{X}'\mathcal{B}\alpha^1.\mathcal{V}\mathcal{X}'\mathcal{C}\alpha^1 = \mathcal{V}\mathcal{X}\{\subseteq\}\alpha^1.(\mathcal{V}\mathcal{X}\{\subseteq\}_*\{\prec\}\alpha^1).(\mathcal{V}\mathcal{X}\{\subseteq\}_*\{\succ\}\alpha^1)) \quad (T_{M526}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_M})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}_*\mathcal{X}\mathcal{E}) + ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_D})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}_*\mathcal{X}\mathcal{E}) \subseteq \\
& (\mathcal{V}\mathcal{X}'\mathcal{B}\alpha^1.\mathcal{V}\mathcal{X}'\mathcal{C}\alpha^1 = \mathcal{V}\mathcal{X}\{\subseteq\}\alpha^1.(\mathcal{V}\mathcal{X}\{\subseteq\}_*\{\prec\}\alpha^1).(\mathcal{V}\mathcal{X}\{\subseteq\}_*\{\succ\}\alpha^1)) \quad (T_{M407}) \text{(Distribution of } \cdot \text{ over } +) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_M})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}_*\mathcal{X}\mathcal{E}) \subseteq (\mathcal{V}\mathcal{X}'\mathcal{B}\alpha^1.\mathcal{V}\mathcal{X}'\mathcal{C}\alpha^1 = \mathcal{V}\mathcal{X}\{\subseteq\}\alpha^1.(\mathcal{V}\mathcal{X}\{\subseteq\}_*\{\prec\}\alpha^1).(\mathcal{V}\mathcal{X}\{\subseteq\}_*\{\succ\}\alpha^1)) \\
& \quad (T_{M072}) \text{(Single Premise Syllogism)}
\end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_M}) (\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subset\}^* \mathcal{X}\mathcal{E}) \subseteq ({}^V\mathcal{X}\alpha^1 \subseteq {}^V\mathcal{X}'\mathcal{B}\alpha^1, {}^V\mathcal{X}'\mathcal{C}\alpha^1) \quad (T_{T245})$$

Proof of ($\mathcal{T}_{T245}$)

$$\begin{aligned}
((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_A})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subset\}_*\mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{X}\alpha^1 = \forall \mathcal{X}'\mathcal{B}\alpha^1.\forall \mathcal{X}'\mathcal{C}\alpha^1) & (T_{S749}) \\
((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_M} + \mathbf{D_D})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subset\}_*\mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{X}\alpha^1 = \forall \mathcal{X}'\mathcal{B}\alpha^1.\forall \mathcal{X}'\mathcal{C}\alpha^1) & (T_{T815}) \\
(((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_M}) + ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_D}))(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subset\}_*\mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{X}\alpha^1 = \forall \mathcal{X}'\mathcal{B}\alpha^1.\forall \mathcal{X}'\mathcal{C}\alpha^1) & (T_{M526}) \\
((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_M})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subset\}_*\mathcal{X}\mathcal{E}) + ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_D})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subset\}_*\mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{X}\alpha^1 = \forall \mathcal{X}'\mathcal{B}\alpha^1.\forall \mathcal{X}'\mathcal{C}\alpha^1) & (T_{M407}) \text{ (Distribution of } \cdot \text{ over } +) \\
((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_M})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subset\}_*\mathcal{X}\mathcal{E}) &\subseteq ((\forall \mathcal{X}\alpha^1 = \forall \mathcal{X}'\mathcal{B}\alpha^1.\forall \mathcal{X}'\mathcal{C}\alpha^1) & (T_{M072}) \text{ (Single Premise Syllogism)}
\end{aligned}$$

$$((\mathcal{X}E)^1 \subseteq D_p)(\alpha^1 \subseteq \mathcal{X}E) \subseteq ({}^v\mathcal{X}{}^vB\alpha^1 = {}^v\mathcal{X}\alpha^1) \quad (T_{T132})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ({}^V\mathcal{X}^V C\alpha^1 = {}^V\mathcal{X}\alpha^1) \quad (T_{T133})$$

Example Proof of ($\mathcal{T}_{T132}$)

$$\begin{array}{ll} ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_D})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ({}^V\mathcal{B}\alpha^1 = \alpha^1) & (\mathcal{T}_{T130}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D_D})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ({}^V\mathcal{X}^V\mathcal{B}\alpha^1 = {}^V\mathcal{X}\alpha^1) & (\mathcal{T}_{R522})(\text{Relational Deduction}) \end{array}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_\mathbf{D})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ({}^v\mathcal{X}^v\mathcal{B}\alpha^1, {}^v\mathcal{X}^v\mathcal{C}\alpha^1 = {}^v\mathcal{X}\alpha^1) \quad (T_{T136})$$

Proof of (T_{T136})

$$\begin{array}{ll}
((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ({}^\vee\mathcal{X}^\vee\mathcal{B}\alpha^1 = {}^\vee\mathcal{X}\alpha^1) & (\mathcal{T}_{7132})(\text{Eqn1}) \\
((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ({}^\vee\mathcal{X}^\vee\mathcal{C}\alpha^1 = {}^\vee\mathcal{X}\alpha^1) & (\mathcal{T}_{7133})(\text{Eqn2}) \\
((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\alpha^1 \subseteq \mathcal{X}\mathcal{E})((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ({}^\vee\mathcal{X}^\vee\mathcal{B}\alpha^1 = {}^\vee\mathcal{X}\alpha^1)({}^\vee\mathcal{X}^\vee\mathcal{C}\alpha^1 = {}^\vee\mathcal{X}\alpha^1) & (\mathcal{T}_{M669})(\text{Combination of Eqns1\&2}) \\
((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ({}^\vee\mathcal{X}^\vee\mathcal{B}\alpha^1 = {}^\vee\mathcal{X}\alpha^1)({}^\vee\mathcal{X}^\vee\mathcal{C}\alpha^1 = {}^\vee\mathcal{X}\alpha^1) & (\mathcal{T}_{M667})(\text{Conjunction Tautology}) \\
((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ({}^\vee\mathcal{X}^\vee\mathcal{B}\alpha^1 {}^\vee\mathcal{X}^\vee\mathcal{C}\alpha^1 = {}^\vee\mathcal{X}\alpha^1 {}^\vee\mathcal{X}\alpha^1) & (\mathcal{T}_{M231})(\text{Combination})
\end{array}$$

(T_{M667})(Conjunction Tautology)

$$(((Q\mathcal{D})^1 \subseteq D_M + P_M)((S\mathcal{D})^1 \subseteq D_D)(\hat{b}^1 \neq \emptyset)(a^1 \subseteq Q\mathcal{D}).(\hat{b}^1 \subseteq Q\mathcal{D}) = E) \subseteq ((a^1 \subseteq \{\sup\}\hat{b}^1) \subseteq (\mathcal{D}a^1 \subseteq \{\prec\}\mathcal{D}\hat{b}^1)) \quad (T_{\tau 405})$$

Proof of (T_{T405})

$$\begin{aligned}
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{\subset\}\hat{b}^1) \subseteq (\mathcal{D}a^1 \subseteq \{>\}\mathcal{D}\hat{b}^1)) \\
& \quad (A_{s30}) \text{ (Axiom of Dimensional Proper Sub-entities)} \\
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(\hat{b}^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq ((\hat{b}^1 \subseteq \{\subset\}a^1) \subseteq (\mathcal{D}\hat{b}^1 \subseteq \{>\}\mathcal{D}a^1)) \\
& \quad \text{Resymbolising} \\
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(\hat{b}^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{\supset\}\hat{b}^1) \subseteq (\mathcal{D}\hat{b}^1 \subseteq \{>\}\mathcal{D}a^1)) \\
& \quad (T_{775}) \\
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(\hat{b}^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{\supset\}\hat{b}^1) \subseteq (\mathcal{D}a^1 \subseteq \{<\}\mathcal{D}\hat{b}^1)) \\
& \quad (T_{776})
\end{aligned}$$

$$(((\mathbf{QD})^1 \subseteq \mathbf{D_M + P_M})((\mathbf{SD})^1 \subseteq \mathbf{D_D})(a^1 \neq \emptyset)(a^1 \subseteq \mathbf{QD}).(\mathbf{b}^1 \subseteq \mathbf{QD}) = \mathbf{E}) \subseteq ((a^1 \subseteq \{\subseteq\})\mathbf{b}^1) \subseteq (\mathcal{D}a^1 \subseteq \{\supseteq\}\mathcal{D}\mathbf{b}^1)) \text{ (} T_{7406} \text{)}$$

$$(((Q\mathcal{D})^1 \subseteq D_M + P_M)((S\mathcal{D})^1 \subseteq D_D)(\mathcal{b}^1 \neq \emptyset)(a^1 \subseteq Q\mathcal{D}).(\mathcal{b}^1 \subseteq Q\mathcal{D}) = E) \subseteq ((a^1 \subseteq \{\supseteq\})\mathcal{b}^1) \subseteq (\mathcal{D}a^1 \subseteq \{\leq\}\mathcal{D}\mathcal{b}^1)) \quad (T_{\tau 407})$$

Example Proof of ($\mathcal{T}_{\mathcal{T}406}$)

$$\begin{aligned}
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{\subset\}\hat{b}^1) \subseteq (Da^1 \subseteq \{\supset\}D\hat{b}^1)) \\
& \quad (A_{S30})(\text{Axiom of Dimensional Proper Sub-entities}) \\
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD)(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 = \hat{b}^1) \subseteq (Da^1 = D\hat{b}^1)) \\
& \quad (T_{7404})(\text{Eqn2}) \\
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E). \\
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq \\
& ((a^1 \subseteq \{\subset\}\hat{b}^1) \subseteq (Da^1 \subseteq \{\supset\}D\hat{b}^1)).((a^1 = \hat{b}^1) \subseteq (Da^1 = D\hat{b}^1)) \quad (T_{669})(\text{Combination of Eqns1\&2}) \\
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq \\
& ((a^1 \subseteq \{\subset\}\hat{b}^1) \subseteq (Da^1 \subseteq \{\supset\}D\hat{b}^1)).((a^1 = \hat{b}^1) \subseteq (Da^1 = D\hat{b}^1)) \quad (T_{667})(\text{Conjunction Tautology}) \\
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq \\
& ((a^1 \subseteq \{\subset\}\hat{b}^1) + (a^1 = \hat{b}^1) \subseteq (Da^1 \subseteq \{\supset\}D\hat{b}^1) + (Da^1 = D\hat{b}^1)) \quad (A_{MX27})(\text{Combination}) \\
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq \\
& ((a^1 \subseteq \{\subset\}\hat{b}^1 + \hat{b}^1) \subseteq (Da^1 \subseteq \{\supset\}D\hat{b}^1 + D\hat{b}^1)) \quad (T_{526}) \\
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq \\
& ((a^1 \subseteq \{\subset\}\hat{b}^1 + \{=\}\hat{b}^1) \subseteq (Da^1 \subseteq \{\supset\}D\hat{b}^1 + \{=\}D\hat{b}^1)) \quad (D_{A144})(\text{Definition of the Reflexive Relationship Set}) \\
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq \\
& ((a^1 \subseteq (\{\subset\} + \{=\})\hat{b}^1) \subseteq (Da^1 \subseteq (\{\supset\} + \{=\})D\hat{b}^1)) \quad (R_{002c})(\text{Distribution of } + \text{ over } *) \\
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq \\
& ((a^1 \subseteq \{\subseteq\}\hat{b}^1) \subseteq (Da^1 \subseteq \{\supseteq\}D\hat{b}^1)) \quad (D_{A147})(\text{Disjunction with } \{=\})
\end{aligned}$$

$$((\chi E)^1 \subseteq D_D) \subseteq ({}^vS\chi a. {}^vQ\chi a \subseteq \{=\} + \{<\} + \{>\})$$

(T_{T457})

Proof of (T_{T457})

$$\begin{array}{ll}
((\mathbf{X}\mathbf{E})^1 \subseteq \mathbf{D}_b)(\mathbf{X}\mathbf{a} \neq \emptyset)(\mathbf{X}\mathbf{a} \neq (\mathbf{X}\mathbf{a})^1) \subseteq ((\mathbf{X}\mathbf{a})^1 \subseteq \mathbf{S}_b \mathbf{S}_o) & (\mathbf{T}_{025b})(\text{Eqn1}) \\
((\mathbf{X}\mathbf{a})^1 \subseteq \mathbf{S}_b \mathbf{S}_o) \subseteq (\forall \mathbf{S}\mathbf{X}\mathbf{a} \cdot \forall \mathbf{Q}\mathbf{X}\mathbf{a} \subseteq \{=\} + \{<\} + \{>\}) & (\mathbf{T}_{722})(\text{Eqn2}) \\
((\mathbf{X}\mathbf{E})^1 \subseteq \mathbf{D}_b)(\mathbf{X}\mathbf{a} \neq \emptyset)(\mathbf{X}\mathbf{a} \neq (\mathbf{X}\mathbf{a})^1) \subseteq (\forall \mathbf{S}\mathbf{X}\mathbf{a} \cdot \forall \mathbf{Q}\mathbf{X}\mathbf{a} \subseteq \{=\} + \{<\} + \{>\}) & (\mathbf{T}_{139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\
(\forall \mathbf{S}\mathbf{X}\mathbf{a} \cdot \{ \neq \} \cdot \forall \mathbf{Q}\mathbf{X}\mathbf{a} \neq \emptyset) \subseteq (\mathbf{X}\mathbf{a} \neq (\mathbf{X}\mathbf{a})^1)(\mathbf{X}\mathbf{a} \neq \emptyset) & (\mathbf{T}_{640}) \\
\sim((\mathbf{X}\mathbf{a} \neq (\mathbf{X}\mathbf{a})^1)(\mathbf{X}\mathbf{a} \neq \emptyset)) \subseteq (\forall \mathbf{S}\mathbf{X}\mathbf{a} \cdot \{ \neq \} \cdot \forall \mathbf{Q}\mathbf{X}\mathbf{a} = \emptyset) & (\mathbf{T}_{012})(\text{Transposition}) \\
(\mathbf{X}\mathbf{a} = (\mathbf{X}\mathbf{a})^1) + (\mathbf{X}\mathbf{a} = \emptyset) \subseteq (\forall \mathbf{S}\mathbf{X}\mathbf{a} \cdot \{ \neq \} \cdot \forall \mathbf{Q}\mathbf{X}\mathbf{a} = \emptyset) & (\mathbf{T}_{025})(\text{DeMorgan's Law}) \\
(\mathbf{X}\mathbf{a} = (\mathbf{X}\mathbf{a})^1) \subseteq (\forall \mathbf{S}\mathbf{X}\mathbf{a} \cdot \{ \neq \} \cdot \forall \mathbf{Q}\mathbf{X}\mathbf{a} = \emptyset) & (\mathbf{T}_{072})(\text{Single Premise Syllogism}) \\
(\mathbf{X}\mathbf{a} = (\mathbf{X}\mathbf{a})^1) \subseteq (\forall \mathbf{S}\mathbf{X}\mathbf{a} \cdot \forall \mathbf{Q}\mathbf{X}\mathbf{a} \subseteq \{=\}) & (\mathbf{T}_{735})(\subseteq \text{Existential Form})
\end{array}$$

$$\begin{aligned}
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} \neq \emptyset)(\mathcal{X}\mathcal{a} = (\mathcal{X}\mathcal{a})^1) \subseteq (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\}) \quad (\mathcal{T}_{M089})(\text{Single Premise Syllogism})(\text{Eqn4}) \\
& \{=\} \subseteq \{=\} + \{<\} + \{>\} \quad (\mathcal{T}_{M083})(\text{Conjunction Elimination})(\text{Eqn5}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} \neq \emptyset)(\mathcal{X}\mathcal{a} = (\mathcal{X}\mathcal{a})^1) \subseteq (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) \\
& \quad (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns4\&5})(\text{Eqn6}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} \neq \emptyset)(\mathcal{X}\mathcal{a} \neq (\mathcal{X}\mathcal{a})^1) + ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} \neq \emptyset)(\mathcal{X}\mathcal{a} = (\mathcal{X}\mathcal{a})^1) \subseteq \\
& (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) + (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) \\
& \quad (\mathcal{A}_{MX27})(\text{Combination of Eqns3\&6}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} \neq \emptyset)(\mathcal{X}\mathcal{a} \neq (\mathcal{X}\mathcal{a})^1) + ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} \neq \emptyset)(\mathcal{X}\mathcal{a} = (\mathcal{X}\mathcal{a})^1) \subseteq \\
& (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) \quad (\mathcal{A}_{MX01})(\text{Disjunction Tautology}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} \neq \emptyset)((\mathcal{X}\mathcal{a} \neq (\mathcal{X}\mathcal{a})^1) + (\mathcal{X}\mathcal{a} = (\mathcal{X}\mathcal{a})^1)) \subseteq (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) \\
& \quad (\mathcal{T}_{M407})(\text{Distribution of } . \text{ over } +) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} \neq \emptyset)\mathcal{E} \subseteq (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) \quad (\mathcal{T}_{M671})(\text{Tertium non Datur}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} \neq \emptyset) \subseteq (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) \quad (\mathcal{A}_{MX17})(\text{Universal Set Conjunction Axiom})(\text{Eqn7}) \\
& \text{Let } \mathcal{X}\mathcal{a} = \emptyset \\
& \forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} = \emptyset \quad (\mathcal{T}_{S238})(\text{Eqn8}) \\
& \emptyset \subseteq \{=\} + \{<\} + \{>\} \quad (\mathcal{T}_{M670})(\text{Null Set Theorem})(\text{Eqn9}) \\
& \forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\} \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn8 into Eqn9}) \\
& \text{Thus } (\mathcal{X}\mathcal{a} = \emptyset) \subseteq (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) \quad (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} = \emptyset) \subseteq (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) \quad (\mathcal{T}_{M089})(\text{Single Premise Syllogism})(\text{Eqn10}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} \neq \emptyset) + ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} = \emptyset) \subseteq \\
& (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) + (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) \\
& \quad (\mathcal{A}_{MX27})(\text{Combination of Eqns7\&10}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} \neq \emptyset) + ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}\mathcal{a} = \emptyset) \subseteq (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) \\
& \quad (\mathcal{A}_{MX01})(\text{Disjunction Tautology}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)((\mathcal{X}\mathcal{a} \neq \emptyset) + (\mathcal{X}\mathcal{a} = \emptyset)) \subseteq (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) \\
& \quad (\mathcal{T}_{M407})(\text{Distribution of } . \text{ over } +) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)\mathcal{E} \subseteq (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) \quad (\mathcal{T}_{M671})(\text{Tertium non Datur}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) \subseteq (\forall \mathcal{S}\mathcal{X}\mathcal{a}.\forall \mathcal{Q}\mathcal{X}\mathcal{a} \subseteq \{=\} + \{<\} + \{>\}) \quad (\mathcal{A}_{MX17})(\text{Universal Set Conjunction Axiom})
\end{aligned}$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 + \mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq ((\{\supseteq\}\mathcal{a}^1.\{\supseteq\}\mathcal{b}^1 = \emptyset) \subseteq (\{\subseteq\}^{\forall}\mathcal{X}\mathcal{a}^1.\{\subseteq\}^{\forall}\mathcal{X}\mathcal{b}^1 = \emptyset)) \quad (\mathcal{T}_{7300})$$

Proof of ($\mathcal{T}_{7300}$)

$$\begin{aligned}
& \text{Let } (\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A \\
& \text{Let } \mathcal{a}^1 + \mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E} \\
& \mathcal{a}^1 + \mathcal{b}^1 \subseteq \mathcal{Q}^{\forall}\mathcal{X} \quad (\mathcal{T}_{R211}) \\
& (\mathcal{a}^1 \subseteq \mathcal{Q}^{\forall}\mathcal{X})(\mathcal{b}^1 \subseteq \mathcal{Q}^{\forall}\mathcal{X}) \quad (\mathcal{T}_{M131b})(\text{Conjunction of Sufficiencies}) \\
& \text{Let } \{\supseteq\}\mathcal{a}^1.\{\supseteq\}\mathcal{b}^1 = \emptyset \\
& \forall \mathcal{X}(\{\supseteq\}\mathcal{a}^1.\{\supseteq\}\mathcal{b}^1) = \emptyset \quad (\mathcal{T}_{S304}) \\
& \forall \mathcal{X}\{\supseteq\}\mathcal{a}^1.\forall \mathcal{X}\{\supseteq\}\mathcal{b}^1 = \emptyset \quad (\mathcal{A}_{R35})(\text{One-to-One Relationship Distribution Axiom}) \\
& \{\subseteq\}^{\forall}\mathcal{X}\mathcal{a}^1.\{\subseteq\}^{\forall}\mathcal{X}\mathcal{b}^1 = \emptyset \quad (\mathcal{A}_{S07}) \\
& \text{Thus } ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 + \mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E})(\{\supseteq\}\mathcal{a}^1.\{\supseteq\}\mathcal{b}^1 = \emptyset) \subseteq (\{\subseteq\}^{\forall}\mathcal{X}\mathcal{a}^1.\{\subseteq\}^{\forall}\mathcal{X}\mathcal{b}^1 = \emptyset) \\
& \quad (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 + \mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq ((\{\supseteq\}\mathcal{a}^1.\{\supseteq\}\mathcal{b}^1 = \emptyset) \subseteq (\{\subseteq\}^{\forall}\mathcal{X}\mathcal{a}^1.\{\subseteq\}^{\forall}\mathcal{X}\mathcal{b}^1 = \emptyset)) \\
& \quad (\mathcal{A}_{MX35})(\text{Exportation/Importation for truth sets})
\end{aligned}$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\{\supseteq\}\mathcal{X}\mathcal{a}^1.\{\supseteq\}\mathcal{X}\mathcal{b}^1 = \emptyset) \subseteq (\{\subseteq\}\mathcal{a}^1.\{\subseteq\}\mathcal{b}^1 = \emptyset)) \quad (\mathcal{T}_{7301})$$

Proof of ($\mathcal{T}_{7301}$)

$$\begin{aligned}
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 + \mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq ((\{\supseteq\}\mathcal{a}^1.\{\supseteq\}\mathcal{b}^1 = \emptyset) \subseteq (\{\subseteq\}^{\forall}\mathcal{X}\mathcal{a}^1.\{\subseteq\}^{\forall}\mathcal{X}\mathcal{b}^1 = \emptyset)) \\
& \quad (\mathcal{T}_{7300})(\text{Eqn1}) \\
& \text{Let } \mathcal{a}^1 = \mathcal{X}\mathcal{a}^1 \quad (\text{Eqn2}) \\
& \text{Let } \mathcal{b}^1 = \mathcal{X}\mathcal{b}^1 \quad (\text{Eqn3}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{X}\mathcal{a}^1 + \mathcal{X}\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq ((\{\supseteq\}\mathcal{X}\mathcal{a}^1.\{\supseteq\}\mathcal{X}\mathcal{b}^1 = \emptyset) \subseteq (\{\subseteq\}^{\forall}\mathcal{X}\mathcal{X}\mathcal{a}^1.\{\subseteq\}^{\forall}\mathcal{X}\mathcal{X}\mathcal{b}^1 = \emptyset))
\end{aligned}$$

$a^1 + b^1 \subseteq E$	(A_{MX19})(Substitution of Eqns2&3 into Eqn1)(Eqn4)
$X(a^1 + b^1) \subseteq XE$	(T_{M744})(Universal Set Theorem)
$Xa^1 + Xb^1 \subseteq XE$	(T_{R196})(Relational Deduction)
$(Xa^1 + Xb^1 \subseteq XE) = E$	(T_{R002b})(Distribution of * over +)
$((X E)^1 \subseteq D_A + P_A) E = E \subseteq ((\{ \subseteq \} X a^1 . \{ \supseteq \} X b^1 = \emptyset) \subseteq (\{ \subseteq \}^V X X a^1 . \{ \subseteq \}^V X X b^1 = \emptyset))$	(A_{MX16})(Order Axiom)(Eqn5)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((\{ \subseteq \} X a^1 . \{ \supseteq \} X b^1 = \emptyset) \subseteq (\{ \subseteq \}^V X X a^1 . \{ \subseteq \}^V X X b^1 = \emptyset))$	(A_{MX19})(Substitution of Eqn5 into Eqn4)
$((X E)^1 \subseteq D_A + P_A) \subseteq (QX = E)$	(A_{MX11})(Universal Set Conjunction Axiom)(Eqn6)
$a^1 \subseteq E$	(A_{S36})(Axiom of Universal Mapping)(Eqn7)
$((X E)^1 \subseteq D_A + P_A) \subseteq (a^1 \subseteq QX)$	(T_{M744})(Universal Set Theorem)(Eqn8)
$(a^1 \subseteq QX) \subseteq (a^1 \subseteq {}^V X X a^1)$	(A_{MX19})(Substitution of Eqn7 into Eqn8)(Eqn9)
$((X E)^1 \subseteq D_A + P_A) \subseteq (a^1 \subseteq {}^V X X a^1)$	(T_{R526})(Eqn10)
$((X E)^1 \subseteq D_A + P_A) \subseteq (\{ \subseteq \} a^1 \subseteq \{ \subseteq \}^V X X a^1)$	(T_{M139})(Hypothetical Syllogism Eqns9&10)
$((X E)^1 \subseteq D_A + P_A) \subseteq (\{ \subseteq \} b^1 \subseteq \{ \subseteq \}^V X X b^1)$	(T_{R196})(Relational Deduction)(Eqn11)
$((X E)^1 \subseteq D_A + P_A) \subseteq (\{ \subseteq \} a^1 . \{ \subseteq \} b^1 \subseteq \{ \subseteq \}^V X X a^1 . \{ \subseteq \}^V X X b^1)$	Resymbolising (Eqn12)
$((X E)^1 \subseteq D_A + P_A) \subseteq (\{ \subseteq \} a^1 . \{ \subseteq \} b^1 \subseteq \{ \subseteq \}^V X X a^1 . \{ \subseteq \}^V X X b^1) = E$	(T_{M669})(Combination of Eqns11&12)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((\{ \subseteq \} a^1 . \{ \subseteq \} b^1 \subseteq \{ \subseteq \}^V X X a^1 . \{ \subseteq \}^V X X b^1) = E)$	(A_{MX16})(Order Axiom)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((\{ \subseteq \} a^1 . \{ \subseteq \} b^1 \subseteq \{ \subseteq \}^V X X a^1 . \{ \subseteq \}^V X X b^1) = E)$	(T_{M094})(Probabilistic Rule)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((\{ \subseteq \}^V X X a^1 . \{ \subseteq \}^V X X b^1 = \emptyset) \subseteq (\{ \subseteq \} a^1 . \{ \subseteq \} b^1 = \emptyset))$	(T_{M097})(Probabilistic Rule)(Eqn13)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((\{ \subseteq \} X a^1 . \{ \supseteq \} X b^1 = \emptyset) \subseteq (\{ \subseteq \} a^1 . \{ \subseteq \} b^1 = \emptyset))$	(T_{M139})(Hypothetical Syllogism Eqns6&13)

$$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((a \subseteq X E) = (\{ \subseteq \} a \subseteq X E)) \quad (T_{T304})$$

$$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((a \subseteq X E) = (\{ \supseteq \} a \subseteq X E)) \quad (T_{T315})$$

Example Proof of (T_{T304})

$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((a \subseteq X E) = (\{ \subseteq \} a \subseteq X E))$	(T_{S801})(Eqn1)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq E$	(T_{M744})(Universal Set Theorem)(Eqn2)
$(a \subseteq X E) = (a \subseteq X E)$	(D_{M00})(Definition of =)
$(a \subseteq X E) = (\{ = \} a \subseteq X E)$	(T_{A080})
$((a \subseteq X E) = (\{ = \} a \subseteq X E)) = E$	(A_{MX16})(Order Axiom)(Eqn3)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((a \subseteq X E) = (\{ = \} a \subseteq X E))$	(A_{MX19})(Substitution of Eqn3 into Eqn2)(Eqn4)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((a \subseteq X E) = (\{ \subseteq \} a \subseteq X E)) \subseteq ((a \subseteq X E) = (\{ = \} a \subseteq X E))$	(T_{M669})(Combination of Eqns1&4)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((a \subseteq X E) = (\{ \subseteq \} a \subseteq X E)) \subseteq ((a \subseteq X E) = (\{ = \} a \subseteq X E))$	(T_{M667})(Conjunction Tautology)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((a \subseteq X E) = (\{ \subseteq \} a \subseteq X E)) \subseteq ((a \subseteq X E) = (\{ \subseteq \} a \subseteq X E)) \subseteq ((a \subseteq X E) = (\{ = \} a \subseteq X E))$	(T_{M669})(Combination)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((a \subseteq X E) = (\{ \subseteq \} a \subseteq X E)) \subseteq ((a \subseteq X E) = (\{ = \} a \subseteq X E))$	(T_{M667})(Conjunction Tautology)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((a \subseteq X E) = (\{ \subseteq \} a + \{ = \} a \subseteq X E + X E))$	(A_{MX21})(Combination)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((a \subseteq X E) = (\{ \subseteq \} a + \{ = \} a \subseteq X E))$	(A_{MX01})(Disjunction Tautology)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((a \subseteq X E) = ((\{ \subseteq \} + \{ = \}) a \subseteq X E))$	(T_{R002b})(Distribution of * over +)
$((X E)^1 \subseteq D_A + P_A) = E \subseteq ((a \subseteq X E) = (\{ \subseteq \} a \subseteq X E))$	(D_{A141})(Disjunction with {=})

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((\{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}b^1 \neq \emptyset) \subseteq (\{\supseteq\}a^1. \{\supseteq\}\mathcal{X}b^1 \neq \emptyset)) \quad (\mathcal{T}_{7306})$$

Proof of ($\mathcal{T}_{7306}$)

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a^1 + b^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((\{\supseteq\}a^1. \{\supseteq\}b^1 = \emptyset) \subseteq (\{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}^V \mathcal{X}b^1 = \emptyset)) \quad (\mathcal{T}_{7300})(\text{Eqn1})$$

$$\text{Let } b^1 = \mathcal{X}b^1 \quad (\text{Eqn2})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a^1 + \mathcal{X}b^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((\{\supseteq\}a^1. \{\supseteq\}\mathcal{X}b^1 = \emptyset) \subseteq (\{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}^V \mathcal{X}\mathcal{X}b^1 = \emptyset)) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a^1 \subseteq \mathcal{X}\mathcal{E})(\mathcal{X}b^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((\{\supseteq\}a^1. \{\supseteq\}\mathcal{X}b^1 = \emptyset) \subseteq (\{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}^V \mathcal{X}\mathcal{X}b^1 = \emptyset)) \quad (\mathcal{T}_{M131b})(\text{Conjunction of Sufficiencies})(\text{Eqn3})$$

$$b^1 \subseteq \mathcal{E} \quad (\mathcal{T}_{M744})(\text{Universal Set Theorem})$$

$$\mathcal{X}b^1 \subseteq \mathcal{X}\mathcal{E} \quad (\mathcal{T}_{R196})(\text{Relational Deduction})$$

$$(\mathcal{X}b^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E} \quad (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn4})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a^1 \subseteq \mathcal{X}\mathcal{E})\mathcal{E} = \mathcal{E}) \subseteq (((\{\supseteq\}a^1. \{\supseteq\}\mathcal{X}b^1 = \emptyset) \subseteq (\{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}^V \mathcal{X}\mathcal{X}b^1 = \emptyset)) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn4 into Eqn3})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((\{\supseteq\}a^1. \{\supseteq\}\mathcal{X}b^1 = \emptyset) \subseteq (\{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}^V \mathcal{X}\mathcal{X}b^1 = \emptyset)) \quad (\mathcal{A}_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn5})$$

$$(b^1 \subseteq \mathcal{Q}\mathcal{X}) \subseteq (b^1 \subseteq {}^V \mathcal{X}\mathcal{X}b^1) \quad (\mathcal{T}_{R526})(\text{Eqn6})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\mathcal{Q}\mathcal{X} = \mathcal{E}) \quad (\mathcal{A}_{S36})(\text{Axiom of Universal Mapping})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\mathcal{Q}\mathcal{X} = \mathcal{E})) = \mathcal{E} \quad (\mathcal{A}_{MX16})(\text{Order Axiom})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq (\mathcal{Q}\mathcal{X} = \mathcal{E}) = \mathcal{E} \quad (\mathcal{T}_{M094})(\text{Probabilistic Rule})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((b^1 \subseteq \mathcal{Q}\mathcal{X}) = (b^1 \subseteq \mathcal{E})) \quad (\mathcal{T}_{M272})(\text{Deductive Rule})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((b^1 \subseteq \mathcal{Q}\mathcal{X}) = \mathcal{E}) \quad (\mathcal{T}_{M001})(\text{Universality of } (\mathcal{T}_{M744}) \text{ Universal Set Theorem})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (b^1 \subseteq \mathcal{Q}\mathcal{X})) = \mathcal{E} \quad (\mathcal{T}_{M094})(\text{Probabilistic Rule})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (b^1 \subseteq \mathcal{Q}\mathcal{X}) \quad (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn7})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (b^1 \subseteq {}^V \mathcal{X}\mathcal{X}b^1) \quad (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns6\&7})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\{\subseteq\}b^1 \subseteq \{\subseteq\}^V \mathcal{X}\mathcal{X}b^1) \quad (\mathcal{T}_{R196})(\text{Relational Deduction})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}b^1 \subseteq \{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}^V \mathcal{X}\mathcal{X}b^1) \quad (\mathcal{T}_{M029})(\text{Deduction})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}b^1 \subseteq \{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}^V \mathcal{X}\mathcal{X}b^1)) = \mathcal{E} \quad (\mathcal{A}_{MX16})(\text{Order Axiom})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}b^1 \subseteq \{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}^V \mathcal{X}\mathcal{X}b^1) = \mathcal{E}) \quad (\mathcal{T}_{M094})(\text{Probabilistic Rule})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}^V \mathcal{X}\mathcal{X}b^1 = \emptyset) \subseteq (\{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}b^1 = \emptyset)) \quad (\mathcal{T}_{M097})(\text{Probabilistic Rule})(\text{Eqn8})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((\{\supseteq\}a^1. \{\supseteq\}\mathcal{X}b^1 = \emptyset) \subseteq (\{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}b^1 = \emptyset)) \quad (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns5\&8})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((\{\subseteq\}^V \mathcal{X}a^1. \{\subseteq\}b^1 \neq \emptyset) \subseteq (\{\supseteq\}a^1. \{\supseteq\}\mathcal{X}b^1 \neq \emptyset)) \quad (\mathcal{T}_{M012})(\text{Transposition})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) = \mathcal{E}) \subseteq ((\alpha^1 \subseteq (\mathcal{B} + \mathcal{C})\mathcal{X}\mathcal{E}) = (\alpha^1 \subseteq \mathcal{X}\mathcal{E}. \{\subseteq\}^* \mathcal{X}\mathcal{E})) \quad (\mathcal{T}_{7309})$$

Proof of ($\mathcal{T}_{7309}$)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) \subseteq (\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}. \{\subseteq\}^* \mathcal{X}\mathcal{E}) \quad (\mathcal{T}_{T166})(\text{Eqn1})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_D) \subseteq (\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}. \{\subseteq\}^* \mathcal{X}\mathcal{E}) \quad (\mathcal{T}_{T172})(\text{Eqn2})$$

$$(\alpha^1 \subseteq \mathcal{S}_D \mathcal{S}_O) \subseteq (\mathcal{B}\alpha = \alpha) \quad (\mathcal{T}_{S893})(\text{Eqn3})$$

$$(\alpha^1 \subseteq \mathcal{S}_D \mathcal{S}_O) \subseteq (\mathcal{C}\alpha = \alpha) \quad (\mathcal{T}_{S894})(\text{Eqn4})$$

$$(\alpha^1 \subseteq \mathcal{S}_D \mathcal{S}_O)(\alpha^1 \subseteq \mathcal{S}_D \mathcal{S}_O) \subseteq (\mathcal{B}\alpha = \alpha)(\mathcal{C}\alpha = \alpha) \quad (\mathcal{T}_{M669})(\text{Combination of Eqns3\&4})$$

$$(\alpha^1 \subseteq \mathcal{S}_D \mathcal{S}_O) \subseteq (\mathcal{B}\alpha = \alpha)(\mathcal{C}\alpha = \alpha) \quad (\mathcal{T}_{M667})(\text{Conjunction Tautology})$$

$$(\alpha^1 \subseteq \mathcal{S}_D \mathcal{S}_O) \subseteq (\mathcal{B}\alpha + \mathcal{C}\alpha = \alpha + \alpha) \quad (\mathcal{T}_{M230})(\text{Combination})$$

$$(\alpha^1 \subseteq \mathcal{S}_D \mathcal{S}_O) \subseteq (\mathcal{B}\alpha + \mathcal{C}\alpha = \alpha) \quad (\mathcal{A}_{MX01})(\text{Disjunction Tautology})(\text{Eqn5})$$

$$\text{Let } \alpha = \mathcal{X}\mathcal{E} \quad (\text{Eqn6})$$

$$\text{Where } (\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_D$$

$$\begin{aligned}
& ((\mathcal{X}\mathcal{E})^1 \subseteq S_D S_O) \subseteq (\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}) & (\mathbf{A}_{MX19}) (\text{Substitution of Eqn6 into Eqn5}) \\
& \text{Thus } ((\mathcal{X}\mathcal{E})^1 \subseteq D_D)((\mathcal{X}\mathcal{E})^1 \subseteq S_D S_O) \subseteq (\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}) & (\mathbf{A}_{MX02}) (\text{Conditional Introduction})(\text{Eqn7}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_D) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq S_D S_O) & (\mathbf{T}_{S723}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_D) = ((\mathcal{X}\mathcal{E})^1 \subseteq D_D)((\mathcal{X}\mathcal{E})^1 \subseteq S_D S_O) & (\mathbf{A}_{MX44}) (\subseteq \text{Equality Form})(\text{Eqn8}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_D) \subseteq (\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}) & (\mathbf{A}_{MX19}) (\text{Substitution of Eqn8 into Eqn7})(\text{Eqn9}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_D) \subseteq (\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) & (\mathbf{A}_{MX19}) (\text{Substitution of Eqn2 into Eqn9})(\text{Eqn10}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_M) + ((\mathcal{X}\mathcal{E})^1 \subseteq D_D) \subseteq & \\
& (\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) + (\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) & (\mathbf{A}_{MX27}) (\text{Combination of Eqns1\&10}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_M) + ((\mathcal{X}\mathcal{E})^1 \subseteq D_D) \subseteq (\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) & (\mathbf{A}_{MX01}) (\text{Disjunction Tautology}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_M + D_D) \subseteq (\mathcal{B}\mathcal{X}\mathcal{E} + \mathcal{C}\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) & (\mathbf{T}_{M526}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_M + D_D) \subseteq ((\mathcal{B} + \mathcal{C})\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) & (\mathbf{T}_{R002b}) (\text{Distribution of } * \text{ over } +) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq D_M + D_D) \subseteq ((\mathcal{B} + \mathcal{C})\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E})) = \mathbf{E} & (\mathbf{A}_{MX16}) (\text{Order Axiom}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq D_M + D_D) = \mathbf{E}) \subseteq (((\mathcal{B} + \mathcal{C})\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) = \mathbf{E}) & (\mathbf{T}_{M094}) (\text{Probabilistic Rule}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq D_M + D_D) = \mathbf{E}) \subseteq ((\alpha^1 \subseteq (\mathcal{B} + \mathcal{C})\mathcal{X}\mathcal{E}) = (\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E})) & (\mathbf{T}_{M272}) (\text{Deductive Rule}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq D_A) = \mathbf{E}) \subseteq ((\alpha^1 \subseteq (\mathcal{B} + \mathcal{C})\mathcal{X}\mathcal{E}) = (\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E})) & (\mathbf{T}_{T815})
\end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq (\{\subseteq\}^V \mathcal{X}\alpha^1 = {}^V \mathcal{X}\alpha^1) \quad (\mathbf{T}_{T310})$$

Proof of ($\mathbf{T}_{T310}$)

$$\begin{aligned}
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) \subseteq ({}^V \mathcal{X}\{\supseteq\}\alpha^1 = \{\subseteq\}^V \mathcal{X}\alpha^1) & (\mathbf{A}_{S07}) (\text{Dimensional Components Axiom})(\text{Eqn1}) \\
& \text{Let } \alpha^1 = \alpha^1 & (\text{Eqn2}) \\
& \text{where } \alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E} & \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) \subseteq ({}^V \mathcal{X}\{\supseteq\}\alpha^1 = \{\subseteq\}^V \mathcal{X}\alpha^1) & (\mathbf{A}_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\
& \text{Thus } ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq ({}^V \mathcal{X}\{\supseteq\}\alpha^1 = \{\subseteq\}^V \mathcal{X}\alpha^1) & (\mathbf{A}_{MX02}) (\text{Conditional Introduction})(\text{Eqn3}) \\
& D_M \subseteq D_A & (\mathbf{T}_{T911}) \\
& D_M \subseteq D_A + P_A & (\mathbf{T}_{M082}) (\text{Disjunction Introduction}) \\
& (D_M \subseteq D_A + P_A) = \mathbf{E} & (\mathbf{A}_{MX16}) (\text{Order Axiom}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) & (\mathbf{T}_{M099}) (\text{Entity Mode / Set Mode Rule}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) & (\mathbf{T}_{M029}) (\text{Deduction})(\text{Eqn4}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq ({}^V \mathcal{X}\{\supseteq\}\alpha^1 = \{\subseteq\}^V \mathcal{X}\alpha^1) & (\mathbf{T}_{M139}) (\text{Hypothetical Syllogism Eqns3\&4})(\text{Eqn5}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_M) \subseteq ((\mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E})^1 \subseteq S_P S_O) & (\mathbf{T}_{T303}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq ((\mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E})^1 \subseteq S_P S_O)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) & (\mathbf{T}_{M029}) (\text{Deduction})(\text{Eqn6}) \\
& ((\mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E})^1 \subseteq S_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq (\alpha^1 = \{\supseteq\}*\alpha^1) & (\mathbf{T}_{T128}) (\text{Eqn7}) \\
& S_P \subseteq S_D & (\mathbf{A}_{R53}) (\text{Axiom of Points}) \\
& S_P S_O \subseteq S_D & (\mathbf{T}_{M089}) (\text{Single Premise Syllogism}) \\
& (S_P S_O \subseteq S_D) = \mathbf{E} & (\mathbf{A}_{MX16}) (\text{Order Axiom}) \\
& ((\mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E})^1 \subseteq S_P S_O) \subseteq ((\mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E})^1 \subseteq S_D) & (\mathbf{T}_{M099}) (\text{Entity Mode / Set Mode Rule}) \\
& ((\mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E})^1 \subseteq S_P S_O)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq ((\mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E})^1 \subseteq S_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) & (\mathbf{T}_{M029}) (\text{Deduction})(\text{Eqn8}) \\
& ((\mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E})^1 \subseteq S_P S_O)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq (\alpha^1 = \{\supseteq\}*\alpha^1) & (\mathbf{T}_{M139}) (\text{Hypothetical Syllogism Eqns7\&8})(\text{Eqn9}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq (\alpha^1 = \{\supseteq\}*\alpha^1) & (\mathbf{T}_{M139}) (\text{Hypothetical Syllogism Eqns6\&9})(\text{Eqn10}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}*\mathcal{X}\mathcal{E}) \subseteq ({}^V \mathcal{X}\alpha^1 = \{\subseteq\}^V \mathcal{X}\alpha^1) & (\mathbf{A}_{MX19}) (\text{Substitution of Eqn10 into Eqn5})
\end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_A)((\mathcal{X}\mathcal{A})^1 = \mathcal{X}\mathcal{A}^1) \subseteq (\mathcal{B}\mathcal{X}\mathcal{A}^1 = \mathcal{X}\mathcal{A}.\{\supseteq\}*\mathcal{X}\mathcal{A})(\{\subseteq\}*\mathcal{X}\mathcal{A}) \quad (\mathbf{T}_{R907})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)((\mathcal{X}a)^1 = \mathcal{X}a^1) \subseteq (C\mathcal{X}a^1 = \mathcal{X}a.(\{<\}_{\mathcal{X}}\mathcal{X}a)(\{<\}_{\mathcal{X}}\mathcal{X}a)) \quad (T_{R908})$$

Example Proof of (T_{R907})

$$\begin{aligned} ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) &\subseteq (\mathcal{B}(\mathcal{X}a)^1 = \mathcal{X}a.(\{>\}_{\mathcal{X}}\mathcal{X}a)(\{<\}_{\mathcal{X}}\mathcal{X}a)) && (T_{T209})(\text{Eqn1}) \\ \text{Let } (\mathcal{X}a)^1 &= \mathcal{X}a^1 && (\text{Eqn2}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) &\subseteq (\mathcal{B}\mathcal{X}a^1 = \mathcal{X}a.(\{>\}_{\mathcal{X}}\mathcal{X}a)(\{<\}_{\mathcal{X}}\mathcal{X}a)) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \text{Thus } (((\mathcal{X}a)^1 = \mathcal{X}a^1) = E) &\subseteq (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) \subseteq (\mathcal{B}\mathcal{X}a^1 = \mathcal{X}a.(\{>\}_{\mathcal{X}}\mathcal{X}a)(\{<\}_{\mathcal{X}}\mathcal{X}a))) && (A_{MX02})(\text{Conditional Introduction}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)((\mathcal{X}a)^1 = \mathcal{X}a^1) &\subseteq (\mathcal{B}\mathcal{X}a^1 = \mathcal{X}a.(\{>\}_{\mathcal{X}}\mathcal{X}a)(\{<\}_{\mathcal{X}}\mathcal{X}a)) && (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq (\mathcal{B} + C)\mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{X} \forall \mathcal{B}\alpha^1 = (\forall \mathcal{X} \{\subseteq\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{<\} * \alpha^1)) \quad (T_{R909})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq (\mathcal{B} + C)\mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{X} \forall C\alpha^1 = (\forall \mathcal{X} \{\subseteq\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{>\} * \alpha^1)) \quad (T_{R910})$$

Example Proof of (T_{R909})

$$\begin{aligned} ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}. \{\subseteq\} * \mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{B}\alpha^1 = (\{\subseteq\} * \alpha^1).(\{\subseteq\} * \{<\} * \alpha^1)) && (T_{T218})(\text{Eqn1}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}. \{\subseteq\} * \mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{X} \forall \mathcal{B}\alpha^1 = \forall \mathcal{X} ((\{\subseteq\} * \alpha^1).(\{\subseteq\} * \{<\} * \alpha^1))) && (T_{R522})(\text{Relational Deduction}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}. \{\subseteq\} * \mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{X} \forall \mathcal{B}\alpha^1 = (\forall \mathcal{X} \{\subseteq\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{<\} * \alpha^1)) && (A_{R35})(\text{One-to-One Relationship Distribution Axiom}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq (\mathcal{B} + C)\mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{X} \forall \mathcal{B}\alpha^1 = (\forall \mathcal{X} \{\subseteq\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{<\} * \alpha^1)) && (T_{T309}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq (\mathcal{B} + C)\mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{X} \forall \mathcal{B}\alpha^1. \forall \mathcal{X} \forall C\alpha^1 = (\forall \mathcal{X} \{\subseteq\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{<\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{>\} * \alpha^1)) \quad (T_{R911})$$

Proof of (T_{R911})

$$\begin{aligned} ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq (\mathcal{B} + C)\mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{X} \forall \mathcal{B}\alpha^1 = (\forall \mathcal{X} \{\subseteq\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{<\} * \alpha^1)) && (T_{R909})(\text{Eqn1}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq (\mathcal{B} + C)\mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{X} \forall C\alpha^1 = (\forall \mathcal{X} \{\subseteq\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{>\} * \alpha^1)) && (T_{R910})(\text{Eqn2}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq (\mathcal{B} + C)\mathcal{X}\mathcal{E}) &((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq (\mathcal{B} + C)\mathcal{X}\mathcal{E}) \subseteq && \\ (\forall \mathcal{X} \forall \mathcal{B}\alpha^1 = (\forall \mathcal{X} \{\subseteq\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{<\} * \alpha^1)) &(\forall \mathcal{X} \forall C\alpha^1 = (\forall \mathcal{X} \{\subseteq\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{>\} * \alpha^1)) && \\ &&& (T_{M669})(\text{Combination of Eqns1\&2}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq (\mathcal{B} + C)\mathcal{X}\mathcal{E}) &\subseteq && \\ (\forall \mathcal{X} \forall \mathcal{B}\alpha^1 = (\forall \mathcal{X} \{\subseteq\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{<\} * \alpha^1)) &(\forall \mathcal{X} \forall C\alpha^1 = (\forall \mathcal{X} \{\subseteq\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{>\} * \alpha^1)) && \\ &&& (T_{M667})(\text{Conjunction Tautology}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq (\mathcal{B} + C)\mathcal{X}\mathcal{E}) &\subseteq && \\ (\forall \mathcal{X} \forall \mathcal{B}\alpha^1. \forall \mathcal{X} \forall C\alpha^1 = (\forall \mathcal{X} \{\subseteq\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{<\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{>\} * \alpha^1)) &&& \\ &&& (T_{M231})(\text{Combination}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq (\mathcal{B} + C)\mathcal{X}\mathcal{E}) &\subseteq && \\ (\forall \mathcal{X} \forall \mathcal{B}\alpha^1. \forall \mathcal{X} \forall C\alpha^1 = (\forall \mathcal{X} \{\subseteq\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{<\} * \alpha^1).(\forall \mathcal{X} \{\subseteq\} * \{>\} * \alpha^1)) &&& \\ &&& (T_{M667})(\text{Conjunction Tautology}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq (\mathcal{B} + C)\mathcal{X}\mathcal{E}) \subseteq (\forall \mathcal{X} \forall \mathcal{B}\alpha^1. \forall \mathcal{X} \forall C\alpha^1 = \forall \mathcal{X} \alpha^1) \quad (T_{R912})$$

Proof of (T_{R912})

$$\begin{aligned} ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}. \{\subseteq\} * \mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{B}\alpha^1. \forall C\alpha^1 = \alpha^1) && (T_{T220}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}. \{\subseteq\} * \mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{X} (\forall \mathcal{B}\alpha^1. \forall C\alpha^1 = \forall \mathcal{X} \alpha^1)) && (T_{R522})(\text{Relational Deduction}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}. \{\subseteq\} * \mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{X} \forall \mathcal{B}\alpha^1. \forall \mathcal{X} \forall C\alpha^1 = \forall \mathcal{X} \alpha^1) && (A_{R35})(\text{One-to-One Relationship Distribution Axiom})(\text{Eqn1}) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) = E) &\subseteq ((\alpha^1 \subseteq (\mathcal{B} + C)\mathcal{X}\mathcal{E}) = (\alpha^1 \subseteq \mathcal{X}\mathcal{E}. \{\subseteq\} * \mathcal{X}\mathcal{E})) && (T_{T309})(\text{Eqn2}) \\ \mathcal{D}_M &\subseteq \mathcal{D}_A && (T_{T911}) \\ (\mathcal{D}_M \subseteq \mathcal{D}_A) &= E && (A_{MX16})(\text{Order Axiom}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) &\subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) && (T_{M099})(\text{Entity Mode / Set Mode Rule}) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A)) &= E && (A_{MX16})(\text{Order Axiom}) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) = E) &\subseteq (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A) = E) && (T_{M094})(\text{Probabilistic Rule})(\text{Eqn3}) \end{aligned}$$

$$\begin{aligned}
((\chi E)^1 \subseteq D_M) = E &\subseteq ((\alpha^1 \subseteq (B + C)\chi E) = (\alpha^1 \subseteq \chi E.\{\subseteq\}^* \chi E)) \\
&\quad (T_{M139})(\text{Hypothetical Syllogism Eqns3\&2})(\text{Eqn4}) \\
((\chi E)^1 \subseteq D_M)(\alpha^1 \subseteq (B + C)\chi E) &\subseteq (\forall \chi \forall B \alpha^1. \forall \chi \forall C \alpha^1 = \forall \chi \alpha^1) \\
&\quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn1})
\end{aligned}$$

$$D_A + P_A \subseteq S_A \quad (T_{T770})$$

Proof of (T_{T770})

$$\begin{aligned}
D_M + P_M &\subseteq S_M && (A_{S12})(\text{Axiom of Mixed Continuum Dimensions})(\text{Eqn1}) \\
D_D + P_D &\subseteq S_D && (A_{R97})(\text{Axiom of Discrete Dimensions})(\text{Eqn2}) \\
D_M + D_D + P_M + P_D &\subseteq S_M + S_D && (A_{MX21})(\text{Combination of Eqns1\&2}) \\
D_M + D_D + P_M + P_D &\subseteq S_A && (D_{S35})(\text{Definition of } S_A) \\
D_A + P_A &\subseteq S_A && (T_{T814}) \& (T_{T815})
\end{aligned}$$

$$(a^1 \subseteq D_M + P_M) \subseteq (a = \{\supset\}^* a^1) \quad (T_{T759})$$

Proof of (T_{T759})

$$\begin{aligned}
(a^1 \subseteq S_A) &\subseteq (a = \{\supset\}^* a^1) && (D_{S28})(\text{Definition of Reverse Abstraction for Members of } S_A) \\
(a^1 \subseteq S_M + S_D) &\subseteq (a = \{\supset\}^* a^1) && (D_{S35})(\text{Definition of } S_A) \\
(a^1 \subseteq S_M) + (a^1 \subseteq S_D) &\subseteq (a = \{\supset\}^* a^1) && (T_{M526}) \\
(a^1 \subseteq S_M) &\subseteq (a = \{\supset\}^* a^1) && (T_{M072})(\text{Single Premise Syllogism})(\text{Eqn1}) \\
D_M + P_M &\subseteq S_M && (A_{S12})(\text{Axiom of Mixed Continuum Dimensions}) \\
(D_M + P_M \subseteq S_M) &= E && (A_{MX16})(\text{Order Axiom}) \\
(a^1 \subseteq D_M + P_M) &\subseteq (a^1 \subseteq S_M) && (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn2}) \\
(a^1 \subseteq D_M + P_M) &\subseteq (a = \{\supset\}^* a^1) && (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})
\end{aligned}$$

$$((\chi E)^1 \subseteq D_M + P_M) \subseteq (\{\supset\}(\chi a)^1 = \chi a) \quad (T_{T774})$$

Proof of (T_{T774})

$$\begin{aligned}
(a^1 \subseteq D_M + P_M) &\subseteq (a = \{\supset\}^* a^1) && (T_{T759})(\text{Eqn1}) \\
\text{Let } a &= \chi a && (\text{Eqn2}) \\
a^1 &= (\chi a)^1 && (T_{M479})(\text{Eqn3}) \\
((\chi E)^1 \subseteq D_M + P_M) &\subseteq (\chi a = \{\supset\}^*(\chi a)^1) && (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})
\end{aligned}$$

$$((\chi E)^1 \subseteq D_M + P_M) \subseteq (\chi \{\subseteq\} a^1 = \{\supset\} \chi a^1) \quad (T_{T377})$$

Proof of (T_{T377})

$$\begin{aligned}
((\chi E)^1 \subseteq D_M + P_M) &= ((\chi a)^1 = \chi a^1) && (A_{S05})(\text{Physical Dimension Aggregation Axiom}) \\
((\chi E)^1 \subseteq D_M + P_M) &\subseteq (\{\supset\}(\chi a)^1 = \{\supset\} \chi a^1) && (T_{R522})(\text{Relational Deduction}) \\
((\chi E)^1 \subseteq D_M + P_M) &\subseteq (\chi a = \{\supset\} \chi a^1) && (T_{T774}) \\
((\chi E)^1 \subseteq D_M + P_M) &\subseteq (\chi \{\subseteq\} a^1 = \{\supset\} \chi a^1) && (A_{R02})(\text{Definition of Disaggregation for Continuum Sets})
\end{aligned}$$

$$(((\chi E)^1 \subseteq D_A + P_A)(a^1 \subseteq \chi E) = E) \subseteq ((\{\supset\} a^1.\{\supset\} \chi b^1 = \emptyset) \subseteq (\{\subseteq\}^* \chi a^1.\{\subseteq\} b^1 = \emptyset)) \quad (T_{T273})$$

Proof of (T_{T273})

$$\begin{aligned}
(((\chi E)^1 \subseteq D_A + P_A)(a^1 + b^1 \subseteq \chi E) = E) &\subseteq ((\{\supset\} a^1.\{\supset\} b^1 = \emptyset) \subseteq (\{\subseteq\}^* \chi a^1.\{\subseteq\}^* \chi b^1 = \emptyset)) \\
&\quad (T_{T300})(\text{Eqn1}) \\
\text{Let } b^1 &= \chi b^1 && (\text{Eqn2}) \\
(b^1 = \chi b^1) &= E && (A_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\
(((\chi E)^1 \subseteq D_A + P_A)(a^1 + \chi b^1 \subseteq \chi E) = E) &\subseteq ((\{\supset\} a^1.\{\supset\} \chi b^1 = \emptyset) \subseteq (\{\subseteq\}^* \chi a^1.\{\subseteq\}^* \chi \chi b^1 = \emptyset)) \\
&\quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn4}) \\
(b^1 \subseteq \chi b^1) &= (b^1 \subseteq \forall \chi b^1) && (T_{R519})(\text{Active/Passive Theorem})
\end{aligned}$$

$(\hat{b}^1 \subseteq \lambda b^1) + (\hat{b}^1 = \lambda b^1) = (b^1 \subseteq {}^v\lambda \hat{b}^1)$	(T_{M268})(Definition of $\subseteq$)
$(\hat{b}^1 = \lambda b^1) \subseteq (b^1 \subseteq {}^v\lambda \hat{b}^1)$	(T_{M072b})(Single Premise Syllogism)(Eqn5)
$E \subseteq (b^1 \subseteq {}^v\lambda \hat{b}^1)$	(A_{MX19})(Substitution of Eqn3 into Eqn5)
$E = (b^1 \subseteq {}^v\lambda \hat{b}^1)$	(T_{M275})(Supersets of the Universal Set)
$b^1 \subseteq {}^v\lambda \hat{b}^1$	(A_{MX16})(Order Axiom)(Eqn6)
$b^1 \subseteq {}^v\lambda \lambda b^1$	(A_{MX19})(Substitution of Eqn2 into Eqn6)
$\{\subseteq\}b^1 \subseteq \{\subseteq\}{}^v\lambda \lambda b^1$	(T_{R196})(Relational Deduction)
$\{\subseteq\}{}^v\lambda a^1.\{\subseteq\}b^1 \subseteq \{\subseteq\}{}^v\lambda a^1.\{\subseteq\}{}^v\lambda \lambda b^1$	(T_{M029})(Deduction)
$(\{\subseteq\}{}^v\lambda a^1.\{\subseteq\}b^1 \subseteq \{\subseteq\}{}^v\lambda a^1.\{\subseteq\}{}^v\lambda \lambda b^1) = E$	(A_{MX16})(Order Axiom)
$(\{\subseteq\}{}^v\lambda a^1.\{\subseteq\}{}^v\lambda \lambda b^1 = \emptyset) \subseteq (\{\subseteq\}{}^v\lambda a^1.\{\subseteq\}b^1 = \emptyset)$	(T_{M097})(Probabilistic Rule)(Eqn7)
$((\lambda E)^1 \subseteq D_A + P_A)(a^1 + \lambda b^1 \subseteq \lambda E) = E \subseteq ((\supseteq)a^1.\{\supseteq\}\lambda b^1 = \emptyset) \subseteq (\{\subseteq\}{}^v\lambda a^1.\{\subseteq\}b^1 = \emptyset)$	(T_{M139})(Hypothetical Syllogism Eqns4&7)
$((\lambda E)^1 \subseteq D_A + P_A)(a^1 \subseteq \lambda E)(\lambda b^1 \subseteq \lambda E) = E \subseteq ((\supseteq)a^1.\{\supseteq\}\lambda b^1 = \emptyset) \subseteq (\{\subseteq\}{}^v\lambda a^1.\{\subseteq\}b^1 = \emptyset)$	(T_{M131b})(Conjunction of Sufficiencies)(Eqn8)
$b^1 \subseteq E$	(T_{M744})(Universal Set Theorem)
$\lambda b^1 \subseteq \lambda E$	(T_{R196})(Relational Deduction)
$(\lambda b^1 \subseteq \lambda E) = E$	(A_{MX16})(Order Axiom)(Eqn9)
$((\lambda E)^1 \subseteq D_A + P_A)(a^1 \subseteq \lambda E)E = E \subseteq ((\supseteq)a^1.\{\supseteq\}\lambda b^1 = \emptyset) \subseteq (\{\subseteq\}{}^v\lambda a^1.\{\subseteq\}b^1 = \emptyset)$	(A_{MX19})(Substitution of Eqn9 into Eqn8)
$((\lambda E)^1 \subseteq D_A + P_A)(a^1 \subseteq \lambda E) = E \subseteq ((\supseteq)a^1.\{\supseteq\}\lambda b^1 = \emptyset) \subseteq (\{\subseteq\}{}^v\lambda a^1.\{\subseteq\}b^1 = \emptyset)$	(A_{MX17})(Universal Set Conjunction Axiom)

$$((\lambda E)^1 \subseteq D_D) \subseteq ({}^vS\lambda E.{}^vQ\lambda E \subseteq \{=\} + \{<\} + \{>\}) \quad (T_{454})$$

Proof of (T_{454})

$((\lambda E)^1 \subseteq D_D) \subseteq ({}^vS\lambda a.{}^vQ\lambda a \subseteq \{=\} + \{<\} + \{>\})$	(T_{457})(Eqn1)
Let $a = E$	(Eqn2)
$((\lambda E)^1 \subseteq D_D) \subseteq ({}^vS\lambda E.{}^vQ\lambda E \subseteq \{=\} + \{<\} + \{>\})$	(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$((\lambda E)^1 \subseteq D_D)(a^1 \subseteq \lambda E)(\hat{b}^1 \subseteq \lambda E)(\hat{b}^1 \neq \emptyset) \subseteq (a^1 \subseteq (\{>\} + \{<\} + \{=\}) * \hat{b}^1) \quad (T_{455})$$

Proof of (T_{455})

$((\lambda E)^1 \subseteq D_D) \subseteq ({}^vS\lambda E.{}^vQ\lambda E \subseteq \{>\} + \{<\} + \{=\})$	(T_{454})(Eqn1)
Let $a^1 \subseteq \lambda E$	
${}^vS a^1 \subseteq {}^vS\lambda E$	(T_{R196})(Relational Deduction)(Eqn2)
Let $\hat{b}^1 \subseteq \lambda E$	
${}^vQ \hat{b}^1 \subseteq {}^vQ\lambda E$	(T_{R196})(Relational Deduction)(Eqn3)
${}^vS a^1.{}^vQ \hat{b}^1 \subseteq {}^vS\lambda E.{}^vQ\lambda E$	(T_{M669})(Combination of Eqns2&3)(Eqn4)
$((\lambda E)^1 \subseteq D_D) \subseteq ({}^vS a^1.{}^vQ \hat{b}^1 \subseteq \{>\} + \{<\} + \{=\})$	(T_{M139})(Hypothetical Syllogism Eqns4&1)
Let $\hat{b}^1 \neq \emptyset$	
$((\lambda E)^1 \subseteq D_D) \subseteq (a^1 \subseteq (\{>\} + \{<\} + \{=\}) * \hat{b}^1)$	(T_{456})
Thus $((a^1 \subseteq \lambda E)(\hat{b}^1 \subseteq \lambda E)(\hat{b}^1 \neq \emptyset) = E) \subseteq (((\lambda E)^1 \subseteq D_D) \subseteq (a^1 \subseteq (\{>\} + \{<\} + \{=\}) * \hat{b}^1))$	(A_{MX02})(Conditional Introduction)
$((\lambda E)^1 \subseteq D_D)(a^1 \subseteq \lambda E)(\hat{b}^1 \subseteq \lambda E)(\hat{b}^1 \neq \emptyset) \subseteq (a^1 \subseteq (\{>\} + \{<\} + \{=\}) * \hat{b}^1)$	(A_{MX35})(Exportation/Importation) for truth sets

$$(((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(\hat{b}^1 \neq \emptyset)(a^1 \subseteq QD)(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{\subseteq\})\hat{b}^1) \subseteq (Da^1 > D\hat{b}^1) \quad (T_{409})$$

$$(((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(\hat{b}^1 \neq \emptyset)(a^1 \subseteq QD)(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{\supseteq\})\hat{b}^1) \subseteq (Da^1 < D\hat{b}^1) \quad (T_{410})$$

$$(((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(\hat{b}^1 \neq \emptyset)(a^1 \subseteq QD)(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{\subseteq\})\hat{b}^1) \subseteq (Da^1 \geq D\hat{b}^1) \quad (T_{411})$$

$$(((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(\hat{b}^1 \neq \emptyset)(a^1 \subseteq QD)(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{\supseteq\})\hat{b}^1) \subseteq (Da^1 \leq D\hat{b}^1) \quad (T_{412})$$

Example Proof of (T_{409})

$$(((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_b)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{\subset\}\hat{b}^1) \subseteq (Da^1 \subseteq \{>\}D\hat{b}^1))$$

(A_{S30})(Axiom of Dimensional Proper Sub-entities)

$$(((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_b)(a^1 \neq \emptyset)(\hat{b}^1 \neq \emptyset)(a^1 \subseteq QD)(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{\subset\}\hat{b}^1) \subseteq (Da^1 > D\hat{b}^1))$$

(T₉₆₂)

$$(((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((\lambda a^1 = \lambda b^1) = (a^1 =_x b^1)) \quad (T_{7458})$$

Proof of (T₇₄₅₈)

$$(((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((\lambda a \langle o \rangle \lambda b) = (a \langle o \rangle_x b)) \quad (D_{S36})(\text{Definition of } \langle o \rangle_x)(\text{Eqn1})$$

Let $o = \{=\}$ (Eqn2)

Let $a = a^1$ (Eqn3)

Let $b = b^1$ (Eqn4)

$$(((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((\lambda a^1 \langle \{=\} \rangle \lambda b^1) = (a^1 \langle \{=\} \rangle_x b^1)) \quad (A_{MX19})(\text{Substitution of Eqns2,3\&4 into Eqn1})$$

$$(((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((\lambda a^1 = \lambda b^1) = (a^1 =_x b^1)) \quad (D_{A229})(\text{Definition of } \langle \rangle)$$

$$(((\lambda E)^1 \subseteq D_A) = E) \subseteq ((\lambda a^1 < \lambda b^1) = (a^1 <_x b^1)) \quad (T_{7459})$$

$$(((\lambda E)^1 \subseteq D_A) = E) \subseteq ((\lambda a^1 > \lambda b^1) = (a^1 >_x b^1)) \quad (T_{7460})$$

$$(((\lambda E)^1 \subseteq D_A) = E) \subseteq ((\lambda a^1 \leq \lambda b^1) = (a^1 \leq_x b^1)) \quad (T_{7644})$$

$$(((\lambda E)^1 \subseteq D_A) = E) \subseteq ((\lambda a^1 \geq \lambda b^1) = (a^1 \geq_x b^1)) \quad (T_{7645})$$

Example Proof of (T₇₄₅₉)

$$(((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((\lambda a \langle o \rangle \lambda b) = (a \langle o \rangle_x b)) \quad (D_{S36})(\text{Definition of } \langle o \rangle_x)(\text{Eqn1})$$

$D_A \subseteq D_A + P_A$ (T_{M082})(Disjunction Introduction)

$(D_A \subseteq D_A + P_A) = E$ (A_{MX16})(Order Axiom)

$((\lambda E)^1 \subseteq D_A) \subseteq ((\lambda E)^1 \subseteq D_A + P_A)$ (T_{M099})(Entity Mode / Set Mode Rule)

$((\lambda E)^1 \subseteq D_A) \subseteq ((\lambda E)^1 \subseteq D_A + P_A) = E$ (A_{MX16})(Order Axiom)

$((\lambda E)^1 \subseteq D_A) = E \subseteq (((\lambda E)^1 \subseteq D_A + P_A) = E)$ (T_{M094})(Probabilistic Rule)(Eqn2)

$((\lambda E)^1 \subseteq D_A) = E \subseteq ((\lambda a \langle o \rangle \lambda b) = (a \langle o \rangle_x b))$ (T_{M139})(Hypothetical Syllogism Eqns2\&1)(Eqn3)

Let $o = \{<\}$ (Eqn4)

Let $a = a^1$ (Eqn5)

Let $b = b^1$ (Eqn6)

$((\lambda E)^1 \subseteq D_A) = E \subseteq ((\lambda a^1 \langle \{<\} \rangle \lambda b^1) = (a^1 \langle \{<\} \rangle_x b^1))$ (A_{MX19})(Substitution of Eqns4,5\&6 into Eqn3)

$((\lambda E)^1 \subseteq D_A) = E \subseteq ((\lambda a^1 < \lambda b^1) = (a^1 <_x b^1))$ (D_{A229})(Definition of $\langle \rangle$)

$$(((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((\lambda a^1 \subseteq o * \lambda b^1) = (a^1 \subseteq o_x * b^1)) \quad (T_{7745})$$

Proof of (T₇₇₄₅)

$$(((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((\lambda a^1 \langle o \rangle \lambda b^1) = (a^1 \langle o \rangle_x b^1))$$

(D_{S36})(Definition of $\langle o \rangle_x$) with $a = a^1$ and $b = b^1$

$$(((\lambda E)^1 \subseteq D_A + P_A)(a^1 \neq \emptyset)(b^1 \neq \emptyset)(\lambda a^1 \neq \emptyset)(\lambda b^1 \neq \emptyset) = E) \subseteq ((\lambda a^1 \subseteq o * \lambda b^1) = (a^1 \subseteq o_x * b^1))$$

(T₇₇₃₆)(Definition of a Relationship)

$$(((\lambda E)^1 \subseteq D_A + P_A)(a^1 \neq \emptyset)(b^1 \neq \emptyset)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((\lambda a^1 \subseteq o * \lambda b^1) = (a^1 \subseteq o_x * b^1))$$

(T₇₂₇₅)

$$(((\lambda E)^1 \subseteq D_A + P_A)(a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((\lambda a^1 \subseteq o * \lambda b^1) = (a^1 \subseteq o_x * b^1))$$

(T_{M667})(Conjunction Tautology)(Eqn1)

$$(((\lambda E)^1 \subseteq D_A + P_A)(a^1 \neq \emptyset)(b^1 \neq \emptyset)(\lambda a^1 \subseteq o * \lambda b^1) = ((\lambda E)^1 \subseteq D_A + P_A)(a^1 \neq \emptyset)(b^1 \neq \emptyset)(a^1 \subseteq o_x * b^1))$$

(T_{M444b})(Biconditional Importation/Exportation)(Eqn2)

Let $a^1 = \emptyset$ (Eqn3)

Let $(\lambda a^1 \subseteq o * \lambda b^1) \neq (a^1 \subseteq o_x * b^1)$ (Eqn4)

$(\lambda \emptyset \subseteq o * \lambda b^1) \neq (\emptyset \subseteq o_x * b^1)$ (A_{MX19})(Substitution of Eqn3 into Eqn4)

$(\emptyset \subseteq o * \lambda b^1) \neq (\emptyset \subseteq o_x * b^1)$ (T_{S181})(Null Object Theorem)

$E \neq E$ (T_{M001})(Universality of (T_{M670}) Null Set Theorem)

$\emptyset$ (T_{M345})

$$\begin{aligned}
& \text{Thus } ((a^1 = \emptyset) = E)((\chi a^1 \subseteq o * \chi b^1) \neq (a^1 \subseteq o_x * b^1)) = \emptyset \\
& \quad (AMX02)(\text{Conditional Introduction}) \\
& ((a^1 = \emptyset) = E) \subseteq ((\chi a^1 \subseteq o * \chi b^1) = (a^1 \subseteq o_x * b^1)) \quad (TM735)(\subseteq \text{Existential Form})(\text{Eqn5}) \\
& \text{Let } b^1 = \emptyset \quad (\text{Eqn6}) \\
& \text{Let } (\chi a^1 \subseteq o * \chi b^1) \neq (a^1 \subseteq o_x * b^1) \quad (\text{Eqn7}) \\
& (\chi a^1 \subseteq o * \chi \emptyset) \neq (a^1 \subseteq o_x * \emptyset) \quad (AMX19)(\text{Substitution of Eqn6 into Eqn7}) \\
& (\chi a^1 \subseteq o * \emptyset) \neq (a^1 \subseteq o_x * \emptyset) \quad (TS181)(\text{Null Object Theorem}) \\
& (\chi a^1 \subseteq \emptyset) \neq (a^1 \subseteq \emptyset) \quad (TS181)(\text{Null Object Theorem}) \\
& (\chi a^1 = \emptyset) \neq (a^1 = \emptyset) \quad (TM052)(\text{Subsets of the Null Set}) \\
& \emptyset \quad (TM024)(\text{Non Existence of Anti-}(TM275)) \\
& \text{Thus } ((b^1 = \emptyset) = E)((\chi a^1 \subseteq o * \chi b^1) \neq (a^1 \subseteq o_x * b^1)) = \emptyset \\
& \quad (AMX02)(\text{Conditional Introduction}) \\
& ((b^1 = \emptyset) = E) \subseteq ((\chi a^1 \subseteq o * \chi b^1) = (a^1 \subseteq o_x * b^1)) \quad (TM735)(\subseteq \text{Existential Form})(\text{Eqn8}) \\
& ((a^1 = \emptyset) = E) + ((b^1 = \emptyset) = E) \subseteq ((\chi a^1 \subseteq o * \chi b^1) = (a^1 \subseteq o_x * b^1)) + ((\chi a^1 \subseteq o * \chi b^1) = (a^1 \subseteq o_x * b^1)) \\
& \quad (AMX21)(\text{Combination of Eqns5\&8}) \\
& ((a^1 = \emptyset) = E) + ((b^1 = \emptyset) = E) \subseteq ((\chi a^1 \subseteq o * \chi b^1) = (a^1 \subseteq o_x * b^1)) \\
& \quad (AMX01)(\text{Disjunction Tautology}) \\
& ((a^1 = \emptyset) + (b^1 = \emptyset) = E) \subseteq ((\chi a^1 \subseteq o * \chi b^1) = (a^1 \subseteq o_x * b^1)) \\
& \quad (TM160)(\text{Probabilistic Disjunction})(\text{Eqn9}) \\
& (((\chi E)^1 \subseteq D_A + P_A)((a^1 = \emptyset) + (b^1 = \emptyset))) \subseteq ((a^1 = \emptyset) + (b^1 = \emptyset)) \\
& \quad (TM083)(\text{Conjunction Elimination}) \\
& (((\chi E)^1 \subseteq D_A + P_A)((a^1 = \emptyset) + (b^1 = \emptyset))) \subseteq ((a^1 = \emptyset) + (b^1 = \emptyset)) = E \\
& \quad (AMX16)(\text{Order Axiom}) \\
& (((\chi E)^1 \subseteq D_A + P_A)((a^1 = \emptyset) + (b^1 = \emptyset)) = E) \subseteq (((a^1 = \emptyset) + (b^1 = \emptyset)) = E) \\
& \quad (TM094)(\text{Probabilistic Rule})(\text{Eqn10}) \\
& (((\chi E)^1 \subseteq D_A + P_A)((a^1 = \emptyset) + (b^1 = \emptyset)) = E) \subseteq ((\chi a^1 \subseteq o * \chi b^1) = (a^1 \subseteq o_x * b^1)) \\
& \quad (TM139)(\text{Hypothetical Syllogism Eqns10\&9}) \\
& (((\chi E)^1 \subseteq D_A + P_A). \sim((a^1 \neq \emptyset)(b^1 \neq \emptyset)) = E) \subseteq ((\chi a^1 \subseteq o * \chi b^1) = (a^1 \subseteq o_x * b^1)) \\
& \quad (TM026)(\text{DeMorgan's Law}) \\
& (((\chi E)^1 \subseteq D_A + P_A). \sim((a^1 \neq \emptyset)(b^1 \neq \emptyset)))(\chi a^1 \subseteq o * \chi b^1) = \\
& (((\chi E)^1 \subseteq D_A + P_A). \sim((a^1 \neq \emptyset)(b^1 \neq \emptyset)))(a^1 \subseteq o_x * b^1) \quad (TM444b)(\text{Biconditional Importation/Exportation})(\text{Eqn11}) \\
& (((\chi E)^1 \subseteq D_A + P_A)(a^1 \neq \emptyset)(b^1 \neq \emptyset)(\chi a^1 \subseteq o * \chi b^1) + ((\chi E)^1 \subseteq D_A + P_A). \sim((a^1 \neq \emptyset)(b^1 \neq \emptyset)))(\chi a^1 \subseteq o * \chi b^1) \\
& = ((\chi E)^1 \subseteq D_A + P_A)(a^1 \neq \emptyset)(b^1 \neq \emptyset)(a^1 \subseteq o_x * b^1) + ((\chi E)^1 \subseteq D_A + P_A). \sim((a^1 \neq \emptyset)(b^1 \neq \emptyset))(a^1 \subseteq o_x * b^1) \\
& \quad (AMX21)(\text{Combination of Eqns2\&11}) \\
& (((\chi E)^1 \subseteq D_A + P_A)(\chi a^1 \subseteq o * \chi b^1)((a^1 \neq \emptyset)(b^1 \neq \emptyset) + \sim((a^1 \neq \emptyset)(b^1 \neq \emptyset))) \\
& = ((\chi E)^1 \subseteq D_A + P_A)(a^1 \subseteq o_x * b^1)((a^1 \neq \emptyset)(b^1 \neq \emptyset) + \sim((a^1 \neq \emptyset)(b^1 \neq \emptyset))) \\
& \quad (TM407)(\text{Distribution of } . \text{ over } +) \\
& (((\chi E)^1 \subseteq D_A + P_A)(\chi a^1 \subseteq o * \chi b^1)E = ((\chi E)^1 \subseteq D_A + P_A)(a^1 \subseteq o_x * b^1)E \\
& \quad (TM671)(\text{Tertium non Datur}) \\
& (((\chi E)^1 \subseteq D_A + P_A)(\chi a^1 \subseteq o * \chi b^1) = ((\chi E)^1 \subseteq D_A + P_A)(a^1 \subseteq o_x * b^1) \\
& \quad (AMX11)(\text{Universal Set Conjunction Axiom}) \\
& (((\chi E)^1 \subseteq D_A + P_A) = E) \subseteq ((\chi a^1 \subseteq o * \chi b^1) = (a^1 \subseteq o_x * b^1)) \\
& \quad (TM444b)(\text{Biconditional Importation/Exportation})
\end{aligned}$$

$$(((\chi E)^1 \subseteq D_A + P_A)(o^1 \subseteq R_B) = E) \subseteq ((a^1 \subseteq {}^v o * b^1) \subseteq (a^1 \subseteq o_x * b^1)) \quad (T743)$$

$$(((\chi E)^1 \subseteq D_A + P_A)(o^1 \subseteq R_B) = E) \subseteq ((a^1 \subseteq {}^v o * b^1) \subseteq (\chi a^1 \subseteq o * \chi b^1)) \quad (T744)$$

Proof of (T743) & (T744)

$$\begin{aligned}
& (o^1 \subseteq R_B) \subseteq ({}^v o = o_A) \quad (AR23)(\text{Reverse Abstract Relationships Axiom})(\text{Eqn1}) \\
& (((\chi E)^1 \subseteq D_A + P_A)(o^1 \subseteq R_B) \subseteq (o_A \subseteq o_x)) \quad (AR84)(\text{Axiom of Abstract Dimensions})(\text{Eqn2}) \\
& (((\chi E)^1 \subseteq D_A + P_A)(o^1 \subseteq R_B) \subseteq ({}^v o \subseteq o_x)) \quad (AMX19)(\text{Substitution of Eqn1 into Eqn2}) \\
& (((\chi E)^1 \subseteq D_A + P_A)(o^1 \subseteq R_B) \subseteq ({}^v o * b^1 \subseteq o_x * b^1)) \quad (TR197)(\text{Relational Deduction}) \\
& (((\chi E)^1 \subseteq D_A + P_A)(o^1 \subseteq R_B) \subseteq ({}^v o * b^1 \subseteq o_x * b^1)) = E \quad (AMX16)(\text{Order Axiom})
\end{aligned}$$

$$\begin{aligned}
& (((\lambda E)^1 \subseteq D_A + P_A)(o^1 \subseteq R_B) = E) \subseteq ((\forall o * b^1 \subseteq o_x * b^1) = E) \\
& \quad \quad \quad (T_{M094})(\text{Probabilistic Rule}) \\
& (((\lambda E)^1 \subseteq D_A + P_A)(o^1 \subseteq R_B) = E) \subseteq ((a^1 \subseteq \forall o * b^1) \subseteq (a^1 \subseteq o_x * b^1)) \\
& \quad \quad \quad (T_{M099})(\text{Entity Mode / Set Mode Rule}) \\
& (((\lambda E)^1 \subseteq D_A + P_A)(o^1 \subseteq R_B) = E) \subseteq ((a^1 \subseteq \forall o * b^1) \subseteq (\lambda a^1 \subseteq o * \lambda b^1)) \\
& \quad \quad \quad (T_{T745})
\end{aligned}$$

$$(((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((a^1 \subseteq \{=\}b^1) \subseteq (\lambda a^1 \subseteq \{=\}\lambda b^1)) \quad (T_{T545})$$

$$(((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((a^1 \subseteq \{<\}b^1) \subseteq (\lambda a^1 \subseteq \{<\}\lambda b^1)) \quad (T_{T543})$$

$$(((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((a^1 \subseteq \{<=\}b^1) \subseteq (\lambda a^1 \subseteq \{<=\}\lambda b^1)) \quad (T_{T546})$$

$$(((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((a^1 \subseteq \{>\}b^1) \subseteq (\lambda a^1 \subseteq \{>\}\lambda b^1)) \quad (T_{T738})$$

$$(((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((a^1 \subseteq \{>=\}b^1) \subseteq (\lambda a^1 \subseteq \{>=\}\lambda b^1)) \quad (T_{T742})$$

Example Proof of (T_{T543})

$$\begin{aligned}
& (((\lambda E)^1 \subseteq D_A + P_A)(o^1 \subseteq R_B) = E) \subseteq ((a^1 \subseteq \forall o * b^1) \subseteq (\lambda a^1 \subseteq o * \lambda b^1)) \\
& \quad \quad \quad (T_{T744})(\text{Eqn1}) \\
& \text{Let } o = \{>\} \\
& \quad \quad \quad (\text{Eqn2}) \\
& (((\lambda E)^1 \subseteq D_A + P_A)(\{>\}^1 \subseteq R_B) = E) \subseteq ((a^1 \subseteq \forall \{>\} * b^1) \subseteq (\lambda a^1 \subseteq \{>\} * \lambda b^1)) \\
& \quad \quad \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\
& (o = \{=\}, \{<\}, \{>\}, \{||\}, \{<=\}, \{>=\}) = (o^1 \subseteq R_B) \\
& \quad \quad \quad (D_{R94})(\text{Definition of a Membership Relationship Set}) \\
& \{>\}^1 \subseteq R_B \\
& \quad \quad \quad \text{Extraction} \\
& (\{>\}^1 \subseteq R_B) = E \\
& \quad \quad \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\
& (((\lambda E)^1 \subseteq D_A + P_A)E = E) \subseteq ((a^1 \subseteq \forall \{>\} * b^1) \subseteq (\lambda a^1 \subseteq \{>\} * \lambda b^1)) \\
& \quad \quad \quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\
& (((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((a^1 \subseteq \forall \{>\} * b^1) \subseteq (\lambda a^1 \subseteq \{>\} * \lambda b^1)) \\
& \quad \quad \quad (A_{MX17})(\text{Universal Set Conjunction Axiom}) \\
& (((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((a^1 \subseteq \{<\} * b^1) \subseteq (\lambda a^1 \subseteq \{<\} * \lambda b^1)) \\
& \quad \quad \quad (D_{A143})(\text{Definition of } \forall \{>\})
\end{aligned}$$

$$(((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 = \hat{b}^1) \subseteq (a^1 =_D \hat{b}^1)) \quad (T_{R775})$$

$$(((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{>\}\hat{b}^1) \subseteq (a^1 <_D \hat{b}^1)) \quad (T_{R776})$$

$$(((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{<\}\hat{b}^1) \subseteq (a^1 >_D \hat{b}^1)) \quad (T_{R777})$$

$$(((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{>=\}\hat{b}^1) \subseteq (a^1 \leq_D \hat{b}^1)) \quad (T_{R778})$$

$$(((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 \subseteq \{<=\}\hat{b}^1) \subseteq (a^1 \geq_D \hat{b}^1)) \quad (T_{R779})$$

Example Proof of (T_{R775})

$$\begin{aligned}
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 = \hat{b}^1) \subseteq (Da^1 = D\hat{b}^1)) \quad (T_{T404}) \\
& (((QD)^1 \subseteq D_M + P_M)((SD)^1 \subseteq D_D)(a^1 \neq \emptyset)(a^1 \subseteq QD).(\hat{b}^1 \subseteq QD) = E) \subseteq ((a^1 = \hat{b}^1) \subseteq (a^1 =_D \hat{b}^1)) \quad (T_{T458})
\end{aligned}$$

$$(((\lambda E)^1 \subseteq D_D) = E) \subseteq ((a^1 \subseteq \{=\}b^1) \subseteq (\lambda a^1 \subseteq \{=\}\lambda b^1)) \quad (T_{T612})$$

Proof of (T_{T612})

$$\begin{aligned}
& (((\lambda E)^1 \subseteq D_A + P_A) = E) \subseteq ((a^1 \subseteq \{=\}b^1) \subseteq (\lambda a^1 \subseteq \{=\}\lambda b^1)) \quad (T_{T545})(\text{Eqn1}) \\
& D_A \subseteq D_A + P_A \quad (T_{M082})(\text{Disjunction Introduction})(\text{Eqn2}) \\
& D_D \subseteq D_A \quad (T_{T885})(\text{Eqn3}) \\
& D_D \subseteq D_A + P_A \quad (T_{M139})(\text{Hypothetical Syllogism Eqns2\&3}) \\
& (D_D \subseteq D_A + P_A) = E \quad (A_{MX16})(\text{Order Axiom}) \\
& ((\lambda E)^1 \subseteq D_D) \subseteq ((\lambda E)^1 \subseteq D_A + P_A) \quad (T_{M099})(\text{Entity Mode / Set Mode Rule}) \\
& ((\lambda E)^1 \subseteq D_D) \subseteq ((\lambda E)^1 \subseteq D_A + P_A) = E \quad (A_{MX16})(\text{Order Axiom}) \\
& (((\lambda E)^1 \subseteq D_D) = E) \subseteq (((\lambda E)^1 \subseteq D_A + P_A) = E) \quad (T_{M094})(\text{Probabilistic Rule})(\text{Eqn4}) \\
& (((\lambda E)^1 \subseteq D_D) = E) \subseteq ((a^1 \subseteq \{=\}b^1) \subseteq (\lambda a^1 \subseteq \{=\}\lambda b^1)) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns4\&1})
\end{aligned}$$

$$(\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x\mathcal{C}^1), (\mathcal{L}_{xy}\mathcal{C}^1), (\mathcal{L}_{xyz}\mathcal{C}^1), \mathcal{A}): ((a^1 \subseteq \{\supset\}b^1) \subseteq (\mathcal{D}\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{D}\mathcal{X}b^1)) \quad (\mathcal{T}_{614b})$$

Proof of ($\mathcal{T}_{614b}$)

$$\begin{aligned} (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x\mathcal{C}^1), (\mathcal{L}_{xy}\mathcal{C}^1), (\mathcal{L}_{xyz}\mathcal{C}^1), \mathcal{A}): ((a^1 \subseteq \{\supset\}b^1) \subseteq (\mathcal{D}\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{D}\mathcal{X}b^1)) & \quad (\mathcal{A}_{R83})(\text{Axiom of Physical Proper Sub-entities}) \\ (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x\mathcal{C}^1), (\mathcal{L}_{xy}\mathcal{C}^1), (\mathcal{L}_{xyz}\mathcal{C}^1), \mathcal{A}): ((b^1 \subseteq \{\supset\}a^1) \subseteq (\mathcal{D}\mathcal{X}b^1 \subseteq \{\supset\}\mathcal{D}\mathcal{X}a^1)) & \quad (\mathcal{T}_{R519})(\text{Active/Passive Theorem}) \\ (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x\mathcal{C}^1), (\mathcal{L}_{xy}\mathcal{C}^1), (\mathcal{L}_{xyz}\mathcal{C}^1), \mathcal{A}): ((a^1 \subseteq \{\supset\}b^1) \subseteq (\mathcal{D}\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{D}\mathcal{X}b^1)) & \quad \text{Resymbolising} \end{aligned}$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{\supset\}b^1) \subseteq (\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{X}b^1)) \quad (\mathcal{T}_{614})$$

Proof of ($\mathcal{T}_{614}$)

$$\begin{aligned} (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{\supset\}b^1) \subseteq (\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{X}b^1)) & \quad (\mathcal{A}_{S33})(\text{Axiom of Physical Proper Sub-entities for Discrete Dimensions}) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((b^1 \subseteq \{\supset\}a^1) \subseteq (\mathcal{X}b^1 \subseteq \{\supset\}\mathcal{X}a^1)) & \quad (\mathcal{T}_{R519})(\text{Active/Passive Theorem}) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{\supset\}b^1) \subseteq (\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{X}b^1)) & \quad \text{Resymbolising} \end{aligned}$$

$$(\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x\mathcal{C}^1), (\mathcal{L}_{xy}\mathcal{C}^1), (\mathcal{L}_{xyz}\mathcal{C}^1), \mathcal{A}): ((a^1 \subseteq \{\subseteq\}b^1) \subseteq (\mathcal{D}\mathcal{X}a^1 \subseteq \{\subseteq\}\mathcal{D}\mathcal{X}b^1)) \quad (\mathcal{T}_{615b})$$

$$(\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x\mathcal{C}^1), (\mathcal{L}_{xy}\mathcal{C}^1), (\mathcal{L}_{xyz}\mathcal{C}^1), \mathcal{A}): ((a^1 \subseteq \{\supset\}b^1) \subseteq (\mathcal{D}\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{D}\mathcal{X}b^1)) \quad (\mathcal{T}_{616b})$$

Example Proof of ($\mathcal{T}_{616b}$)

$$\begin{aligned} \text{Let } a^1 \subseteq b^1 & \\ \mathcal{D}\mathcal{X}a^1 \subseteq \mathcal{D}\mathcal{X}b^1 & \\ \text{Thus } (a^1 \subseteq b^1) \subseteq (\mathcal{D}\mathcal{X}a^1 \subseteq \mathcal{D}\mathcal{X}b^1) & \\ (a^1 \subseteq \{=\}b^1) \subseteq (\mathcal{D}\mathcal{X}a^1 \subseteq \{=\}\mathcal{D}\mathcal{X}b^1) & \quad (\text{Eqn1}) \\ (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x\mathcal{C}^1), (\mathcal{L}_{xy}\mathcal{C}^1), (\mathcal{L}_{xyz}\mathcal{C}^1), \mathcal{A}): ((a^1 \subseteq \{\supset\}b^1) \subseteq (\mathcal{D}\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{D}\mathcal{X}b^1)) & \quad (\mathcal{T}_{614b})(\text{Eqn2}) \\ (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x\mathcal{C}^1), (\mathcal{L}_{xy}\mathcal{C}^1), (\mathcal{L}_{xyz}\mathcal{C}^1), \mathcal{A}): & \\ ((a^1 \subseteq \{\supset\}b^1) + (a^1 \subseteq \{=\}b^1) \subseteq (\mathcal{D}\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{D}\mathcal{X}b^1) + (\mathcal{D}\mathcal{X}a^1 \subseteq \{=\}\mathcal{D}\mathcal{X}b^1)) & \quad (\mathcal{A}_{MX21})(\text{Combination of Eqns1\&2}) \\ (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x\mathcal{C}^1), (\mathcal{L}_{xy}\mathcal{C}^1), (\mathcal{L}_{xyz}\mathcal{C}^1), \mathcal{A}): ((a^1 \subseteq \{\supset\}b^1 + \{=\}b^1) \subseteq (\mathcal{D}\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{D}\mathcal{X}b^1 + \{=\}\mathcal{D}\mathcal{X}b^1)) & \quad (\mathcal{T}_{M526}) \\ (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x\mathcal{C}^1), (\mathcal{L}_{xy}\mathcal{C}^1), (\mathcal{L}_{xyz}\mathcal{C}^1), \mathcal{A}): ((a^1 \subseteq (\{\supset\} + \{=\})b^1) \subseteq (\mathcal{D}\mathcal{X}a^1 \subseteq (\{\supset\} + \{=\})\mathcal{D}\mathcal{X}b^1)) & \quad (\mathcal{T}_{R002b})(\text{Distribution of } * \text{ over } +) \\ (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x\mathcal{C}^1), (\mathcal{L}_{xy}\mathcal{C}^1), (\mathcal{L}_{xyz}\mathcal{C}^1), \mathcal{A}): ((a^1 \subseteq \{\supset\}b^1) \subseteq (\mathcal{D}\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{D}\mathcal{X}b^1)) & \quad (\mathcal{D}_{A141})(\text{Disjunction with } \{=\}) \end{aligned}$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{\subseteq\}b^1) \subseteq (\mathcal{X}a^1 \subseteq \{\subseteq\}\mathcal{X}b^1)) \quad (\mathcal{T}_{615})$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{\supset\}b^1) \subseteq (\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{X}b^1)) \quad (\mathcal{T}_{616})$$

Example Proof of ($\mathcal{T}_{616}$)

$$\begin{aligned} (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{=\}b^1) \subseteq (\mathcal{X}a^1 \subseteq \{=\}\mathcal{X}b^1)) & \quad (\mathcal{T}_{612})(\text{Eqn1}) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{\supset\}b^1) \subseteq (\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{X}b^1)) & \quad (\mathcal{T}_{614})(\text{Eqn2}) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{\supset\}b^1) + (a^1 \subseteq \{=\}b^1) \subseteq (\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{X}b^1) + (\mathcal{X}a^1 \subseteq \{=\}\mathcal{X}b^1)) & \quad (\mathcal{A}_{MX21})(\text{Combination of Eqns1\&2}) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{\supset\}b^1 + \{=\}b^1) \subseteq (\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{X}b^1 + \{=\}\mathcal{X}b^1)) & \quad (\mathcal{T}_{M526}) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq (\{\supset\} + \{=\})b^1) \subseteq (\mathcal{X}a^1 \subseteq (\{\supset\} + \{=\})\mathcal{X}b^1)) & \quad (\mathcal{T}_{R002b})(\text{Distribution of } * \text{ over } +) \\ (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{\supset\}b^1) \subseteq (\mathcal{X}a^1 \subseteq \{\supset\}\mathcal{X}b^1)) & \quad (\mathcal{D}_{A141})(\text{Disjunction with } \{=\}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) \subseteq (\mathcal{X}a = \mathcal{X}a.\{\subset\}*\mathcal{X}a)$$

(T₆₂₈)Proof of (T₆₂₈)

$$(\beta^1 \subseteq \mathcal{S}_0\mathcal{S}_0)(\alpha \subseteq \beta) \subseteq (\alpha = \alpha.\{\subset\}*\alpha)$$

(T₇₃₇)

$$(a^1 \subseteq \mathcal{S}_0\mathcal{S}_0)(b \subseteq a) \subseteq (b = b.\{\subset\}*\mathcal{S}_0b)$$

Resymbolising (Eqn1)

$$(a^1 \subseteq \mathcal{D}_0) \subseteq (a^1 \subseteq \mathcal{S}_0\mathcal{S}_0)$$

(T₅₉₈₉)

$$(a^1 \subseteq \mathcal{D}_0)(b \subseteq a) \subseteq (a^1 \subseteq \mathcal{S}_0\mathcal{S}_0)(b \subseteq a)$$

(T₆₀₂₉)(Deduction)(Eqn2)

$$(a^1 \subseteq \mathcal{D}_0)(b \subseteq a) \subseteq (b = b.\{\subset\}*\mathcal{S}_0b)$$

(T₆₁₃₉)(Hypothetical Syllogism Eqns2&1)(Eqn3)

$$\text{Let } a = \mathcal{X}\mathcal{E}$$

(Eqn4)

$$\text{Let } b = \mathcal{X}a$$

(Eqn5)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq (\mathcal{X}a = \mathcal{X}a.\{\subset\}*\mathcal{X}a)$$

(A_{MX19})(Substitution of Eqns4&5 into Eqn3)(Eqn6)

$$a \subseteq \mathcal{E}$$

(T₆₇₄₄)(Universal Set Theorem)

$$\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}$$

(T₆₁₉₆)(Relational Deduction)

$$(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}$$

(A_{MX16})(Order Axiom)(Eqn7)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)\mathcal{E} \subseteq (\mathcal{X}a = \mathcal{X}a.\{\subset\}*\mathcal{X}a)$$

(A_{MX19})(Substitution of Eqn7 into Eqn6)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) \subseteq (\mathcal{X}a = \mathcal{X}a.\{\subset\}*\mathcal{X}a)$$

(A_{MX11})(Universal Set Conjunction Axiom)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) \subseteq (\mathcal{X}a^1 \subseteq \{\subset\}*\mathcal{X}a)$$

(T₆₄₆)Proof of (T₆₄₆)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) \subseteq (\mathcal{X}a = \mathcal{X}a.\{\subset\}*\mathcal{X}a)$$

(T₆₂₈)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) \subseteq (\mathcal{X}a^1 \subseteq \{\subset\}*\mathcal{X}a)$$

(A_{MX44})(Equality Form)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0).(a \subseteq \mathcal{X}\mathcal{E}) \subseteq (a.\{\subset\}a = \emptyset) \quad (T_{633})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0).(a \subseteq \mathcal{X}\mathcal{E}) \subseteq (a.\{\supset\}a = \emptyset) \quad (T_{647})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0).(a \subseteq \mathcal{X}\mathcal{E}) \subseteq (a.\{\parallel\}a = \emptyset) \quad (T_{634})$$

Example Proof of (T₆₃₃)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) \subseteq (\mathcal{X}a = \mathcal{X}a.\{\subset\}*\mathcal{X}a)$$

(T₆₂₈)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) \subseteq (\mathcal{X}a \subseteq \{\subset\}*\mathcal{X}a)$$

(A_{MX44})(Equality Form)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) \subseteq (\mathcal{X}a.\{\subset\}*\mathcal{X}a = \emptyset)$$

(T₆₇₃₅)(Existential Form)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0).\mathcal{E} \subseteq (\mathcal{X}a.\{\subset\}*\mathcal{X}a = \emptyset)$$

(A_{MX11})(Universal Set Conjunction Axiom)(Eqn1)

$$a \subseteq \mathcal{E}$$

(T₆₇₄₄)(Universal Set Theorem)

$$\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}$$

(T₆₁₉₆)(Relational Deduction)

$$(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}$$

(A_{MX16})(Order Axiom)(Eqn2)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0).(\mathcal{X}a \subseteq \mathcal{X}\mathcal{E}) \subseteq (\mathcal{X}a.\{\subset\}*\mathcal{X}a = \emptyset)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)

$$\text{Let } a = \mathcal{X}a$$

(Eqn4)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0).(a \subseteq \mathcal{X}\mathcal{E}) \subseteq (a.\{\subset\}a = \emptyset)$$

(A_{MX19})(Substitution of Eqn4 into Eqn3)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0).(a \subseteq \mathcal{X}\mathcal{E}) \subseteq (a.\{\supset\}a = a) \quad (T_{630})$$

Proof of (T₆₃₀)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0).(a \subseteq \mathcal{X}\mathcal{E}) \subseteq (a.\{\supset\}a = \emptyset)$$

(T₆₄₇)(Eqn1)

$$a.a = a$$

(T₆₆₆₇)(Conjunction Tautology)

$$a.\{=\}a = a$$

(T₆₀₈₀)(Eqn2)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0).(a \subseteq \mathcal{X}\mathcal{E}) \subseteq (a.\{\supset\}a + a.\{=\}a = \emptyset + a)$$

(A_{MX21})(Combination of Eqns1&2)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0).(a \subseteq \mathcal{X}\mathcal{E}) \subseteq (a.\{\supset\}a + a.\{=\}a = a)$$

(T₆₆₆₈)(Null Set Disjunction)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0).(a \subseteq \mathcal{X}\mathcal{E}) \subseteq (a.\{\supset\}a + \{=\}a = a)$$

(T₆₄₀₇)(Distribution of . over +)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0).(a \subseteq \mathcal{X}\mathcal{E}) \subseteq (a.\{\supset\} + \{=\}a = a)$$

(T_{6002c})(Distribution of + over *)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0).(a \subseteq \mathcal{X}\mathcal{E}) \subseteq (a.\{\supset\}a = a)$$

(D₆₁₄₁)(Disjunction with {=})

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0 + \mathcal{P}_0) \subseteq ((\mathcal{X}a)^1 \neq \mathcal{X}a^1) \quad (T_{626})$$

Proof of (T₆₂₆)

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M + \mathbf{P}_M) = ((\mathcal{X}\mathbf{a})^1 = \mathcal{X}\mathbf{a}^1)$	(A ₅₀₅)(Physical Dimension Aggregation Axiom)
$((\mathcal{X}\mathcal{E})^1 \not\subseteq \mathbf{D}_M + \mathbf{P}_M) = ((\mathcal{X}\mathbf{a})^1 \neq \mathcal{X}\mathbf{a}^1)$	(T ₆₀₀₈)(Inverse of an Equality)(Eqn1)
$\mathbf{D}_A \subseteq \sim \mathbf{P}_A$	(A _{R82})(Axiom of Dimensions and Polydimensions)
$\mathbf{D}_M + \mathbf{P}_M \subseteq \sim(\mathbf{D}_D + \mathbf{P}_D)$	(T ₈₁₄)&(T ₈₁₅)
$((\mathbf{D}_M + \mathbf{P}_M \subseteq \sim(\mathbf{D}_D + \mathbf{P}_D)) = \mathbf{E}) = \mathbf{E}$	(A _{MX16})(Order Axiom)(Eqn2)
$((\mathbf{D}_M + \mathbf{P}_M \subseteq \sim(\mathbf{D}_D + \mathbf{P}_D)) = \mathbf{E}) \subseteq (((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M + \mathbf{P}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \not\subseteq \mathbf{D}_D + \mathbf{P}_D))$	(T ₃₅₄)(Entity Mode / Set Mode Rule)(Eqn3)
$\mathbf{E} \subseteq (((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M + \mathbf{P}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \not\subseteq \mathbf{D}_D + \mathbf{P}_D))$	(A _{MX19})(Substitution of Eqn2 into Eqn3)
$\mathbf{E} = (((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M + \mathbf{P}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \not\subseteq \mathbf{D}_D + \mathbf{P}_D))$	(T ₂₇₅)(Supersets of the Universal Set)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M + \mathbf{P}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \not\subseteq \mathbf{D}_D + \mathbf{P}_D)$	(A _{MX16})(Order Axiom)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D + \mathbf{P}_D) \subseteq ((\mathcal{X}\mathcal{E})^1 \not\subseteq \mathbf{D}_M + \mathbf{P}_M)$	(T ₆₀₁₂)(Transposition)(Eqn4)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M + \mathbf{P}_M) \subseteq ((\mathcal{X}\mathbf{a})^1 \neq \mathcal{X}\mathbf{a}^1)$	(A _{MX19})(Substitution of Eqn1 into Eqn4)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D) \subseteq (\{\subset\} * \mathcal{X}\mathbf{a} \subseteq \sim \mathcal{X}\mathcal{E}) \quad (\text{T}_{632})$$

Proof of (T₆₃₂)

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D) \subseteq (\mathcal{X}\mathbf{a} = \mathcal{X}\mathbf{a}.\{\subset\} * \mathcal{X}\mathbf{a})$	(T ₆₂₈)
Let $\mathbf{a} = \mathbf{E}$	(Eqn2)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D) \subseteq (\mathcal{X}\mathcal{E} = \mathcal{X}\mathcal{E}.\{\subset\} * \mathcal{X}\mathcal{E})$	(A _{MX19})(Substitution of Eqn2 into Eqn1)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D) \subseteq (\mathcal{X}\mathcal{E} \subseteq \{\subset\} * \mathcal{X}\mathcal{E})$	(A _{MX44})($\subseteq$ Equality Form)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D) \subseteq (\{\subset\} * \mathcal{X}\mathcal{E} \subseteq \sim \mathcal{X}\mathcal{E})$	(T ₆₀₁₂)(Transposition)(Eqn3)
$\mathbf{a} \subseteq \mathbf{E}$	(T ₇₄₄)(Universal Set Theorem)
$\mathcal{X}\mathbf{a} \subseteq \mathcal{X}\mathcal{E}$	(T _{R196})(Relational Deduction)
$\{\subset\} * \mathcal{X}\mathbf{a} \subseteq \{\subset\} * \mathcal{X}\mathcal{E}$	(T _{R196})(Relational Deduction)(Eqn4)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D) \subseteq (\{\subset\} * \mathcal{X}\mathbf{a} \subseteq \sim \mathcal{X}\mathcal{E})$	(T ₆₁₃₉)(Hypothetical Syllogism Eqns3&4)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D)(\mathbf{a} \subseteq \mathcal{X}\mathcal{E})(\mathbf{a} \neq \emptyset)(\mathbf{a} \neq \mathbf{a}^1) \subseteq (\mathcal{B}\mathbf{a} = \mathbf{a}) \quad (\text{T}_{635})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D)(\mathbf{a} \subseteq \mathcal{X}\mathcal{E})(\mathbf{a} \neq \emptyset)(\mathbf{a} \neq \mathbf{a}^1) \subseteq (\mathcal{C}\mathbf{a} = \mathbf{a}) \quad (\text{T}_{637})$$

Example Proof of (T₆₃₅)

Let $(\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D$	
$(\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_D \mathbf{S}_D$	(T ₅₇₂₃)
Let $\mathbf{a} \subseteq \mathcal{X}\mathcal{E}$	
Let $\mathbf{a} \neq \emptyset$	
Let $\mathbf{a} \neq \mathbf{a}^1$	
$\mathbf{a}^1 \subseteq \mathbf{S}_D \mathbf{S}_D$	(T ₇₉₅₇)
$\mathcal{B}\mathbf{a} = \mathbf{a}$	(T ₅₈₉₃)
Thus $((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D)(\mathbf{a} \subseteq \mathcal{X}\mathcal{E})(\mathbf{a} \neq \emptyset)(\mathbf{a} \neq \mathbf{a}^1) \subseteq (\mathcal{B}\mathbf{a} = \mathbf{a})$	(A _{MX02})(Conditional Introduction)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D)(\mathbf{a} \subseteq \mathcal{X}\mathcal{E})(\mathbf{a} \neq \emptyset)(\mathbf{a} \neq \mathbf{a}^1) \subseteq (\mathbf{a} \subseteq {}^v \mathcal{B}\mathbf{a}) \quad (\text{T}_{639})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D)(\mathbf{a} \subseteq \mathcal{X}\mathcal{E})(\mathbf{a} \neq \emptyset)(\mathbf{a} \neq \mathbf{a}^1) \subseteq (\mathbf{a} \subseteq {}^v \mathcal{C}\mathbf{a}) \quad (\text{T}_{641})$$

Example Proof of (T₆₃₉)

Let $(\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D$	
$(\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{S}_D \mathbf{S}_D$	(T ₅₇₂₃)
Let $\mathbf{a} \subseteq \mathcal{X}\mathcal{E}$	
Let $\mathbf{a} \neq \emptyset$	
Let $\mathbf{a} \neq \mathbf{a}^1$	
$\mathbf{a}^1 \subseteq \mathbf{S}_D \mathbf{S}_D$	(T ₇₉₅₇)
$(\mathbf{a} \subseteq {}^v \mathcal{B}\mathbf{a})$	(T ₇₂₅₂)
Thus $((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_D)(\mathbf{a} \subseteq \mathcal{X}\mathcal{E})(\mathbf{a} \neq \emptyset)(\mathbf{a} \neq \mathbf{a}^1) \subseteq (\mathbf{a} \subseteq {}^v \mathcal{B}\mathbf{a})$	(A _{MX02})(Conditional Introduction)

$$(((\lambda E)^1 \subseteq D_0) = E) \subseteq ((a^1 \subseteq \{=\}E^1) \subseteq (\lambda a^1 \subseteq \{=\}\lambda E^1)) \quad (T_{R780})$$

$$(((\lambda E)^1 \subseteq D_0) = E) \subseteq ((a^1 \subseteq \{<\}E^1) \subseteq (\lambda a^1 \subseteq \{<\}\lambda E^1)) \quad (T_{R782})$$

$$(((\lambda E)^1 \subseteq D_0) = E) \subseteq ((a^1 \subseteq \{=\}\emptyset^1) \subseteq (\lambda a^1 \subseteq \{=\}\lambda \emptyset^1)) \quad (T_{R781})$$

$$(((\lambda E)^1 \subseteq D_0) = E) \subseteq ((a^1 \subseteq \{>\}\emptyset^1) \subseteq (\lambda a^1 \subseteq \{>\}\lambda \emptyset^1)) \quad (T_{R783})$$

Example Proof of (T_{R780})

$$(((\lambda E)^1 \subseteq D_0) = E) \subseteq ((a^1 \subseteq \{=\}b^1) \subseteq (\lambda a^1 \subseteq \{=\}\lambda b^1)) \quad (T_{T612})(Eqn1)$$

$$\text{Let } b = E \quad (Eqn2)$$

$$b^1 = E^1 \quad (T_{M488})(Eqn3)$$

$$(((\lambda E)^1 \subseteq D_0) = E) \subseteq ((a^1 \subseteq \{=\}E^1) \subseteq (\lambda a^1 \subseteq \{=\}\lambda E^1)) \quad (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})$$

$$((\lambda E)^1 \subseteq D_0)(a^1 \neq \emptyset)(b^1 \neq \emptyset) \subseteq (a^1 >_x b^1) + (a^1 =_x b^1) + (a^1 <_x b^1) \quad (T_{R784})$$

Proof of (T_{R784})

$$((\lambda E)^1 \subseteq D_0)(a^1 \subseteq \lambda E)(b^1 \subseteq \lambda E)(b^1 \neq \emptyset) \subseteq (a^1 \subseteq (\{>\} + \{<\} + \{=\}) * b^1) \quad (T_{T455})$$

$$(Eqn1)$$

$$\text{Let } a^1 = \lambda a^1 \quad (Eqn2)$$

$$\text{Let } b^1 = \lambda b^1 \quad (Eqn3)$$

$$((\lambda E)^1 \subseteq D_0)(\lambda a^1 \subseteq \lambda E)(\lambda b^1 \subseteq \lambda E)(\lambda b^1 \neq \emptyset) \subseteq (\lambda a^1 \subseteq (\{>\} + \{<\} + \{=\}) * \lambda b^1)$$

$$(A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})(Eqn4)$$

$$a^1 \subseteq E \quad (T_{M744})(\text{Universal Set Theorem})$$

$$\lambda a^1 \subseteq \lambda E \quad (T_{R196})(\text{Relational Deduction})$$

$$(\lambda a^1 \subseteq \lambda E) = E \quad (A_{MX16})(\text{Order Axiom})(Eqn5)$$

$$b^1 \subseteq E \quad (T_{M744})(\text{Universal Set Theorem})$$

$$\lambda b^1 \subseteq \lambda E \quad (T_{R196})(\text{Relational Deduction})$$

$$(\lambda b^1 \subseteq \lambda E) = E \quad (A_{MX16})(\text{Order Axiom})(Eqn6)$$

$$((\lambda E)^1 \subseteq D_0)EE(\lambda b^1 \neq \emptyset) \subseteq (\lambda a^1 \subseteq (\{>\} + \{<\} + \{=\}) * \lambda b^1)$$

$$(A_{MX19})(\text{Substitution of Eqns5\&6 into Eqn4})$$

$$((\lambda E)^1 \subseteq D_0)(\lambda b^1 \neq \emptyset) \subseteq (\lambda a^1 \subseteq (\{>\} + \{<\} + \{=\}) * \lambda b^1)$$

$$(A_{MX17})(\text{Universal Set Conjunction Axiom})$$

$$((\lambda E)^1 \subseteq D_0)(b^1 \neq \emptyset) \subseteq (\lambda a^1 \subseteq (\{>\} + \{<\} + \{=\}) * \lambda b^1)$$

$$(T_{T275})$$

$$((\lambda E)^1 \subseteq D_0)(b^1 \neq \emptyset) \subseteq (\lambda a^1 \subseteq \{>\} * \lambda b^1 + \{<\} * \lambda b^1 + \{=\} * \lambda b^1)$$

$$(T_{R002c})(\text{Distribution of } + \text{ over } *)$$

$$((\lambda E)^1 \subseteq D_0)(b^1 \neq \emptyset) \subseteq (\lambda a^1 \subseteq \{>\} * \lambda b^1) + (\lambda a^1 \subseteq \{<\} * \lambda b^1) + (\lambda a^1 \subseteq \{=\} * \lambda b^1)$$

$$(T_{M526})$$

$$((\lambda E)^1 \subseteq D_0)(a^1 \neq \emptyset)(b^1 \neq \emptyset) \subseteq (\lambda a^1 > \lambda b^1) + (\lambda a^1 = \lambda b^1) + (\lambda a^1 < \lambda b^1)$$

$$(T_{S411})(\text{Definition of an Ordering Relation})$$

$$((\lambda E)^1 \subseteq D_0)(a^1 \neq \emptyset)(b^1 \neq \emptyset) \subseteq (a^1 >_x b^1) + (a^1 =_x b^1) + (a^1 <_x b^1)$$

$$(T_{T458}), (T_{T459}), (T_{T460})$$

$$(a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq ((ab)^1 = \emptyset) \quad (T_{T735})$$

Strictly this is a theorem of membership theory. However, it is derived using relationship theory and is only used in the proofs of (T_{R785}) and (T_{R785b}).

Proof of (T_{T735})

$$\text{Let } a^1 = \emptyset \quad (Eqn1)$$

$$ab \subseteq a \quad (T_{M083})(\text{Conjunction Elimination})$$

$$(b^1 \neq \emptyset) \subseteq ((ab)^1 \subseteq \{\subseteq\}a^1) \quad (T_{S412})(Eqn2)$$

$$(b^1 \neq \emptyset) \subseteq ((ab)^1 \subseteq \{\subseteq\}\emptyset) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(b^1 \neq \emptyset) \subseteq ((ab)^1 \subseteq \emptyset) \quad (T_{S181})(\text{Null Object Theorem})$$

$$(b^1 \neq \emptyset) \subseteq ((ab)^1 = \emptyset) \quad (T_{M052})(\text{Subsets of the Null Set})$$

Thus $(a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq ((ab)^1 = \emptyset) \quad (A_{MX02})(\text{Conditional Introduction})$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A})(b^1 \neq \emptyset) \subseteq ((ab)^1 \leq_{DX} a^1) \quad (T_{R785b})$

Proof of (T_{R785b})

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): ((a^1 \subseteq \{\subseteq\} b^1) \subseteq (DX a^1 \subseteq \{\leq\} DX b^1))$

$(T_{T615b})(\text{Eqn1})$

Let $a^1 = (ab)^1$

(Eqn2)

Let $b^1 = a^1$

(Eqn3)

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): (((ab)^1 \subseteq \{\subseteq\} a^1) \subseteq (DX(ab)^1 \subseteq \{\leq\} DX a^1))$

$(A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): ((ab)^1 \subseteq \{\subseteq\} a^1) \subseteq (DX(ab)^1 \subseteq \{\leq\} DX a^1)$

$(A_{MX35})(\text{Exportation/Importation})(\text{Eqn4})$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): (a^1 \neq \emptyset)(b^1 \neq \emptyset)((ab)^1 \subseteq \{\subseteq\} a^1) \subseteq$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): ((ab)^1 \subseteq \{\subseteq\} a^1)$

$(T_{M083})(\text{Conjunction Elimination})(\text{Eqn5})$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): (a^1 \neq \emptyset)(b^1 \neq \emptyset)((ab)^1 \subseteq \{\subseteq\} a^1) \subseteq (DX(ab)^1 \subseteq \{\leq\} DX a^1)$

$(T_{M139})(\text{Hypothetical Syllogism Eqns5\&4})(\text{Eqn6})$

$ab \subseteq a$

$(T_{M083})(\text{Conjunction Elimination})$

$(a^1 \neq \emptyset) \subseteq ((ab)^1 \subseteq \{\subseteq\} a^1)$

(T_{S412})

$(a^1 \neq \emptyset) = (a^1 \neq \emptyset)((ab)^1 \subseteq \{\subseteq\} a^1)$

$(A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn7})$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): (a^1 \neq \emptyset) \subseteq (DX(ab)^1 \subseteq \{\leq\} DX a^1)$

$(A_{MX19})(\text{Substitution of Eqn7 into Eqn6})(\text{Eqn8})$

Let $a^1 = \emptyset$

(Eqn9)

Let $b^1 \neq \emptyset$

$(ab)^1 = \emptyset$

$(T_{T735})(\text{Eqn10})$

Let $DX(ab)^1 \not\subseteq \{\leq\} DX a^1$

(Eqn11)

$DX \emptyset \not\subseteq \{\leq\} DX \emptyset$

$(A_{MX19})(\text{Substitution of Eqns9\&10 into Eqn11})$

$\emptyset \not\subseteq \emptyset$

$(T_{S181})(\text{Null Object Theorem})$

$(\emptyset \sim \subseteq \emptyset)(\emptyset \neq \emptyset)$

(T_{M198})

$(\emptyset \sim \subseteq \emptyset)\emptyset$

(T_{M346})

$\emptyset$

$(T_{M050})(\text{Null Set Conjunction})$

Thus $(a^1 = \emptyset)(b^1 \neq \emptyset)(DX(ab)^1 \not\subseteq \{\leq\} DX a^1) = \emptyset$

$(A_{MX02})(\text{Conditional Introduction})$

$(a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq (DX(ab)^1 \subseteq \{\leq\} DX a^1)$

$(T_{M735})(\subseteq \text{Existential Form})(\text{Eqn12})$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): (a^1 \neq \emptyset)(a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq (a^1 = \emptyset)(b^1 \neq \emptyset)$

$(T_{M083})(\text{Conjunction Elimination})(\text{Eqn13})$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): (a^1 \neq \emptyset)(a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq (DX(ab)^1 \subseteq \{\leq\} DX a^1)$

$(T_{M139})(\text{Hypothetical Syllogism Eqns13\&12})(\text{Eqn14})$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): (a^1 \neq \emptyset)(b^1 \neq \emptyset) +$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): (a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq$

$(DX(ab)^1 \subseteq \{\leq\} DX a^1) + (DX(ab)^1 \subseteq \{\leq\} DX a^1) \quad (A_{MX21})(\text{Combination of Eqns8\&14})$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): (a^1 \neq \emptyset)(b^1 \neq \emptyset) +$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): (a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq (DX(ab)^1 \subseteq \{\leq\} DX a^1)$

$(A_{MX01})(\text{Disjunction Tautology})$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): ((a^1 \neq \emptyset) + (a^1 = \emptyset))(b^1 \neq \emptyset) \subseteq (DX(ab)^1 \subseteq \{\leq\} DX a^1)$

$(T_{M407})(\text{Distribution of } \cdot \text{ over } +)$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): E(b^1 \neq \emptyset) \subseteq (DX(ab)^1 \subseteq \{\leq\} DX a^1)$

$(T_{M671})(\text{Tertium non Datur})$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): (b^1 \neq \emptyset) \subseteq (DX(ab)^1 \subseteq \{\leq\} DX a^1)$

$(A_{MX11})(\text{Universal Set Conjunction Axiom})$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): (b^1 \neq \emptyset) \subseteq (DX(ab)^1 \leq DX a^1)$

$(T_{T736})(\text{Definition of a Relationship})$

$(X = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): (b^1 \neq \emptyset) \subseteq ((ab)^1 \leq_{DX} a^1)$

(T_{T644})

$$((\chi E)^1 \subseteq D_0)(b^1 \neq \emptyset) \subseteq ((ab)^1 \leq_x a^1) \quad (T_{R785})$$

Proof of (T_{R785})

$$\begin{aligned}
 &(((\chi E)^1 \subseteq D_0) = E) \subseteq ((a^1 \subseteq \{\subseteq\} b^1) \subseteq (\chi a^1 \subseteq \{\leq\} \chi b^1)) \\
 &\quad (T_{T615})(Eqn1) \\
 \text{Let } a^1 &= (ab)^1 & (Eqn2) \\
 \text{Let } b^1 &= a^1 & (Eqn3) \\
 &(((\chi E)^1 \subseteq D_0) = E) \subseteq (((ab)^1 \subseteq \{\subseteq\} a^1) \subseteq (\chi(ab)^1 \subseteq \{\leq\} \chi a^1)) \\
 &\quad (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\
 &((\chi E)^1 \subseteq D_0)((ab)^1 \subseteq \{\subseteq\} a^1) \subseteq (\chi(ab)^1 \subseteq \{\leq\} \chi a^1) & (A_{MX35})(\text{Exportation/Importation})(Eqn4) \\
 &((\chi E)^1 \subseteq D_0)(a^1 \neq \emptyset)(b^1 \neq \emptyset)((ab)^1 \subseteq \{\subseteq\} a^1) \subseteq ((\chi E)^1 \subseteq D_0)((ab)^1 \subseteq \{\subseteq\} a^1) \\
 &\quad (T_{M083})(\text{Conjunction Elimination})(Eqn5) \\
 &((\chi E)^1 \subseteq D_0)(a^1 \neq \emptyset)(b^1 \neq \emptyset)((ab)^1 \subseteq \{\subseteq\} a^1) \subseteq (\chi(ab)^1 \subseteq \{\leq\} \chi a^1) \\
 &\quad (T_{M139})(\text{Hypothetical Syllogism Eqns5\&4})(Eqn6) \\
 &ab \subseteq a & (T_{M083})(\text{Conjunction Elimination}) \\
 &(a^1 \neq \emptyset) \subseteq ((ab)^1 \subseteq \{\subseteq\} a^1) & (T_{S412}) \\
 &(a^1 \neq \emptyset) = (a^1 \neq \emptyset)((ab)^1 \subseteq \{\subseteq\} a^1) & (A_{MX44})(\subseteq \text{Equality Form})(Eqn7) \\
 &((\chi E)^1 \subseteq D_0)(a^1 \neq \emptyset) \subseteq (\chi(ab)^1 \subseteq \{\leq\} \chi a^1) & (A_{MX19})(\text{Substitution of Eqn7 into Eqn6})(Eqn8) \\
 \text{Let } a^1 &= \emptyset & (Eqn9) \\
 \text{Let } b^1 &= \emptyset \\
 &(ab)^1 = \emptyset & (T_{T735})(Eqn10) \\
 \text{Let } \chi(ab)^1 &\notin \{\leq\} \chi a^1 & (Eqn11) \\
 \chi \emptyset &\notin \{\leq\} \chi \emptyset & (A_{MX19})(\text{Substitution of Eqns9\&10 into Eqn11}) \\
 \emptyset &\notin \emptyset & (T_{S181})(\text{Null Object Theorem}) \\
 (\emptyset \sim \subset \emptyset)(\emptyset \neq \emptyset) & & (T_{M198}) \\
 (\emptyset \sim \subset \emptyset) \emptyset & & (T_{M346}) \\
 \emptyset & & (T_{M050})(\text{Null Set Conjunction}) \\
 \text{Thus } (a^1 = \emptyset)(b^1 \neq \emptyset)(\chi(ab)^1 \notin \{\leq\} \chi a^1) &= \emptyset & (A_{MX02})(\text{Conditional Introduction}) \\
 (a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq (\chi(ab)^1 \subseteq \{\leq\} \chi a^1) & & (T_{M735})(\subseteq \text{Existential Form})(Eqn12) \\
 ((\chi E)^1 \subseteq D_0)(a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq (a^1 = \emptyset)(b^1 \neq \emptyset) & & (T_{M083})(\text{Conjunction Elimination})(Eqn13) \\
 ((\chi E)^1 \subseteq D_0)(a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq (\chi(ab)^1 \subseteq \{\leq\} \chi a^1) & & (T_{M139})(\text{Hypothetical Syllogism Eqns13\&12})(Eqn14) \\
 ((\chi E)^1 \subseteq D_0)(a^1 \neq \emptyset)(b^1 \neq \emptyset) + ((\chi E)^1 \subseteq D_0)(a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq (\chi(ab)^1 \subseteq \{\leq\} \chi a^1) + (\chi(ab)^1 \subseteq \{\leq\} \chi a^1) \\
 & & (A_{MX27})(\text{Combination of Eqns8\&14}) \\
 ((\chi E)^1 \subseteq D_0)(a^1 \neq \emptyset)(b^1 \neq \emptyset) + ((\chi E)^1 \subseteq D_0)(a^1 = \emptyset)(b^1 \neq \emptyset) \subseteq (\chi(ab)^1 \subseteq \{\leq\} \chi a^1) \\
 & & (A_{MX01})(\text{Disjunction Tautology}) \\
 ((\chi E)^1 \subseteq D_0)((a^1 \neq \emptyset) + (a^1 = \emptyset))(b^1 \neq \emptyset) \subseteq (\chi(ab)^1 \subseteq \{\leq\} \chi a^1) \\
 & & (T_{M407})(\text{Distribution of } \cdot \text{ over } +) \\
 ((\chi E)^1 \subseteq D_0)E(b^1 \neq \emptyset) \subseteq (\chi(ab)^1 \subseteq \{\leq\} \chi a^1) & & (T_{M671})(\text{Tertium non Datur}) \\
 ((\chi E)^1 \subseteq D_0)(b^1 \neq \emptyset) \subseteq (\chi(ab)^1 \subseteq \{\leq\} \chi a^1) & & (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\
 ((\chi E)^1 \subseteq D_0)(b^1 \neq \emptyset) \subseteq (\chi(ab)^1 \leq \chi a^1) & & (T_{T736})(\text{Definition of a Relationship}) \\
 ((\chi E)^1 \subseteq D_0)(b^1 \neq \emptyset) \subseteq ((ab)^1 \leq_x a^1) & & (T_{T644})
 \end{aligned}$$

$$(\chi = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A})(a^1 \neq \emptyset)((a + b)^1 \neq \emptyset) \subseteq (a^1 \leq_{D\chi} (a + b)^1) \quad (T_{R786b})$$

Proof of (T_{R786b})

$$\begin{aligned}
 &(\chi = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): ((a^1 \subseteq \{\subseteq\} b^1) \subseteq (D\chi a^1 \subseteq \{\leq\} D\chi b^1)) \\
 &\quad (T_{T615b})(Eqn1) \\
 \text{Let } b^1 &= (a + b)^1 & (Eqn2) \\
 &(\chi = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}): ((a^1 \subseteq \{\subseteq\} (a + b)^1) \subseteq (D\chi a^1 \subseteq \{\leq\} D\chi (a + b)^1)) \\
 &\quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
 &(\chi = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A})(a^1 \subseteq \{\subseteq\} (a + b)^1) \subseteq (D\chi a^1 \subseteq \{\leq\} D\chi (a + b)^1) \\
 &\quad (A_{MX35})(\text{Exportation/Importation}) \\
 &(\chi = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A}) (a^1 \neq \emptyset)((a + b)^1 \neq \emptyset)(a^1 \subseteq \{\subseteq\} (a + b)^1) \subseteq
 \end{aligned}$$

$$\begin{aligned}
& (\mathcal{D}\mathcal{X}a^1 \subseteq \{\leq\} \mathcal{D}\mathcal{X}(a+b)^1) & (\mathcal{T}_{M089})(\text{Single Premise Syllogism})(\text{Eqn3}) \\
& a \subseteq a+b & (\mathcal{T}_{M082})(\text{Disjunction Introduction}) \\
& ((a+b)^1 \neq \emptyset) \subseteq ((a^1 \subseteq \{\subseteq\}(a+b)^1) & (\mathcal{T}_{S412}) \\
& ((a+b)^1 \neq \emptyset) = ((a+b)^1 \neq \emptyset)((a^1 \subseteq \{\subseteq\}(a+b)^1) & (\mathcal{A}_{MX44})(\subseteq \text{Equality Form})(\text{Eqn4}) \\
& (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A})(a^1 \neq \emptyset)((a+b)^1 \neq \emptyset) \subseteq (\mathcal{D}\mathcal{X}a^1 \subseteq \{\leq\} \mathcal{D}\mathcal{X}(a+b)^1) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\
& (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A})(a^1 \neq \emptyset)((a+b)^1 \neq \emptyset) \subseteq (\mathcal{D}\mathcal{X}a^1 \leq \mathcal{D}\mathcal{X}(a+b)^1) & (\mathcal{T}_{T736})(\text{Definition of a Relationship}) \\
& (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A})(a^1 \neq \emptyset)((a+b)^1 \neq \emptyset) \subseteq (a^1 \leq_{\mathcal{D}\mathcal{X}}(a+b)^1) & (\mathcal{T}_{T644})
\end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(a^1 \neq \emptyset)((a+b)^1 \neq \emptyset) \subseteq (a^1 \leq_x (a+b)^1) \quad (\mathcal{T}_{R786})$$

Proof of ($\mathcal{T}_{R786}$)

$$\begin{aligned}
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{\subseteq\}b^1) \subseteq (\mathcal{X}a^1 \subseteq \{\leq\}\mathcal{X}b^1)) & (\mathcal{T}_{T615})(\text{Eqn1}) \\
& \text{Let } b^1 = (a+b)^1 & (\text{Eqn2}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{\subseteq\}(a+b)^1) \subseteq (\mathcal{X}a^1 \subseteq \{\leq\}\mathcal{X}(a+b)^1)) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(a^1 \subseteq \{\subseteq\}(a+b)^1) \subseteq (\mathcal{X}a^1 \subseteq \{\leq\}\mathcal{X}(a+b)^1) & (\mathcal{A}_{MX35})(\text{Exportation/Importation}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(a^1 \neq \emptyset)((a+b)^1 \neq \emptyset)(a^1 \subseteq \{\subseteq\}(a+b)^1) \subseteq (\mathcal{X}a^1 \subseteq \{\leq\}\mathcal{X}(a+b)^1) & (\mathcal{T}_{M089})(\text{Single Premise Syllogism})(\text{Eqn3}) \\
& a \subseteq a+b & (\mathcal{T}_{M082})(\text{Disjunction Introduction}) \\
& ((a+b)^1 \neq \emptyset) \subseteq ((a^1 \subseteq \{\subseteq\}(a+b)^1) & (\mathcal{T}_{S412}) \\
& ((a+b)^1 \neq \emptyset) = ((a+b)^1 \neq \emptyset)((a^1 \subseteq \{\subseteq\}(a+b)^1) & (\mathcal{A}_{MX44})(\subseteq \text{Equality Form})(\text{Eqn4}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(a^1 \neq \emptyset)((a+b)^1 \neq \emptyset) \subseteq (\mathcal{X}a^1 \subseteq \{\leq\}\mathcal{X}(a+b)^1) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(a^1 \neq \emptyset)((a+b)^1 \neq \emptyset) \subseteq (\mathcal{X}a^1 \leq \mathcal{X}(a+b)^1) & (\mathcal{T}_{T736})(\text{Definition of a Relationship}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(a^1 \neq \emptyset)((a+b)^1 \neq \emptyset) \subseteq (a^1 \leq_x (a+b)^1) & (\mathcal{T}_{T644})
\end{aligned}$$

$$(\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A})((a=b)^1 \neq \emptyset)((a=b) + (a \subset b))^1 \neq \emptyset \subseteq ((a=b)^1 \leq_{\mathcal{D}\mathcal{X}}(a \subseteq b)^1) \quad (\mathcal{T}_{R787b})$$

Proof of ($\mathcal{T}_{R787b}$)

$$\begin{aligned}
& (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A})(a^1 \neq \emptyset)((a+b)^1 \neq \emptyset) \subseteq (a^1 \leq_{\mathcal{D}\mathcal{X}}(a+b)^1) & (\mathcal{T}_{R786b}) \\
& (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A})((a=b)^1 \neq \emptyset)((a=b) + (a \subset b))^1 \neq \emptyset \subseteq & \\
& ((a=b)^1 \leq_{\mathcal{D}\mathcal{X}}((a=b) + (a \subset b))^1) & \text{Resymbolising} \\
& (\mathcal{X} = \mathcal{M}, \mathcal{T}, (\mathcal{L}_x c^1), (\mathcal{L}_{xy} c^1), (\mathcal{L}_{xyz} c^1), \mathcal{A})((a=b)^1 \neq \emptyset)((a=b) + (a \subset b))^1 \neq \emptyset \subseteq & \\
& ((a=b)^1 \leq_{\mathcal{D}\mathcal{X}}(a \subseteq b)^1) & (\mathcal{T}_{M266})(\text{Definition of } \subseteq)
\end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)((a=b)^1 \neq \emptyset)((a \subseteq b)^1 \neq \emptyset) \subseteq ((a=b)^1 \leq_x (a \subseteq b)^1) \quad (\mathcal{T}_{R787})$$

Proof of ($\mathcal{T}_{R787}$)

$$\begin{aligned}
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)(a^1 \neq \emptyset)((a+b)^1 \neq \emptyset) \subseteq (a^1 \leq_x (a+b)^1) & (\mathcal{T}_{R786}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)((a=b)^1 \neq \emptyset)((a=b) + (a \subset b))^1 \neq \emptyset \subseteq ((a=b)^1 \leq_x ((a=b) + (a \subset b))^1) & \\
& & \text{Resymbolising} \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0)((a=b)^1 \neq \emptyset)((a \subseteq b)^1 \neq \emptyset) \subseteq ((a=b)^1 \leq_x (a \subseteq b)^1) & (\mathcal{T}_{M266})(\text{Definition of } \subseteq)
\end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq ((\mathcal{X}\mathcal{E}.\{\subset\}^* \mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_0) \quad (\mathcal{T}_{T705})$$

Proof of ($\mathcal{T}_{T705}$)

$$\begin{aligned}
& (a^1 \subseteq \mathcal{S}_A) \subseteq ((a.\{\subset\}^* a)^1 \subseteq \mathcal{S}_0) & (\mathcal{A}_{R78})(\text{Axiom of Abstract Discrete Sets})(\text{Eqn1}) \\
& \mathcal{D}_A + \mathcal{P}_A \subseteq \mathcal{S}_A & (\mathcal{T}_{T770}) \\
& (\mathcal{D}_A + \mathcal{P}_A \subseteq \mathcal{S}_A) = \mathcal{E} & (\mathcal{A}_{MX16})(\text{Order Axiom}) \\
& (a^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (a^1 \subseteq \mathcal{S}_A) & (\mathcal{T}_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn2}) \\
& (a^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq ((a.\{\subset\}^* a)^1 \subseteq \mathcal{S}_0) & (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\
& \text{Let } a = \mathcal{X}\mathcal{E} & (\text{Eqn4})
\end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq ((\mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_D) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn4 into Eqn3})$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{P}_A) \subseteq ((\mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_D \mathcal{S}_U) \quad (\mathcal{T}_{7708})$$

Proof of ($\mathcal{T}_{7708}$)

$$\begin{aligned} ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) &\subseteq ((\mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_D) & (\mathcal{T}_{7705})(\text{Eqn1}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{P}_A) &\subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) & (\mathcal{T}_{M072})(\text{Single Premise Syllogism})(\text{Eqn2}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{P}_A) &\subseteq ((\mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_D) & (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{P}_A) &\subseteq ((\mathcal{X}\mathcal{a})^1 \subseteq \mathcal{S}_U) & (\mathcal{T}_{7919})(\text{Eqn4}) \\ \text{Let } (\mathcal{X}\mathcal{a})^1 &= (\mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E})^1 & (\text{Eqn5}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{P}_A) &\subseteq ((\mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_U) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn5 into Eqn4})(\text{Eqn6}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{P}_A) &\subseteq ((\mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_D \mathcal{S}_U) & (\mathcal{T}_{M132})(\text{Composition of Eqns3\&6}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E}) \subseteq (\alpha^1 = \{\supset\}^* \alpha^1) \quad (\mathcal{T}_{7707})$$

Proof of ($\mathcal{T}_{7707}$)

$$\begin{aligned} (\alpha^1 \subseteq \mathcal{S}_D)(\beta^1 \subseteq \alpha) &\subseteq (\beta^1 = \{\supset\}^* \beta^1) & (\mathcal{T}_{7128})(\text{Eqn1}) \\ \text{Let } \alpha &\subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E} & (\text{Eqn2}) \\ \text{Let } \beta &= \alpha & (\text{Eqn3}) \\ ((\mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E}) &\subseteq (\alpha^1 = \{\supset\}^* \alpha^1) & (\mathcal{A}_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E}) \subseteq (\alpha^1 = \{\supset\}^* \alpha^1) \quad (\mathcal{T}_{7729})$$

Proof of ($\mathcal{T}_{7729}$)

$$\begin{aligned} ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) &\subseteq ((\mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_D) & (\mathcal{T}_{7705}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E}) &\subseteq ((\mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E}) & (\mathcal{T}_{M029})(\text{Deduction})(\text{Eqn1}) \\ ((\mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E})^1 \subseteq \mathcal{S}_D)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E}) &\subseteq (\alpha^1 = \{\supset\}^* \alpha^1) & (\mathcal{T}_{7707})(\text{Eqn2}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^* \mathcal{X}\mathcal{E}) &\subseteq (\alpha^1 = \{\supset\}^* \alpha^1) & (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{S}_C) \subseteq (\mathcal{A}\mathcal{a}^1 \subseteq \{\supset\}^* \mathcal{X}\mathcal{a}^1) \quad (\mathcal{T}_{7771})$$

Proof of ($\mathcal{T}_{7771}$)

$$\begin{aligned} ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) &\subseteq (\forall \mathcal{A}\mathcal{X}\mathcal{a}^1 = (\forall \mathcal{X}\mathcal{X}\mathcal{a}^1)^1) & (\mathcal{A}_{S34})(\text{Axiom of Defining and Measurable Characteristics}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) &\subseteq (\{\subseteq\}^* \forall \mathcal{A}\mathcal{X}\mathcal{a}^1 = \{\subseteq\}^* (\forall \mathcal{X}\mathcal{X}\mathcal{a}^1)^1) & (\mathcal{T}_{R522})(\text{Relational Deduction}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{S}_C) &\subseteq (\{\subseteq\}^* \forall \mathcal{A}\mathcal{X}\mathcal{a}^1 = \forall \mathcal{X}\mathcal{X}\mathcal{a}^1) & (\mathcal{A}_{R02})(\text{Definition of Disaggregation for Continuum Sets}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{S}_C) &\subseteq (\forall \mathcal{A}\{\supset\}^* \mathcal{X}\mathcal{a}^1 = \forall \mathcal{X}\mathcal{X}\mathcal{a}^1) & (\mathcal{T}_{7829}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{S}_C) &\subseteq (\mathcal{A}^* \mathcal{A}\{\supset\}^* \mathcal{X}\mathcal{a}^1 = \mathcal{A}^* \mathcal{X}\mathcal{X}\mathcal{a}^1) & (\mathcal{T}_{R522})(\text{Relational Deduction}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{S}_C) &\subseteq (\{\supset\}^* \mathcal{X}\mathcal{a}^1 = \mathcal{A}^* \mathcal{X}\mathcal{X}\mathcal{a}^1) & (\mathcal{T}_{7878})(\text{Eqn1}) \\ \mathcal{a}^1 &\subseteq \forall \mathcal{X}\mathcal{X}\mathcal{a}^1 & (\mathcal{T}_{7348}) \\ \mathcal{A}\mathcal{a}^1 &\subseteq \mathcal{A}^* \mathcal{X}\mathcal{X}\mathcal{a}^1 & (\mathcal{T}_{R196})(\text{Relational Deduction})(\text{Eqn2}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{S}_C) &\subseteq (\mathcal{A}\mathcal{a}^1 \subseteq \{\supset\}^* \mathcal{X}\mathcal{a}^1) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \end{aligned}$$

$$(\mathcal{a}^1 \neq \emptyset) = (\forall \mathcal{A}\mathcal{T}\mathcal{a}^1 \neq \emptyset) \quad (\mathcal{T}_{7602})$$

Proof of ($\mathcal{T}_{7602}$)

$$\begin{aligned} \text{Let } \mathcal{a}^1 &= \emptyset \\ \mathcal{T}\mathcal{a}^1 &= \mathcal{T}\emptyset & (\mathcal{T}_{R522})(\text{Relational Deduction}) \end{aligned}$$

$\mathcal{T}a^1 = \emptyset$	($\mathcal{T}_{S181}$)(Null Object Theorem)
$\forall \mathcal{A}\mathcal{T}a^1 = \forall \mathcal{A}\emptyset$	($\mathcal{T}_{R522}$)(Relational Deduction)
$\forall \mathcal{A}\mathcal{T}a^1 = \emptyset$	($\mathcal{T}_{S181}$)(Null Object Theorem)
Thus $(a^1 = \emptyset) \subseteq (\forall \mathcal{A}\mathcal{T}a^1 = \emptyset)$	($\mathcal{A}_{MX02}$)(Conditional Introduction)(Eqn1)
Let $\forall \mathcal{A}\mathcal{T}a^1 = \emptyset$	
$\mathcal{A}^{\forall} \mathcal{A}\mathcal{T}a^1 = \mathcal{A}\emptyset$	($\mathcal{T}_{R522}$)(Relational Deduction)
$\mathcal{A}^{\forall} \mathcal{A}\mathcal{T}a^1 = \emptyset$	($\mathcal{T}_{S181}$)(Null Object Theorem)
$\forall \mathcal{A}\mathcal{A}\mathcal{T}a^1 = \emptyset$	($\mathcal{T}_{T877}$)
$\mathcal{T}a^1 = \emptyset$	($\mathcal{T}_{T878}$)
$\forall \mathcal{T}\mathcal{T}a^1 = \forall \mathcal{T}\emptyset$	($\mathcal{T}_{R522}$)(Relational Deduction)
$\forall \mathcal{T}\mathcal{T}a^1 = \emptyset$	($\mathcal{T}_{S181}$)(Null Object Theorem)
Thus $(\forall \mathcal{A}\mathcal{T}a^1 = \emptyset) \subseteq (\forall \mathcal{T}\mathcal{T}a^1 = \emptyset)$	($\mathcal{A}_{MX02}$)(Conditional Introduction)(Eqn2)
$((a \subseteq \mathcal{T}\mathcal{E}) = \mathcal{E}) \subseteq ((a \neq \emptyset) = (\forall \mathcal{T}a \neq \emptyset))$	($\mathcal{T}_{T274}\mathcal{T}$)(Eqn3)
Let $a = \mathcal{T}a^1$	(Eqn4)
$((\mathcal{T}a^1 \subseteq \mathcal{T}\mathcal{E}) = \mathcal{E}) \subseteq ((\mathcal{T}a^1 \neq \emptyset) = (\forall \mathcal{T}\mathcal{T}a^1 \neq \emptyset))$	($\mathcal{A}_{MX19}$)(Substitution of Eqn4 into Eqn3)
$((\mathcal{T}a^1 \subseteq \mathcal{T}\mathcal{E}) = \mathcal{E}) \subseteq ((\mathcal{T}a^1 = \emptyset) = (\forall \mathcal{T}\mathcal{T}a^1 = \emptyset))$	($\mathcal{T}_{M008}$)(Inverse of an Equality)(Eqn4)
$a^1 \subseteq \mathcal{E}$	($\mathcal{T}_{M744}$)(Universal Set Theorem)
$\mathcal{T}a^1 \subseteq \mathcal{T}\mathcal{E}$	($\mathcal{T}_{R522}$)(Relational Deduction)
$(\mathcal{T}a^1 \subseteq \mathcal{T}\mathcal{E}) = \mathcal{E}$	($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn5)
$(\mathcal{E} = \mathcal{E}) \subseteq ((\mathcal{T}a^1 = \emptyset) = (\forall \mathcal{T}\mathcal{T}a^1 = \emptyset))$	($\mathcal{A}_{MX19}$)(Substitution of Eqn5 into Eqn4)
$\mathcal{E} \subseteq ((\mathcal{T}a^1 = \emptyset) = (\forall \mathcal{T}\mathcal{T}a^1 = \emptyset))$	($\mathcal{T}_{M342}$)
$\mathcal{E} = ((\mathcal{T}a^1 = \emptyset) = (\forall \mathcal{T}\mathcal{T}a^1 = \emptyset))$	($\mathcal{T}_{M275}$)(Supersets of the Universal Set)
$(\mathcal{T}a^1 = \emptyset) = (\forall \mathcal{T}\mathcal{T}a^1 = \emptyset)$	($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn6)
$(\forall \mathcal{A}\mathcal{T}a^1 = \emptyset) \subseteq (\mathcal{T}a^1 = \emptyset)$	($\mathcal{A}_{MX19}$)(Substitution of Eqn6 into Eqn2)(Eqn7)
$(a \neq \emptyset) = (\mathcal{T}a \neq \emptyset)$	($\mathcal{T}_{T275}\mathcal{T}$)
$(a = \emptyset) = (\mathcal{T}a = \emptyset)$	($\mathcal{T}_{M008}$)(Inverse of an Equality)(Eqn8)
$(\forall \mathcal{A}\mathcal{T}a^1 = \emptyset) \subseteq (a^1 = \emptyset)$	($\mathcal{A}_{MX19}$)(Substitution of Eqn8 into Eqn7)(Eqn9)
$(a^1 = \emptyset) = (\forall \mathcal{A}\mathcal{T}a^1 = \emptyset)$	($\mathcal{T}_{M430}$)(Biconditional Introduction Eqns1&9)
$(a^1 \neq \emptyset) = (\forall \mathcal{A}\mathcal{T}a^1 \neq \emptyset)$	($\mathcal{T}_{M008}$)(Inverse of an Equality)

12.3 Proofs of Theorems in the @ Function

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (((@(\forall \mathcal{X}a^1)^1)b^1 = \{\subseteq\}(\forall \mathcal{X}a^1)^1.\{\subseteq\}b^1.\{\{\subseteq\}*(\{\subseteq\}(\forall \mathcal{X}a^1)^1.\{\subseteq\}b^1)) \quad (\mathcal{T}_{T285})$$

Proof of ($\mathcal{T}_{T285}$)

$(@a^1)b^1 = \{\subseteq\}a^1.\{\subseteq\}b^1.\{\{\subseteq\}*(\{\subseteq\}a^1.\{\subseteq\}b^1)$	($\mathcal{D}_{R05}$)(Definition of @)(Eqn1)
Let $a^1 = (\forall \mathcal{X}a^1)^1$	(Eqn2)
where $a^1 \subseteq \mathcal{X}\mathcal{E}$	
and where $(\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A$	
$((@(\forall \mathcal{X}a^1)^1)b^1 = \{\subseteq\}(\forall \mathcal{X}a^1)^1.\{\subseteq\}b^1.\{\{\subseteq\}*(\{\subseteq\}(\forall \mathcal{X}a^1)^1.\{\subseteq\}b^1)$	($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)
Thus $((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (((@(\forall \mathcal{X}a^1)^1)b^1 = \{\subseteq\}(\forall \mathcal{X}a^1)^1.\{\subseteq\}b^1.\{\{\subseteq\}*(\{\subseteq\}(\forall \mathcal{X}a^1)^1.\{\subseteq\}b^1))$	($\mathcal{A}_{MX02}$)(Conditional Introduction)

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (((@(\{=\}_x a^1)^1)b^1 = \{\subseteq\}(\{=\}_x a^1)^1.\{\subseteq\}b^1.\{\{\subseteq\}*(\{\subseteq\}(\{=\}_x a^1)^1.\{\subseteq\}b^1)) \quad (\mathcal{T}_{T286})$$

Proof of ($\mathcal{T}_{T286}$)

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (((@(\forall \mathcal{X}a^1)^1)b^1 = \{\subseteq\}(\forall \mathcal{X}a^1)^1.\{\subseteq\}b^1.\{\{\subseteq\}*(\{\subseteq\}(\forall \mathcal{X}a^1)^1.\{\subseteq\}b^1))$	($\mathcal{T}_{T285}$)(Eqn1)
Let $a^1 = \mathcal{X}a^1$	(Eqn2)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{X}a^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (((@(\forall \mathcal{X}\mathcal{X}a^1)^1)b^1 = \{\subseteq\}(\forall \mathcal{X}\mathcal{X}a^1)^1.\{\subseteq\}b^1.\{\{\subseteq\}*(\{\subseteq\}(\forall \mathcal{X}\mathcal{X}a^1)^1.\{\subseteq\}b^1))$	($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)(Eqn3)

$$\begin{aligned}
a^1 &\subseteq E & (T_{M744})(\text{Universal Set Theorem}) \\
\chi a^1 &\subseteq \chi E & (T_{R196})(\text{Relational Deduction}) \\
(\chi a^1 \subseteq \chi E) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\
((\chi E)^1 \subseteq D_A + P_A)E &\subseteq (((@(\chi \chi a^1)^1)b^1 = \{\subseteq\}(\chi \chi a^1)^1.\{\subseteq\}b^1.\{\subseteq\}(\chi \chi a^1)^1.\{\subseteq\}b^1)) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\
((\chi E)^1 \subseteq D_A + P_A) &\subseteq (((@(\chi \chi a^1)^1)b^1 = \{\subseteq\}(\chi \chi a^1)^1.\{\subseteq\}b^1.\{\subseteq\}(\chi \chi a^1)^1.\{\subseteq\}b^1)) & (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\
((\chi VE)^1 \subseteq D_A + P_A) &\subseteq (((@(\chi \{=\} \chi a^1)^1)b^1 = \{\subseteq\}(\chi \{=\} \chi a^1)^1.\{\subseteq\}b^1.\{\subseteq\}(\chi \{=\} \chi a^1)^1.\{\subseteq\}b^1)) & (T_{A080}) \\
((\chi VE)^1 \subseteq D_A + P_A) &\subseteq (((@(\{=\}_x a^1)^1)b^1 = \{\subseteq\}(\{=\}_x a^1)^1.\{\subseteq\}b^1.\{\subseteq\}(\{=\}_x a^1)^1.\{\subseteq\}b^1)) & (T_{R571})(\text{Definition of a mapped relationship})
\end{aligned}$$

12.4 Proofs of Theorems in the @_x Function

$$((\chi E)^1 \subseteq D_A + P_A)(b^1 \subseteq \chi E) \subseteq ((@_x b^1) * a^1 = (@(\chi b^1)^1)a^1) \quad (T_{7908})$$

Proof of (T₇₉₀₈)

$$\begin{aligned}
((\chi E)^1 \subseteq D_A + P_A)(b^1 \subseteq \chi E) &\subseteq ((@_x b^1) * a^1 = \{\subseteq\}a^1.\{\subseteq\}(\chi b^1)^1.\{\subseteq\}(\{=\}_x a^1.\{\subseteq\}(\chi b^1)^1)) & (D_{R35})(\text{Definition of } @_x)(\text{Eqn1}) \\
(@a^1)b^1 &= \{\subseteq\}a^1.\{\subseteq\}b^1.\{\subseteq\}(\{=\}_x a^1.\{\subseteq\}b^1) & (D_{R05})(\text{Definition of } @) \\
(@b^1)a^1 &= \{\subseteq\}a^1.\{\subseteq\}b^1.\{\subseteq\}(\{=\}_x a^1.\{\subseteq\}b^1) & (T_{7276})(\text{Eqn2}) \\
\text{Let } b^1 &= (\chi b^1)^1 & (\text{Eqn3}) \\
(@(\chi b^1)^1)a^1 &= \{\subseteq\}a^1.\{\subseteq\}(\chi b^1)^1.\{\subseteq\}(\{=\}_x a^1.\{\subseteq\}(\chi b^1)^1) & (A_{MX19})(\text{Substitution of Eqn3 into Eqn2})(\text{Eqn4}) \\
((\chi E)^1 \subseteq D_A + P_A)(b^1 \subseteq \chi E) &\subseteq ((@_x b^1) * a^1 = (@(\chi b^1)^1)a^1) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn1})
\end{aligned}$$

$$((\chi E)^1 \subseteq D_A + P_A)(\chi a^1 \neq \emptyset)(\chi a^1 \neq (\chi a^1)^1) \subseteq ((\chi a^1)^1 \subseteq S_c S_u) \quad (T_{7731})$$

Proof of (T₇₇₃₁)

$$\begin{aligned}
(a^1 \subseteq S_c S_u)(b \subseteq a)(b \neq \emptyset)(b \neq b^1) &\subseteq (b^1 \subseteq S_c S_u) & (T_{7730})(\text{Eqn1}) \\
\text{Let } a &= \chi E & (\text{Eqn2}) \\
\text{Let } b &= \chi a^1 & (\text{Eqn3}) \\
((\chi E)^1 \subseteq S_c S_u)(\chi a^1 \subseteq \chi E)(\chi a^1 \neq \emptyset)(\chi a^1 \neq (\chi a^1)^1) &\subseteq ((\chi a^1)^1 \subseteq S_c S_u) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})(\text{Eqn4}) \\
a^1 &\subseteq E & (T_{M744})(\text{Universal Set Theorem}) \\
\chi a^1 &\subseteq \chi E & (T_{R196})(\text{Relational Deduction}) \\
(\chi a^1 \subseteq \chi E) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn5}) \\
((\chi E)^1 \subseteq S_c S_u)E(\chi a^1 \neq \emptyset)(\chi a^1 \neq (\chi a^1)^1) &\subseteq ((\chi a^1)^1 \subseteq S_c S_u) & (A_{MX19})(\text{Substitution of Eqn5 into Eqn4}) \\
((\chi E)^1 \subseteq S_c S_u)(\chi a^1 \neq \emptyset)(\chi a^1 \neq (\chi a^1)^1) &\subseteq ((\chi a^1)^1 \subseteq S_c S_u) & (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn6}) \\
((\chi E)^1 \subseteq D_A + P_A) &\subseteq ((Q\chi)^1 \subseteq S_c S_u) & (A_{R79})(\text{Axiom of the Physical Universe}) \\
((\chi E)^1 \subseteq D_A + P_A) &\subseteq ((\chi E)^1 \subseteq S_c S_u) & (T_{R211}) \\
((\chi E)^1 \subseteq D_A + P_A)(\chi a^1 \neq \emptyset)(\chi a^1 \neq (\chi a^1)^1) &\subseteq ((\chi E)^1 \subseteq S_c S_u)(\chi a^1 \neq \emptyset)(\chi a^1 \neq (\chi a^1)^1) & (T_{M029})(\text{Deduction})(\text{Eqn7}) \\
((\chi E)^1 \subseteq D_A + P_A)(\chi a^1 \neq \emptyset)(\chi a^1 \neq (\chi a^1)^1) &\subseteq ((\chi a^1)^1 \subseteq S_c S_u) & (T_{M139})(\text{Hypothetical Syllogism Eqns7\&6})
\end{aligned}$$

$$((\chi E)^1 \subseteq D_A + P_A)(a^1 \subseteq \chi E) \subseteq ((\chi a^1)^1 \neq \emptyset^1) \quad (T_{S598})$$

$$((\chi E)^1 \subseteq D_A + P_A)(a^1 \subseteq \chi E) \subseteq ((\chi a^1)^1 \neq E^1) \quad (T_{S599})$$

Example Proof of (T₅₉₈)

$$\begin{aligned}
 &(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{Y} = \emptyset^1, \mathcal{E}^1) = \mathcal{E}) \subseteq ((\mathcal{a} = \mathcal{Y}) \subseteq (\mathcal{X} * \mathcal{a}^1 = \emptyset)).((\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \neq \mathcal{Y})) \\
 &\hspace{15em} (\mathcal{A}_{R92})(\emptyset^1 \text{ and } \mathcal{E}^1 \text{ do not Map}) \\
 &(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{Y} = \emptyset^1, \mathcal{E}^1) = \mathcal{E}) \subseteq ((\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \neq \mathcal{Y})) \\
 &\hspace{15em} (\mathcal{T}_{M073})(\text{Single Premise Syllogism}) \\
 &(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \neq \emptyset^1)) \hspace{1em} \text{Extraction} \\
 &((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \neq \emptyset^1) \hspace{1em} (\mathcal{A}_{MX35})(\text{Exportation/Importation for truth sets})
 \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \{\subseteq\}\mathcal{V}\mathcal{X}\mathcal{a}^1) \hspace{1em} (\mathcal{T}_{7733})$$

Proof of (T₇₇₃₃)

$$\begin{aligned}
 &((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \subseteq ((\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \subseteq \mathcal{S}_c\mathcal{S}_u) \\
 &\hspace{15em} (\mathcal{T}_{7731})(\text{Eqn1}) \\
 &((\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \subseteq \mathcal{S}_c\mathcal{S}_u) \subseteq ((\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \subseteq \mathcal{S}_c) \hspace{1em} (\mathcal{T}_{M073})(\text{Single Premise Syllogism})(\text{Eqn2}) \\
 &((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \subseteq ((\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \subseteq \mathcal{S}_c) \\
 &\hspace{15em} (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3}) \\
 &(\mathcal{a}^1 \subseteq \mathcal{S}_c) \subseteq (\{\subseteq\} * \mathcal{a}^1 = \{\subseteq\} * \mathcal{a}) \hspace{1em} (\mathcal{T}_{S378})(\text{Eqn4}) \\
 &\text{Let } \mathcal{a} = \mathcal{V}\mathcal{X}\mathcal{a}^1 \hspace{15em} (\text{Eqn5}) \\
 &((\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \subseteq \mathcal{S}_c) \subseteq (\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) \hspace{1em} (\mathcal{A}_{MX19})(\text{Substitution of Eqn5 into Eqn4})(\text{Eqn6}) \\
 &((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \subseteq (\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) \\
 &\hspace{15em} (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns3\&6})(\text{Eqn7}) \\
 &\text{Let } \mathcal{V}\mathcal{X}\mathcal{a}^1 = \emptyset \hspace{15em} (\text{Eqn8}) \\
 &((\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \emptyset) + ((\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \emptyset^1) \hspace{1em} (\mathcal{T}_{M496}) \\
 &((\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \emptyset) + \emptyset \hspace{15em} (\mathcal{T}_{M024})(\text{Non-existence of anti-}(\mathcal{T}_{598})) \\
 &(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \emptyset \hspace{15em} (\mathcal{T}_{M668})(\text{Null Set Disjunction})(\text{Eqn9}) \\
 &\text{Let } \{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \neq \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1 \hspace{1em} (\text{Eqn10}) \\
 &\{\subseteq\} * \emptyset \neq \{\subseteq\} * \emptyset \hspace{15em} (\mathcal{A}_{MX19})(\text{Substitution of Eqns8\&9 into Eqn10}) \\
 &\emptyset \neq \emptyset \hspace{15em} (\mathcal{T}_{S181})(\text{Null Object Theorem}) \\
 &\emptyset \hspace{15em} (\mathcal{T}_{M346}) \\
 &\text{Thus } (\mathcal{V}\mathcal{X}\mathcal{a}^1 = \emptyset)(\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \neq \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) = \emptyset \hspace{1em} (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\
 &(\mathcal{V}\mathcal{X}\mathcal{a}^1 = \emptyset) \subseteq (\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) \hspace{1em} (\mathcal{T}_{M735})(\subseteq \text{Existential Form})(\text{Eqn11}) \\
 &\text{Let } \mathcal{V}\mathcal{X}\mathcal{a}^1 = (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \hspace{15em} (\text{Eqn12}) \\
 &\text{Let } \{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \neq \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1 \hspace{1em} (\text{Eqn13}) \\
 &\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \neq \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1 \hspace{15em} (\mathcal{A}_{MX19})(\text{Substitution of Eqn12 into Eqn13}) \\
 &\emptyset \hspace{15em} (\mathcal{T}_{M024})(\text{Non-existence of Anti-}(\mathcal{D}_{M00}) \text{ Definition of } =) \\
 &\text{Thus } (\mathcal{V}\mathcal{X}\mathcal{a}^1 = (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1)(\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 \neq \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) = \emptyset \hspace{1em} (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\
 &(\mathcal{V}\mathcal{X}\mathcal{a}^1 = (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \subseteq (\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) \hspace{1em} (\mathcal{T}_{M735})(\subseteq \text{Existential Form})(\text{Eqn14}) \\
 &(\mathcal{V}\mathcal{X}\mathcal{a}^1 = \emptyset) + (\mathcal{V}\mathcal{X}\mathcal{a}^1 = (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \subseteq (\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) + (\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) \\
 &\hspace{15em} (\mathcal{A}_{MX21})(\text{Combination of Eqns11\&14}) \\
 &(\mathcal{V}\mathcal{X}\mathcal{a}^1 = \emptyset) + (\mathcal{V}\mathcal{X}\mathcal{a}^1 = (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \subseteq (\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) \\
 &\hspace{15em} (\mathcal{A}_{MX01})(\text{Disjunction Tautology}) \\
 &\sim(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \subseteq (\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) \\
 &\hspace{15em} (\mathcal{T}_{M025})(\text{DeMorgan's Law}) \\
 &((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A).\sim(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \subseteq (\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) \\
 &\hspace{15em} (\mathcal{T}_{M089})(\text{Single Premise Syllogism})(\text{Eqn15}) \\
 &((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) + ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A).\sim(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1)) \\
 &\subseteq (\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) + (\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) \\
 &\hspace{15em} (\mathcal{A}_{MX21})(\text{Combination of Eqns7\&15}) \\
 &((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) + ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A).\sim(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1)) \\
 &\subseteq (\{\subseteq\} * (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 = \{\subseteq\} * \mathcal{V}\mathcal{X}\mathcal{a}^1) \hspace{1em} (\mathcal{A}_{MX01})(\text{Disjunction Tautology}) \\
 &((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)((\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) + \sim(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq \emptyset)(\mathcal{V}\mathcal{X}\mathcal{a}^1 \neq (\mathcal{V}\mathcal{X}\mathcal{a}^1)^1))
 \end{aligned}$$

$$\begin{aligned}
&\subseteq (\{\subseteq\} * (\forall \mathcal{X}a^1)^1 = \{\subseteq\} * \forall \mathcal{X}a^1) && (T_{M407})(\text{Distribution of } . \text{ over } *) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A)E \subseteq (\{\subseteq\} * (\forall \mathcal{X}a^1)^1 = \{\subseteq\} * \forall \mathcal{X}a^1) && (T_{M671})(\text{Tertium non Datur}) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A) \subseteq (\{\subseteq\} * (\forall \mathcal{X}a^1)^1 = \{\subseteq\} * \forall \mathcal{X}a^1) && (A_{MX11})(\text{Universal Set Conjunction Axiom})
\end{aligned}$$

$$((\mathcal{X}E)^1 \subseteq D_A + P_A)(\mathcal{b}^1 \subseteq \mathcal{X}E) \subseteq ((@_x \mathcal{b}^1) * a^1 = \{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1} . (\{\subseteq\} * (\{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1}))) \quad (T_{T732})$$

Proof of (T_{T732})

$$\begin{aligned}
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(\mathcal{b}^1 \subseteq \mathcal{X}E) \subseteq ((@_x \mathcal{b}^1) * a^1 = \{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1} . (\{\subseteq\} * (\{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1}))) \\
&\quad (D_{R35})(\text{Definition of the } @_x \text{ Function})(\text{Eqn1}) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(\mathcal{b}^1 \subseteq \mathcal{X}E) \subseteq ((@_x \mathcal{b}^1) * a^1 = \{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1} . (\{\subseteq\} * (\{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1}))) \\
&\quad (T_{T733})
\end{aligned}$$

$$((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq ((@_x a^1) * E^1 = \forall \mathcal{X}a^1) \quad (T_{T295})$$

Proof of (T_{T295})

$$\begin{aligned}
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(\mathcal{b}^1 \subseteq \mathcal{X}E) \subseteq ((@_x \mathcal{b}^1) * a^1 = \{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1} . (\{\subseteq\} * (\{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1}))) \\
&\quad (T_{T732})(\text{Eqn1}) \\
&\text{Let } a^1 = E^1 && (\text{Eqn2}) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(\mathcal{b}^1 \subseteq \mathcal{X}E) \subseteq ((@_x \mathcal{b}^1) * E^1 = \{\subseteq\} E^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1} . (\{\subseteq\} * (\{\subseteq\} E^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1}))) \\
&\quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(\mathcal{b}^1 \subseteq \mathcal{X}E) \subseteq ((@_x \mathcal{b}^1) * E^1 = E . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1} . (\{\subseteq\} * (E . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1}))) \\
&\quad (T_{A644}) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq ((@_x a^1) * E^1 = E . \{\subseteq\}^{\forall \mathcal{X} a^1} . (\{\subseteq\} * (E . \{\subseteq\}^{\forall \mathcal{X} a^1}))) \\
&\quad \text{Resymbolising} \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq ((@_x a^1) * E^1 = \{\subseteq\}^{\forall \mathcal{X} a^1} . (\{\subseteq\} * (\{\subseteq\}^{\forall \mathcal{X} a^1}))) \\
&\quad (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq ((@_x a^1) * E^1 = \forall \mathcal{X}a^1) && (T_{T271})
\end{aligned}$$

$$((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq ((@_x a^1) * E = \{\subseteq\}^{\forall \mathcal{X} a^1}) \quad (T_{S735})$$

$$((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq (S(@_x a^1) = \{\subseteq\}^{\forall \mathcal{X} a^1}) \quad (T_{T254})$$

Proof of (T_{S735}) & (T_{T254})

$$\begin{aligned}
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq ((@_x a^1) * E^1 = \forall \mathcal{X}a^1) && (T_{T295}) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq (\{\subseteq\}(@_x a^1) * E^1 = \{\subseteq\}^{\forall \mathcal{X} a^1}) && (T_{R196})(\text{Relational Deduction}) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq ((@_x a^1) * \{\subseteq\} E^1 = \{\subseteq\}^{\forall \mathcal{X} a^1}) && (A_{S10})(@_x \text{ Function Axiom}) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq ((@_x a^1) * E = \{\subseteq\}^{\forall \mathcal{X} a^1}) && (T_{A644}) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq (S(@_x a^1) = \{\subseteq\}^{\forall \mathcal{X} a^1}) && (T_{R210})
\end{aligned}$$

$$((\mathcal{X}E)^1 \subseteq D_A + P_A) \subseteq ((@_x \mathcal{X}a^1) a^1 = a^1) \quad (T_{T257})$$

Proof of (T_{T257})

$$\begin{aligned}
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(\mathcal{b}^1 \subseteq \mathcal{X}E) \subseteq ((@_x \mathcal{b}^1) * a^1 = \{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1} . (\{\subseteq\} * (\{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{b}^1}))) \\
&\quad (T_{T732})(\text{Eqn1}) \\
&\text{Let } \mathcal{b}^1 = \mathcal{X}a^1 && (\text{Eqn2}) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A)(\mathcal{X}a^1 \subseteq \mathcal{X}E) \subseteq ((@_x \mathcal{X}a^1) * a^1 = \{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{X}a^1} . (\{\subseteq\} * (\{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{X}a^1}))) \\
&\quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\
&a^1 \subseteq E && (T_{M744})(\text{Universal Set Theorem}) \\
&\mathcal{X}a^1 \subseteq \mathcal{X}E && (T_{R196})(\text{Relational Deduction}) \\
&(\mathcal{X}a^1 \subseteq \mathcal{X}E) = E && (A_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A)E \subseteq ((@_x \mathcal{X}a^1) * a^1 = \{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{X}a^1} . (\{\subseteq\} * (\{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{X}a^1}))) \\
&\quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\
&((\mathcal{X}E)^1 \subseteq D_A + P_A) \subseteq ((@_x \mathcal{X}a^1) * a^1 = \{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{X}a^1} . (\{\subseteq\} * (\{\subseteq\} a^1 . \{\subseteq\}^{\forall \mathcal{X} \mathcal{X}a^1}))) \\
&\quad (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn5}) \\
&a^1 \subseteq E && (T_{M744})(\text{Universal Set Theorem})(\text{Eqn6})
\end{aligned}$$

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\mathcal{Q}\mathcal{X} = \mathcal{E})$	($\mathcal{A}_{S36}$)(Axiom of Universal Mapping)(Eqn7)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\mathcal{a}^1 \subseteq \mathcal{Q}\mathcal{X})$	($\mathcal{A}_{MX19}$)(Substitution of Eqn7 into Eqn6)(Eqn8)
$(\mathcal{a}^1 \subseteq \mathcal{Q}\mathcal{X}) \subseteq (\mathcal{a}^1 \subseteq \vee \mathcal{X}\mathcal{X}\mathcal{a}^1)$	($\mathcal{T}_{R526}$)(Eqn9)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\mathcal{a}^1 \subseteq \vee \mathcal{X}\mathcal{X}\mathcal{a}^1)$	($\mathcal{T}_{M139}$)(Hypothetical Syllogism Eqns8&9)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\{\subseteq\}\mathcal{a}^1 \subseteq \{\subseteq\} \vee \mathcal{X}\mathcal{X}\mathcal{a}^1)$	($\mathcal{T}_{R196}$)(Relational Deduction)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\{\subseteq\}\mathcal{a}^1 = \{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{X}\mathcal{a}^1)$	($\mathcal{A}_{MX44}$)($\subseteq$ Equality Form)(Eqn10)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq ((@_x \mathcal{X}\mathcal{a}^1) * \mathcal{a}^1 = \{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\}\mathcal{a}^1 \cdot (\{\subseteq\} \cdot \{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\}\mathcal{a}^1))$	($\mathcal{A}_{MX19}$)(Substitution of Eqn10 into Eqn5)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq ((@_x \mathcal{X}\mathcal{a}^1) * \mathcal{a}^1 = \{\subseteq\}\mathcal{a}^1 \cdot (\{\subseteq\} * \{\subseteq\}\mathcal{a}^1))$	($\mathcal{T}_{M667}$)(Conjunction Tautology)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq ((@_x \mathcal{X}\mathcal{a}^1) * \mathcal{a}^1 = \mathcal{a}^1)$	($\mathcal{T}_{T271}$)

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq ((\{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{b}^1) = \emptyset) = ((@_x \mathcal{b}^1) * \mathcal{a}^1 = \emptyset)) \quad (\mathcal{T}_{T297})$$

Proof of ($\mathcal{T}_{T297}$)

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((@_x \mathcal{b}^1) * \mathcal{a}^1 = \{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{b}^1 \cdot (\{\subseteq\} * (\{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{b}^1)))$	($\mathcal{T}_{T732}$)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((@_x \mathcal{b}^1) * \mathcal{a}^1 = \{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{b}^1)$	($\mathcal{T}_{M073b}$)(Single Premise Syllogism)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((@_x \mathcal{b}^1) * \mathcal{a}^1 = \{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{b}^1) = \mathcal{E}$	($\mathcal{A}_{MX16}$)(Order Axiom)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E} \subseteq (((@_x \mathcal{b}^1) * \mathcal{a}^1 = \{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{b}^1) = \mathcal{E})$	($\mathcal{T}_{M094}$)(Probabilistic Rule)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E} \subseteq ((\{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{b}^1) = \emptyset) = ((@_x \mathcal{b}^1) * \mathcal{a}^1 = \emptyset))$	($\mathcal{T}_{M097}$)(Probabilistic Rule)

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{X}\mathcal{b}^1) = \emptyset) = ((@_x \mathcal{X}\mathcal{b}^1) * \mathcal{a}^1 = \emptyset)) \quad (\mathcal{T}_{T293})$$

Proof of ($\mathcal{T}_{T293}$)

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E} \subseteq ((\{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{b}^1) = \emptyset) = ((@_x \mathcal{b}^1) * \mathcal{a}^1 = \emptyset))$	($\mathcal{T}_{T297}$)(Eqn1)
Let $\mathcal{b}^1 = \mathcal{X}\mathcal{b}^1$	(Eqn2)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{X}\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E} \subseteq ((\{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{X}\mathcal{b}^1) = \emptyset) = ((@_x \mathcal{X}\mathcal{b}^1) * \mathcal{a}^1 = \emptyset))$	($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)(Eqn3)
$\mathcal{b}^1 \subseteq \mathcal{E}$	($\mathcal{T}_{M744}$)(Universal Set Theorem)
$\mathcal{X}\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}$	($\mathcal{T}_{R196}$)(Relational Deduction)
$(\mathcal{X}\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}$	($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn4)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)\mathcal{E} = \mathcal{E} \subseteq ((\{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{X}\mathcal{b}^1) = \emptyset) = ((@_x \mathcal{X}\mathcal{b}^1) * \mathcal{a}^1 = \emptyset))$	($\mathcal{A}_{MX19}$)(Substitution of Eqn4 into Eqn3)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E} \subseteq ((\{\subseteq\}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{X}\mathcal{b}^1) = \emptyset) = ((@_x \mathcal{X}\mathcal{b}^1) * \mathcal{a}^1 = \emptyset))$	($\mathcal{A}_{MX11}$)(Universal Set Conjunction Axiom)

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\{\supset\}\mathcal{X}\mathcal{a}^1 \cdot \{\supset\}\mathcal{X}\mathcal{b}^1 = \emptyset) \subseteq ((@_x \mathcal{X}\mathcal{b}^1) * \mathcal{a}^1 = \emptyset)) \quad (\mathcal{T}_{T298})$$

Proof of ($\mathcal{T}_{T298}$)

$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 + \mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E} \subseteq ((\{\supset\}\mathcal{a}^1 \cdot \{\supset\}\mathcal{b}^1 = \emptyset) \subseteq (\{\subseteq\} \vee \mathcal{X}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{b}^1 = \emptyset))$	($\mathcal{T}_{T300}$)(Eqn1)
Let $\mathcal{a}^1 = \mathcal{X}\mathcal{a}^1$	(Eqn2)
Let $\mathcal{b}^1 = \mathcal{X}\mathcal{b}^1$	(Eqn3)
$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{X}\mathcal{a}^1 + \mathcal{X}\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E} \subseteq ((\{\supset\}\mathcal{X}\mathcal{a}^1 \cdot \{\supset\}\mathcal{X}\mathcal{b}^1 = \emptyset) \subseteq ((\{\subseteq\} \vee \mathcal{X}\mathcal{X}\mathcal{a}^1 \cdot \{\subseteq\} \vee \mathcal{X}\mathcal{X}\mathcal{b}^1 = \emptyset))$	($\mathcal{A}_{MX19}$)(Substitution of Eqns2&3 into Eqn1)(Eqn4)
$\mathcal{a}^1 + \mathcal{b}^1 \subseteq \mathcal{E}$	($\mathcal{T}_{M744}$)(Universal Set Theorem)
$\mathcal{X}(\mathcal{a}^1 + \mathcal{b}^1) \subseteq \mathcal{X}\mathcal{E}$	($\mathcal{T}_{R196}$)(Relational Deduction)
$\mathcal{X}\mathcal{a}^1 + \mathcal{X}\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}$	($\mathcal{T}_{R002b}$)(Distribution of $*$ over $+$)
$(\mathcal{X}\mathcal{a}^1 + \mathcal{X}\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}$	($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn5)

$$\begin{aligned}
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \mathcal{E} = \mathcal{E}) \subseteq ((\{\supset\}\mathcal{X}\mathcal{a}^1, \{\supset\}\mathcal{X}\mathcal{b}^1 = \emptyset) \subseteq ((\{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{b}^1 = \emptyset)) \\
& \hspace{15em} (\mathbf{A}_{MX19}) (\text{Substitution of Eqn5 into Eqn4}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\{\supset\}\mathcal{X}\mathcal{a}^1, \{\supset\}\mathcal{X}\mathcal{b}^1 = \emptyset) \subseteq ((\{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{b}^1 = \emptyset)) \\
& \hspace{15em} (\mathbf{A}_{MX11}) (\text{Universal Set Conjunction Axiom}) (\text{Eqn6}) \\
& \mathcal{a}^1 \subseteq \mathcal{E} \hspace{15em} (\mathbf{T}_{M744}) (\text{Universal Set Theorem}) (\text{Eqn7}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\mathcal{Q}\mathcal{X} = \mathcal{E}) \hspace{15em} (\mathbf{A}_{S36}) (\text{Axiom of Universal Mapping}) (\text{Eqn8}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\mathcal{a}^1 \subseteq \mathcal{Q}\mathcal{X}) \hspace{15em} (\mathbf{A}_{MX19}) (\text{Substitution of Eqn8 into Eqn7}) (\text{Eqn9}) \\
& (\mathcal{a}^1 \subseteq \mathcal{Q}\mathcal{X}) \subseteq (\mathcal{a}^1 \subseteq {}^V \mathcal{X}\mathcal{X}\mathcal{a}^1) \hspace{15em} (\mathbf{T}_{R526}) (\text{Eqn10}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\mathcal{a}^1 \subseteq {}^V \mathcal{X}\mathcal{X}\mathcal{a}^1) \hspace{15em} (\mathbf{T}_{M139}) (\text{Hypothetical Syllogism Eqns9\&10}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\{\subseteq\}\mathcal{a}^1 \subseteq \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{a}^1) \hspace{15em} (\mathbf{T}_{R196}) (\text{Relational Deduction}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\{\subseteq\}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{b}^1 \subseteq \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{b}^1) \\
& \hspace{15em} (\mathbf{T}_{M029}) (\text{Deduction}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\{\subseteq\}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{b}^1 \subseteq \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{b}^1)) = \mathcal{E} \\
& \hspace{15em} (\mathbf{A}_{MX16}) (\text{Order Axiom}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\{\subseteq\}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{b}^1 \subseteq \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{b}^1) = \mathcal{E}) \\
& \hspace{15em} (\mathbf{T}_{M094}) (\text{Probabilistic Rule}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{b}^1 = \emptyset) \subseteq (\{\subseteq\}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{b}^1 = \emptyset)) \\
& \hspace{15em} (\mathbf{T}_{M097}) (\text{Probabilistic Rule}) (\text{Eqn11}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\{\supset\}\mathcal{X}\mathcal{a}^1, \{\supset\}\mathcal{X}\mathcal{b}^1 = \emptyset) \subseteq ((\{\subseteq\}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{b}^1 = \emptyset)) \\
& \hspace{15em} (\mathbf{T}_{M139}) (\text{Hypothetical Syllogism Eqns6\&11}) (\text{Eqn12}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\{\subseteq\}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{X}\mathcal{b}^1) = \emptyset) = ((@_x \mathcal{X}\mathcal{b}^1) * \mathcal{a}^1 = \emptyset) \\
& \hspace{15em} (\mathbf{T}_{T293}) (\text{Eqn13}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\{\supset\}\mathcal{X}\mathcal{a}^1, \{\supset\}\mathcal{X}\mathcal{b}^1 = \emptyset) \subseteq ((@_x \mathcal{X}\mathcal{b}^1) * \mathcal{a}^1 = \emptyset)) \\
& \hspace{15em} (\mathbf{A}_{MX19}) (\text{Substitution of Eqn13 into Eqn12})
\end{aligned}$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) (\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((@_x \mathcal{b}^1) * \mathcal{a}^1 \neq \emptyset) \subseteq (\{\supset\}\mathcal{b}^1, \{\supset\}\mathcal{X}\mathcal{a}^1 \neq \emptyset)) \quad (\mathbf{T}_{T307})$$

Proof of (T_{T307})

$$\begin{aligned}
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) (\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq ((\{\subseteq\}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{b}^1) = \emptyset) = ((@_x \mathcal{b}^1) * \mathcal{a}^1 = \emptyset) \\
& \hspace{15em} (\mathbf{T}_{T297}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) (\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((@_x \mathcal{b}^1) * \mathcal{a}^1 \neq \emptyset) = (\{\subseteq\}\mathcal{a}^1, \{\subseteq\}^V \mathcal{X}\mathcal{b}^1 \neq \emptyset)) \\
& \hspace{15em} (\mathbf{T}_{M008}) (\text{Inverse of an Equality}) (\text{Eqn1}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) (\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq ((\{\subseteq\}^V \mathcal{X}\mathcal{b}^1, \{\subseteq\}\mathcal{a}^1 \neq \emptyset) \subseteq (\{\supset\}\mathcal{b}^1, \{\supset\}\mathcal{X}\mathcal{a}^1 \neq \emptyset)) \\
& \hspace{15em} (\mathbf{T}_{T306}) (\text{Eqn2}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) (\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((@_x \mathcal{b}^1) * \mathcal{a}^1 \neq \emptyset) \subseteq (\{\supset\}\mathcal{b}^1, \{\supset\}\mathcal{X}\mathcal{a}^1 \neq \emptyset)) \\
& \hspace{15em} (\mathbf{A}_{MX19}) (\text{Substitution of Eqn1 into Eqn2})
\end{aligned}$$

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) (\beta^1 \subseteq \mathcal{X}\mathcal{E}, \{\subset\} * \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((@_x \beta^1) * \mathcal{a}^1 \neq \emptyset) \subseteq (\{\supset\}\beta^1, \{\supset\}\mathcal{X}\mathcal{a}^1 \neq \emptyset)) \quad (\mathbf{T}_{T308})$$

Proof of (T_{T308})

$$\begin{aligned}
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) (\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((@_x \mathcal{b}^1) * \mathcal{a}^1 \neq \emptyset) \subseteq (\{\supset\}\mathcal{b}^1, \{\supset\}\mathcal{X}\mathcal{a}^1 \neq \emptyset)) \\
& \hspace{15em} (\mathbf{T}_{T307}) (\text{Eqn1}) \\
& \text{Let } \mathcal{b}^1 = \beta^1 \hspace{15em} (\text{Eqn2}) \\
& \text{where } \beta^1 \subseteq \mathcal{X}\mathcal{E}, \{\subset\} * \mathcal{X}\mathcal{E} \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) (\beta^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((@_x \beta^1) * \mathcal{a}^1 \neq \emptyset) \subseteq (\{\supset\}\beta^1, \{\supset\}\mathcal{X}\mathcal{a}^1 \neq \emptyset)) \\
& \hspace{15em} (\mathbf{A}_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\
& \text{Thus } (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) (\beta^1 \subseteq \mathcal{X}\mathcal{E}, \{\subset\} * \mathcal{X}\mathcal{E}) (\beta^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((@_x \beta^1) * \mathcal{a}^1 \neq \emptyset) \subseteq (\{\supset\}\beta^1, \{\supset\}\mathcal{X}\mathcal{a}^1 \neq \emptyset)) \\
& \hspace{15em} (\mathbf{A}_{MX02}) (\text{Conditional Introduction}) (\text{Eqn3}) \\
& (\beta^1 \subseteq \mathcal{X}\mathcal{E}, \{\subset\} * \mathcal{X}\mathcal{E}) \subseteq (\beta^1 \subseteq \mathcal{X}\mathcal{E}) \hspace{15em} (\mathbf{T}_{M073}) (\text{Single Premise Syllogism}) \\
& (\beta^1 \subseteq \mathcal{X}\mathcal{E}, \{\subset\} * \mathcal{X}\mathcal{E}) = (\beta^1 \subseteq \mathcal{X}\mathcal{E}, \{\supset\} * \mathcal{X}\mathcal{E}) (\beta^1 \subseteq \mathcal{X}\mathcal{E}) \hspace{15em} (\mathbf{A}_{MX44}) (\subseteq \text{Equality Form}) (\text{Eqn4}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) (\beta^1 \subseteq \mathcal{X}\mathcal{E}, \{\subset\} * \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((@_x \beta^1) * \mathcal{a}^1 \neq \emptyset) \subseteq (\{\supset\}\beta^1, \{\supset\}\mathcal{X}\mathcal{a}^1 \neq \emptyset)) \\
& \hspace{15em} (\mathbf{A}_{MX19}) (\text{Substitution of Eqn4 into Eqn3})
\end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (\mathcal{X}(@_x a^1) * E = \{\supset\} a^1) \quad (T_{7302})$$

Proof of (T₇₃₀₂)

$$\begin{aligned} ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) &\subseteq ((@_x a^1) * E = \{\supset\}^v \mathcal{X} a^1) & (T_{5735})(\text{Eqn1}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) &\subseteq (\forall \mathcal{X} \{\supset\} a^1 = \{\supset\}^v \mathcal{X} a^1) & (A_{507})(\text{Dimensional Components Axiom}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) &\subseteq (\forall \mathcal{X} \{\supset\} a^1 = \{\supset\}^v \mathcal{X} a^1) & (T_{M089})(\text{Single Premise Syllogism})(\text{Eqn2}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) &\subseteq ((@_x a^1) * E = \forall \mathcal{X} \{\supset\} a^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) &\subseteq (\mathcal{X}(@_x a^1) * E = \mathcal{X}^v \mathcal{X} \{\supset\} a^1) & (T_{R196})(\text{Relational Deduction}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) &\subseteq (\mathcal{X}(@_x a^1) * E = \{\supset\} a^1) & (T_{5784}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E}) \subseteq (\mathcal{X}(@_x \alpha^1) * E = \alpha^1) \quad (T_{R879})$$

Proof of (T_{R879})

$$\begin{aligned} ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) &\subseteq (\mathcal{X}(@_x a^1) * E = \{\supset\} a^1) & (T_{7302})(\text{Eqn1}) \\ \text{Let } a^1 &= \alpha^1 & (\text{Eqn2}) \\ \text{where } \alpha^1 &\subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E} \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) &\subseteq (\mathcal{X}(@_x \alpha^1) * E = \{\supset\} \alpha^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \text{Thus } ((\mathcal{X}\mathcal{E})^1 \subseteq D_M + P_M)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) &\subseteq (\mathcal{X}(@_x \alpha^1) * E = \{\supset\} \alpha^1) & (A_{MX02})(\text{Conditional Introduction})(\text{Eqn3}) \\ (\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E}) &\subseteq (\alpha^1 \subseteq \mathcal{X}\mathcal{E}) & (T_{M073})(\text{Single Premise Syllogism}) \\ (\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E}) &= (\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E})(\alpha^1 \subseteq \mathcal{X}\mathcal{E}) & (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn4}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E}) &\subseteq (\mathcal{X}(@_x \alpha^1) * E = \{\supset\} \alpha^1) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})(\text{Eqn5}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E}) &\subseteq (\mathcal{X}(@_x \alpha^1) * E = \alpha^1) & (T_{729}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E}) \subseteq (\mathcal{X}S(@_x \alpha^1) * E = \alpha^1) \quad (T_{R880})$$

Proof of (T_{R880})

$$\begin{aligned} ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E}) &\subseteq (\mathcal{X}(@_x \alpha^1) * E = \alpha^1) & (T_{R879}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(\alpha^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E}) &\subseteq (\mathcal{X}S(@_x \alpha^1) * E = \alpha^1) & (T_{R210}) \end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(b^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E})((@_x b^1) a^1 \neq \emptyset) \subseteq (\mathcal{X}(@_x b^1) * a^1 = b^1) \quad (T_{5738})$$

Proof of (T₅₇₃₈)

$$\begin{aligned} ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(b^1 \subseteq \mathcal{X}\mathcal{E}) &\subseteq ((@_x b^1) * a^1 = \{\supset\} a^1. \{\supset\}^v \mathcal{X} b^1. \{\supset\} * \{\supset\} a^1. \{\supset\}^v \mathcal{X} b^1)) & (D_{R35})(\text{Definition of } @_x)(\text{Eqn1}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_A) &\subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) & (T_{M150})(\text{Single Premise Syllogism})(\text{Eqn2}) \\ D_M &\subseteq D_A & (T_{7911}) \\ (D_M \subseteq D_A) &= E & (A_{MX16})(\text{Order Axiom}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_M) &\subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq D_A) & (T_{M099})(\text{Entity Mode / Set Mode Rule})(\text{Eqn3}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_M) &\subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A) & (T_{M139})(\text{Hypothetical Syllogism Eqns3\&2})(\text{Eqn4}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(b^1 \subseteq \mathcal{X}\mathcal{E}) &\subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}\mathcal{E}) & (T_{M029})(\text{Deduction})(\text{Eqn5}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(b^1 \subseteq \mathcal{X}\mathcal{E}) &\subseteq ((@_x b^1) * a^1 = \{\supset\} a^1. \{\supset\}^v \mathcal{X} b^1. \{\supset\} * \{\supset\} a^1. \{\supset\}^v \mathcal{X} b^1)) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&5})(\text{Eqn6}) \\ (b^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E}) &\subseteq (b^1 \subseteq \mathcal{X}\mathcal{E}) & (T_{M073})(\text{Single Premise Syllogism}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(b^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E}) &\subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq D_A)(b^1 \subseteq \mathcal{X}\mathcal{E}) & (T_{M029})(\text{Deduction})(\text{Eqn7}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(b^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E}) &\subseteq ((@_x b^1) * a^1 = \{\supset\} a^1. \{\supset\}^v \mathcal{X} b^1. \{\supset\} * \{\supset\} a^1. \{\supset\}^v \mathcal{X} b^1)) & (T_{M139})(\text{Hypothetical Syllogism Eqns6\&7}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(b^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E}) &\subseteq ((@_x b^1) * a^1 = \{\supset\} a^1. \forall \mathcal{X} b^1. \{\supset\} * \{\supset\} a^1. \forall \mathcal{X} b^1)) & (T_{7310})(\text{Eqn8}) \\ \{\supset\} a^1. \forall \mathcal{X} b^1. \{\supset\} * \{\supset\} a^1. \forall \mathcal{X} b^1 &\subseteq \forall \mathcal{X} b^1 & (T_{M083})(\text{Conjunction Elimination})(\text{Eqn9}) \\ ((\mathcal{X}\mathcal{E})^1 \subseteq D_M)(b^1 \subseteq \mathcal{X}\mathcal{E}.\{\supset\} * \mathcal{X}\mathcal{E}) &\subseteq ((@_x b^1) * a^1 \subseteq \forall \mathcal{X} b^1) \end{aligned}$$

$$\begin{aligned}
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^*\mathcal{X}\mathcal{E}) \subseteq (\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 \subseteq \mathcal{X}'\mathcal{X}\mathcal{b}^1) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn8 into Eqn9}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^*\mathcal{X}\mathcal{E}) \subseteq (\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 \subseteq \mathcal{b}^1) & (\mathbf{T}_{R196})(\text{Relational Deduction})(\text{Eqn10}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) = \mathcal{E}) \subseteq ((\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^*\mathcal{X}\mathcal{E}) \subseteq (\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 \subseteq \mathcal{b}^1)) & (\mathbf{T}_{S784}) \\
& & (\mathbf{A}_{MX35})(\text{Exportation/Importation}) \text{ for truth sets (Eqn11)} \\
\text{Let } (\mathcal{@}_x\mathcal{b}^1)\mathcal{a}^1 \neq \emptyset & \\
((\mathcal{@}_x\mathcal{b}^1)\mathcal{a}^1 \neq \emptyset) = \mathcal{E} & (\mathbf{A}_{MX16})(\text{Order Axiom})(\text{Eqn12}) \\
(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) \subseteq ((\mathcal{a} \neq \emptyset) = (\mathcal{X}\mathcal{a} \neq \emptyset)) & (\mathbf{T}_{T275})(\text{Eqn13}) \\
((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) & (\text{Eqn4}) \\
(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)) = \mathcal{E} & (\mathbf{A}_{MX16})(\text{Order Axiom}) \\
(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) = \mathcal{E}) \subseteq (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) = \mathcal{E}) & (\mathbf{T}_{M094})(\text{Probabilistic Rule})(\text{Eqn14}) \\
(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) = \mathcal{E}) \subseteq ((\mathcal{a} \neq \emptyset) = (\mathcal{X}\mathcal{a} \neq \emptyset)) & (\mathbf{T}_{M139})(\text{Hypothetical Syllogism Eqns14\&13})(\text{Eqn15}) \\
\text{Let } \mathcal{a} = (\mathcal{@}_x\mathcal{b}^1)\mathcal{a}^1 & (\text{Eqn16}) \\
(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) = \mathcal{E}) \subseteq (((\mathcal{@}_x\mathcal{b}^1)\mathcal{a}^1 \neq \emptyset) = (\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)\mathcal{a}^1 \neq \emptyset)) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn16 into Eqn15})(\text{Eqn17}) \\
(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) = \mathcal{E}) \subseteq (\mathcal{E} = (\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)\mathcal{a}^1 \neq \emptyset)) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn12 into Eqn17})(\text{Eqn18}) \\
(\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)\mathcal{a}^1 \neq \emptyset)(\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 \subseteq \mathcal{b}^1) \subseteq (\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 = \mathcal{b}^1) & (\mathbf{T}_{M577})(\text{Eqn19}) \\
(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) = \mathcal{E}) \subseteq (\mathcal{E}(\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 \subseteq \mathcal{b}^1) \subseteq (\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 = \mathcal{b}^1)) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn18 into Eqn19}) \\
(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) = \mathcal{E}) \subseteq ((\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 \subseteq \mathcal{b}^1) \subseteq (\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 = \mathcal{b}^1)) & (\mathbf{A}_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn20}) \\
(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M) = \mathcal{E}) \subseteq ((\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^*\mathcal{X}\mathcal{E}) \subseteq (\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 = \mathcal{b}^1)) & (\mathbf{T}_{M139})(\text{Hypothetical Syllogism Eqns11\&20}) \\
((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^*\mathcal{X}\mathcal{E}) \subseteq (\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 = \mathcal{b}^1) & (\mathbf{A}_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\
\text{Thus } ((\mathcal{@}_x\mathcal{b}^1)\mathcal{a}^1 \neq \emptyset) \subseteq (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^*\mathcal{X}\mathcal{E}) \subseteq (\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 = \mathcal{b}^1)) & (\mathbf{A}_{MX02})(\text{Conditional Introduction}) \\
((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}.\{\subseteq\}^*\mathcal{X}\mathcal{E})((\mathcal{@}_x\mathcal{b}^1)\mathcal{a}^1 \neq \emptyset) \subseteq (\mathcal{X}(\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 = \mathcal{b}^1) & (\mathbf{A}_{MX35})(\text{Exportation/Importation}) \text{ for truth sets}
\end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq ((\mathcal{@}_x\mathcal{X}\mathcal{b}^1)*\mathcal{a}^1 = \{\subseteq\}\mathcal{a}^1.\{\subseteq\}^*(\mathcal{X}\mathcal{X}\mathcal{b}^1)^1.\{\subseteq\}^*(\{\subseteq\}\mathcal{a}^1.\{\subseteq\}^*(\mathcal{X}\mathcal{X}\mathcal{b}^1)^1))) \quad (\mathbf{T}_{R886})$$

Proof of ($\mathbf{T}_{R886}$)

$$\begin{aligned}
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 = \{\subseteq\}\mathcal{a}^1.\{\subseteq\}^*(\mathcal{X}\mathcal{b}^1)^1.\{\subseteq\}^*(\{\subseteq\}\mathcal{a}^1.\{\subseteq\}^*(\mathcal{X}\mathcal{b}^1)^1))) & (\mathbf{D}_{R35})(\text{Definition of } \mathcal{@}_x)(\text{Eqn1}) \\
& & (\text{Eqn2}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{X}\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((\mathcal{@}_x\mathcal{X}\mathcal{b}^1)*\mathcal{a}^1 = \{\subseteq\}\mathcal{a}^1.\{\subseteq\}^*(\mathcal{X}\mathcal{X}\mathcal{b}^1)^1.\{\subseteq\}^*(\{\subseteq\}\mathcal{a}^1.\{\subseteq\}^*(\mathcal{X}\mathcal{X}\mathcal{b}^1)^1))) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\
& & (\mathbf{T}_{M744})(\text{Universal Set Theorem}) \\
& \mathcal{b}^1 \subseteq \mathcal{E} & (\mathbf{T}_{R196})(\text{Relational Deduction}) \\
& \mathcal{X}\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E} & (\mathbf{A}_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\
& (\mathcal{X}\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E} & (\mathbf{A}_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)\mathcal{E} \subseteq ((\mathcal{@}_x\mathcal{X}\mathcal{b}^1)*\mathcal{a}^1 = \{\subseteq\}\mathcal{a}^1.\{\subseteq\}^*(\mathcal{X}\mathcal{X}\mathcal{b}^1)^1.\{\subseteq\}^*(\{\subseteq\}\mathcal{a}^1.\{\subseteq\}^*(\mathcal{X}\mathcal{X}\mathcal{b}^1)^1))) & (\mathbf{A}_{MX11})(\text{Universal Set Conjunction Axiom}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq ((\mathcal{@}_x\mathcal{X}\mathcal{b}^1)*\mathcal{a}^1 = \{\subseteq\}\mathcal{a}^1.\{\subseteq\}^*(\mathcal{X}\mathcal{X}\mathcal{b}^1)^1.\{\subseteq\}^*(\{\subseteq\}\mathcal{a}^1.\{\subseteq\}^*(\mathcal{X}\mathcal{X}\mathcal{b}^1)^1))) &
\end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M + \mathcal{P}_M)((\mathcal{Y}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\mathcal{X}(\mathcal{@}_y\mathcal{Y}\mathcal{b}^1)*\mathcal{a}^1 \subseteq \{\supseteq\}\mathcal{X}\mathcal{a}^1) \quad (\mathbf{T}_{R892})$$

Proof of ($\mathbf{T}_{R892}$)

$$\begin{aligned}
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((\mathcal{@}_x\mathcal{b}^1)*\mathcal{a}^1 = \{\subseteq\}\mathcal{a}^1.\{\subseteq\}^*(\mathcal{X}\mathcal{b}^1)^1.\{\subseteq\}^*(\{\subseteq\}\mathcal{a}^1.\{\subseteq\}^*(\mathcal{X}\mathcal{b}^1)^1))) & (\mathbf{D}_{R35})(\text{Definition of } \mathcal{@}_x)
\end{aligned}$$

$$\begin{aligned}
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((@_x \mathcal{b}^1) * \mathcal{a}^1 = \{\subseteq\} \mathcal{a}^1) \\
& \quad (T_{M073b})(\text{Single Premise Syllogism}) \\
& ((\mathcal{Y}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{b}^1 \subseteq \mathcal{Y}\mathcal{E}) \subseteq ((@_y \mathcal{b}^1) * \mathcal{a}^1 \subseteq \{\subseteq\} \mathcal{a}^1) \\
& \quad \text{Resymbolising (Eqn1)} \\
& \text{Let } \mathcal{b}^1 = \mathcal{Y}\mathcal{b}^1 \quad (\text{Eqn2}) \\
& ((\mathcal{Y}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{Y}\mathcal{b}^1 \subseteq \mathcal{Y}\mathcal{E}) \subseteq ((@_y \mathcal{Y}\mathcal{b}^1) * \mathcal{a}^1 \subseteq \{\subseteq\} \mathcal{a}^1) \\
& \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn4}) \\
& \mathcal{b}^1 \subseteq \mathcal{E} \quad (T_{M744})(\text{Universal Set Theorem}) \\
& \mathcal{Y}\mathcal{b}^1 \subseteq \mathcal{Y}\mathcal{E} \quad (T_{R196})(\text{Relational Deduction}) \\
& (\mathcal{Y}\mathcal{b}^1 \subseteq \mathcal{Y}\mathcal{E}) = \mathcal{E} \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn5}) \\
& ((\mathcal{Y}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)\mathcal{E} \subseteq ((@_y \mathcal{Y}\mathcal{b}^1) * \mathcal{a}^1 \subseteq \{\subseteq\} \mathcal{a}^1) \quad (A_{MX19})(\text{Substitution of Eqn5 into Eqn4}) \\
& ((\mathcal{Y}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq ((@_y \mathcal{Y}) * \mathcal{a}^1 \subseteq \{\subseteq\} \mathcal{a}^1) \quad (A_{MX11})(\text{Universal Set Conjunction Axiom}) \\
& ((\mathcal{Y}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\mathcal{X}(@_y \mathcal{Y}\mathcal{b}^1) * \mathcal{a}^1 \subseteq \mathcal{X}\{\subseteq\} \mathcal{a}^1) \quad (T_{R196})(\text{Relational Deduction}) \\
& \text{Let } ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M + \mathcal{P}_M) \\
& ((\mathcal{Y}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\mathcal{X}(@_y \mathcal{Y}\mathcal{b}^1) * \mathcal{a}^1 \subseteq \{\supseteq\} \mathcal{X}\mathcal{a}^1) \quad (T_{T377}) \\
& \text{Thus } ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_M + \mathcal{P}_M) ((\mathcal{Y}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq (\mathcal{X}(@_y \mathcal{Y}\mathcal{b}^1) * \mathcal{a}^1 \subseteq \{\supseteq\} \mathcal{X}\mathcal{a}^1) \\
& \quad (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E} \subseteq ((\{\supseteq\} \mathcal{a}^1 . \{\supseteq\} \mathcal{X}\mathcal{b}^1 = \emptyset) \subseteq ((@(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \mathcal{b}^1 = \emptyset)) \quad (T_{T268})$$

Proof of (T_{T268})

$$\begin{aligned}
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (((@(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \mathcal{b}^1 = \{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 . \{\subseteq\} \mathcal{b}^1 . (\{\subseteq\} * (\{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 . \{\subseteq\} \mathcal{b}^1))) \\
& \quad (T_{T285}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (((@(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \mathcal{b}^1 \subseteq \{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 . \{\subseteq\} \mathcal{b}^1) \\
& \quad (T_{M073b})(\text{Single Premise Syllogism}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq (((@(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \mathcal{b}^1 \subseteq \{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 . \{\subseteq\} \mathcal{b}^1)) = \mathcal{E} \\
& \quad (A_{MX16})(\text{Order Axiom}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq (((@(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \mathcal{b}^1 \subseteq \{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 . \{\subseteq\} \mathcal{b}^1) = \mathcal{E}) \\
& \quad (T_{M094})(\text{Probabilistic Rule}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq ((\{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 . \{\subseteq\} \mathcal{b}^1 = \emptyset) \subseteq ((@(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \mathcal{b}^1 = \emptyset)) \\
& \quad (T_{M097})(\text{Probabilistic Rule})(\text{Eqn1}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq ((\{\supseteq\} \mathcal{a}^1 . \{\supseteq\} \mathcal{X}\mathcal{b}^1 = \emptyset) \subseteq (\{\subseteq\} \mathcal{V}\mathcal{X}\mathcal{a}^1 . \{\subseteq\} \mathcal{b}^1 = \emptyset)) \\
& \quad (T_{T273}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq ((\{\supseteq\} \mathcal{a}^1 . \{\supseteq\} \mathcal{X}\mathcal{b}^1 = \emptyset) \subseteq (\{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1 . \{\subseteq\} \mathcal{b}^1 = \emptyset)) \\
& \quad (T_{T733})(\text{Eqn2}) \\
& (((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E}) \subseteq ((\{\supseteq\} \mathcal{a}^1 . \{\supseteq\} \mathcal{X}\mathcal{b}^1 = \emptyset) \subseteq ((@(\mathcal{V}\mathcal{X}\mathcal{a}^1)^1) \mathcal{b}^1 = \emptyset)) \\
& \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})
\end{aligned}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq ((@_x \mathcal{X}\mathcal{a}^1) \mathcal{E}^1 = (\mathcal{V}\mathcal{X}\mathcal{X}\mathcal{a}^1)^1) \quad (T_{T419})$$

Proof of (T_{T419})

$$\begin{aligned}
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{b}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((@_x \mathcal{b}^1) * \mathcal{a}^1 = \{\subseteq\} \mathcal{a}^1 . \{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{b}^1)^1 . (\{\subseteq\} * (\{\subseteq\} \mathcal{a}^1 . \{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{b}^1)^1))) \\
& \quad (D_{R35})(\text{Definition of } @_x)(\text{Eqn1}) \\
& \text{Let } \mathcal{b}^1 = \mathcal{X}\mathcal{a}^1 \quad (\text{Eqn2}) \\
& \text{Let } \mathcal{a}^1 = \mathcal{E}^1 \quad (\text{Eqn3}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)(\mathcal{X}\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) \subseteq ((@_x \mathcal{X}\mathcal{a}^1) * \mathcal{E}^1 = \{\subseteq\} \mathcal{E}^1 . \{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{X}\mathcal{a}^1)^1 . (\{\subseteq\} * (\{\subseteq\} \mathcal{E}^1 . \{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{X}\mathcal{a}^1)^1))) \\
& \quad (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})(\text{Eqn4}) \\
& \mathcal{a}^1 \subseteq \mathcal{E} \quad (T_{M744})(\text{Universal Set Theorem}) \\
& \mathcal{X}\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E} \quad (T_{R196})(\text{Relational Deduction}) \\
& (\mathcal{X}\mathcal{a}^1 \subseteq \mathcal{X}\mathcal{E}) = \mathcal{E} \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn5}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A)\mathcal{E} \subseteq ((@_x \mathcal{X}\mathcal{a}^1) * \mathcal{E}^1 = \{\subseteq\} \mathcal{E}^1 . \{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{X}\mathcal{a}^1)^1 . (\{\subseteq\} * (\{\subseteq\} \mathcal{E}^1 . \{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{X}\mathcal{a}^1)^1))) \\
& \quad (A_{MX19})(\text{Substitution of Eqn5 into Eqn4}) \\
& ((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_A + \mathcal{P}_A) \subseteq ((@_x \mathcal{X}\mathcal{a}^1) * \mathcal{E}^1 = \{\subseteq\} \mathcal{E}^1 . \{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{X}\mathcal{a}^1)^1 . (\{\subseteq\} * (\{\subseteq\} \mathcal{E}^1 . \{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{X}\mathcal{a}^1)^1))) \\
& \quad (A_{MX11})(\text{Universal Set Conjunction Axiom})
\end{aligned}$$

$$\begin{aligned}
((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) &\subseteq ((@_x \mathcal{X}a^1) * \mathcal{E}^1 = \mathcal{E}.\{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{X}a^1)^1.(\{\subseteq\}*(\mathcal{E}.\{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{X}a^1)^1))) \\
&\quad (\mathbf{A}_{R02})(\text{Definition of Disaggregation for Continuum Sets}) \\
((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) &\subseteq ((@_x \mathcal{X}a^1) * \mathcal{E}^1 = \{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{X}a^1)^1.(\{\subseteq\}*(\mathcal{E}.\{\subseteq\}(\mathcal{V}\mathcal{X}\mathcal{X}a^1)^1))) \\
&\quad (\mathbf{A}_{MX11})(\text{Universal Set Conjunction Axiom}) \\
((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) &\subseteq ((@_x \mathcal{X}a^1) * \mathcal{E}^1 = (\mathcal{V}\mathcal{X}\mathcal{X}a^1)^1) \quad (\mathbf{T}_{T271})
\end{aligned}$$

Chapter 13 Proofs of Theorems of Space, Time and Space-time

13.1 Theorems Derived by Substitution into General Theory of Dimensions

Many of the theorems of time, space, and space-time can be derived by simple substitution into the general theory of dimensions. They are, therefore, listed with the theorems of general dimensions. For example, from the general equation:

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_A + \mathbf{P}_A) \subseteq ((@_x \mathcal{X}a^1)a^1 = a^1) \quad (\mathcal{T}_{257})$$

the following can be derived as shown in the example proof below:

$$(@_{\mathcal{T}} \mathcal{T}a^1)a^1 = a^1 \quad (\mathcal{T}_{257}\mathcal{T})$$

$$(@_{Oc^1} (Oc^1)a^1)a^1 = a^1 \quad (\mathcal{T}_{257}\mathcal{O})$$

$$(@_{\mathcal{A}} \mathcal{A}a^1)a^1 = a^1 \quad (\mathcal{T}_{257}\mathcal{A})$$

The latter, however, is not normally used because it simplifies to:

$$(@a^1)a^1 = a^1 \quad (\mathcal{R}_{984})$$

The same is true of many other theorems of the space-time polydimension because this is the same as the abstract aspect of the universe.

Note that when theorems are derived in this way the same reference number is used followed by reference to the dimension in question, e.g., $(\mathcal{T}_{257}\mathcal{T})$.

Some examples are given below:

$$(\mathcal{T}\mathcal{E})^1 \subseteq \mathbf{D}_M \quad (\mathcal{T}_{287}) \text{ Time is a Mixed Continuum Dimension}$$

Proof of $(\mathcal{T}_{287})$

$$(\mathcal{X} = \mathcal{M}, \mathcal{T}, \mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M)$$

$$(\mathcal{A}_{S19}) \text{ (Spatio-temporal Dimensions are Continua)}$$

$$(\mathcal{T}\mathcal{E})^1 \subseteq \mathbf{D}_M$$

Extraction

$$((\mathcal{O}a^1)\mathcal{E})^1 \subseteq \mathbf{P}_M$$

$$(\mathcal{T}_{288})$$

$$(\mathcal{A}\mathcal{E})^1 \subseteq \mathbf{P}_M$$

$$(\mathcal{T}_{289}) \text{ Space-time is a Mixed Continuum Polydimension}$$

Example Proof of $(\mathcal{T}_{288})$

$$(\mathcal{X} = \mathcal{A}, \mathcal{O}b^1, \mathcal{L}_{xyz}b^1) \subseteq ((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{P}_M)$$

$$(\mathcal{A}_{R89}) \text{ (Axiom of Polydimensions)}$$

$$((\mathcal{O}b^1)\mathcal{E})^1 \subseteq \mathbf{P}_M$$

Extraction

$$(@_{\mathcal{T}} \mathcal{T}a^1)a^1 = a^1$$

$$(\mathcal{T}_{257}\mathcal{T})$$

Example Proof of $(\mathcal{T}_{257}\mathcal{T})$

$$(\mathcal{T}\mathcal{E})^1 \subseteq \mathbf{D}_M$$

$$(\mathcal{T}_{287}) \text{ (Time is a Mixed Continuum Dimension) (Eqn1)}$$

$$\mathbf{D}_M \subseteq \mathbf{D}_M + \mathbf{D}_D$$

$$(\mathcal{T}_{M082}) \text{ (Disjunction Introduction) (Eqn2)}$$

$$(\mathcal{T}\mathcal{E})^1 \subseteq \mathbf{D}_M + \mathbf{D}_D$$

$$(\mathcal{T}_{M139}) \text{ (Hypothetical Syllogism Eqns1&2)}$$

$$((\mathcal{T}\mathcal{E})^1 \subseteq \mathbf{D}_M + \mathbf{D}_D) = \mathbf{E}$$

$$(\mathcal{A}_{MX16}) \text{ (Order Axiom) (Eqn3)}$$

$$((\mathcal{X}\mathcal{E})^1 \subseteq \mathbf{D}_M + \mathbf{D}_D) \subseteq ((@_x \mathcal{X}a^1)a^1 = a^1)$$

$$(\mathcal{T}_{257}) \text{ (Eqn4)}$$

$$\text{Let } \mathcal{X} = \mathcal{T}$$

$$\text{(Eqn5)}$$

$$((\mathcal{T}\mathcal{E})^1 \subseteq \mathbf{D}_M + \mathbf{D}_D) \subseteq ((@_{\mathcal{T}} \mathcal{T}a^1)a^1 = a^1)$$

$$(\mathcal{A}_{MX19}) \text{ (Substitution of Eqn5 into Eqn4) (Eqn6)}$$

$$\mathbf{E} \subseteq ((@_{\mathcal{T}} \mathcal{T}a^1)a^1 = a^1)$$

$$(\mathcal{A}_{MX19}) \text{ (Substitution of Eqn3 into Eqn6)}$$

$$\mathbf{E} = ((@_{\mathcal{T}} \mathcal{T}a^1)a^1 = a^1)$$

$$(\mathcal{T}_{M275}) \text{ (Supersets of the Universal Set)}$$

$$(@_{\mathcal{T}} \mathcal{T}a^1)a^1 = a^1$$

$$(\mathcal{A}_{MX16}) \text{ (Order Axiom)}$$

$$\{\subseteq\}a^1 \subseteq \{\supset\}_T a^1 \quad (T_{7382}T)$$

Example Proof of ($T_{7382}T$)

$$\begin{aligned} ((\mathcal{X}E)^1 \subseteq D_M) &\subseteq (\{\subseteq\}a^1 \subseteq \{\supset\}_T a^1) & (T_{7382})(\text{Eqn1}) \\ \text{Let } \mathcal{X} &= T & (\text{Eqn2}) \\ ((TE)^1 \subseteq D_M) &\subseteq (\{\subseteq\}a^1 \subseteq \{\supset\}_T a^1) & (A_{MX19})(\text{Substitution of Eqn6 into Eqn1}) \\ E &\subseteq (\{\subseteq\}a^1 \subseteq \{\supset\}_T a^1) & (T_{M001})(\text{Universality of } (T_{7287}) \text{ Time is a Mixed Continuum Dimension}) \\ E &= (\{\subseteq\}a^1 \subseteq \{\supset\}_T a^1) & (T_{M275})(\text{Supersets of the Universal Set}) \\ \{\subseteq\}a^1 &\subseteq \{\supset\}_T a^1 & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$(Ta)^1 = Ta^1 \quad (A_{S05}T)$$

$$((Ob^1)a)^1 = (Ob^1)a^1 \quad (A_{S05}O)$$

$$(Aa)^1 = Aa^1 \quad (A_{S05}A)$$

Example Proof of ($A_{S05}T$)

$$\begin{aligned} ((\mathcal{X}E)^1 \subseteq D_M + P_M) &= ((\mathcal{X}a)^1 = \mathcal{X}a^1) & (A_{S05})(\text{Physical Dimension Aggregation Axiom})(\text{Eqn1}) \\ \text{Let } \mathcal{X} &= T & (\text{Eqn2}) \\ ((TE)^1 \subseteq D_M + P_M) &= ((Ta)^1 = Ta^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\ (TE)^1 &\subseteq D_M & (T_{7287})(\text{Time is a Mixed Continuum Dimension}) \\ (TE)^1 &\subseteq D_M + P_M & (T_{M150})(\text{Single Premise Syllogism}) \\ ((TE)^1 \subseteq D_M + P_M) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\ E &= ((Ta)^1 = Ta^1) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\ (Ta)^1 &= Ta^1 & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$T\{\subseteq\}a^1 = \{\supset\}Ta^1 \quad (T_{7377}T)$$

$$(Ob^1)\{\subseteq\}a^1 = \{\supset\}(Ob^1)a^1 \quad (T_{7377}O)$$

$$A\{\subseteq\}a^1 = \{\supset\}Aa^1 \quad (T_{7377}A)$$

Proof of ($T_{7377}T$)

$$\begin{aligned} ((\mathcal{X}E)^1 \subseteq D_M + P_M) &\subseteq (\mathcal{X}\{\subseteq\}a^1 = \{\supset\}\mathcal{X}a^1) & (T_{7377})(\text{Eqn1}) \\ \text{Let } \mathcal{X} &= T & (\text{Eqn2}) \\ ((TE)^1 \subseteq D_M + P_M) &\subseteq (T\{\subseteq\}a^1 = \{\supset\}Ta^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\ (TE)^1 &\subseteq D_M & (T_{7287})(\text{Time is a Mixed Continuum Dimension}) \\ (TE)^1 &\subseteq D_M + P_M & (T_{M150})(\text{Single Premise Syllogism}) \\ ((TE)^1 \subseteq D_M + P_M) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\ E &\subseteq (T\{\subseteq\}a^1 = \{\supset\}Ta^1) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\ E &= (T\{\subseteq\}a^1 = \{\supset\}Ta^1) & (T_{M275})(\text{Supersets of the Universal Set}) \\ T\{\subseteq\}a^1 &= \{\supset\}Ta^1 & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$(\mathcal{b}^1 \subseteq (Oc^1)E) \subseteq ((@_{Oc^1}\mathcal{b}^1)*a^1 = \{\subseteq\}a^1.\{\subseteq\}^V(Oc^1)\mathcal{b}^1.\{\subseteq\}^*(\{\subseteq\}a^1.\{\subseteq\}^V(Oc^1)\mathcal{b}^1))) \quad (T_{7732}O)$$

Proof of ($T_{7732}O$)

$$\begin{aligned} ((\mathcal{X}E)^1 \subseteq D_A + P_A)(\mathcal{b}^1 \subseteq \mathcal{X}E) &\subseteq ((@_x\mathcal{b}^1)*a^1 = \{\subseteq\}a^1.\{\subseteq\}^V\mathcal{X}\mathcal{b}^1.\{\subseteq\}^*(\{\subseteq\}a^1.\{\subseteq\}^V\mathcal{X}\mathcal{b}^1))) & (T_{7732})(\text{Eqn1}) \\ \text{Let } \mathcal{X} &= Oc^1 & (\text{Eqn2}) \\ (((Oc^1)E)^1 \subseteq D_A + P_A)(\mathcal{b}^1 \subseteq (Oc^1)E) &\subseteq ((@_{Oc^1}\mathcal{b}^1)*a^1 = \{\subseteq\}a^1.\{\subseteq\}^V(Oc^1)\mathcal{b}^1.\{\subseteq\}^*(\{\subseteq\}a^1.\{\subseteq\}^V(Oc^1)\mathcal{b}^1))) & (A_{MX19})(\text{Substitution of Eqns2 into Eqn1})(\text{Eqn3}) \\ ((Oc^1)E)^1 &\subseteq P_M & (T_{7288})(\text{Eqn4}) \end{aligned}$$

$$\begin{aligned}
P_M \subseteq P_A & \quad (T_{884})(\text{Eqn5}) \\
((Oc^1)E)^1 \subseteq P_A & \quad (T_{139})(\text{Hypothetical Syllogism Eqns4\&5}) \\
(((Oc^1)E)^1 \subseteq D_A + P_A) & \quad (T_{150})(\text{Single Premise Syllogism}) \\
(((Oc^1)E)^1 \subseteq D_A + P_A) = E & \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn6}) \\
E(\mathcal{B}^1 \subseteq (Oc^1)E) \subseteq ((@_{Oc^1}\mathcal{B}^1)*a^1 = \{\subseteq\}a^1.\{\subseteq\}^V(Oc^1)\mathcal{B}^1.\{\subseteq\}*(\{\subseteq\}a^1.\{\subseteq\}^V(Oc^1)\mathcal{B}^1))) & \quad (A_{MX19})(\text{Substitution of Eqn6 into Eqn3}) \\
(\mathcal{B}^1 \subseteq (Oc^1)E) \subseteq ((@_{Oc^1}\mathcal{B}^1)*a^1 = \{\subseteq\}a^1.\{\subseteq\}^V(Oc^1)\mathcal{B}^1.\{\subseteq\}*(\{\subseteq\}a^1.\{\subseteq\}^V(Oc^1)\mathcal{B}^1))) & \quad (A_{MX11})(\text{Universal Set Conjunction Axiom})
\end{aligned}$$

$$(@_{\mathcal{A}}\mathcal{B}^1)*a^1 = (@^V\mathcal{A}\mathcal{B}^1)a^1 \quad (T_{418})$$

$$(@_{\mathcal{A}}\mathcal{B}^1)*a^1 = \{\subseteq\}^V\mathcal{A}\mathcal{B}^1.\{\subseteq\}a^1.\{\subseteq\}*(\{\subseteq\}^V\mathcal{A}\mathcal{B}^1.\{\subseteq\}a^1) \quad (T_{417})$$

$$(@_{\mathcal{A}}(Oa^1)b^1)a^1 = \{\subseteq\}^V\mathcal{A}(Oa^1)b^1.\{\subseteq\}a^1.\{\subseteq\}*(\{\subseteq\}^V\mathcal{A}(Oa^1)b^1.\{\subseteq\}a^1) \quad (T_{417}O) \text{ Definition of a Natural Component}$$

Proof of (T_{418}) , (T_{417}) & $(T_{417}O)$

$$\begin{aligned}
(@a^1)b^1 &= \{\subseteq\}a^1.\{\subseteq\}b^1.\{\subseteq\}*(\{\subseteq\}a^1.\{\subseteq\}b^1) & (D_{R05})(\text{Definition of } @)(\text{Eqn1}) \\
\text{Let } a^1 &= {}^V\mathcal{A}\mathcal{B}^1 & (\text{Eqn2}) \\
\text{Let } b^1 &= a^1 & (\text{Eqn3}) \\
(@^V\mathcal{A}\mathcal{B}^1)a^1 &= \{\subseteq\}^V\mathcal{A}\mathcal{B}^1.\{\subseteq\}a^1.\{\subseteq\}*(\{\subseteq\}^V\mathcal{A}\mathcal{B}^1.\{\subseteq\}a^1) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})(\text{Eqn4}) \\
((\mathcal{X}E)^1 \subseteq D_A + P_A)(\mathcal{B}^1 \subseteq \mathcal{X}E) &\subseteq ((@_{\mathcal{X}}\mathcal{B}^1)*a^1 = (@^V\mathcal{X}\mathcal{B}^1)a^1) & (T_{908})(\text{Eqn5}) \\
\text{Let } \mathcal{X} &= \mathcal{A} & (\text{Eqn6}) \\
((\mathcal{A}E)^1 \subseteq D_A + P_A)(\mathcal{B}^1 \subseteq \mathcal{A}E) &\subseteq ((@_{\mathcal{A}}\mathcal{B}^1)*a^1 = (@^V\mathcal{A}\mathcal{B}^1)a^1) & (A_{MX19})(\text{Substitution of Eqn6 into Eqn5})(\text{Eqn7}) \\
(\mathcal{A}E)^1 &\subseteq P_M & (T_{289})(\text{Space-time is a Mixed Continuum Polydimension})(\text{Eqn8}) \\
P_M &\subseteq P_A & (T_{884})(\text{Eqn9}) \\
P_A &\subseteq D_A + P_A & (T_{M082})(\text{Disjunction Introduction})(\text{Eqn10}) \\
P_M &\subseteq D_A + P_A & (T_{M139})(\text{Hypothetical Syllogism Eqns9\&10})(\text{Eqn11}) \\
(\mathcal{A}E)^1 &\subseteq D_A + P_A & (T_{M139})(\text{Hypothetical Syllogism Eqns8\&11}) \\
((\mathcal{A}E)^1 \subseteq D_A + P_A) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn12}) \\
E(\mathcal{B}^1 \subseteq \mathcal{A}E) &\subseteq ((@_{\mathcal{A}}\mathcal{B}^1)*a^1 = (@^V\mathcal{A}\mathcal{B}^1)a^1) & (A_{MX19})(\text{Substitution of Eqn12 into Eqn7}) \\
(\mathcal{B}^1 \subseteq \mathcal{A}E) &\subseteq ((@_{\mathcal{A}}\mathcal{B}^1)*a^1 = (@^V\mathcal{A}\mathcal{B}^1)a^1) & (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn13}) \\
\mathcal{A}^1 &\subseteq R_P R_S R_U & (A_{S86})(\text{Nature of } \mathcal{A}) \\
\mathcal{A}^1 &\subseteq R_P & (T_{M073})(\text{Single Premise Syllogism}) \\
(\mathcal{A}^1 \subseteq R_P) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn14}) \\
(\mathcal{A}^1 \subseteq R_P) &\subseteq (\mathcal{A}^1 \subseteq R_O)((^V\mathcal{A})^1 \subseteq R_O) & (D_{R31})(\text{Definition of a Pairing Relationship Set}) \\
(\mathcal{A}^1 \subseteq R_P) &\subseteq ((^V\mathcal{A})^1 \subseteq R_O) & (T_{M073})(\text{Single Premise Syllogism})(\text{Eqn15}) \\
E &\subseteq ((^V\mathcal{A})^1 \subseteq R_O) & (A_{MX19})(\text{Substitution of Eqn14 into Eqn15}) \\
E &= ((^V\mathcal{A})^1 \subseteq R_O) & (T_{M275})(\text{Supersets of the Universal Set})(\text{Eqn16}) \\
(((^V\mathcal{A})^1 \subseteq R_O) = E) &= ((\mathcal{B}^1 \subseteq Q^V\mathcal{A})(\mathcal{B}^1 \neq \emptyset) \subseteq (^V\mathcal{A}*\mathcal{B}^1 = (^V\mathcal{A}*\mathcal{B}^1)^1)) & (D_{R30})(\text{Definition of a One-to-One Relationship Set})(\text{Eqn17}) \\
(E = E) &= ((\mathcal{B}^1 \subseteq Q^V\mathcal{A})(\mathcal{B}^1 \neq \emptyset) \subseteq (^V\mathcal{A}*\mathcal{B}^1 = (^V\mathcal{A}*\mathcal{B}^1)^1)) & (A_{MX19})(\text{Substitution of Eqn16 into Eqn17}) \\
E &= ((\mathcal{B}^1 \subseteq Q^V\mathcal{A})(\mathcal{B}^1 \neq \emptyset) \subseteq (^V\mathcal{A}*\mathcal{B}^1 = (^V\mathcal{A}*\mathcal{B}^1)^1)) & (T_{M342}) \\
(\mathcal{B}^1 \subseteq Q^V\mathcal{A})(\mathcal{B}^1 \neq \emptyset) &\subseteq (^V\mathcal{A}*\mathcal{B}^1 = (^V\mathcal{A}*\mathcal{B}^1)^1) & (A_{MX16})(\text{Order Axiom})(\text{Eqn18}) \\
(\mathcal{B}^1 \subseteq \mathcal{A}E)(\mathcal{B}^1 \subseteq Q^V\mathcal{A})(\mathcal{B}^1 \neq \emptyset) &\subseteq ((@_{\mathcal{A}}\mathcal{B}^1)*a^1 = (@^V\mathcal{A}\mathcal{B}^1)a^1) & (A_{MX19})(\text{Substitution of Eqn18 into Eqn13})(\text{Eqn19}) \\
Q^V\mathcal{A} &= \mathcal{A}*E & (T_{R211})(\text{Eqn20}) \\
(\mathcal{B}^1 \subseteq \mathcal{A}E)(\mathcal{B}^1 \subseteq \mathcal{A}E)(\mathcal{B}^1 \neq \emptyset) &\subseteq ((@_{\mathcal{A}}\mathcal{B}^1)*a^1 = (@^V\mathcal{A}\mathcal{B}^1)a^1) & (A_{MX19})(\text{Substitution of Eqn20 into Eqn19}) \\
(\mathcal{B}^1 \subseteq \mathcal{A}E)(\mathcal{B}^1 \neq \emptyset) &\subseteq ((@_{\mathcal{A}}\mathcal{B}^1)*a^1 = (@^V\mathcal{A}\mathcal{B}^1)a^1) & (T_{M667})(\text{Conjunction Tautology})
\end{aligned}$$

$(b^1 \subseteq E)(b^1 \neq \emptyset) \subseteq ((@_A b^1) * a^1 = (@^V A b^1) a^1)$	(A_{S87})(Aspect Opposite of E)
$E(b^1 \neq \emptyset) \subseteq ((@_A b^1) * a^1 = (@^V A b^1) a^1)$	(T_{M001})(Universality of (T_{M744}) Universal Set Theorem)
$(b^1 \neq \emptyset) \subseteq ((@_A b^1) * a^1 = (@^V A b^1) a^1)$	(A_{MX11})(Universal Set Conjunction Axiom)(Eqn21)
Let $b^1 = \emptyset$	(Eqn22)
Let $(@_A b^1) * a^1 \neq (@^V A b^1) a^1$	(Eqn23)
$(@_A \emptyset) * a^1 \neq (@^V A \emptyset) a^1$	(A_{MX19})(Substitution of Eqn22 into Eqn23)
$(@_A \emptyset) * a^1 \neq (@ \emptyset) a^1$	(T_{S181})(Null Object Theorem)
$\emptyset * a^1 \neq \emptyset * a^1$	(T_{S181})(Null Object Theorem)
$\emptyset \neq \emptyset$	(T_{R072})(Null Function Theorem)
$\emptyset$	(T_{M346})
Thus $(b^1 = \emptyset)((@_A b^1) * a^1 \neq (@^V A b^1) a^1) = \emptyset$	(A_{MX02})(Conditional Introduction)
$(b^1 = \emptyset) \subseteq ((@_A b^1) * a^1 = (@^V A b^1) a^1)$	(T_{M735})($\subseteq$ Existential Form)(Eqn24)
$(b^1 = \emptyset) + (b^1 \neq \emptyset) \subseteq ((@_A b^1) * a^1 = (@^V A b^1) a^1) + ((@_A b^1) * a^1 = (@^V A b^1) a^1)$	(A_{MX21})(Combination of Eqns21&24)
$(b^1 = \emptyset) + (b^1 \neq \emptyset) \subseteq ((@_A b^1) * a^1 = (@^V A b^1) a^1)$	(A_{MX01})(Disjunction Tautology)
$E \subseteq ((@_A b^1) * a^1 = (@^V A b^1) a^1)$	(T_{M671})(Tertium non Datur)
$E = ((@_A b^1) * a^1 = (@^V A b^1) a^1)$	(T_{M275})(Supersets of the Universal Set)
$(@_A b^1) * a^1 = (@^V A b^1) a^1$	(A_{MX16})(Order Axiom)(Eqn25)
$(@_A b^1) * a^1 = \{\subseteq\}^V A b^1. \{\subseteq\} a^1. \{\subseteq\} * (\{\subseteq\}^V A b^1. \{\subseteq\} a^1)$	(A_{MX19})(Substitution of Eqn25 into Eqn4)(Eqn26)
Let $b^1 = (Oa^1)b^1$	(Eqn27)
$(@_A (Oa^1)b^1) * a^1 = \{\subseteq\}^V A (Oa^1)b^1. \{\subseteq\} a^1. \{\subseteq\} * (\{\subseteq\}^V A (Oa^1)b^1. \{\subseteq\} a^1)$	(A_{MX19})(Substitution of Eqn27 into Eqn26)

13.2 Proofs of Special Theorems of Space, Time and Space-Time

There are, however, characteristics particular to the dimensions of space, time and space-time which allow additional special theorems to be developed. These are described below.

$$\{>\}_T * E = E \quad (T_{T349})$$

$$\{<\}_T * E = E \quad (T_{T350})$$

Example Proof of (T_{T349})

$(X = \{>\}, \{<\}) \subseteq (X T E = T E)$	(A_{S14})(Axiom of Unbounded Time)
$\{>\} T E = T E$	Extraction
${}^V T \{>\} T E = {}^V T T E$	(T_{R522})(Relational Deduction)
$\{>\}_T E = {}^V T T E$	(T_{R571})(Definition of a Mapped Relationship)(Eqn1)
$(E \subseteq Q T) \subseteq (E \subseteq {}^V T T E)$	(T_{R527})
$(E \subseteq E) \subseteq (E \subseteq {}^V T T E)$	(A_{S36T})
$(E \subset E) + (E = E) \subseteq (E \subseteq {}^V T T E)$	(T_{M266})(Definition of $\subseteq$)
$(E \subset E) + E \subseteq (E \subseteq {}^V T T E)$	(T_{M342})
$E \subseteq (E \subseteq {}^V T T E)$	(T_{M055})(Universal Set Disjunction)
$E = (E \subseteq {}^V T T E)$	(T_{M275})(Supersets of the Universal Set)
$E \subseteq {}^V T T E$	(A_{MX16})(Order Axiom)(Eqn2)
$E \subseteq \{>\}_T E$	(A_{MX19})(Substitution of Eqn1 into Eqn2)
$E = \{>\}_T E$	(T_{M275})(Supersets of the Universal Set)

$$B_T E^1 = \emptyset \quad (T_{T351})$$

$$C_T E^1 = \emptyset \quad (T_{T352})$$

Example Proof of (T_{T351})

$$\begin{array}{ll}
\mathcal{B}_T E^1 = \{=\}_T E.(\{>\}_T E).(\{\supset\}_T E) & (T_{R725})(Eqn1) \\
\{>\}_T E = E & (T_{T349}) \\
\{>\}_T E = \sim E & (T_{M008})(\text{Inverse of an Equality}) \\
\{>\}_T E = \emptyset & (D_{M02})(\text{Definition of } \emptyset)(Eqn2) \\
\mathcal{B}_T E^1 = \{=\}_T E.\emptyset.(\{\supset\}_T E) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
\mathcal{B}_T E^1 = \emptyset & (T_{M050})(\text{Null Set Conjunction})
\end{array}$$

$$E^1 \subseteq P_M \quad (T_{T278})$$

Proof of (T_{T278})

$$\begin{array}{ll}
(\mathcal{A}E)^1 \subseteq P_M & (T_{T289}) \\
E^1 \subseteq P_M & (A_{S87})(\text{Aspect Opposite of } E)
\end{array}$$

$$\mathcal{A}^1 \subseteq R_P \quad (T_{T281})$$

Proof of (T_{T281})

$$\begin{array}{ll}
\mathcal{A}^1 \subseteq R_P R_S R_U & (A_{S86})(\text{Nature of } \mathcal{A}) \\
\mathcal{A}^1 \subseteq R_P & (T_{M073})(\text{Single Premise Syllogism})
\end{array}$$

$$((t^1 \subseteq TE).((@_T t^1)a \neq \emptyset) = E) \subseteq (((@_T t^1)a \subseteq (@_T t^1)b) = (\forall Sa.\{=\}. \forall Qb = \emptyset)) \quad (T_{T898})$$

$$((t^1 \subseteq TE).((@_T t^1)a \neq \emptyset) = E) \subseteq ((@_T t^1)a \subseteq (@_T t^1)b) = (\forall Sa.\{=\}. \forall Qb = \emptyset) \quad (T_{T899})$$

Example Proof of (T_{T898})

$$\begin{array}{ll}
\text{Let } (t^1 \subseteq TE).((@_T t^1)a \neq \emptyset) & \\
\text{Let } (@_T t^1)a \subseteq (@_T t^1)b & (Eqn1) \\
\text{Let } TE = t^1 + \dots + t^1 & (Eqn2) \\
((@_T t^1)a \subseteq (@_T t^1)b) \dots ((@_T t^1)a \subseteq (@_T t^1)b) & (T_{M669})(\text{Combination of Eqn1 for } r = 1 \text{ to } E) \\
(@_T t^1)a + \dots + (@_T t^1)a \subseteq (@_T t^1)b + \dots + (@_T t^1)b & (T_{M131b})(\text{Conjunction of Sufficiencies}) \\
((@_T t^1) + \dots + (@_T t^1))a \subseteq ((@_T t^1) + \dots + (@_T t^1))b & (T_{R002c})(\text{Distribution of } + \text{ over } *) \\
(@_T (t^1 + \dots + t^1))a \subseteq (@_T (t^1 + \dots + t^1))b & (T_{R002b})(\text{Distribution of } * \text{ over } +)(Eqn3) \\
(@_T TE)a \subseteq (@_T TE)b & (A_{MX19})(\text{Substitution of Eqn2 into Eqn3}) \\
\forall Sa.\{=\} \subseteq \forall Qb.\{=\} & (A_{S94})(\text{Axiom of Temporal Sections}) \\
\forall Sa.\{=\}.\sim(\forall Qb.\{=\}) = \emptyset & (T_{M735})(\subseteq \text{ Existential Form}) \\
\forall Sa.\{=\}.\sim\forall Qb + \sim\{=\} = \emptyset & (T_{M025})(\text{DeMorgan's Law}) \\
\forall Sa.\{=\}.\sim\forall Qb + \forall Sa.\{=\}.\sim\{=\} = \emptyset & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
\forall Sa.\{=\}.\sim\forall Qb + \forall Sa.\emptyset = \emptyset & (T_{M621})(\text{Complement Conjunction}) \\
\forall Sa.\{=\}.\sim\forall Qb + \emptyset = \emptyset & (T_{M050})(\text{Null Set Conjunction}) \\
\forall Sa.\{=\}.\sim\forall Qb = \emptyset & (T_{M668})(\text{Null Set Disjunction}) \\
\forall Sa.\{=\}.\forall Qb = \emptyset & (T_{R037})(\text{Sub-Relationship Negative Theorem}) \\
\text{Thus } ((@_T t^1)a \subseteq (@_T t^1)b) = (\forall Sa.\{=\}.\forall Qb = \emptyset) & (A_{MX02})(\text{Conditional Introduction}) \\
& \text{since this part of the proof is reversible.} \\
\text{Thus } ((t^1 \subseteq TE).((@_T t^1)a \neq \emptyset) = E) \subseteq (((@_T t^1)a \subseteq (@_T t^1)b) = (\forall Sa.\{=\}.\forall Qb = \emptyset)) & \\
& (A_{MX02})(\text{Conditional Introduction}) \\
((t^1 \subseteq TE).((@_T t^1)a \neq \emptyset) = E) \subseteq (((@_T t^1)a \subseteq (@_T t^1)b) = (\forall Sa.\{=\}.\forall Qb = \emptyset)) & \\
& \text{Resymbolising}
\end{array}$$

$$((t^1 \subseteq TE).((@_T t^1)a \neq \emptyset) \subseteq ((@_T t^1)a \not\subseteq (@_T t^1)b)) = (\forall Sa.\{=\}.\forall Qb \neq \emptyset) \quad (T_{T900})$$

$$((t^1 \subseteq TE).((@_T t^1)a \neq \emptyset) \subseteq ((@_T t^1)a \not\subseteq (@_T t^1)b)) = (\forall Sa.\{=\}.\forall Qb \neq \emptyset) \quad (T_{T901})$$

Example Proof of (T_{T900})

$$\begin{array}{ll}
((t^1 \subseteq TE).((@_T t^1)a \neq \emptyset) \subseteq ((@_T t^1)a \subseteq (@_T t^1)b)) = (\forall Sa.\{=\}.\forall Qb = \emptyset) & (T_{T898}) \\
((t^1 \subseteq TE).((@_T t^1)a \neq \emptyset) \subseteq ((@_T t^1)a \not\subseteq (@_T t^1)b)) = (\forall Sa.\{=\}.\forall Qb \neq \emptyset) & (T_{M008})(\text{Inverse of an Equality})
\end{array}$$

13.3 Proofs of Theorems of Time & Tense

$$\mathcal{T}(\forall \text{Sa}^1.c. \forall \text{Qb}^1) \subseteq \mathcal{T}^v \text{Sa}^1 \quad (\mathcal{T}_{R921})$$

$$\mathcal{T}(\forall \text{Sa}^1.c. \forall \text{Qb}^1) \subseteq \mathcal{T}c \quad (\mathcal{T}_{R923})$$

$$\mathcal{T}(\forall \text{Sa}^1.c. \forall \text{Qb}^1) \subseteq \mathcal{T}^v \text{Qb}^1 \quad (\mathcal{T}_{R924})$$

Example Proof of ($\mathcal{T}_{R921}$)

$$\forall \text{Sa}^1.c. \forall \text{Qb}^1 \subseteq \forall \text{Sa}^1 \quad (\mathcal{T}_{M083})(\text{Conjunction Elimination})$$

$$\mathcal{T}(\forall \text{Sa}^1.c. \forall \text{Qb}^1) \subseteq \mathcal{T}^v \text{Sa}^1 \quad (\mathcal{T}_{R196})(\text{Relational Deduction})$$

$$(x^1 \subseteq R_E) \subseteq (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1) \quad (\mathcal{T}_{T357})$$

$$(x^1 \subseteq R_E) \subseteq (\mathcal{T}x^1 = \mathcal{T}(@Qx^1)x^1) \quad (\mathcal{T}_{T358})$$

Example Proof of ($\mathcal{T}_{T357}$)

$$(x^1 \subseteq R_E) = (\mathcal{T}(@Sx^1)x^1 = \mathcal{T}(@Qx^1)x^1)$$

$$x^1 = ((@Sx^1)x^1 + (@Qx^1)x^1)^1$$

$$\text{Let } x^1 \neq \emptyset$$

$$(@x^1)x^1 = x^1$$

$$(@x^1)x^1 \neq \emptyset$$

$$\text{Let } x^1 = \forall \text{Sa}^1.x^1. \forall \text{Qb}^1$$

$$Sx^1 = S(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1)$$

$$Sx^1 = a^1$$

$$Qx^1 = Q(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1)$$

$$Qx^1 = b^1$$

$$(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1 \subseteq R_E) = (\mathcal{T}(@a^1)(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) = \mathcal{T}(@b^1)(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1))$$

($\mathcal{D}_{R37}$)(Definition of an Event Relationship)(Eqn1)

($\mathcal{A}_{S13}$)(Axiom of Physical Relationships)(Eqn2)

(Eqn3)

($\mathcal{T}_{R984}$)(Eqn4)

($\mathcal{A}_{MX19}$)(Substitution of Eqn4 into Eqn3)

(Eqn5)

($\mathcal{T}_{R522}$)(Relational Deduction)

($\mathcal{T}_{T359}$)(Eqn6) since $x^1 \neq \emptyset$

($\mathcal{T}_{R522}$)(Relational Deduction)

($\mathcal{T}_{T360}$)(Eqn7)

($\mathcal{A}_{MX19}$)(Substitution of Eqns5,6&7 into Eqn1)(Eqn8)

$$\forall \text{Sa}^1.x^1. \forall \text{Qb}^1 = ((@a^1)(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) + (@b^1)(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1))^1$$

($\mathcal{A}_{MX19}$)(Substitution of Eqns5,6&7 into Eqn2)

$$\mathcal{T}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) = \mathcal{T}((@a^1)(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) + (@b^1)(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1))^1$$

($\mathcal{T}_{R522}$)(Relational Deduction)

$$\mathcal{T}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) = (\mathcal{T}((@a^1)(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) + (@b^1)(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1)))^1$$

($\mathcal{A}_{S05T}$)

$$\mathcal{T}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) = (\mathcal{T}(@a^1)(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) + \mathcal{T}(@b^1)(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1))^1$$

($\mathcal{T}_{R002b}$)(Distribution of * over +)(Eqn9)

$$(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1 \subseteq R_E) = (\mathcal{T}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) = (\mathcal{T}(@a^1)(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) + \mathcal{T}(@b^1)(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1))^1)$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn8 into Eqn9)

$$(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1 \subseteq R_E) = (\mathcal{T}(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1) = (\mathcal{T}(@a^1)(\forall \text{Sa}^1.x^1. \forall \text{Qb}^1))^1)$$

($\mathcal{A}_{MX01}$)(Disjunction Tautology)(Eqn10)

$$(x^1 \subseteq R_E) = (\mathcal{T}x^1 = \mathcal{T}((@Sx^1)x^1)^1)$$

($\mathcal{A}_{MX19}$)(Substitution of Eqns5&6 into Eqn10)

$$(x^1 \subseteq R_E) = (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1)$$

($\mathcal{T}_{T361}$) since $(@x^1)x^1 \neq \emptyset$

$$\text{Thus } ((x^1 \neq \emptyset) = E) \subseteq ((x^1 \subseteq R_E) = (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1)) \quad (\mathcal{A}_{MX02})(\text{Conditional Introduction})$$

$$(x^1 \neq \emptyset)(x^1 \subseteq R_E) = (x^1 \neq \emptyset)(\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1)$$

($\mathcal{T}_{M444b}$)(Exportation from a Biconditional)

$$(x^1 \neq \emptyset)(x^1 \subseteq R_E) \subseteq (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1)$$

($\mathcal{T}_{M073b}$)(Single Premise Syllogism)(Eqn11)

$$\text{Let } x^1 = \emptyset$$

(Eqn12)

$$\text{Let } (x^1 \subseteq R_E) \neq (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1)$$

(Eqn13)

$$(\emptyset \subseteq R_E) \neq (\mathcal{T}\emptyset = \mathcal{T}(@\emptyset)\emptyset)$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn12 into Eqn13)

$$(\emptyset \subseteq R_E) \neq (\emptyset = \emptyset)$$

($\mathcal{T}_{S181}$)(Null Object Theorem)

$$E \neq (\emptyset = \emptyset)$$

($\mathcal{T}_{M001}$)(Universality of ($\mathcal{T}_{M670}$) Null Set Theorem)

$$E \neq E$$

($\mathcal{T}_{M343}$)

$$\emptyset$$

($\mathcal{T}_{M345}$)

$$\text{Thus } ((x^1 = \emptyset) = E)((x^1 \subseteq R_E) \neq (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1)) = \emptyset$$

($\mathcal{A}_{MX02}$)(Conditional Introduction)

$$((x^1 = \emptyset) = E) \subseteq ((x^1 \subseteq R_E) = (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1))$$

($\mathcal{T}_{M735}$)($\subseteq$ Existential Form)

$$\begin{aligned}
(x^1 = \emptyset)(x^1 \subseteq R_E) &= (x^1 = \emptyset)(\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1) & (T_{M444b})(\text{Exportation from a Biconditional}) \\
(x^1 = \emptyset)(x^1 \subseteq R_E) &\subseteq (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1) & (T_{M073b})(\text{Single Premise Syllogism})(\text{Eqn14}) \\
(x^1 \neq \emptyset)(x^1 \subseteq R_E) + (x^1 = \emptyset)(x^1 \subseteq R_E) &\subseteq (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1) + (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1) \\
& & (A_{MX21})(\text{Combination of Eqns11\&14}) \\
(x^1 \neq \emptyset)(x^1 \subseteq R_E) + (x^1 = \emptyset)(x^1 \subseteq R_E) &\subseteq (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1) & (A_{MX01})(\text{Disjunction Tautology}) \\
((x^1 \neq \emptyset) + (x^1 = \emptyset))(x^1 \subseteq R_E) &\subseteq (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1) & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
E(x^1 \subseteq R_E) &\subseteq (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1) & (T_{M671})(\text{Tertium non Datur}) \\
(x^1 \subseteq R_E) &\subseteq (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1) & (A_{MX11})(\text{Universal Set Conjunction Axiom})
\end{aligned}$$

$$(x^1 \neq \emptyset) = ((@Sx^1)x^1 \neq \emptyset) \quad (T_{T372})$$

$$(x^1 \neq \emptyset) = ((@Qx^1)x^1 \neq \emptyset) \quad (T_{T373})$$

Example Proof of (T_{T372})

$$\begin{aligned}
X &= (S, Q, V, {}^vS, {}^vQ): (a^1 \neq \emptyset) = (X * a^1 \neq \emptyset) & (A_{R42})(\text{Matching Axiom}) \\
(a^1 \neq \emptyset) &= (Sa^1 \neq \emptyset) & \text{Extraction} \\
(x^1 \neq \emptyset) &= (Sx^1 \neq \emptyset) & \text{Resymbolising (Eqn1)} \\
(@Sx^1)x^1 &\subseteq \{\subseteq\}Sx^1 & (T_{T369}) \\
((@Sx^1)x^1 \subseteq \{\subseteq\}Sx^1) &= E & (A_{MX16})(\text{Order Axiom}) \\
((@Sx^1)x^1 \neq \emptyset) &\subseteq (\{\subseteq\}Sx^1 \neq \emptyset) & (T_{M096})(\text{Probabilistic Rule})(\text{Eqn2}) \\
(\{\subseteq\}Sx^1 \neq \emptyset) &\subseteq (Sx^1 \neq \emptyset) & (T_{R170})(\text{Eqn3}) \\
((@Sx^1)x^1 \neq \emptyset) &\subseteq (Sx^1 \neq \emptyset) & (T_{M139})(\text{Hypothetical Syllogism Eqns2\&3})(\text{Eqn4}) \\
((@Sx^1)x^1 \neq \emptyset) &\subseteq (x^1 \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn4})(\text{Eqn5}) \\
(X = S, Q) = E &\subseteq ((x^1 \neq \emptyset) \subseteq ((@Sx^1)x^1 \neq \emptyset)) & (A_{S16})(2^{\text{nd}} \text{ Axiom of Physical Relationships}) \\
(x^1 \neq \emptyset) &\subseteq ((@Sx^1)x^1 \neq \emptyset) & \text{Extraction (Eqn6)} \\
(x^1 \neq \emptyset) &= ((@Sx^1)x^1 \neq \emptyset) & (T_{M430})(\text{Biconditional Introduction Eqns5\&6})
\end{aligned}$$

$$((@a^1)({}^vSa^1.x^1.{}^vQb^1) \neq \emptyset) = ({}^vSa^1.x^1.{}^vQb^1 \neq \emptyset) \quad (T_{T374})$$

Example Proof of (T_{T374})

$$\begin{aligned}
(x^1 \neq \emptyset) &= ((@Sx^1)x^1 \neq \emptyset) & (T_{T372})(\text{Eqn1}) \\
\text{Let } x^1 &= {}^vSa^1.x^1.{}^vQb^1 & (\text{Eqn2}) \\
({}^vSa^1.x^1.{}^vQb^1 \neq \emptyset) &= ((@S({}^vSa^1.x^1.{}^vQb^1))({}^vSa^1.x^1.{}^vQb^1) \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})(\text{Eqn3}) \\
({}^vSa^1.x^1.{}^vQb^1 \neq \emptyset) &= (S({}^vSa^1.x^1.{}^vQb^1) = a^1) & (T_{T359})(\text{Eqn4}) \\
({}^vSa^1.x^1.{}^vQb^1 \neq \emptyset) &= ((@a^1)({}^vSa^1.x^1.{}^vQb^1) \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn3})
\end{aligned}$$

$$((@a^1)({}^vSa^1.c.{}^vQb^1) \neq \emptyset) = ({}^vSa^1.c.{}^vQb^1 \neq \emptyset) \quad (T_{T328})$$

Proof of (T_{T328})

$$\begin{aligned}
({}^vSa^1.x_r.{}^vQb^1 \neq \emptyset) &= ((@a^1)({}^vSa^1.x_r.{}^vQb^1) \neq \emptyset) & (T_{T374})(\text{Eqn1}) \\
({}^vSa^1.x_1.{}^vQb^1 \neq \emptyset) + \dots + ({}^vSa^1.x_n.{}^vQb^1 \neq \emptyset) &= ((@a^1)({}^vSa^1.x_1.{}^vQb^1) \neq \emptyset) + \dots + ((@a^1)({}^vSa^1.x_n.{}^vQb^1) \neq \emptyset) \\
& & (A_{MX21})(\text{Combination of Eqn1 for } r = 1 \text{ to } n) \\
({}^vSa^1.x_1.{}^vQb^1 + \dots + {}^vSa^1.x_n.{}^vQb^1 \neq \emptyset) &= ((@a^1)({}^vSa^1.x_1.{}^vQb^1) + \dots + ((@a^1)({}^vSa^1.x_n.{}^vQb^1) \neq \emptyset) \\
& & (T_{M238})(\text{Probabilistic Disjunction}) \\
({}^vSa^1.x_1.{}^vQb^1 + \dots + {}^vSa^1.x_n.{}^vQb^1 \neq \emptyset) &= ((@a^1)({}^vSa^1.x_1.{}^vQb^1 + \dots + {}^vSa^1.x_n.{}^vQb^1) \neq \emptyset) \\
& & (A_{S64})(\text{Distribution of } * \text{ over } +)(\text{Eqn2}) \\
\text{Let } {}^vSa^1.x_1.{}^vQb^1 + \dots + {}^vSa^1.x_n.{}^vQb^1 &= {}^vSa^1.c.{}^vQb^1 & (\text{Eqn3}) \\
({}^vSa^1.c.{}^vQb^1 \neq \emptyset) &= ((@a^1)({}^vSa^1.c.{}^vQb^1) \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn3})
\end{aligned}$$

$$(x^1 \subseteq R_E) = (\mathcal{T}x^1 = \mathcal{T}\{\subseteq\}Sx^1.\mathcal{T}x^1.\mathcal{T}\{\subseteq\}Qx^1) \quad (T_{T375})$$

Proof of (T_{T375})

$$\begin{aligned}
(x^1 \subseteq R_E) &= (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1) & (T_{T357}) \\
(x^1 \subseteq R_E) &= (\mathcal{T}x^1 = \mathcal{T}(@Sx^1)x^1.\mathcal{T}(@Sx^1)x^1) & (T_{M667})(\text{Conjunction Tautology})(\text{Eqn1})
\end{aligned}$$

$$\begin{aligned}
(x^1 \subseteq R_E) &= (T(@Sx^1)x^1 = T(@Qx^1)x^1) & (D_{R37})(\text{Definition of an Event Relationship})(\text{Eqn2}) \\
(x^1 \subseteq R_E) &= (Tx^1 = T(@Sx^1)x^1 \cdot T(@Qx^1)x^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\
((x^1 \subseteq R_E) = (Tx^1 = T(@Sx^1)x^1 \cdot T(@Qx^1)x^1)) &= E & (A_{MX16})(\text{Order Axiom}) \\
((x^1 \subseteq R_E) = E) &= ((Tx^1 = T(@Sx^1)x^1 \cdot T(@Qx^1)x^1) = E) & (T_{M094})(\text{Probabilistic Rule}) \\
((x^1 \subseteq R_E) = E) &= ((Tx^1 = T(@Sx^1)x^1 \cdot T(@Qx^1)x^1) + (Tx^1 \subseteq T(@Sx^1)x^1 \cdot T(@Qx^1)x^1) = E) & (T_{M159}) \\
((x^1 \subseteq R_E) = E) &= ((Tx^1 \subseteq T(@Sx^1)x^1 \cdot T(@Qx^1)x^1) = E) & (T_{M266})(\text{Definition of } \subseteq)(\text{Eqn4}) \\
(@Sx^1)x^1 \subseteq \{\subseteq\}Sx^1 & & (T_{369}) \\
T(@Sx^1)x^1 \subseteq T\{\subseteq\}Sx^1 & & (T_{R196})(\text{Relational Deduction})(\text{Eqn5}) \\
(@Qx^1)x^1 \subseteq \{\subseteq\}Qx^1 & & (T_{369}) \\
T(@Qx^1)x^1 \subseteq T\{\subseteq\}Qx^1 & & (T_{R196})(\text{Relational Deduction})(\text{Eqn6}) \\
T(@Sx^1)x^1 \cdot T(@Qx^1)x^1 \subseteq T\{\subseteq\}Sx^1 \cdot T\{\subseteq\}Qx^1 & & (T_{M669})(\text{Combination of Eqns5\&6}) \\
(T(@Sx^1)x^1 \cdot T(@Qx^1)x^1 \subseteq T\{\subseteq\}Sx^1 \cdot T\{\subseteq\}Qx^1) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn7}) \\
(x^1 \subseteq a)(a \subseteq b) \subseteq (x^1 a = x^1 b) & & (T_{M579})(\text{Eqn8}) \\
\text{Let } (x^1 \subseteq R_E) & & \\
\text{Let } x^1 = Tx^1 & & (\text{Eqn9}) \\
\text{Let } a = T(@Sx^1)x^1 \cdot T(@Qx^1)x^1 & & (\text{Eqn10}) \\
\text{Let } b = T\{\subseteq\}Sx^1 \cdot T\{\subseteq\}Qx^1 & & (\text{Eqn11}) \\
(Tx^1 \subseteq T(@Sx^1)x^1 \cdot T(@Qx^1)x^1) \cdot (T(@Sx^1)x^1 \cdot T(@Qx^1)x^1 \subseteq T\{\subseteq\}Sx^1 \cdot T\{\subseteq\}Qx^1) \subseteq & & \\
(T(@Sx^1)x^1 \cdot Tx^1 \cdot T(@Qx^1)x^1 = T\{\subseteq\}Sx^1 \cdot Tx^1 \cdot T\{\subseteq\}Qx^1) & & (A_{MX19})(\text{Substitution of Eqns9,10\&11 into Eqn8})(\text{Eqn12}) \\
E \cdot E \subseteq (T(@Sx^1)x^1 \cdot Tx^1 \cdot T(@Qx^1)x^1 = T\{\subseteq\}Sx^1 \cdot Tx^1 \cdot T\{\subseteq\}Qx^1) & & (A_{MX19})(\text{Substitution of Eqns4\&7 into Eqn12}) \\
E \subseteq (T(@Sx^1)x^1 \cdot Tx^1 \cdot T(@Qx^1)x^1 = T\{\subseteq\}Sx^1 \cdot Tx^1 \cdot T\{\subseteq\}Qx^1) & & (T_{M667})(\text{Conjunction Tautology}) \\
E = (T(@Sx^1)x^1 \cdot Tx^1 \cdot T(@Qx^1)x^1 = T\{\subseteq\}Sx^1 \cdot Tx^1 \cdot T\{\subseteq\}Qx^1) & & (T_{M275})(\text{Supersets of the Universal Set}) \\
T(@Sx^1)x^1 \cdot Tx^1 \cdot T(@Qx^1)x^1 = T\{\subseteq\}Sx^1 \cdot Tx^1 \cdot T\{\subseteq\}Qx^1 & & (A_{MX16})(\text{Order Axiom}) \\
\text{Thus } (x^1 \subseteq R_E) \subseteq (T(@Sx^1)x^1 \cdot Tx^1 \cdot T(@Qx^1)x^1 = T\{\subseteq\}Sx^1 \cdot Tx^1 \cdot T\{\subseteq\}Qx^1) & & (A_{MX02})(\text{Conditional Introduction})(\text{Eqn13}) \\
(x^1 \subseteq R_E) \subseteq (Tx^1 \cdot Tx^1 = T\{\subseteq\}Sx^1 \cdot Tx^1 \cdot T\{\subseteq\}Qx^1) & & (A_{MX19})(\text{Substitution of Eqn3 into Eqn13}) \\
(x^1 \subseteq R_E) = (Tx^1 = T\{\subseteq\}Sx^1 \cdot Tx^1 \cdot T\{\subseteq\}Qx^1) & & (T_{M667})(\text{Conjunction Tautology})
\end{aligned}$$

$$((\forall Sa^1 \cdot c \cdot \forall Qb^1)^1 \subseteq R_E) \subseteq (T(\forall Sa^1 \cdot c \cdot \forall Qb^1) \subseteq \{\supseteq\}Ta^1 \cdot Tc \cdot \{\supseteq\}Tb^1) \quad (T_{327})$$

Proof of (T₃₂₇)

$$\begin{aligned}
(x^1 \subseteq R_E) &= (Tx^1 = T\{\subseteq\}Sx^1 \cdot Tx^1 \cdot T\{\subseteq\}Qx^1) & (T_{375}) \\
(x^1 \subseteq R_E) &= (Tx^1 = \{\supseteq\}TSx^1 \cdot Tx^1 \cdot \{\supseteq\}TQx^1) & (T_{377}T)(\text{Eqn1}) \\
(x_1^1 \subseteq R_E) \dots (x_n^1 \subseteq R_E) &= (Tx_1^1 = \{\supseteq\}TSx_1^1 \cdot Tx_1^1 \cdot \{\supseteq\}TQx_1^1) \dots (Tx_n^1 = \{\supseteq\}TSx_n^1 \cdot Tx_n^1 \cdot \{\supseteq\}TQx_n^1) & (T_{M669})(\text{Combination of Eqn1 for } r=1 \text{ to } n) \\
(x_1^1 + \dots + x_n^1 \subseteq R_E) &= (Tx_1^1 = \{\supseteq\}TSx_1^1 \cdot Tx_1^1 \cdot \{\supseteq\}TQx_1^1) \dots (Tx_n^1 = \{\supseteq\}TSx_n^1 \cdot Tx_n^1 \cdot \{\supseteq\}TQx_n^1) & (T_{M131b})(\text{Conjunction of Sufficiencies}) \\
(x_1^1 + \dots + x_n^1 \subseteq R_E) &\subseteq (Tx_1^1 + \dots + Tx_n^1 = \{\supseteq\}TSx_1^1 \cdot Tx_1^1 \cdot \{\supseteq\}TQx_1^1 + \dots + \{\supseteq\}TSx_n^1 \cdot Tx_n^1 \cdot \{\supseteq\}TQx_n^1) & (T_{M230})(\text{Combination}) \\
(x_1^1 + \dots + x_n^1 \subseteq R_E) &\subseteq (T(x_1^1 + \dots + x_n^1) = \{\supseteq\}TSx_1^1 \cdot Tx_1^1 \cdot \{\supseteq\}TQx_1^1 + \dots + \{\supseteq\}TSx_n^1 \cdot Tx_n^1 \cdot \{\supseteq\}TQx_n^1) & (T_{R002b})(\text{Distribution of } * \text{ over } +)(\text{Eqn2}) \\
\text{For } r = 1 \text{ to } n & & \\
\text{Let } x_r^1 \neq \emptyset & & \\
\text{Let } x_r^1 = \forall Sa^1 \cdot x_r^1 \cdot \forall Qb^1 & & (\text{Eqn3}) \\
Sx_r^1 = S(\forall Sa^1 \cdot x_r^1 \cdot \forall Qb^1) & & (T_{R522})(\text{Relational Deduction}) \\
Sx_r^1 = a^1 & & (T_{359})(\text{Eqn4}) \text{ since } x_r^1 \neq \emptyset \\
Qx_r^1 = Q(\forall Sa^1 \cdot x_r^1 \cdot \forall Qb^1) & & (T_{R522})(\text{Relational Deduction})
\end{aligned}$$

$$\begin{aligned}
& Qx_r^1 = b^1 && (T_{360})(\text{Eqn5}) \\
& ((\forall Sa^1.x_1^1.\forall Qb^1)+++(\forall Sa^1.x_n^1.\forall Qb^1) \subseteq R_E) \subseteq (\mathcal{T}((\forall Sa^1.x_1^1.\forall Qb^1)+++(\forall Sa^1.x_n^1.\forall Qb^1)) = \\
& \{\supset\}Ta^1.\mathcal{T}(\forall Sa^1.x_r^1.\forall Qb^1).\{\supset\}Tb^1+++ \{\supset\}Ta^1.\mathcal{T}(\forall Sa^1.x_r^1.\forall Qb^1).\{\supset\}Tb^1) \\
& && (A_{MX19})(\text{Substitution of Eqns3,4\&5 into Eqn2}) \\
& ((\forall Sa^1.x_1^1.\forall Qb^1)+++(\forall Sa^1.x_n^1.\forall Qb^1) \subseteq R_E) \subseteq (\mathcal{T}((\forall Sa^1.x_1^1.\forall Qb^1)+++(\forall Sa^1.x_n^1.\forall Qb^1)) = \\
& \{\supset\}Ta^1.(\mathcal{T}(\forall Sa^1.x_1^1.\forall Qb^1)+++ \mathcal{T}(\forall Sa^1.x_n^1.\forall Qb^1)).\{\supset\}Tb^1) \\
& && (T_{M407})(\text{Distribution of } \cdot \text{ over } +) \\
& ((\forall Sa^1.x_1^1.\forall Qb^1)+++(\forall Sa^1.x_n^1.\forall Qb^1) \subseteq R_E) \subseteq (\mathcal{T}((\forall Sa^1.x_1^1.\forall Qb^1)+++(\forall Sa^1.x_n^1.\forall Qb^1)) = \\
& \{\supset\}Ta^1.\mathcal{T}((\forall Sa^1.x_1^1.\forall Qb^1)+++(\forall Sa^1.x_n^1.\forall Qb^1)).\{\supset\}Tb^1) \\
& && (T_{R002b})(\text{Distribution of } * \text{ over } +)(\text{Eqn6}) \\
& \text{Let } \forall Sa^1.x_1^1.\forall Qb^1+++ \forall Sa^1.x_n^1.\forall Qb^1 = \forall Sa^1.c.\forall Qb^1 && (\text{Eqn7}) \\
& (\forall Sa^1.c.\forall Qb^1 \subseteq R_E) \subseteq (\mathcal{T}(\forall Sa^1.c.\forall Qb^1) = \{\supset\}Ta^1.\mathcal{T}(\forall Sa^1.c.\forall Qb^1).\{\supset\}Tb^1) \\
& && (A_{MX19})(\text{Substitution of Eqn7 into Eqn6}) \\
& ((\forall Sa^1.c.\forall Qb^1)^1 \subseteq R_E) \subseteq (\mathcal{T}(\forall Sa^1.c.\forall Qb^1) = \{\supset\}Ta^1.\mathcal{T}(\forall Sa^1.c.\forall Qb^1).\{\supset\}Tb^1) \\
& && (A_{S17})(\text{Axiom of Event Relationships})(\text{Eqn8}) \\
& \forall Sa^1.c.\forall Qb^1 \subseteq c && (T_{M083})(\text{Conjunction Elimination}) \\
& \mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \mathcal{T}c && (T_{R196})(\text{Relational Deduction}) \\
& \{\supset\}Ta^1.\mathcal{T}(\forall Sa^1.c.\forall Qb^1).\{\supset\}Tb^1 \subseteq \{\supset\}Ta^1.\mathcal{T}c.\{\supset\}Tb^1 \\
& && (T_{M029})(\text{Deduction})(\text{Eqn9}) \\
& ((\forall Sa^1.c.\forall Qb^1)^1 \subseteq R_E) \subseteq (\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \{\supset\}Ta^1.\mathcal{T}c.\{\supset\}Tb^1) \\
& && (T_{M700})(\text{Substitution Syllogism Eqns8\&9})
\end{aligned}$$

$$(c^1 \subseteq R_E) \subseteq ((\forall Sa^1.c.\forall Qb^1)^1 \subseteq R_E) \quad (T_{319})$$

Proof of (T₃₁₉)

$$\begin{aligned}
& \forall Sa^1.c.\forall Qb^1 \subseteq c && (T_{M083})(\text{Conjunction Elimination}) \\
& (c \subseteq d) \subseteq (\forall Sa^1.c.\forall Qb^1 \subseteq d) && (T_{M364})(\text{Deductive Rule}) \\
& (c^1 \subseteq \{\supset\}d^1) \subseteq ((\forall Sa^1.c.\forall Qb^1)^1 \subseteq \{\supset\}d^1) && (T_{R653})(\text{Definition of a Relationship})(\text{Eqn1}) \\
& \text{Let } \{\supset\}d^1 = R_E && (\text{Eqn2}) \\
& (c^1 \subseteq R_E) \subseteq ((\forall Sa^1.c.\forall Qb^1)^1 \subseteq R_E) && (A_{MX19})(\text{Substitution of Eqns2 into Eqn1})
\end{aligned}$$

$$(c^1 \subseteq R_E) \subseteq (\{\supset\}_\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \{\supset\}_\mathcal{T}a^1) \quad (T_{R926})$$

$$(c^1 \subseteq R_E) \subseteq (\{\supset\}_\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \{\supset\}_\mathcal{T}c) \quad (T_{R927})$$

$$(c^1 \subseteq R_E) \subseteq (\{\supset\}_\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \{\supset\}_\mathcal{T}b^1) \quad (T_{R928})$$

Example Proof of (T₉₂₆)

$$\begin{aligned}
& ((\forall Sa^1.c.\forall Qb^1)^1 \subseteq R_E) \subseteq (\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \{\supset\}Ta^1.\mathcal{T}c.\{\supset\}Tb^1) \\
& && (T_{327}) \\
& (c^1 \subseteq R_E) \subseteq (\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \{\supset\}Ta^1.\mathcal{T}c.\{\supset\}Tb^1) && (T_{319}) \\
& (c^1 \subseteq R_E) \subseteq (\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \{\supset\}Ta^1) && (T_{M073})(\text{Single Premise Syllogism}) \\
& (c^1 \subseteq R_E) \subseteq (\{\supset\}_\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \{\supset\}_\mathcal{T}Ta^1) && (T_{R196})(\text{Relational Deduction}) \\
& (c^1 \subseteq R_E) \subseteq (\{\supset\}_\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \{\supset\}_\mathcal{T}Ta^1) && (T_{A817}) \\
& (c^1 \subseteq R_E) \subseteq (\forall \mathcal{T}\{\supset\}_\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \forall \mathcal{T}\{\supset\}_\mathcal{T}Ta^1) && (T_{R196})(\text{Relational Deduction}) \\
& (c^1 \subseteq R_E) \subseteq (\{\supset\}_\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \{\supset\}_\mathcal{T}Ta^1) && (T_{R571})(\text{Definition of a Mapped Relationship})
\end{aligned}$$

$$\forall Sa^1.c.\forall Qb^1 \subseteq \{\supset\}_\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \quad (T_{320})$$

Proof of (T₃₂₀)

$$\begin{aligned}
& \{\supset\} = \{\supset\} + \{=\} && (D_{A141})(\text{Disjunction with } \{=\}) \\
& \{=\} \subseteq \{\supset\} && (T_{M072b})(\text{Single Premise Syllogism}) \\
& \{=\}\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \{\supset\}\mathcal{T}(\forall Sa^1.c.\forall Qb^1) && (T_{R197})(\text{Relational Deduction}) \\
& \mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \{\supset\}\mathcal{T}(\forall Sa^1.c.\forall Qb^1) && (T_{A080}) \\
& \forall \mathcal{T}\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \forall \mathcal{T}\{\supset\}_\mathcal{T}(\forall Sa^1.c.\forall Qb^1) && (T_{R196})(\text{Relational Deduction}) \\
& \forall \mathcal{T}\mathcal{T}(\forall Sa^1.c.\forall Qb^1) \subseteq \{\supset\}_\mathcal{T}(\forall Sa^1.c.\forall Qb^1) && (T_{R571})(\text{Definition of a Mapped Relationship})(\text{Eqn1})
\end{aligned}$$

$$\begin{aligned}
(c^1 \subseteq R_E) &\subseteq (\forall Sa^1.c. \forall Qb^1 \subseteq \{\supset\}_\tau a^1) & (T_{R929}) \\
(c^1 \subseteq R_E) &\subseteq (\{\supset\}_\tau (\forall Sa^1.c. \forall Qb^1) \subseteq \{\{\supset\}_\tau \supset\}_\tau a^1) & (T_{R196})(\text{Relational Deduction}) \\
((c^1 \subseteq R_E) &\subseteq (\{\supset\}_\tau (\forall Sa^1.c. \forall Qb^1) \subseteq \{\{\supset\}_\tau \supset\}_\tau a^1)) = E & (A_{MX16})(\text{Order Axiom}) \\
((c^1 \subseteq R_E) = E) &\subseteq ((\{\supset\}_\tau (\forall Sa^1.c. \forall Qb^1) \subseteq (\{\supset\}_\tau \supset\}_\tau a^1)) = E) & (T_{M094})(\text{Probabilistic Rule}) \\
((c^1 \subseteq R_E) = E) &\subseteq ((\{\supset\}_\tau (\forall Sa^1.c. \forall Qb^1) \neq \emptyset) \subseteq (\{\supset\}_\tau \supset\}_\tau a^1) \neq \emptyset) & (T_{M096})(\text{Probabilistic Rule})
\end{aligned}$$

$$(\{\supset\}_\tau (@a^1)(\forall Sa^1.c. \forall Qb^1) \neq \emptyset) \subseteq (\forall Sa^1 \neq \emptyset) \quad (T_{T334})$$

$$(\{\supset\}_\tau (@a^1)(\forall Sa^1.c. \forall Qb^1) \neq \emptyset) \subseteq (c \neq \emptyset) \quad (T_{T335})$$

$$(\{\supset\}_\tau (@a^1)(\forall Sa^1.c. \forall Qb^1) \neq \emptyset) \subseteq (\forall Qb^1 \neq \emptyset) \quad (T_{T336})$$

Example Proof of (T_{T334})

$$\begin{aligned}
(\{\supset\}_\tau (@a^1)(\forall Sa^1.c. \forall Qb^1) \neq \emptyset) &\subseteq (\forall Sa^1.c. \forall Qb^1 \neq \emptyset) & (T_{R170})(\text{Eqn1}) \\
(\forall Sa^1 \neq \emptyset) &\subseteq (\forall Sa^1.c. \forall Qb^1 \neq \emptyset) & (T_{M350}) \\
(\forall Sa^1.c. \forall Qb^1 \neq \emptyset) &\subseteq (\forall Sa^1 \neq \emptyset) & (T_{M012})(\text{Transposition})(\text{Eqn2}) \\
(\{\supset\}_\tau (@a^1)(\forall Sa^1.c. \forall Qb^1) \neq \emptyset) &\subseteq (\forall Sa^1 \neq \emptyset) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})
\end{aligned}$$

$$(\{\supset\}_\tau (@a^1)(\forall Sa^1.c. \forall Qb^1) \neq \emptyset) \subseteq (\{\supset\}_\tau \supset\}_\tau a^1 \neq \emptyset) \quad (T_{T337})$$

Proof of (T_{T337})

$$\begin{aligned}
(@a^1)(\forall Sa^1.c. \forall Qb^1) &\subseteq \{\subseteq\} a^1 & (T_{T369})(\text{Eqn1}) \\
\{\subseteq\} a^1 &\subseteq \{\supset\}_\tau a^1 & (T_{T382T})(\text{Eqn2}) \\
(@a^1)(\forall Sa^1.c. \forall Qb^1) &\subseteq \{\supset\}_\tau a^1 & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\
\{\supset\}_\tau (@a^1)(\forall Sa^1.c. \forall Qb^1) &\subseteq \{\supset\}_\tau \supset\}_\tau a^1 & (T_{R196})(\text{Relational Deduction}) \\
(\{\supset\}_\tau (@a^1)(\forall Sa^1.c. \forall Qb^1) &\subseteq \{\supset\}_\tau \supset\}_\tau a^1) = E & (A_{MX16})(\text{Order Axiom}) \\
(\{\supset\}_\tau (@a^1)(\forall Sa^1.c. \forall Qb^1) \neq \emptyset) &\subseteq (\{\supset\}_\tau \supset\}_\tau a^1 \neq \emptyset) & (T_{M096})(\text{Probabilistic Rule})
\end{aligned}$$

$$(\{\supset\}_\tau (@a^1)(\forall Sa^1.c. \forall Qb^1) \neq \emptyset) \subseteq (a^1 \neq \emptyset) \quad (T_{T380})$$

$$(\{\supset\}_\tau (@a^1)(\forall Sa^1.c. \forall Qb^1) \neq \emptyset) \subseteq (b^1 \neq \emptyset) \quad (T_{T338})$$

Example Proof of (T_{T338})

$$\begin{aligned}
(\{\supset\}_\tau (@a^1)(\forall Sa^1.c. \forall Qb^1) \neq \emptyset) &\subseteq (\forall Qb^1 \neq \emptyset) & (T_{T336})(\text{Eqn1}) \\
(\forall Qb^1 \neq \emptyset) &\subseteq (b^1 \neq \emptyset) & (T_{R170})(\text{Eqn2}) \\
(\{\supset\}_\tau (@a^1)(\forall Sa^1.c. \forall Qb^1) \neq \emptyset) &\subseteq (b^1 \neq \emptyset) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})
\end{aligned}$$

$$((\forall Sa^1.c. \forall Qb^1)^1 \subseteq R_E) \subseteq (\mathcal{T}(@a^1)(\forall Sa^1.c. \forall Qb^1) = \mathcal{T}(\forall Sa^1.c. \forall Qb^1)) \quad (T_{T325})$$

$$((\forall Sa^1.c. \forall Qb^1)^1 \subseteq R_E) \subseteq (\mathcal{T}(@b^1)(\forall Sa^1.c. \forall Qb^1) = \mathcal{T}(\forall Sa^1.c. \forall Qb^1)) \quad (T_{T326})$$

Example Proof of (T_{T325})

$$\begin{aligned}
(x_r^1 \subseteq R_E) &\subseteq (\mathcal{T}(@Sx_r^1)x_r^1 = \mathcal{T}x_r^1) & (T_{T357})(\text{Eqn1}) \\
\text{Let } x_r^1 &\neq \emptyset & \\
\text{Let } x_r^1 &= \forall Sa^1.x_r^1. \forall Qb^1 & (\text{Eqn2}) \\
Sx_r^1 &= S(\forall Sa^1.x_r^1. \forall Qb^1) & (T_{R522})(\text{Relational Deduction}) \\
Sx_r^1 &= a^1 & (T_{T359})(\text{Eqn3}) \text{ since } x_r^1 \neq \emptyset \\
(\forall Sa^1.x_r^1. \forall Qb^1 \subseteq R_E) &\subseteq (\mathcal{T}(@Sa^1)(\forall Sa^1.x_r^1. \forall Qb^1) = \mathcal{T}(\forall Sa^1.x_r^1. \forall Qb^1)) & \\
& & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})(\text{Eqn4}) \\
(\forall Sa^1.x_1^1. \forall Qb^1 \subseteq R_E) \dots (\forall Sa^1.x_n^1. \forall Qb^1 \subseteq R_E) &\subseteq & \\
(\mathcal{T}(@a^1)(\forall Sa^1.x_1^1. \forall Qb^1) = \mathcal{T}(\forall Sa^1.x_1^1. \forall Qb^1)) \dots (\mathcal{T}(@a^1)(\forall Sa^1.x_n^1. \forall Qb^1) = \mathcal{T}(\forall Sa^1.x_n^1. \forall Qb^1)) & & \\
& & (T_{M669})(\text{Combination of Eqn4 for } r = 1 \text{ to } n) \\
(\forall Sa^1.x_1^1. \forall Qb^1 \dots \forall Sa^1.x_n^1. \forall Qb^1 \subseteq R_E) &\subseteq & \\
(\mathcal{T}(@a^1)(\forall Sa^1.x_1^1. \forall Qb^1) = \mathcal{T}(\forall Sa^1.x_1^1. \forall Qb^1)) \dots (\mathcal{T}(@a^1)(\forall Sa^1.x_n^1. \forall Qb^1) = \mathcal{T}(\forall Sa^1.x_n^1. \forall Qb^1)) & & \\
& & (T_{M131b})(\text{Conjunction of Sufficiencies}) \\
(\forall Sa^1.x_1^1. \forall Qb^1 \dots \forall Sa^1.x_n^1. \forall Qb^1 \subseteq R_E) &\subseteq & \\
(\mathcal{T}(@a^1)(\forall Sa^1.x_1^1. \forall Qb^1) \dots \mathcal{T}(@a^1)(\forall Sa^1.x_n^1. \forall Qb^1) = \mathcal{T}(\forall Sa^1.x_1^1. \forall Qb^1)) \dots \mathcal{T}(\forall Sa^1.x_n^1. \forall Qb^1) & &
\end{aligned}$$

$$\begin{aligned}
& (\mathcal{T}_{M230})(\text{Combination}) \\
& (\forall \text{Sa}^1.x_1^1.\forall \text{Qb}^1+++ \forall \text{Sa}^1.x_n^1.\forall \text{Qb}^1 \subseteq \text{Re}) \subseteq \\
& (\mathcal{T}(@a^1)(\forall \text{Sa}^1.x_1^1.\forall \text{Qb}^1)+++(@a^1)(\forall \text{Sa}^1.x_n^1.\forall \text{Qb}^1)) = \mathcal{T}(\forall \text{Sa}^1.x_1^1.\forall \text{Qb}^1+++ \forall \text{Sa}^1.x_n^1.\forall \text{Qb}^1) \\
& (\mathcal{T}_{R002b})(\text{Distribution of } * \text{ over } +) \\
& (\forall \text{Sa}^1.x_1^1.\forall \text{Qb}^1+++ \forall \text{Sa}^1.x_n^1.\forall \text{Qb}^1 \subseteq \text{Re}) \subseteq \\
& (\mathcal{T}(@a^1)(\forall \text{Sa}^1.x_1^1.\forall \text{Qb}^1+++ \forall \text{Sa}^1.x_n^1.\forall \text{Qb}^1)) = \mathcal{T}(\forall \text{Sa}^1.x_1^1.\forall \text{Qb}^1+++ \forall \text{Sa}^1.x_n^1.\forall \text{Qb}^1) \\
& (\mathcal{T}_{R002b})(\text{Distribution of } * \text{ over } +)(\text{Eqn5}) \\
& \text{Let } \forall \text{Sa}^1.x_1^1.\forall \text{Qb}^1+++ \forall \text{Sa}^1.x_n^1.\forall \text{Qb}^1 = \forall \text{Sa}^1.c.\forall \text{Qb}^1 \quad (\text{Eqn6}) \\
& (\forall \text{Sa}^1.c.\forall \text{Qb}^1 \subseteq \text{Re}) \subseteq (\mathcal{T}(@a^1)(\forall \text{Sa}^1.c.\forall \text{Qb}^1)) = \mathcal{T}(\forall \text{Sa}^1.c.\forall \text{Qb}^1) \\
& (\mathcal{A}_{MX19})(\text{Substitution of Eqn6 into Eqn5}) \\
& ((\forall \text{Sa}^1.c.\forall \text{Qb}^1)^1 \subseteq \text{Re}) \subseteq (\mathcal{T}(@a^1)(\forall \text{Sa}^1.c.\forall \text{Qb}^1)) = \mathcal{T}(\forall \text{Sa}^1.c.\forall \text{Qb}^1) \\
& (\mathcal{A}_{S17})(\text{Axiom of Event Relationships})
\end{aligned}$$

$$\begin{aligned}
(c^1 \subseteq \text{Re}) & \subseteq (\mathcal{B}_{\mathcal{T}}(@a^1)(\forall \text{Sa}^1.c.\forall \text{Qb}^1) = \mathcal{B}_{\mathcal{T}}(\forall \text{Sa}^1.c.\forall \text{Qb}^1)) & (\mathcal{T}_{T353}) \\
(c^1 \subseteq \text{Re}) & \subseteq (C_{\mathcal{T}}(@a^1)(\forall \text{Sa}^1.c.\forall \text{Qb}^1) = C_{\mathcal{T}}(\forall \text{Sa}^1.c.\forall \text{Qb}^1)) & (\mathcal{T}_{T354}) \\
(c^1 \subseteq \text{Re}) & \subseteq (\{>\}_{\mathcal{T}}(@a^1)(\forall \text{Sa}^1.c.\forall \text{Qb}^1) = \{>\}_{\mathcal{T}}(\forall \text{Sa}^1.c.\forall \text{Qb}^1)) & (\mathcal{T}_{T355}) \\
(c^1 \subseteq \text{Re}) & \subseteq (\{<\}_{\mathcal{T}}(@a^1)(\forall \text{Sa}^1.c.\forall \text{Qb}^1) = \{<\}_{\mathcal{T}}(\forall \text{Sa}^1.c.\forall \text{Qb}^1)) & (\mathcal{T}_{T356})
\end{aligned}$$

Example Proof of ($\mathcal{T}_{T353}$)

$$\begin{aligned}
(c^1 \subseteq \text{Re}) & \subseteq ((\forall \text{Sa}^1.c.\forall \text{Qb}^1)^1 \subseteq \text{Re}) & (\mathcal{T}_{T319})(\text{Eqn1}) \\
((\forall \text{Sa}^1.c.\forall \text{Qb}^1)^1 \subseteq \text{Re}) & \subseteq (\mathcal{T}(@a^1)(\forall \text{Sa}^1.c.\forall \text{Qb}^1) = \mathcal{T}(\forall \text{Sa}^1.c.\forall \text{Qb}^1)) & (\mathcal{T}_{T325})(\text{Eqn2}) \\
(c^1 \subseteq \text{Re}) & \subseteq (\mathcal{T}(@a^1)(\forall \text{Sa}^1.c.\forall \text{Qb}^1) = \mathcal{T}(\forall \text{Sa}^1.c.\forall \text{Qb}^1)) & (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\
(c^1 \subseteq \text{Re}) & \subseteq (\mathcal{B}\mathcal{T}(@a^1)(\forall \text{Sa}^1.c.\forall \text{Qb}^1) = \mathcal{B}\mathcal{T}(\forall \text{Sa}^1.c.\forall \text{Qb}^1)) & (\mathcal{T}_{R522b})(\text{Relational Deduction}) \\
(c^1 \subseteq \text{Re}) & \subseteq (\mathcal{T}\mathcal{B}\mathcal{T}(@a^1)(\forall \text{Sa}^1.c.\forall \text{Qb}^1) = \mathcal{T}\mathcal{B}\mathcal{T}(\forall \text{Sa}^1.c.\forall \text{Qb}^1)) & (\mathcal{T}_{R522b})(\text{Relational Deduction}) \\
(c^1 \subseteq \text{Re}) & \subseteq (\mathcal{B}_{\mathcal{T}}(@a^1)(\forall \text{Sa}^1.c.\forall \text{Qb}^1) = \mathcal{B}_{\mathcal{T}}(\forall \text{Sa}^1.c.\forall \text{Qb}^1)) & (\mathcal{T}_{R571})(\text{Definition of a Mapped Relationship})
\end{aligned}$$

$$(x^1 \not\subseteq \text{Re}) = (\mathcal{T}(@\text{Sx}^1)x^1 \neq \mathcal{T}(@\text{Qx}^1)x^1) \quad (\mathcal{T}_{T322}) \text{ Definition of an Trans-temporal Relationship}$$

Proof of ($\mathcal{T}_{T322}$)

$$\begin{aligned}
(x^1 \subseteq \text{Re}) & = (\mathcal{T}(@\text{Sx}^1)x^1 = \mathcal{T}(@\text{Qx}^1)x^1) & (\mathcal{D}_{R37})(\text{Definition of an Event Relationship}) \\
(x^1 \not\subseteq \text{Re}) & = (\mathcal{T}(@\text{Sx}^1)x^1 \neq \mathcal{T}(@\text{Qx}^1)x^1) & (\mathcal{T}_{M008})(\text{Inverse of an Equality})
\end{aligned}$$

$$(x^1 \subseteq \forall \text{Sa}^1.c.\forall \text{Qb}^1)(\forall \text{Sa}^1.x_1^1.\forall \text{Qb}^1 \neq \emptyset) \subseteq (x^1 = ((@a^1)x^1 + (@b^1)x^1)^1) \quad (\mathcal{T}_{T321})$$

Proof of ($\mathcal{T}_{T321}$)

$$\begin{aligned}
x^1 & = ((@a^1)x^1 + (@b^1)x^1)^1 & (\mathcal{A}_{S13})(\text{Axiom of Physical Relationships})(\text{Eqn1}) \\
\text{Let } x^1 & \subseteq \forall \text{Sa}^1.c.\forall \text{Qb}^1 \\
\text{Let } \forall \text{Sa}^1.x_1^1.\forall \text{Qb}^1 & \neq \emptyset \\
x^1 & \neq \emptyset & (\mathcal{T}_{M350})\&(\mathcal{T}_{M012})(\text{Transposition}) \\
\text{Sx}^1 & \subseteq \text{S}(\forall \text{Sa}^1.c.\forall \text{Qb}^1) & (\mathcal{T}_{R196})(\text{Relational Deduction}) \\
\text{Sx}^1 & \subseteq a^1 & (\mathcal{T}_{T359}) \\
\text{Sx}^1 & = a^1 & (\mathcal{T}_{M577})(\text{Eqn2}) \\
\text{Qx}^1 & \subseteq \text{Q}(\forall \text{Sa}^1.c.\forall \text{Qb}^1) & (\mathcal{T}_{R196})(\text{Relational Deduction}) \\
\text{Qx}^1 & \subseteq b^1 & (\mathcal{T}_{T360}) \\
\text{Qx}^1 & = b^1 & (\mathcal{T}_{M577})(\text{Eqn3}) \\
x^1 & = ((@a^1)x^1 + (@b^1)x^1)^1 & (\mathcal{A}_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\
\text{Thus } (x^1 \subseteq \forall \text{Sa}^1.c.\forall \text{Qb}^1)(\forall \text{Sa}^1.x_1^1.\forall \text{Qb}^1 \neq \emptyset) & \subseteq (x^1 = ((@a^1)x^1 + (@b^1)x^1)^1) & (\mathcal{A}_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

13.4 Proofs of Theorems of Conventional Tense Logic

Conventional Tense Logic uses propositions or truth sets. These can be symbolised $\{p\}$, $\{q\}$, $\{r\}$, etc. rather than being expressed in their entirety. For example, $\{p\} = (a \subseteq b)$.

$\{p\} = E \subseteq (\{p\} \subseteq \{>\}_{T^*} \sim \{p\})$	(T_{7339}) First Rule of Tense Logic
$\{p\} = E \subseteq (\{p\} \subseteq \{<\}_{T^*} \sim \{p\})$	(T_{7341}) Second Rule of Tense Logic
$(\sim \{p\} = E) \subseteq (\sim \{p\} \subseteq \{>\}_{T^*} \{p\})$	(T_{7340})
$(\sim \{p\} = E) \subseteq (\sim \{p\} \subseteq \{<\}_{T^*} \{p\})$	(T_{7342})

Example Proof of (T_{7339})

Let $\{p\} = E$	(Eqn1)
Let $\{p\} \not\subseteq \{>\}_{T^*} \sim \{p\}$	(Eqn2)
$E \not\subseteq \{>\}_{T^*} E$	($AMX19$)(Substitution of Eqn1 into Eqn2)
$E \not\subseteq \{>\}_{T^*} \emptyset$	($DM02$)(Definition of $\emptyset$)
$E \not\subseteq \sim \emptyset$	($TS181$)(Null Object Theorem)
$E \not\subseteq E$	($TM054$)(Negative of Null Set)
$(E \sim \subseteq E). (E \neq E)$	($TM198$)
$(E \sim \subseteq E). \emptyset$	($TM345$)
$\emptyset$	($TM050$)(Null Set Conjunction)
Thus $(\{p\} = E). (\{p\} \not\subseteq \{>\}_{T^*} \sim \{p\}) = \emptyset$	($AMX02$)(Conditional Introduction)
$\{p\} = E \subseteq (\{p\} \subseteq \{>\}_{T^*} \sim \{p\})$	($TM735$)($\subseteq$ Existential Form)

$\{>\}_{T^*} \sim \{p\} \subseteq \{>\}_{T^*} \{p\}$	(T_{7347})
$\{<\}_{T^*} \sim \{p\} \subseteq \{<\}_{T^*} \{p\}$	(T_{7343}) Fifth Rule of Tense Logic

Example Proof of (T_{7347})

$\{>\}_{T^*} E = E$	(T_{7349})
$\{>\}_{T^*} (\sim \{p\} + \{p\}) = E$	($TM671$)(Tertium non Datur)
$(\{>\}_{T^*} \sim \{p\}) + (\{>\}_{T^*} \{p\}) = E$	($TR002b$)(Distribution of $*$ over $+$)
$\sim((\{>\}_{T^*} \sim \{p\}) + (\{>\}_{T^*} \{p\})) = \sim E$	($TM008$)(Inverse of an Equality)
$(\{>\}_{T^*} \sim \{p\}). (\{>\}_{T^*} \{p\}) = \sim E$	($TM026$)(DeMorgan's Law)
$(\{>\}_{T^*} \sim \{p\}). (\{>\}_{T^*} \{p\}) = \emptyset$	($DM02$)(Definition of $\emptyset$)
$\{>\}_{T^*} \sim \{p\} \subseteq \{>\}_{T^*} \{p\}$	($TM735$)($\subseteq$ Existential Form)

$\{p\} \subseteq \{>\}_{T^*} \{<\}_{T^*} \{p\}$	(T_{7344}) Seventh Rule of Tense Logic
$\{p\} \subseteq \{<\}_{T^*} \{>\}_{T^*} \{p\}$	(T_{7346}) Eighth Rule of Tense Logic

Example Proof of (T_{7344})

$b \subseteq {}^V c * c * b$	(T_{7348})
$\{p\} \subseteq {}^V \{<\}_{T^*} \{<\}_{T^*} \{p\}$	Resymbolising
$\{p\} \subseteq \{>\}_{T^*} \{<\}_{T^*} \{p\}$	($DA143$)(Symmetrical Opposite of an Asymmetrical Relationship Set)

$\{>\}_{T^*} \sim \{p\} \subseteq \{q\} \subseteq (\{>\}_{T^*} \sim \{p\} \subseteq \{>\}_{T^*} \sim \{q\})$	($TR933$) Ninth Rule of Tense Logic
$\{<\}_{T^*} \sim \{p\} \subseteq \{q\} \subseteq (\{<\}_{T^*} \sim \{p\} \subseteq \{<\}_{T^*} \sim \{q\})$	($TR934$) Tenth Rule of Tense Logic

Example Proof of ($TR933$)

$((a \subseteq b) = E) = ((a = E) \subseteq (b = E))$	($TM094$)(Probabilistic Rule)
$(a \subseteq b)(a = E) \subseteq (b = E)$	($AMX35$)(Exportation/Importation) for truth sets (Eqn1)
Let $(a = E) = \{p\}$	(Eqn2)*
Let $(b = E) = \{q\}$	(Eqn3)*
Let $(a \subseteq b) = \{\{p\} \subseteq \{q\}\}$	(Eqn4)*

$\{\{p\} \subseteq \{q\}\} \cdot \{p\} \subseteq \{q\}$	($\mathbf{A}_{MX19}$)(Substitution of Eqns2,3&4 into Eqn1)
$\sim\{q\} \subseteq \sim(\{\{p\} \subseteq \{q\}\} \cdot \{p\})$	($\mathbf{T}_{M012}$)(Transposition)
$\{>\}_{T^*} \sim \{q\} \subseteq \{>\}_{T^*} \sim (\{\{p\} \subseteq \{q\}\} + \sim\{p\})$	($\mathbf{T}_{R196}$)(Relational Deduction)
$\{>\}_{T^*} \sim \{q\} \subseteq \{>\}_{T^*} \sim (\{\{p\} \subseteq \{q\}\} + \{>\}_{T^*} \sim \{p\})$	($\mathbf{T}_{R002b}$)(Distribution of * over +)
$\sim(\{>\}_{T^*} \sim (\{\{p\} \subseteq \{q\}\} + \{>\}_{T^*} \sim \{p\})) \subseteq \{>\}_{T^*} \sim \{q\}$	($\mathbf{T}_{M012}$)(Transposition)
$\{>\}_{T^*} \sim (\{\{p\} \subseteq \{q\}\} \cdot \{>\}_{T^*} \sim \{p\}) \subseteq \{>\}_{T^*} \sim \{q\}$	($\mathbf{T}_{M026}$)(DeMorgan's Law)
$\{>\}_{T^*} \sim (\{\{p\} \subseteq \{q\}\} \subseteq (\{>\}_{T^*} \sim \{p\})) \subseteq \{>\}_{T^*} \sim \{q\}$	($\mathbf{A}_{MX35}$)(Exportation/Importation) for truth sets*

* Note that truth sets are all members of the set of circumstances and are of the same order, therefore. Thus, it is not necessary to adjust the order of a statement by appending = E.

$$\{>\}_{T^*}\{p\} = \{>\}_{T^*}\{>\}_{T^*}\{p\} \quad (\mathbf{T}_{T346})$$

$$\{<\}_{T^*}\{p\} = \{<\}_{T^*}\{<\}_{T^*}\{p\} \quad (\mathbf{T}_{R932}) \text{ Sixth Rule of Tense Logic}$$

Example Proof of ($\mathbf{T}_{RT346}$)

$(a^1 \subseteq S_M S_O + S_P S_O) \subseteq (\{<\}_{T^*} \{<\}_{T^*} a = \{<\}_{T^*} a)$	($\mathbf{T}_{S925}$)
$(a^1 \subseteq S_M S_O + S_P S_O) \subseteq (\{<\}_{T^*} \{<\}_{T^*} a = \{<\}_{T^*} a)$	($\mathbf{A}_{R57}$)(Binary Relationship Interchangeability Axiom)
$(\{p\}^1 \subseteq S_M S_O + S_P S_O) \subseteq (\{<\}_{T^*} \{<\}_{T^*} \{p\} = \{<\}_{T^*} \{p\})$	Resymbolising (Eqn1)
$\mathbf{D}_M + \mathbf{P}_M \subseteq S_M$	($\mathbf{A}_{S12}$)(Axiom of Mixed Continuum Dimensions)
$\mathbf{P}_M \subseteq S_M$	($\mathbf{T}_{M072}$)(Single Premise Syllogism)
$\mathbf{P}_M \subseteq S_M + S_P$	($\mathbf{T}_{M150}$)(Single Premise Syllogism)
$(\mathbf{P}_M \subseteq S_M + S_P) = \mathbf{E}$	($\mathbf{A}_{MX16}$)(Order Axiom)
$(\{p\}^1 \subseteq \mathbf{P}_M) \subseteq (\{p\}^1 \subseteq S_M + S_P)$	($\mathbf{T}_{M099}$)(Entity Mode / Set Mode Rule)(Eqn2)
$(\{p\}^1 \subseteq \mathbf{P}_M) \subseteq (\{<\}_{T^*} \{<\}_{T^*} \{p\} = \{<\}_{T^*} \{p\})$	($\mathbf{T}_{M139}$)(Hypothetical Syllogism Eqns2&1)(Eqn3)
Let $\{p\}$	
$\{p\} = \mathbf{E}$	($\mathbf{A}_{MX16}$)(Order Axiom)
$\{p\}^1 = \mathbf{E}^1$	($\mathbf{T}_{M479}$)(Eqn4)
$\mathbf{E}^1 \subseteq \mathbf{P}_M$	($\mathbf{T}_{T278}$)(Eqn5)
$\{p\}^1 \subseteq \mathbf{P}_M$	($\mathbf{A}_{MX19}$)(Substitution of Eqn4 into Eqn5)
$(\{p\}^1 \subseteq \mathbf{P}_M) = \mathbf{E}$	($\mathbf{A}_{MX16}$)(Order Axiom)(Eqn6)
$\mathbf{E} \subseteq (\{<\}_{T^*} \{<\}_{T^*} \{p\} = \{<\}_{T^*} \{p\})$	($\mathbf{A}_{MX19}$)(Substitution of Eqn6 into Eqn3)
$\mathbf{E} = (\{<\}_{T^*} \{<\}_{T^*} \{p\} = \{<\}_{T^*} \{p\})$	($\mathbf{T}_{M275}$)(Supersets of the Universal Set)
$\{<\}_{T^*} \{<\}_{T^*} \{p\} = \{<\}_{T^*} \{p\}$	($\mathbf{A}_{MX16}$)(Order Axiom)

13.5 Proofs of Theorems of Continuity

$$\mathbf{M}_{CS} \cdot \mathbf{M}_{CT} \subseteq \mathbf{M}_{CS} \quad (\mathbf{T}_{R888})$$

$$\mathbf{M}_{CS} \cdot \mathbf{M}_{CT} \subseteq \mathbf{M}_{CT} \quad (\mathbf{T}_{R889})$$

Example Proof of ($\mathbf{T}_{R888}$)

$$\mathbf{M}_{CS} \cdot \mathbf{M}_{CT} \subseteq \mathbf{M}_{CS} \quad (\mathbf{T}_{M083}) \text{ (Conjunction Elimination)}$$

$$\mathbf{M}_{CP} \subseteq \mathbf{M}_{CT} \quad (\mathbf{T}_{R891})$$

$$\sim \mathbf{M}_{CT} \subseteq \sim \mathbf{M}_{CP} \quad (\mathbf{T}_{R890})$$

Proof of ($\mathbf{T}_{R891}$) & ($\mathbf{T}_{R890}$)

$(a^1 \subseteq \mathbf{M}_{CP}) = ((@_T \mathcal{I} \mathbf{E}) a^1 \not\subseteq \sim \mathbf{M}_{CS}) \cdot (@_A (O a^1) \mathbf{E}) a^1 \cdot {}^V \mathcal{I} \mathbf{A}^1 \cdot \mathbf{M}_{CT} \neq \emptyset$	($\mathbf{D}_{R83}$)(Definition of a Spatio-temporally Continuous Physical Entity)(Eqn1)
$(a^1 \subseteq \mathbf{M}_{CP}) \subseteq (@_A (O a^1) \mathbf{E}) a^1 \cdot {}^V \mathcal{I} \mathbf{A}^1 \cdot \mathbf{M}_{CT} \neq \emptyset$	($\mathbf{T}_{M073b}$)(Single Premise Syllogism)
$(a^1 \subseteq \mathbf{M}_{CP}) \subseteq ({}^V \mathcal{I} \mathbf{A}^1 \neq \emptyset)$	($\mathbf{T}_{M350}$)($\mathbf{T}_{M012}$)
$(a^1 \subseteq \mathbf{M}_{CP}) \subseteq (a^1 \neq \emptyset)$	($\mathbf{T}_{R171b}$)(Eqn2)

$$\begin{aligned}
(a^1 \subseteq M_{CT}) &= (@_A(Oa^1)E)a^1. \forall T \neg Ta^1. M_{CT} \neq \emptyset & (D_{R81})(\text{Definition of a Temporally Continuous Physical Entity})(\text{Eqn3}) \\
(a^1 \subseteq M_{CP}) &= ((@_T TE)a^1 \not\subseteq \sim M_{CS}). (a^1 \subseteq M_{CT}) & (A_{MX19})(\text{Substitution of Eqn3 into Eqn1}) \\
(a^1 \subseteq M_{CP}) &\subseteq (a^1 \subseteq M_{CT}) & (T_{M073b})(\text{Single Premise Syllogism}) \\
(M_{CP} \subseteq M_{CT}) &= E & (T_{M099})(\text{Entity Mode / Set Mode Rule}) \text{ since } a^1 \neq \emptyset \\
M_{CP} &\subseteq M_{CT} & (A_{MX16})(\text{Order Axiom}) \\
\sim M_{CT} &\subseteq \sim M_{CP} & (T_{M012})(\text{Transposition})
\end{aligned}$$

13.6 Proofs of Theorems of Relationships

$$\forall Sa.\{=\} = (@_T TE)a \quad (T_{7703})$$

$$\forall Qa.\{=\} = (@_T TE)a \quad (T_{7704})$$

Example Proof of (T₇₇₀₃)

$$\begin{aligned}
(X = \forall S, \forall Q) &\subseteq (Xa.\{=\} = (@_T TE)a) & (A_{S94})(\text{Axiom of Temporal Sections}) \\
\forall Sa.\{=\} &= (@_T TE)a & \text{Extraction}
\end{aligned}$$

$$\forall Sa^1 = \{\supset\}(@_T TE)a^1 \quad (T_{S764})$$

$$\forall Qa^1 = \{\supset\}(@_T TE)a^1 \quad (T_{S765})$$

Example Proof of (T_{S764})

$$\begin{aligned}
(X = S, Q) &\subseteq (\forall Xa^1 = \{\supset\}(@_T TE)a^1) & (D_{R32})(\text{Physical Definition of } \forall S \text{ \& } \forall Q) \\
\forall Sa^1 &= \{\supset\}(@_T TE)a^1 & \text{Extraction}
\end{aligned}$$

$$\forall Sa^1 = (\{+\}E)(@_T TE)a^1 \quad (T_{S766})$$

$$\forall Qa^1 = (\{+\}E)(@_T TE)a^1 \quad (T_{S767})$$

Example Proof of (T_{S766})

$$\begin{aligned}
\forall Sa^1 &= \{\supset\}(@_T TE)a^1 & (T_{S764}) \\
\forall Sa^1 &= (\{+\}E)(@_T TE)a^1 & (T_{A178})
\end{aligned}$$

Chapter 14 Proofs of Theorems of Causality & Capability

14.1 Proofs

$$(a^1 \neq \emptyset)(a^1 \neq E^1) = (Ba^1 \neq \emptyset) \quad (T_{S775})$$

$$(a^1 \neq \emptyset)(a^1 \neq E^1) = (Ca^1 \neq \emptyset) \quad (T_{S776})$$

Example Proof of (T_{S775})

$$((X = B, C) = E) \subseteq ((a^1 \neq \emptyset)(a^1 \neq E^1) = (Xa^1 \neq \emptyset)) \quad (A_{R44})(\text{Axiom of Spatio-temporal Beginnings and Ends})$$

$$(a^1 \neq \emptyset)(a^1 \neq E^1) = (Ba^1 \neq \emptyset)$$

Extraction

$$Ba^1 = ((@_T B T a^1) a^1) \quad (T_{T383})$$

$$Ca^1 = (@_T C T a^1) a^1 \quad (T_{T384})$$

Example Proof of (T_{T383})

$$(a = a_1^1 \text{+++} a_n^1) \subseteq (Ba = (@_T B T a_1^1) a_1^1 \text{+++} (@_T B T a_n^1) a_n^1)$$

(D_{R67})(Definition of **B**)(Eqn1)

$$\text{For } r = 2 \text{ to } n \text{ let } a_r^1 = \emptyset$$

(Eqn2)

$$(a = a_1^1 \text{+++} \emptyset) \subseteq (Ba = (@_T B T a_1^1) a_1^1 \text{+++} (@_T B T \emptyset) \emptyset)$$

(A_{MX19})(For $r = 2$ to n Substitution of Eqn2 into Eqn1))

$$(a = a_1^1) \subseteq (Ba = (@_T B T a_1^1) a_1^1 \text{+++} (@_T B T \emptyset) \emptyset)$$

(T_{M668})(Null Set Disjunction)

$$(a = a_1^1) \subseteq (Ba = (@_T B T a_1^1) a_1^1 \text{+++} \emptyset)$$

(T_{S181})(Null Object Theorem)

$$(a = a_1^1) \subseteq (Ba = (@_T B T a_1^1) a_1^1)$$

(T_{M668})(Null Set Disjunction)(Eqn3)

$$\text{Let } a = a_1^1$$

(Eqn4)

$$(a = a_1^1) = E$$

(A_{MX16})(Order Axiom)(Eqn5)

$$E \subseteq (Ba_1^1 = (@_T B T a_1^1) a_1^1)$$

(A_{MX19})(Substitution of Eqns4& into Eqn3)

$$E = (Ba_1^1 = (@_T B T a_1^1) a_1^1)$$

(T_{M275})(Supersets of the Universal Set)

$$Ba_1^1 = (@_T B T a_1^1) a_1^1$$

(A_{MX16})(Order Axiom)

$$Ba^1 = (@_T B T a^1) a^1$$

Resymbolising

$$B^1 \subseteq R_0 \quad (T_{S777})$$

$$C^1 \subseteq R_0 \quad (T_{S778})$$

Example Proof of (T_{S777})

$$Ba^1 = (@_T B T a^1) a^1$$

(T_{T383})(Eqn1)

$$(@_T T a^1)^1 \subseteq R_0$$

(A_{R91T})(Eqn2)

$$\text{Let } T a^1 = B T a^1$$

(Eqn3)

$$(@_T B T a^1)^1 \subseteq R_0$$

(A_{MX19})(Substitution of Eqn3 into Eqn2)

$$((@_T B T a^1)^1 \subseteq R_0) = E$$

(A_{MX16})(Order Axiom)(Eqn4)

$$(((@_T B T a^1)^1 \subseteq R_0) = E) = ((a^1 \subseteq Q(@_T B T a^1))(a^1 \neq \emptyset) \subseteq ((@_T B T a^1) a^1 = ((@_T B T a^1) a^1)^1))$$

(D_{R30})(Definition of a One-to-One Relationship Set)(Eqn5)

$$(E = E) = ((a^1 \subseteq Q(@_T B T a^1))(a^1 \neq \emptyset) \subseteq ((@_T B T a^1) a^1 = ((@_T B T a^1) a^1)^1))$$

(A_{MX19})(Substitution of Eqn4 into Eqn5)

$$E = ((a^1 \subseteq Q(@_T B T a^1))(a^1 \neq \emptyset) \subseteq ((@_T B T a^1) a^1 = ((@_T B T a^1) a^1)^1))$$

(T_{M342})

$$(a^1 \subseteq Q(@_T B T a^1))(a^1 \neq \emptyset) \subseteq ((@_T B T a^1) a^1 = ((@_T B T a^1) a^1)^1)$$

(A_{MX16})(Order Axiom)(Eqn6)

$$((@_T B T a^1) a^1 \neq \emptyset) \subseteq (a^1 \subseteq Q(@_T B T a^1))$$

(T_{S410b})

$$((@_T B T a^1) a^1 \neq \emptyset)(a^1 \neq \emptyset) \subseteq (a^1 \subseteq Q(@_T B T a^1))(a^1 \neq \emptyset)$$

(T_{M029})(Deduction)(Eqn7)

$$(((@_T B T a^1) a^1 \neq \emptyset)(a^1 \neq \emptyset) \subseteq ((@_T B T a^1) a^1 = ((@_T B T a^1) a^1)^1))$$

$(Ba^1 \neq \emptyset)(a^1 \neq \emptyset) \subseteq (Ba^1 = (Ba^1)^1)$	(T_{M139}) (Hypothetical Syllogism Eqns7&6)(Eqn8)
$((a^1 \neq \emptyset) = E) \subseteq ((B * a^1 \neq \emptyset) = (a^1 \subseteq QB))$	(A_{MX19}) (Substitution of Eqn1 into Eqn8)(Eqn9)
$(Ba^1 \neq \emptyset)(a^1 \neq \emptyset) = (a^1 \subseteq QB)(a^1 \neq \emptyset)$	(T_{7702})
$(a^1 \subseteq QB)(a^1 \neq \emptyset) \subseteq (Ba^1 = (Ba^1)^1)$	(T_{M444b}) (Biconditional Importation/Exportation)(Eqn10)
$((a^1 \subseteq QB)(a^1 \neq \emptyset) \subseteq (Ba^1 = (Ba^1)^1) = E$	(A_{MX19}) (Substitution of Eqn10 into Eqn9)
$((B^1 \subseteq Ro) = E) = ((a^1 \subseteq QB)(a^1 \neq \emptyset) \subseteq (Ba^1 = (Ba^1)^1))$	(A_{MX16}) (Order Axiom)(Eqn11)
$((B^1 \subseteq Ro) = E) = E$	(D_{R30}) (Definition of a One-to-One Relationship Set)(Eqn12)
$(B^1 \subseteq Ro) = E$	(A_{MX19}) (Substitution of Eqn11 into Eqn12)
$B^1 \subseteq Ro$	(A_{MX16}) (Order Axiom)
	(A_{MX16}) (Order Axiom)

$$((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq ((@_x a^1) * b^1 = (@^v \mathcal{X}a^1)b^1) \quad (T_{623})$$

Proof of (T_{623})

$((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq ((@_x a^1) * b^1 = \{\subseteq\}b^1 \cdot \{\subseteq\}^v \mathcal{X}a^1 \cdot (\{\subseteq\} * (\{\subseteq\}b^1 \cdot \{\subseteq\}^v \mathcal{X}a^1)))$	(T_{732}) (Eqn1)
$(@^v \mathcal{X}a^1)b^1 = \{\subseteq\}b^1 \cdot \{\subseteq\}^v \mathcal{X}a^1 \cdot (\{\subseteq\} * (\{\subseteq\}b^1 \cdot \{\subseteq\}^v \mathcal{X}a^1))$	(D_{R05}) (Definition of @)(Eqn2)
$((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq ((@_x a^1) * b^1 = (@^v \mathcal{X}a^1)b^1)$	(A_{MX19}) (Substitution of Eqn2 into Eqn1)

$$(@_T B\mathcal{T}a^1) * b^1 = (@_{\mathcal{T}a^1})b^1 \quad (T_{631})$$

$$(@_T C\mathcal{T}a^1) * b^1 = (@_{C\mathcal{T}a^1})b^1 \quad (T_{629})$$

Example Proof of (T_{631})

$((\mathcal{X}E)^1 \subseteq D_A + P_A)(a^1 \subseteq \mathcal{X}E) \subseteq ((@_x a^1) * b^1 = (@^v \mathcal{X}a^1)b^1)$	(T_{623}) (Eqn1)
Let $\mathcal{X} = \mathcal{T}$	(Eqn2)
Let $a^1 = B\mathcal{T}a^1$	(Eqn3)
$((\mathcal{T}E)^1 \subseteq D_A + P_A)(B\mathcal{T}a^1 \subseteq \mathcal{T}E) \subseteq ((@_T B\mathcal{T}a^1) * b^1 = (@^v \mathcal{T}B\mathcal{T}a^1)b^1)$	(A_{MX19}) (Substitution of Eqns2&3 into Eqn1)
$((\mathcal{T}E)^1 \subseteq D_A + P_A)(B\mathcal{T}a^1 \subseteq \mathcal{T}E) \subseteq ((@_T B\mathcal{T}a^1) * b^1 = (@_{B\mathcal{T}a^1})b^1)$	(T_{R571}) (Definition of a Mapped Relationship)(Eqn4)
$(\mathcal{T}E)^1 \subseteq D_M$	(T_{7287}) (Time is a Mixed Continuum Dimension)(Eqn5)
$D_M \subseteq D_A$	(T_{7911}) (Eqn6)
$(\mathcal{T}E)^1 \subseteq D_A$	(T_{M139}) (Hypothetical Syllogism Eqns5&6)
$(\mathcal{T}E)^1 \subseteq D_A + P_A$	(T_{M150}) (Single Premise Syllogism)
$((\mathcal{T}E)^1 \subseteq D_A + P_A) = E$	(A_{MX16}) (Order Axiom)(Eqn7)
$E(B\mathcal{T}a^1 \subseteq \mathcal{T}E) \subseteq ((@_T B\mathcal{T}a^1) * b^1 = (@_{B\mathcal{T}a^1})b^1)$	(T_{M744}) (Universal Set Theorem)
$a^1 \subseteq E$	(T_{R196}) (Relational Deduction)
$\mathcal{T}a^1 \subseteq \mathcal{T}E$	(T_{S805T})
$B\mathcal{T}a^1 \subseteq \mathcal{T}E$	(A_{MX16}) (Order Axiom)(Eqn8)
$(B\mathcal{T}a^1 \subseteq \mathcal{T}E) = E$	(A_{MX19}) (Substitution of Eqns7&8 into Eqn4)
$EE \subseteq ((@_T B\mathcal{T}a^1) * b^1 = (@_{B\mathcal{T}a^1})b^1)$	(T_{M667}) (Conjunction Tautology)
$E \subseteq ((@_T B\mathcal{T}a^1) * b^1 = (@_{B\mathcal{T}a^1})b^1)$	(T_{M275}) (Supersets of the Universal Set)
$E = ((@_T B\mathcal{T}a^1) * b^1 = (@_{B\mathcal{T}a^1})b^1)$	(A_{MX16}) (Order Axiom)
$(@_T B\mathcal{T}a^1) * b^1 = (@_{B\mathcal{T}a^1})b^1$	

$$((@a)E^1 \subseteq \{\supset\} * b) \cdot ((@b)E^1 \subseteq \{\supset\} * c) \subseteq ((@a)E^1 \subseteq \{\supset\} * c) \quad (T_{613})$$

Proof of (T_{613})

Let $a \subseteq \{\supset\} * b$	(Eqn1)
Let $b \subseteq \{\supset\} * c$	
$\{\supset\} * b \subseteq \{\supset\} * \{\supset\} * c$	(T_{R196}) (Relational Deduction)(Eqn2)
$a \subseteq \{\supset\} * \{\supset\} * c$	(T_{M139}) (Hypothetical Syllogism Eqns1&2)(Eqn3)

$$\{\supset\}*\{\supset\}*c \subseteq \{\supset\}*c$$

(T_{S177})(Definition of a Transitive Relationship Set)(Eqn4)

$$a \subseteq \{\supset\}*c$$

(T_{M139})(Hypothetical Syllogism Eqns3&4)

Thus $(a \subseteq \{\supset\}*b). (b \subseteq \{\supset\}*c) \subseteq (a \subseteq \{\supset\}*c)$ (A_{MX02})(Conditional Introduction)

$$((@a)E^1 \subseteq \{\supset\}*b). ((@b)E^1 \subseteq \{\supset\}*c) \subseteq ((@a)E^1 \subseteq \{\supset\}*c) \quad (T_{T689})$$

$$(b \subseteq \{\subseteq\}(@a)E^1). (c \subseteq \{\subseteq\}(@b)E^1) \subseteq (c \subseteq \{\subseteq\}(@a)E^1) \quad (T_{T621})$$

Proof of (T_{T621})

$$((@a)E^1 \subseteq \{\supset\}*b). ((@b)E^1 \subseteq \{\supset\}*c) \subseteq ((@a)E^1 \subseteq \{\supset\}*c) \quad (T_{T613})$$

$$(b \subseteq {}^V\{\supset\}(@a)E^1). (c \subseteq {}^V\{\supset\}(@b)E^1) \subseteq (c \subseteq {}^V\{\supset\}(@a)E^1) \quad (T_{R457})(\text{Active/Passive Theorem})$$

$$(b \subseteq \{\subseteq\}(@a)E^1). (c \subseteq \{\subseteq\}(@b)E^1) \subseteq (c \subseteq \{\subseteq\}(@a)E^1) \quad (T_{T741}) \text{ Definition of } {}^V\{\supset\}$$

$$((@a)E^1 \not\subseteq \{\supset\}_*Bb) = (Bb \not\subseteq \{\subseteq\}_*(@a)E^1) \quad (T_{T604})$$

Proof of (T_{T604})

$$(a \not\subseteq c_*b) = (b \not\subseteq {}^Vc_*a) \quad (T_{R458})(\text{Active/Passive Theorem})(\text{Eqn1})$$

$$\text{Let } a = (@a)E^1 \quad (\text{Eqn2})$$

$$\text{Let } b = Bb \quad (\text{Eqn3})$$

$$\text{Let } c = \{\supset\} \quad (\text{Eqn4})$$

$$((@a)E^1 \not\subseteq \{\supset\}_*Bb) = (Bb \not\subseteq {}^V\{\supset\}_*(@a)E^1)$$

(A_{MX19})(Substitution of Eqns2,3&4 into Eqn1)

$$((@a)E^1 \not\subseteq \{\supset\}_*Bb) = (Bb \not\subseteq \{\subseteq\}_*(@a)E^1)$$

(D_{A143})(Symmetrical Opposite of an Asymmetrical Relationship Set)

Chapter 15 Proofs of Theorems of Knowledge and Communication

15.1 Proofs of Theorems of Knowledge

$(a = \emptyset) = ("a" \subseteq {}^vH_{s*}J^1)$ (T_{7709}) Definition of not Knowing of

Proof of (T_{7709})

$(a \neq \emptyset) = ("a" \subseteq {}^vH_{s*}J^1)$ (D_{R01})(Definition of Knowing of)

$(a = \emptyset) = ("a" \not\subseteq {}^vH_{s*}J^1)$ (T_{M008})(Inverse of an Equality)

$(a = \emptyset) = ("a" \subseteq {}^vH_{s*}J^1)$ (T_{M604})

$(("a" \subseteq {}^vH_{s*d^1}) \subseteq {}^vH_{s*d^1}) = E$ (T_{7734})

Proof of (T_{7734})

$(("a" \subseteq {}^vH_{s*d^1}) \subseteq {}^vH_{s*d^1}) = \emptyset$ (A_{S91})(Axiom of Unknown Knowns)

$(("a" \subseteq {}^vH_{s*d^1}) \not\subseteq {}^vH_{s*d^1}) = \sim \emptyset$ (T_{M008})(Inverse of an Equality)

$(("a" \subseteq {}^vH_{s*d^1}) \not\subseteq {}^vH_{s*d^1}) = E$ (T_{M054})(Negative of Null Set)

$(("a" \subseteq {}^vH_{s*d^1}) \subseteq {}^vH_{s*d^1}) = E$ (T_{M604})

$(a \subseteq {}^vH_{s*d^1}) = ((a \subseteq {}^vH_{s*d^1}) \subseteq {}^vH_{s*d^1})$ (T_{7746})

Proof of (T_{7746})

$\{p\} = ("p" \subseteq {}^vH_{s*}J^1)$ (D_{R02})(Definition of Knowing that)(Eqn1)

Let $\{p\} = ("a" \subseteq {}^vH_{s*}J^1)$ (Eqn2)

$(a \subseteq {}^vH_{s*}J^1) = ((a \subseteq {}^vH_{s*}J^1) \subseteq {}^vH_{s*}J^1)$ (A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)

Let $J^1 = d^1$ (Eqn4)

$(a \subseteq {}^vH_{s*d^1}) = ((a \subseteq {}^vH_{s*d^1}) \subseteq {}^vH_{s*d^1})$ (A_{MX19})(Substitution of Eqn4 into Eqn3)

$(\{p\} \neq E) = ("p" \neq E \subseteq {}^vH_{s*}J^1)$ (T_{S041})

$(\{p\} \neq \emptyset) = ("p" \neq \emptyset \subseteq {}^vH_{s*}J^1)$ (T_{S043})

$(\{p\} = \emptyset) = ("p" = \emptyset \subseteq {}^vH_{s*}J^1)$ (T_{S042})

Example Proof of (T_{S041})

$\{p\} = ("p" \subseteq {}^vH_{s*}J^1)$ (D_{R02})(Definition of Knowing that)(Eqn1)

Let $\{p\} = (\{p\} \neq E)$ (Eqn2)

$(\{p\} \neq E) = ("p" \neq E \subseteq {}^vH_{s*}J^1)$ (A_{MX19})(Substitution of Eqn2 into Eqn1)

$"a" \neq \emptyset$ (T_{7398}) Knowledge exists

Proof of (T_{7398})

$"a" = "a"^{*1}$ (D_{S24})(Definition of "a")(Eqn1)

Let $"a" = \emptyset$ (Eqn2)

$\emptyset = \emptyset^1$ (A_{MX19})(Substitution of Eqn2 into Eqn1)

$\emptyset$ (T_{M024})(Non existence of anti-(T_{M780}))

Thus $(a = \emptyset) = \emptyset$ (A_{MX02})(Conditional Introduction)

$(a \neq \emptyset) = \sim \emptyset$ (T_{M008})(Inverse of an Equality)

$(a \neq \emptyset) = E$ (T_{M054})(Negative of Null Set)

$"a" \neq \emptyset$ (A_{MX16})(Order Axiom)

$(b \subseteq {}^vH_{s*a^1}) = (b \not\subseteq {}^vH_{s*a^1})$ (T_{S049})

Proof of (T₅₀₄₉)

$$((a^1 \neq \emptyset) = E) \subseteq ((a^1 \not\subseteq b) = (a^1 \subseteq \underline{b}))$$

(T₀₉₀)(Inverse Unit Subset Rule)(Eqn1)

$$\text{Let } a^1 = "b"$$

(Eqn2)

$$\text{Let } b = {}^V H_{S*} a^1$$

(Eqn3)

$$(("b" \neq \emptyset) = E) \subseteq ((("b" \not\subseteq {}^V H_{S*} a^1) = ("b" \subseteq {}^V H_{S*} a^1))$$

(A_{MX19})(Substitution of Eqns2&3 into Eqn1)

$$(E = E) \subseteq ((("b" \not\subseteq {}^V H_{S*} a^1) = ("b" \subseteq {}^V H_{S*} a^1))$$

(T₀₉₀)(Universality of (T₇₃₉₈) Knowledge exists)

$$E \subseteq ((("b" \not\subseteq {}^V H_{S*} a^1) = ("b" \subseteq {}^V H_{S*} a^1))$$

(T_{M342})

$$E = ((("b" \not\subseteq {}^V H_{S*} a^1) = ("b" \subseteq {}^V H_{S*} a^1))$$

(T_{M275})(Supersets of the Universal Set)

$$("b" \not\subseteq {}^V H_{S*} a^1) = ("b" \subseteq {}^V H_{S*} a^1)$$

(A_{MX16})(Order Axiom)

$$((x^1 \subseteq Aa^1) = E) \subseteq ((("x^1" \subseteq {}^V H_{S*} d^1) = ("a^1" \subseteq {}^V H_{S*} d^1))$$

(T₇₅₀₇)

Proof of (T₇₅₀₇)

$$((x^1 \subseteq Aa^1) = E) \subseteq ((a^1 \neq \emptyset) = (x^1 \neq \emptyset))$$

(A_{S04})(2nd Axiom of Intrinsic Features)

$$((x^1 \subseteq Aa^1) = E) \subseteq ((x^1 = \emptyset) = (a^1 = \emptyset))$$

(T_{M008})(Inverse of an Equality)

$$((x^1 \subseteq Aa^1) = E) \subseteq ((("x^1" \subseteq {}^V H_{S*} J^1) = ("a^1" \subseteq {}^V H_{S*} J^1))$$

(T₇₇₀₉)(Definition of not Knowing of)(Eqn1)

$$\text{Let } J^1 = d^1$$

(Eqn2)

$$((x^1 \subseteq Aa^1) = E) \subseteq ((("x^1" \subseteq {}^V H_{S*} d^1) = ("a^1" \subseteq {}^V H_{S*} d^1))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$((x^1 \subseteq Aa^1) = E) \subseteq ((("x^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1) = ("a^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1))$$

(T₇₅₁₅)

Proof of (T₇₅₁₅)

$$((x^1 \subseteq Aa^1) = E) \subseteq ((a^1 \neq \emptyset) = (x^1 \neq \emptyset))$$

(A_{S04})(2nd Axiom of Intrinsic Features)

$$((x^1 \subseteq Aa^1) = E) \subseteq ((x^1 \neq \emptyset) = (a^1 \neq \emptyset))$$

(T_{M197})(Commutativity of =)

$$((x^1 \subseteq Aa^1) = E) \subseteq ((("x^1 \neq \emptyset" \subseteq {}^V H_{S*} J^1) = ("a^1 \neq \emptyset" \subseteq {}^V H_{S*} J^1))$$

(D_{R07})(Definition of Knowing of)(Eqn1)

$$\text{Let } J^1 = d^1$$

(Eqn2)

$$((x^1 \subseteq Aa^1) = E) \subseteq ((("x^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1) = ("a^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(x^1 \neq \emptyset)(x^1 \subseteq {}^V K*d^1) \subseteq ("x^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1)$$

(T₇₄₀₈)

Proof of (T₇₄₀₈)

$$(a \not\subseteq {}^V K*d^1) \subseteq ("a^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1)$$

(A_{S99})(Knowing Implies Knowing that...)(Eqn1)

$$\text{Let } a = x^1$$

(Eqn2)

$$(x^1 \not\subseteq {}^V K*d^1) \subseteq ("x^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)

$$((x^1 \neq \emptyset) = E) \subseteq ((x^1 \not\subseteq {}^V K*d^1) = (x^1 \subseteq {}^V K*d^1))$$

(T_{M090})(Eqn4)

$$((x^1 \neq \emptyset) = E) \subseteq ((x^1 \subseteq {}^V K*d^1) \subseteq ("x^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1))$$

(A_{MX19})(Substitution of Eqn4 into Eqn3)

$$(x^1 \neq \emptyset)(x^1 \subseteq {}^V K*d^1) \subseteq ("x^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1)$$

(A_{MX35})(Exportation/Importation)

$$((x^1 \subseteq Aa^1) = E) \subseteq (x^1 \neq \emptyset)(x^1 \subseteq {}^V K*d^1) \subseteq ("a^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1))$$

(T₇₅₂₈)

Proof of (T₇₅₂₈)

$$(x^1 \neq \emptyset)(x^1 \subseteq {}^V K*d^1) \subseteq ("x^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1)$$

(T₇₄₀₈)(Eqn1)

$$((x^1 \subseteq Aa^1) = E) \subseteq ((a^1 \neq \emptyset) = (x^1 \neq \emptyset))$$

(A_{S04})(2nd Axiom of Intrinsic Features)(Eqn2)

$$((x^1 \subseteq Aa^1) = E) \subseteq (x^1 \neq \emptyset)(x^1 \subseteq {}^V K*d^1) \subseteq ("a^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$({}^V \mathcal{A}Ta^1 \neq \emptyset)({}^V \mathcal{A}Ta^1 \subseteq {}^V K*d^1) \subseteq ("{}^V \mathcal{A}Ta^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1)$$

(T₇₆₀₀)

Proof of (T₇₆₀₀)

$$(x^1 \neq \emptyset)(x^1 \subseteq {}^V K*d^1) \subseteq ("x^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1)$$

(T₇₄₀₈)(Eqn1)

$$\text{Let } x^1 = {}^V \mathcal{A}Ta^1$$

(Eqn2)

$$({}^V \mathcal{A}Ta^1 \neq \emptyset)({}^V \mathcal{A}Ta^1 \subseteq {}^V K*d^1) \subseteq ("{}^V \mathcal{A}Ta^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$("{}^V \mathcal{A}Ta^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1) = ("a^1 \neq \emptyset" \subseteq {}^V H_{S*} d^1)$$

(T₇₆₀₃)

$$(\forall \mathcal{A}\mathcal{T}a^1 \neq \emptyset)(\forall \mathcal{A}\mathcal{T}a^1 \subseteq \forall K*d^1) \subseteq ("a^1 \neq \emptyset" \subseteq \forall H_s*d^1) \quad (T_{605})$$

$$("a^1 \neq \emptyset" \subseteq \forall H_s*d^1)(\forall \mathcal{A}\mathcal{T}a^1 \subseteq \forall K*d^1) \subseteq (" \forall \mathcal{A}\mathcal{T}a^1 \subseteq \forall K*d^1" \subseteq \forall H_s*d^1) \quad (T_{609})$$

Example Proof of (T₆₀₃)

$$((x^1 \subseteq Aa^1) = E) \subseteq ((x^1 \neq \emptyset) \subseteq \forall H_s*d^1) = ("a^1 \neq \emptyset" \subseteq \forall H_s*d^1))$$

(T₅₁₅)(Eqn1)

$$\text{Let } x^1 = \forall \mathcal{A}\mathcal{T}a^1$$

(Eqn2)

$$((\forall \mathcal{A}\mathcal{T}a^1 \subseteq Aa^1) = E) \subseteq ((\forall \mathcal{A}\mathcal{T}a^1 \neq \emptyset) \subseteq \forall H_s*d^1) = ("a^1 \neq \emptyset" \subseteq \forall H_s*d^1))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)

$$(x^1 = Sa^1, Qa^1, \{\supset\}a^1, Ba^1, Ca^1, \forall \mathcal{A}\mathcal{T}a^1, \forall \mathcal{A}(\mathcal{L}_{x^1c^1})a^1, \forall \mathcal{A}(\mathcal{L}_{xy^1c^1})a^1, \forall \mathcal{A}(\mathcal{O}c^1)a^1, \forall \mathcal{A}Ma^1, \{\supset\}Ba^1, \forall B\{\supset\}a^1) \subseteq (x^1 \subseteq Aa^1)$$

(A_{S82})(1st Axiom of Intrinsic Features)

$$\forall \mathcal{A}\mathcal{T}a^1 \subseteq Aa^1$$

Extraction

$$(\forall \mathcal{A}\mathcal{T}a^1 \subseteq Aa^1) = E$$

(A_{MX16})(Order Axiom)(Eqn4)

$$(E = E) \subseteq ((\forall \mathcal{A}\mathcal{T}a^1 \neq \emptyset) \subseteq \forall H_s*d^1) = ("a^1 \neq \emptyset" \subseteq \forall H_s*d^1))$$

(A_{MX19})(Substitution of Eqn4 into Eqn3)

$$E \subseteq ((\forall \mathcal{A}\mathcal{T}a^1 \neq \emptyset) \subseteq \forall H_s*d^1) = ("a^1 \neq \emptyset" \subseteq \forall H_s*d^1))$$

(T_{M342})

$$E = ((\forall \mathcal{A}\mathcal{T}a^1 \neq \emptyset) \subseteq \forall H_s*d^1) = ("a^1 \neq \emptyset" \subseteq \forall H_s*d^1))$$

(T_{M275})(Supersets of the Universal Set)

$$(\forall \mathcal{A}\mathcal{T}a^1 \neq \emptyset) \subseteq \forall H_s*d^1 = ("a^1 \neq \emptyset" \subseteq \forall H_s*d^1)$$

(A_{MX16})(Order Axiom)

$$(a = b) = ("a" = "b")$$

(T_{S590})

Proof of (T_{S590})

$$\text{Let } a = b$$

(Eqn1)

$$"a" = "a"$$

(D_{M00})(Definition of =)(Eqn2)

$$"a" = "b"$$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

$$\text{Thus } (a = b) \subseteq ("a" = "b")$$

(A_{MX02})(Conditional Introduction)(Eqn3)

$$\text{Let } "a" = "b"$$

$$("a" \subseteq \forall H_s*j^1) = ("b" \subseteq \forall H_s*j^1)$$

(T_{M791})

$$a = b$$

(D_{R02})(Definition of Knowing That)

$$\text{Thus } ("a" = "b") \subseteq (a = b)$$

(A_{MX02})(Conditional Introduction)(Eqn4)

$$(a = b) = ("a" = "b")$$

(T_{M430})(Biconditional Introduction Eqns3&4)

$$(a = a^1) = ("a" = "a^1")$$

(T_{T432})

$$(a \neq a^1) = ("a" \neq "a^1")$$

(T_{T914})

Proof of (T_{T432}) & (T_{T914})

$$(a = b) = ("a" = "b")$$

(T_{S590})(Eqn1)

$$\text{Let } b = a^1$$

(Eqn2)

$$(a = a^1) = ("a" = "a^1")$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(a \neq a^1) = ("a" \neq "a^1")$$

(T_{M008})(Inverse of an Equality)

$$"\{p\}" = "\{p\} = E"$$

(T_{S588})

Proof of (T_{S588})

$$("\{p\}" = "\{p\} = E") = (" \{p\} = "\{p\} = E")$$

(D_{M00})(Definition of =)

$$("\{p\}" = "\{p\} = E") = (\{p\} = (\{p\} = E))$$

(T_{S590})

$$((\{p\} = "\{p\} = E") = E) = ((\{p\} = E) = (\{p\} = E))$$

(A_{MX16})(Order Axiom) to balance terms of the equation.

$$((\{p\} = "\{p\} = E") = E) = E$$

(T_{M007})(Universality of (D_{M00}) Definition of =)

$$("\{p\}" = "\{p\} = E") = E$$

(A_{MX16})(Order Axiom)

$$"\{p\}" = "\{p\} = E"$$

(A_{MX16})(Order Axiom)

$$"\sim\{p\}" = "\{p\} = \emptyset"$$

(T_{S589})

Proof of (T_{S589})

$$"\{p\}" = "\{p\} = E"$$

(T_{S588})

$$"\sim\{p\}" = "\sim\{p\} = E"$$

Resymbolising

$$"\sim\{p\}" = "\{p\} = \sim E"$$

(T_{M008})(Inverse of an Equality)

$$\sim\{p\} = \{p\} = \emptyset \quad (D_{M02})(\text{Definition of } \emptyset)$$

$$(d^1 \subseteq SH_s)(d^1 \neq \emptyset) \subseteq (\forall H_s * d^1 \neq \emptyset) \quad (T_{7400}) \text{ Agents and Media Hold Knowledge}$$

Proof of (T₇₄₀₀)

$$\begin{aligned} (b^1 \subseteq Qc)(b^1 \neq \emptyset) &\subseteq (c * b^1 \neq \emptyset) && (T_{S410c})(\text{Eqn1}) \\ \text{Let } b^1 = d^1 &&& (\text{Eqn2}) \\ \text{Let } c = \forall H_s &&& (\text{Eqn3}) \\ (d^1 \subseteq Q \forall H_s)(d^1 \neq \emptyset) &\subseteq (\forall H_s * d^1 \neq \emptyset) && (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\ (d^1 \subseteq SH_s)(d^1 \neq \emptyset) &\subseteq (\forall H_s * d^1 \neq \emptyset) && (T_{R015})(\text{Reversibility of Matching Entities}) \end{aligned}$$

$$(\text{"Fa"} \subseteq \forall H_s * d^1) \subseteq (\text{"a"} \subseteq \forall H_s * d^1) \quad (T_{7535})$$

Proof of (T₇₅₃₅)

$$\begin{aligned} (\text{"Fa"} \subseteq \forall H_s * d^1) &\subseteq (\text{"a"} \subseteq \forall H_s * d^1) && (A_{S95})(\text{Knowledge of... Axiom})(\text{Eqn1}) \\ \text{Let } a = a^1 &&& (\text{Eqn2}) \\ (\text{"Fa"} \subseteq \forall H_s * d^1) &\subseteq (\text{"a"} \subseteq \forall H_s * d^1) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(\text{"Sa"} \subseteq \forall H_s * d^1) \subseteq (\text{"a"} \subseteq \forall H_s * d^1) \quad (T_{7477})$$

$$(\text{"Qa"} \subseteq \forall H_s * d^1) \subseteq (\text{"a"} \subseteq \forall H_s * d^1) \quad (T_{7453})$$

$$(\forall a^1 \subseteq \forall H_s * d^1) \subseteq (\text{"a"} \subseteq \forall H_s * d^1) \quad (T_{7451})$$

Example Proof of (T₇₄₇₇)

$$\begin{aligned} (\text{"Fa"} \subseteq \forall H_s * d^1) &\subseteq (\text{"a"} \subseteq \forall H_s * d^1) && (T_{7535})(\text{Eqn1}) \\ \text{Let } Fa^1 = Sa^1 &&& (\text{Eqn2}) \\ (\text{"Sa"} \subseteq \forall H_s * d^1) &\subseteq (\text{"a"} \subseteq \forall H_s * d^1) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(\text{"a"} \subseteq \forall H_s * d^1) \subseteq (\text{"Fa"} \subseteq \forall H_s * d^1) \quad (T_{7437})$$

Proof of (T₇₄₃₇)

$$\begin{aligned} (\text{"Fa"} \subseteq \forall H_s * d^1) &\subseteq (\text{"a"} \subseteq \forall H_s * d^1) && (A_{S95})(\text{Knowledge of... Axiom}) \\ (\text{"a"} \not\subseteq \forall H_s * d^1) &\subseteq (\text{"Fa"} \not\subseteq \forall H_s * d^1) && (T_{M008})(\text{Inverse of an Equality}) \\ (\text{"a"} \subseteq \forall H_s * d^1) &\subseteq (\text{"Fa"} \subseteq \forall H_s * d^1) && (T_{S049}) \end{aligned}$$

$$(\text{"a"} \subseteq \forall H_s * d^1) \subseteq (\text{"a"} \subseteq \forall H_s * e^1 \subseteq \forall H_s * d^1) \quad (T_{S066})$$

$$(\text{"a"} \subseteq \forall H_s * d^1) \subseteq (\text{"a"} \subseteq \forall H_s * e^1 \subseteq \forall H_s * d^1) \quad (T_{7445})$$

Example Proof of (T_{S066})

$$\begin{aligned} (\text{"a"} \subseteq \forall H_s * d^1) &\subseteq (\text{"Fa"} \subseteq \forall H_s * d^1) && (T_{7437})(\text{Eqn1}) \\ \text{Let "Fa"} = \text{"a"} \subseteq \forall H_s * e^1 &&& (\text{Eqn2}) \\ (\text{"a"} \subseteq \forall H_s * d^1) &\subseteq (\text{"a"} \subseteq \forall H_s * e^1 \subseteq \forall H_s * d^1) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(\text{"a"} \subseteq \forall H_s * e^1 \subseteq \forall H_s * d^1) \subseteq (\text{"a"} \subseteq \forall H_s * d^1) \quad (T_{7441})$$

$$(\text{"a"} \subseteq \forall H_s * e^1 \subseteq \forall H_s * d^1) \subseteq (\text{"a"} \subseteq \forall H_s * d^1) \quad (T_{7442})$$

Example Proof of (T₇₄₄₁)

$$\begin{aligned} (\text{"Fa"} \subseteq \forall H_s * d^1) &\subseteq (\text{"a"} \subseteq \forall H_s * d^1) && (A_{S95})(\text{Knowledge of... Axiom}) \\ \text{Let "Fa"} = \text{"a"} \subseteq \forall H_s * e^1 &&& (\text{Eqn2}) \\ (\text{"a"} \subseteq \forall H_s * e^1 \subseteq \forall H_s * d^1) &\subseteq (\text{"a"} \subseteq \forall H_s * d^1) && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(\text{"a"} \subseteq \forall H_s * d^1) \subseteq (\text{"a"} \subseteq \forall H_s * e^1 \subseteq \forall H_s * d^1) \quad (T_{S068})$$

$$(\text{"a"} \subseteq \forall H_s * d^1) \subseteq (\text{"a"} \subseteq \forall H_s * e^1 \subseteq \forall H_s * d^1) \quad (T_{7439})$$

Example Proof of (T_{S068})

$$\begin{aligned} (\text{"a"} \subseteq \forall H_s * d^1) &\subseteq (\text{"a"} \subseteq \forall H_s * e^1 \subseteq \forall H_s * d^1) && (T_{S066})(\text{Eqn1}) \\ \text{Let "a"} = \text{"a"} &&& (\text{Eqn2}) \end{aligned}$$

$$("a^1" \subseteq {}^V H_{S*d^1}) \subseteq ("a^1" \subseteq {}^V H_{S*e^1} \subseteq {}^V H_{S*d^1}) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$("a^1" \subseteq {}^V H_{S*d^1}) \subseteq ("Sa^1" \subseteq {}^V H_{S*d^1}) \quad (T_{S034})$$

$$("a^1" \subseteq {}^V H_{S*d^1}) \subseteq ("Qa^1" \subseteq {}^V H_{S*d^1}) \quad (T_{S075})$$

$$("a^1" \subseteq {}^V H_{S*d^1}) \subseteq ("Sa^1" \subseteq {}^V H_{S*d^1}) \quad (T_{T449})$$

$$("a^1" \subseteq {}^V H_{S*d^1}) \subseteq ("Qa^1" \subseteq {}^V H_{S*d^1}) \quad (T_{T448})$$

$$("a^1" \subseteq {}^V H_{S*d^1}) \subseteq ("a^1" \subseteq {}^V H_{S*d^1}) \quad (T_{T447})$$

Example Proof of (T_{S034})

$$("a" \subseteq {}^V H_{S*d^1}) \subseteq ("Fa" \subseteq {}^V H_{S*d^1}) \quad (T_{T437})(\text{Eqn1})$$

$$\text{Let } a = a^1 \quad (\text{Eqn2})$$

$$\text{Let } Fa^1 = {}^V Sa^1 \quad (\text{Eqn3})$$

$$("a^1" \subseteq {}^V H_{S*d^1}) \subseteq ("Sa^1" \subseteq {}^V H_{S*d^1}) \quad (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})$$

$$("Sa^1" \subseteq {}^V H_{S*d^1}) \subseteq ("a^1" \subseteq {}^V H_{S*d^1}) \quad (T_{S073})$$

$$("Qa^1" \subseteq {}^V H_{S*d^1}) \subseteq ("a^1" \subseteq {}^V H_{S*d^1}) \quad (T_{S074})$$

Example Proof of (T_{S073})

$$("Fa" \subseteq {}^V H_{S*d^1}) \subseteq ("a" \subseteq {}^V H_{S*d^1}) \quad (A_{S95})(\text{Knowledge of... Axiom})(\text{Eqn1})$$

$$\text{Let } a = a^1 \quad (\text{Eqn2})$$

$$\text{Let } Fa^1 = {}^V Sa^1 \quad (\text{Eqn2})$$

$$("Sa^1" \subseteq {}^V H_{S*d^1}) \subseteq ("a^1" \subseteq {}^V H_{S*d^1}) \quad (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})$$

$$("Sd^1" \subseteq {}^V H_{S*d^1}) \subseteq ("d^1" \subseteq {}^V H_{S*d^1}) \quad (T_{S035})$$

$$("d^1" \subseteq {}^V H_{S*d^1}) \subseteq ("Sd^1" \subseteq {}^V H_{S*d^1}) \quad (T_{S036})$$

Proof of (T_{S035}) & (T_{S036})

$$("Sa^1" \subseteq {}^V H_{S*d^1}) \subseteq ("a^1" \subseteq {}^V H_{S*d^1}) \quad (T_{S073})(\text{Eqn1})$$

$$\text{Let } a^1 = d^1 \quad (\text{Eqn2})$$

$$("Sd^1" \subseteq {}^V H_{S*d^1}) \subseteq ("d^1" \subseteq {}^V H_{S*d^1}) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$("d^1" \not\subseteq {}^V H_{S*d^1}) \subseteq ("Sd^1" \not\subseteq {}^V H_{S*d^1}) \quad (T_{M012})(\text{Transposition})$$

$$("d^1" \subseteq {}^V H_{S*d^1}) \subseteq ("Sd^1" \subseteq {}^V H_{S*d^1}) \quad (T_{M090})(\text{Inverse Unit Subset Rule}) \text{ NB. "d^1" \& "Sd^1" exist.}$$

$$("r \neq \emptyset" \subseteq {}^V H_{S*d^1}) + ("r = \emptyset" \subseteq {}^V H_{S*d^1}) \subseteq ("r" \subseteq {}^V H_{S*d^1}) \quad (T_{T710}) \text{ Knowing that... implies Knowing of...}$$

Proof of (T_{T710})

$$("Fr" \subseteq {}^V H_{S*d^1}) \subseteq ("r" \subseteq {}^V H_{S*d^1}) \quad (A_{S95})(\text{Knowledge of... Axiom})(\text{Eqn1})$$

$$\text{Let } Fr = (r = \emptyset) \quad (\text{Eqn2})$$

$$("r = \emptyset" \subseteq {}^V H_{S*d^1}) \subseteq ("r" \subseteq {}^V H_{S*d^1}) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3})$$

$$\text{Let } Fr = (r \neq \emptyset) \quad (\text{Eqn4})$$

$$("r \neq \emptyset" \subseteq {}^V H_{S*d^1}) \subseteq ("r" \subseteq {}^V H_{S*d^1}) \quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn1})(\text{Eqn5})$$

$$("r \neq \emptyset" \subseteq {}^V H_{S*d^1}) + ("r = \emptyset" \subseteq {}^V H_{S*d^1}) \subseteq ("r" \subseteq {}^V H_{S*d^1}) + ("r" \subseteq {}^V H_{S*d^1})$$

$$(A_{MX21})(\text{Combination of Eqns3\&5})$$

$$("r \neq \emptyset" \subseteq {}^V H_{S*d^1}) + ("r = \emptyset" \subseteq {}^V H_{S*d^1}) \subseteq ("r" \subseteq {}^V H_{S*d^1})$$

$$(A_{MX01})(\text{Disjunction Tautology})$$

$$("r" \subseteq {}^V H_{S*d^1}) \subseteq ("r \neq \emptyset" \subseteq {}^V H_{S*d^1}). ("r = \emptyset" \subseteq {}^V H_{S*d^1}) \quad (T_{T324})$$

Proof of (T_{R324})

$$("r \neq \emptyset" \subseteq {}^V H_{S*d^1}) + ("r = \emptyset" \subseteq {}^V H_{S*d^1}) \subseteq ("r" \subseteq {}^V H_{S*d^1}) \quad (T_{T710})(\text{Knowing that... implies Knowing of...})$$

$$("r" \not\subseteq {}^V H_{S*d^1}) \subseteq \sim(("r \neq \emptyset" \subseteq {}^V H_{S*d^1}) + ("r = \emptyset" \subseteq {}^V H_{S*d^1}))$$

$$(T_{M012})(\text{Transposition})$$

$$("r" \not\subseteq {}^V H_{S*d^1}) \subseteq ("r \neq \emptyset" \not\subseteq {}^V H_{S*d^1}). ("r = \emptyset" \not\subseteq {}^V H_{S*d^1}) \quad (T_{M026})(\text{DeMorgan's Law})$$

$$("r" \subseteq {}^V H_{S*d^1}) \subseteq ("r \neq \emptyset" \subseteq {}^V H_{S*d^1}). ("r = \emptyset" \subseteq {}^V H_{S*d^1}) \quad (T_{S049})$$

$$(\text{"a"} \subseteq {}^V\text{H}_S * d^1) = (\text{"a"}^1 \subseteq {}^V\text{H}_S * d^1) \quad (T_{S077})$$

Proof of (T_{S077})

$$\begin{aligned} (\text{"Fa"} \subseteq {}^V\text{H}_S * d^1) &\subseteq (\text{"a"} \subseteq {}^V\text{H}_S * d^1) & (A_{S95})(\text{Knowledge of... Axiom})(\text{Eqn1}) \\ \text{Let Fa} &= a^1 & (\text{Eqn2}) \\ (\text{"a"}^1 \subseteq {}^V\text{H}_S * d^1) &\subseteq (\text{"a"} \subseteq {}^V\text{H}_S * d^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\ (\text{"Fa"}^1 \subseteq {}^V\text{H}_S * d^1) &\subseteq (\text{"a"}^1 \subseteq {}^V\text{H}_S * d^1) & (A_{S95})(\text{Knowledge of... Axiom})(\text{Eqn4}) \\ \text{Let Fa}^1 &= a & (\text{Eqn5}) \\ (\text{"a"} \subseteq {}^V\text{H}_S * d^1) &\subseteq (\text{"a"}^1 \subseteq {}^V\text{H}_S * d^1) & (A_{MX19})(\text{Substitution of Eqn5 into Eqn4})(\text{Eqn6}) \\ (\text{"a"} \subseteq {}^V\text{H}_S * d^1) &= (\text{"a"}^1 \subseteq {}^V\text{H}_S * d^1) & (T_{M430})(\text{Biconditional Introduction Eqns3\&6}) \end{aligned}$$

$$(\text{"a"} \subseteq {}^V\text{H}_S * d^1) = (\text{"a"}^1 \subseteq {}^V\text{H}_S * d^1) \quad (T_{S078})$$

Proof of (T_{S078})

$$\begin{aligned} (\text{"a"} \subseteq {}^V\text{H}_S * d^1) &= (\text{"a"}^1 \subseteq {}^V\text{H}_S * d^1) & (T_{S077}) \\ (\text{"a"} \not\subseteq {}^V\text{H}_S * d^1) &= (\text{"a"}^1 \not\subseteq {}^V\text{H}_S * d^1) & (T_{M008})(\text{Inverse of an Equality}) \\ (\text{"a"} \subseteq {}^V\text{H}_S * d^1) &= (\text{"a"}^1 \subseteq {}^V\text{H}_S * d^1) & (T_{S049}) \end{aligned}$$

$$(a^1 \neq \emptyset)(a^1 \subseteq {}^V\text{K} * d^1) \subseteq (\text{"a"}^1 \neq \emptyset \subseteq {}^V\text{H}_S * d^1) \quad (T_{T554})$$

Proof of (T_{T554})

$$\begin{aligned} (a \not\subseteq {}^V\text{K} * d^1) &\subseteq (\text{"a"}^1 \neq \emptyset \subseteq {}^V\text{H}_S * d^1) & (A_{S99})(\text{Knowing Implies Knowing that...})(\text{Eqn1}) \\ \text{Let a} &= a^1 & (\text{Eqn2}) \\ (a^1 \not\subseteq {}^V\text{K} * d^1) &\subseteq (\text{"a"}^{11} \neq \emptyset \subseteq {}^V\text{H}_S * d^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ ((a^1 \neq \emptyset) = E) &\subseteq ((a^1 \not\subseteq {}^V\text{K} * d^1) \subseteq (\text{"a"}^1 \neq \emptyset \subseteq {}^V\text{H}_S * d^1)) & (D_{M17})(\text{Definition of } a^{11}) \\ ((a^1 \neq \emptyset) = E) &\subseteq ((a^1 \subseteq {}^V\text{K} * d^1) \subseteq (\text{"a"}^1 \neq \emptyset \subseteq {}^V\text{H}_S * d^1)) & (T_{M090})(\text{Inverse Unit Subset Rule}) \\ (a^1 \neq \emptyset)(a^1 \subseteq {}^V\text{K} * d^1) &\subseteq (\text{"a"}^1 \neq \emptyset \subseteq {}^V\text{H}_S * d^1) & (A_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \end{aligned}$$

$$(d^1 \subseteq \text{SK}) \subseteq (\text{"d"}^1 \subseteq {}^V\text{H}_S * d^1) \quad (T_{T558})$$

Proof of (T_{T558})

$$\begin{aligned} (a \subseteq {}^V\text{K} * d^1) &\subseteq (\text{"a"} \subseteq {}^V\text{H}_S * d^1) & (A_{S97})(\text{Knowing Implies Knowing of...})(\text{Eqn1}) \\ \text{Let a} &= d^1 & (\text{Eqn2}) \\ (d^1 \subseteq {}^V\text{K} * d^1) &\subseteq (\text{"d"}^1 \subseteq {}^V\text{H}_S * d^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\ (d^1 \subseteq \text{SK}) &\subseteq (d^1 \subseteq {}^V\text{K} * d^1) & (A_{S85})(\text{Axiom of Self Knowledge})(\text{Eqn4}) \\ (d^1 \subseteq \text{SK}) &\subseteq (\text{"d"}^1 \subseteq {}^V\text{H}_S * d^1) & (T_{M139})(\text{Hypothetical Syllogism Eqns4\&3}) \end{aligned}$$

$$\text{H}_S * \{\text{"p"}\} \subseteq \text{H}_S * \sim\{\text{"p"}\} \quad (T_{T259})$$

Proof of (T_{T259})

$$\begin{aligned} (a \subseteq \text{H}_S * \{\text{"p"}\}) &\subseteq (a \subseteq \text{H}_S * \sim\{\text{"p"}\}) & (A_{S22})(\text{Epistemic Axiom})(\text{Eqn1}) \\ \text{Let a} &= E & (\text{Eqn2}) \\ (E \subseteq \text{H}_S * \{\text{"p"}\}) &\subseteq (E \subseteq \text{H}_S * \sim\{\text{"p"}\}) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (E = \text{H}_S * \{\text{"p"}\}) &\subseteq (E = \text{H}_S * \sim\{\text{"p"}\}) & (T_{M275})(\text{Supersets of the Universal Set}) \\ (\text{H}_S * \{\text{"p"}\} \subseteq \text{H}_S * \sim\{\text{"p"}\}) &= E & (T_{M094})(\text{Probabilistic Rule}) \\ \text{H}_S * \{\text{"p"}\} &\subseteq \text{H}_S * \sim\{\text{"p"}\} & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$(\{\text{"p"}\} \subseteq {}^V\text{H}_S * d^1) \subseteq \sim(\sim\{\text{"p"}\} \subseteq {}^V\text{H}_S * d^1) \quad (T_{S053})$$

Proof of (T_{S053})

$$\begin{aligned} \text{H}_S * \{\text{"p"}\} &\subseteq \text{H}_S * \sim\{\text{"p"}\} & (T_{T259}) \\ (d^1 \subseteq \text{H}_S * \{\text{"p"}\}) &\subseteq (d^1 \subseteq \text{H}_S * \sim\{\text{"p"}\}) & (T_{M272})(\text{Deductive Rule}) \\ (\{\text{"p"}\} \subseteq {}^V\text{H}_S * d^1) &\subseteq (d^1 \subseteq \text{H}_S * \sim\{\text{"p"}\}) & (T_{R519})(\text{Active/Passive Theorem}) \\ (\{\text{"p"}\} \subseteq {}^V\text{H}_S * d^1) &\subseteq (\sim\{\text{"p"}\} \subseteq {}^V\text{H}_S * d^1) & (T_{R612})(\text{Active/Passive Theorem}) \\ (\{\text{"p"}\} \subseteq {}^V\text{H}_S * d^1) &\subseteq (\sim\{\text{"p"}\} \not\subseteq {}^V\text{H}_S * d^1) & (T_{S049}) \\ (\{\text{"p"}\} \subseteq {}^V\text{H}_S * d^1) &\subseteq \sim(\sim\{\text{"p"}\} \subseteq {}^V\text{H}_S * d^1) & (T_{M259})(\text{Definition of } \not\subseteq) \end{aligned}$$

$$(\{p\} \subseteq {}^V H_S * J^1).(\{p\} \subseteq {}^V H_S * d^1) \subseteq \sim((\sim\{p\} \subseteq {}^V H_S * J^1).(\sim\{p\} \subseteq {}^V H_S * d^1)) \quad (T_{S076})$$

$$(\sim\{p\} \subseteq {}^V H_S * J^1).(\sim\{p\} \subseteq {}^V H_S * d^1) \subseteq \sim((\{p\} \subseteq {}^V H_S * J^1).(\{p\} \subseteq {}^V H_S * d^1)) \quad (T_{S052})$$

Example Proof of (T_{S076})

$$(\{p\} \subseteq {}^V H_S * d^1) \subseteq \sim(\sim\{p\} \subseteq {}^V H_S * d^1) \quad (T_{S053})$$

$$(\{p\} \subseteq {}^V H_S * J^1).(\{p\} \subseteq {}^V H_S * d^1) \subseteq \sim(\sim\{p\} \subseteq {}^V H_S * d^1)$$

(T_{M089})(Single Premise Syllogism)

$$(\{p\} \subseteq {}^V H_S * J^1).(\{p\} \subseteq {}^V H_S * d^1) \subseteq \sim(\sim\{p\} \subseteq {}^V H_S * d^1) + \sim(\sim\{p\} \subseteq {}^V H_S * d^1)$$

(T_{M150})(Single Premise Syllogism)

$$(\{p\} \subseteq {}^V H_S * J^1).(\{p\} \subseteq {}^V H_S * d^1) \subseteq \sim((\sim\{p\} \subseteq {}^V H_S * d^1).(\sim\{p\} \subseteq {}^V H_S * d^1))$$

(T_{M025})(DeMorgan's Law)

$$(\{a\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * J^1.(\{a\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * d^1 \subseteq (\{a\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * J^1 \quad (T_{S070})$$

Proof of (T_{S070})

$$ab \subseteq a \quad (T_{M083})(Eqn1)$$

$$\text{Let } a = (\{a\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * J^1 \quad (Eqn2)$$

$$\text{Let } b = (\{a\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * d^1 \quad (Eqn3)$$

$$(\{a\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * J^1.(\{a\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * d^1 \subseteq (\{a\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * J^1$$

(A_{MX19})(Substitution of Eqns2&3 into Eqn1)

$$(\{p\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * J^1.(\{p\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * d^1 \subseteq (\{p\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * J^1 \quad (T_{S32})$$

Proof of (T_{S32})

$$ab \subseteq a \quad (T_{M083})(Eqn1)$$

$$\text{Let } a = (\{p\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * J^1 \quad (Eqn2)$$

$$\text{Let } b = (\{p\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * d^1 \quad (Eqn3)$$

$$(\{p\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * J^1.(\{p\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * d^1 \subseteq (\{p\} \subseteq {}^V H_S * d^1) \subseteq {}^V H_S * J^1$$

(A_{MX19})(Substitution of Eqns2&3 into Eqn1)

$$(\{p\} = F_1 a_1 \dots F_n a_n) = (\{p\} = "F_1 a_1 \dots F_n a_n") \quad (T_{S596})$$

Proof of (T_{S596})

$$\text{Let } \{p\} = F_1 a_1 \dots F_n a_n \quad (Eqn1)$$

$$\{p\} = \{p\} \quad (D_{M00})(\text{Definition of } =)(Eqn2)$$

$$\{p\} = "F_1 a_1 \dots F_n a_n" \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$$

$$\text{Thus } (\{p\} = F_1 a_1 \dots F_n a_n) \subseteq (\{p\} = "F_1 a_1 \dots F_n a_n") \quad (A_{MX02})(\text{Conditional Introduction})(Eqn3)$$

$$\text{Let } \{p\} = "F_1 a_1 \dots F_n a_n"$$

$$(\{p\} \subseteq {}^V H_S * J^1) = ("F_1 a_1 \dots F_n a_n" \subseteq {}^V H_S * J^1) \quad (T_{M791})$$

$$\{p\} = F_1 a_1 \dots F_n a_n \quad (D_{R02})(\text{Definition of Knowing That})$$

$$\text{Thus } (\{p\} = "F_1 a_1 \dots F_n a_n") \subseteq (\{p\} = F_1 a_1 \dots F_n a_n) \quad (A_{MX02})(\text{Conditional Introduction})(Eqn4)$$

$$(\{p\} = F_1 a_1 \dots F_n a_n) = (\{p\} = "F_1 a_1 \dots F_n a_n") \quad (T_{M430})(\text{Biconditional Introduction Eqns3\&4})$$

$$(((\{p\} \subseteq \{q\}) \subseteq {}^V H_S * n^1) = E) = E \subseteq (((\{p\} \subseteq {}^V H_S * n^1) = E) \subseteq ((\{q\} \subseteq {}^V H_S * n^1) \neq \emptyset)) \quad (T_{S037})$$

Proof of (T_{S037})

$$(((\{p_1\} \dots \{p_n\} \subseteq \{c\}) = E) = E) \subseteq (((\{p_1\} \subseteq {}^V H_S * n^1) \dots (\{p_n\} \subseteq {}^V H_S * n^1) = E) \subseteq ((\{c\} \subseteq {}^V H_S * n^1) \neq \emptyset))$$

(A_{R63})(Axiom of Possible Deduction)(Eqn1)

$$\text{Let } n = 2 \quad (Eqn2)$$

$$\text{Let } \{p_1\} = \{p\} \subseteq \{q\} \quad (Eqn3)$$

$$\text{Let } \{p_2\} = \{p\} \quad (Eqn4)$$

$$\text{Let } \{c\} = \{q\} \quad (Eqn5)$$

$$\{p_1\} = \{p\} \subseteq \{q\} \quad (T_{S596})(Eqn6)$$

$$\{p_2\} = \{p\} \quad (T_{S596})(Eqn7)$$

$$\{c\} = \{q\} \quad (T_{S596})(Eqn8)$$

$$(((\{p\} \subseteq \{q\}).\{p\} \subseteq \{q\}) = E) = E \subseteq (((\{p\} \subseteq \{q\}) \subseteq {}^V H_S * n^1).(\{p\} \subseteq {}^V H_S * n^1) = E) \subseteq$$

$$((\{q\} \subseteq {}^V H_S * n^1) \neq \emptyset) \quad (A_{MX19})(\text{Substitution of Eqns2,3,4,5,6,7\&8 into Eqn1})$$

$$(((\{p\} \subseteq \{q\}).\{p\} = E) \subseteq (\{q\} = E)) = E \subseteq (((\{p\} \subseteq \{q\}) \subseteq {}^V H_S * n^1).(\{p\} \subseteq {}^V H_S * n^1) = E) \subseteq$$

$((\{q\} \subseteq {}^V H_{S * n^1}) \neq \emptyset)$	(Eqn9)
Let $\{p\} \subseteq \{q\}$	(Eqn10)
Let $\{p\} = E$	(Eqn11)
$E \subseteq \{q\}$	(A_{MX19})(Substitution of Eqn11 into Eqn10)
$E = \{q\}$	(T_{M275})(Supersets of the Universal Set)
$\{q\} = E$	(T_{M197})(Commutativity of $=$)
Thus $(\{p\} \subseteq \{q\}).\{p\} = E \subseteq (\{q\} = E)$	(A_{MX02})(Conditional Introduction)
$((\{p\} \subseteq \{q\}).\{p\} = E \subseteq (\{q\} = E)) = E = E$	(A_{MX16})(Order Axiom)(Eqn12)
$E \subseteq (((\{p\} \subseteq \{q\}) \subseteq {}^V H_{S * n^1}).(\{p\} \subseteq {}^V H_{S * n^1}) = E) \subseteq ((\{q\} \subseteq {}^V H_{S * n^1}) \neq \emptyset)$	(A_{MX19})(Substitution of Eqn12 into Eqn9)
$E = (((\{p\} \subseteq \{q\}) \subseteq {}^V H_{S * n^1}).(\{p\} \subseteq {}^V H_{S * n^1}) = E) \subseteq ((\{q\} \subseteq {}^V H_{S * n^1}) \neq \emptyset)$	(T_{M275})(Supersets of the Universal Set)
$((\{p\} \subseteq \{q\}) \subseteq {}^V H_{S * n^1}).(\{p\} \subseteq {}^V H_{S * n^1}) = E \subseteq ((\{q\} \subseteq {}^V H_{S * n^1}) \neq \emptyset)$	(A_{MX16})(Order Axiom)
$((\{p\} \subseteq \{q\}) \subseteq {}^V H_{S * n^1}) = E.((\{p\} \subseteq {}^V H_{S * n^1}) = E) \subseteq ((\{q\} \subseteq {}^V H_{S * n^1}) \neq \emptyset)$	(T_{M108})(Conjunction Introduction)
$((\{p\} \subseteq \{q\}) \subseteq {}^V H_{S * n^1}) = E = E \subseteq (((\{p\} \subseteq {}^V H_{S * n^1}) = E) \subseteq ((\{q\} \subseteq {}^V H_{S * n^1}) \neq \emptyset))$	(A_{MX35})(Exportation/Importation) for truth sets

15.2 Proofs of Theorems of Communication

$(\{a^1 \neq \emptyset\} \subseteq {}^V H_{S J^1}).(({}^V K d^1)^1 \subseteq {}^V (I * a^1) * ({}^V H_{S * E})^1) \subseteq (\{a^1 \neq \emptyset\} \subseteq {}^V H_{S J^1}) \quad (T_{S097})$

Proof of (T_{S097})

$ab \subseteq a$	(T_{M083})(Eqn1)
Let $a = (\{a^1 \neq \emptyset\} \subseteq {}^V H_{S J^1})$	(Eqn2)
Let $b = (({}^V K d^1)^1 \subseteq {}^V (I * a^1) * ({}^V H_{S * E})^1)$	(Eqn3)
$(\{a^1 \neq \emptyset\} \subseteq {}^V H_{S J^1}).(({}^V K d^1)^1 \subseteq {}^V (I * a^1) * ({}^V H_{S * E})^1) \subseteq (\{a^1 \neq \emptyset\} \subseteq {}^V H_{S J^1})$	(A_{MX19})(Substitution of Eqns2&3 into Eqn1)

$(\{a\} \subseteq {}^V H_{S * e^1}).(({}^V H_{S d^1})^1 \subseteq {}^V (I * a) * ({}^V H_{S e^1})^1) \subseteq (\{a\} \subseteq {}^V H_{S * e^1}) \quad (T_{S098})$

Proof of (T_{S098})

$ab \subseteq a$	(T_{M083})(Eqn1)
Let $a = (\{a\} \subseteq {}^V H_{S * e^1})$	(Eqn2)
Let $b = (({}^V H_{S d^1})^1 \subseteq {}^V (I * a) * ({}^V H_{S e^1})^1)$	(Eqn3)
$(\{a\} \subseteq {}^V H_{S * e^1}).(({}^V H_{S d^1})^1 \subseteq {}^V (I * a) * ({}^V H_{S e^1})^1) \subseteq (\{a\} \subseteq {}^V H_{S * e^1})$	(A_{MX19})(Substitution of Eqns2&3 into Eqn1)

Chapter 16 Proofs of Theorems of Necessity and Desirability

16.1 Proofs

$$(\{p\} \subseteq {}^vH_s * d^1) = (d^1 \subseteq H_s * \{p\}) \quad (T_{S165})$$

$$(\{p\} \subseteq {}^vH_w * d^1) = (d^1 \subseteq H_w * \{p\}) \quad (T_{T386})$$

$$(\{p\} \subseteq {}^vW_s * d^1) = (d^1 \subseteq W_s * \{p\}) \quad (T_{S166})$$

$$(\{p\} \subseteq {}^vW_w * d^1) = (d^1 \subseteq W_w * \{p\}) \quad (T_{S167})$$

Example Proof of (T_{S165})

$$(a^1 \subseteq c * b^1) = (b^1 \subseteq {}^v c * a^1) \quad (T_{R519})(\text{Active/Passive Theorem})(\text{Eqn1})$$

$$\text{Let } a^1 = \{p\} \quad (\text{Eqn2})$$

$$\text{Let } c = {}^v H_s \quad (\text{Eqn3})$$

$$\text{Let } b^1 = d^1 \quad (\text{Eqn4})$$

$$(\{p\} \subseteq {}^v H_s * d^1) = (d^1 \subseteq H_s * \{p\}) \quad (A_{MX19})(\text{Substitution of Eqns2,3\&4 into Eqn1})$$

$${}^v H_s * d^1 \subseteq {}^v H_w * d^1 \quad (T_{S142})$$

$${}^v W_s * d^1 \subseteq {}^v W_w * d^1 \quad (T_{S143})$$

$${}^v H_w * d^1 \subseteq {}^v H_s * d^1 \quad (T_{S144})$$

$${}^v W_w * d^1 \subseteq {}^v W_s * d^1 \quad (T_{S145})$$

Example Proof of (T_{S142}) & (T_{S144})

$$H_s \subseteq H_w \quad (A_{S20})(H \text{ Axiom})$$

$${}^v H_s \subseteq {}^v H_w \quad (T_{R022a})(\text{Relationship Opposite of a Proposition})$$

$${}^v H_s * d^1 \subseteq {}^v H_w * d^1 \quad (T_{R197})(\text{Relational Deduction})$$

$${}^v H_w * d^1 \subseteq {}^v H_s * d^1 \quad (T_{M012})(\text{Transposition})$$

$${}^v H_w * d^1 \sim \subset {}^v H_s * d^1 \quad (T_{S146})$$

$${}^v W_w * d^1 \sim \subset {}^v W_s * d^1 \quad (T_{S147})$$

$${}^v H_s * d^1 \sim \subset {}^v H_w * d^1 \quad (T_{S148})$$

$${}^v W_s * d^1 \sim \subset {}^v W_w * d^1 \quad (T_{S149})$$

Example Proof of (T_{S146}) & (T_{S148})

$${}^v H_s * d^1 \subseteq {}^v H_w * d^1 \quad (T_{S142})$$

$${}^v H_s * d^1 \sim \supset {}^v H_w * d^1 \quad (T_{M562})$$

$${}^v H_w * d^1 \sim \subset {}^v H_s * d^1 \quad (T_{M195})(\subset \text{ Reverse of } \supset)$$

$${}^v H_s * d^1 \sim \subset {}^v H_w * d^1 \quad (T_{M008})(\text{Inverse of an Equality}) \& (T_{M010})(\text{Inverse of a Proper Subset})$$

$$(\{p\} \subseteq {}^v H_s * d^1) \subseteq (\{p\} \subseteq {}^v H_w * d^1) \quad (T_{S150})$$

$$(\{p\} \subseteq {}^v W_s * d^1) \subseteq (\{p\} \subseteq {}^v W_w * d^1) \quad (T_{S151})$$

$$(\{p\} \subseteq {}^v H_w * d^1) \subseteq (\{p\} \subseteq {}^v H_s * d^1) \quad (T_{S152})$$

$$(\{p\} \subseteq {}^v W_w * d^1) \subseteq (\{p\} \subseteq {}^v W_s * d^1) \quad (T_{S153})$$

$$(\sim \{p\} \subseteq {}^v H_s * d^1) \subseteq (\sim \{p\} \subseteq {}^v H_w * d^1) \quad (T_{S157})$$

$$(\sim \{p\} \subseteq {}^v W_s * d^1) \subseteq (\sim \{p\} \subseteq {}^v W_w * d^1) \quad (T_{S158})$$

$$(\sim \{p\} \subseteq {}^v H_w * d^1) \subseteq (\sim \{p\} \subseteq {}^v H_s * d^1) \quad (T_{S159})$$

$$(\sim \{p\} \subseteq {}^v W_w * d^1) \subseteq (\sim \{p\} \subseteq {}^v W_s * d^1) \quad (T_{S160})$$

Example Proof of (T_{S150})

$${}^v H_s * d^1 \subseteq {}^v H_w * d^1 \quad (T_{S142})$$

$$(\{p\} \subseteq {}^v H_{S*} d^1) \subseteq (\{p\} \subseteq {}^v H_{W*} d^1) \quad (T_{M272})(\text{Deductive Rule})$$

$$(\{p\} \subseteq {}^v H_{W*} d^1) \sim \subset (\{p\} \subseteq {}^v H_{S*} d^1) \quad (T_{S154})$$

$$(\{p\} \subseteq {}^v W_{W*} d^1) \sim \subset (\{p\} \subseteq {}^v W_{S*} d^1) \quad (T_{S155})$$

$$(\{p\} \subseteq {}^v H_{S*} d^1) \sim \subset (\{p\} \subseteq {}^v H_{W*} d^1) \quad (T_{S141})$$

$$(\{p\} \subseteq {}^v W_{S*} d^1) \sim \subset (\{p\} \subseteq {}^v W_{W*} d^1) \quad (T_{S156})$$

$$(\sim\{p\} \subseteq {}^v H_{W*} d^1) \sim \subset (\sim\{p\} \subseteq {}^v H_{S*} d^1) \quad (T_{S161})$$

$$(\sim\{p\} \subseteq {}^v W_{W*} d^1) \sim \subset (\sim\{p\} \subseteq {}^v W_{S*} d^1) \quad (T_{S162})$$

$$(\sim\{p\} \subseteq {}^v H_{S*} d^1) \sim \subset (\sim\{p\} \subseteq {}^v H_{W*} d^1) \quad (T_{S163})$$

$$(\sim\{p\} \subseteq {}^v W_{S*} d^1) \sim \subset (\sim\{p\} \subseteq {}^v W_{W*} d^1) \quad (T_{S164})$$

Example Proof of (T_{S154})

$$(\{p\} \subseteq {}^v H_{S*} d^1) \subseteq (\{p\} \subseteq {}^v H_{W*} d^1) \quad (T_{S150})$$

$$(\{p\} \subseteq {}^v H_{S*} d^1) \sim \supset (\{p\} \subseteq {}^v H_{W*} d^1) \quad (T_{M562})$$

$$(\{p\} \subseteq {}^v H_{W*} d^1) \sim \subset (\{p\} \subseteq {}^v H_{S*} d^1) \quad (T_{M195})(\subset \text{ Reverse of } \supset) \text{ \& } (T_{M008})(\text{Inverse of an Equality})$$

$$(\{r\} \subseteq {}^v H_{S*} d^1) \subseteq (\{r \neq \emptyset\} \subseteq {}^v H_{W*} d^1) \quad (T_{S649})$$

$$(\{r\} \subseteq {}^v H_{S*} d^1) \subseteq (\{r = \emptyset\} \subseteq {}^v H_{W*} d^1) \quad (T_{S663})$$

Example Proof of (T_{S649})

$$(\{r\} \subseteq {}^v H_{S*} d^1) \subseteq (\{r \neq \emptyset\} \subseteq {}^v H_{S*} d^1). (\{r = \emptyset\} \subseteq {}^v H_{S*} d^1) \quad (T_{T324})$$

$$(\{r\} \subseteq {}^v H_{S*} d^1) \subseteq (\{r \neq \emptyset\} \subseteq {}^v H_{S*} d^1) \quad (T_{M073})(\text{Single Premise Syllogism})$$

Chapter 17 Proofs of Theorems of Quantity and Number

17.1 Introductory and Null Quantity Theorems

A. Introductory Theorems

$$\{a\}^1 + \{b\}^1 = (\{+\} * \{b\}^1) * \{a\}^1 \quad (T_{T471})$$

$$\{a\}^1 - \{b\}^1 = (\{-\} * \{b\}^1) * \{a\}^1 \quad (T_{T472})$$

$$\{a\}^1 \times \{b\}^1 = (\{x\} * \{b\}^1) * \{a\}^1 \quad (T_{T473})$$

$$\{a\}^1 \div \{b\}^1 = (\{\div\} * \{b\}^1) * \{a\}^1 \quad (T_{T474})$$

$$\{a\}^1 \wedge \{b\}^1 = (\{\wedge\} * \{b\}^1) * \{a\}^1 \quad (T_{T475})$$

$$\{a\}^1 \vee \{b\}^1 = (\{\vee\} * \{b\}^1) * \{a\}^1 \quad (T_{T476})$$

Example Proof of (T_{T471})

$$(\circ = \{+\}, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\{a\}^1 \circ \{b\}^1 = (\circ * \{b\}^1) * \{a\}^1)$$

(D_{R20})(Definition of an Arithmetical Operator)
Extraction

$$\{a\}^1 + \{b\}^1 = (\{+\} * \{b\}^1) * \{a\}^1$$

B. Null Quantity Theorems

$$(a = E) \subseteq (\mathcal{N}a^1 = \emptyset) \quad (T_{S786})$$

Proof of (T_{S786})

$$(a = E) \subseteq (\mathcal{N}a^1 = \{\infty\}^1)$$

(D_{R25})(Definition of Infinity)(Eqn1)

$$\{\infty\}^1 = \emptyset$$

(A_{R90})(Infinity does not Exist)(Eqn2)

$$(a = E) \subseteq (\mathcal{N}a^1 = \emptyset)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(a^1 = \emptyset) \subseteq (\mathcal{N}a^1 = \emptyset) \quad (T_{S798})$$

Proof of (T_{S798})

$$\text{Let } a^1 = \emptyset$$

(Eqn1)

$$\mathcal{N}\emptyset = \emptyset$$

(T_{S187})(Null Object Theorem)(Eqn2)

$$\mathcal{N}a^1 = \emptyset$$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

$$\text{Thus } (a^1 = \emptyset) \subseteq (\mathcal{N}a^1 = \emptyset)$$

(A_{MX02})(Conditional Introduction)

$$(a = \emptyset^1) \subseteq (\mathcal{N}a^1 = \emptyset) \quad (T_{S809})$$

$$(a = E^1) \subseteq (\mathcal{N}a^1 = \emptyset) \quad (T_{S787})$$

Proof of (T_{S809})

$$((\mathcal{X}E)^1 \subseteq D_A + P_A)(a = \emptyset^1) \subseteq (\mathcal{X} * a^1 = \emptyset)$$

(T_{S796})($\emptyset^1$ does not Map)(Eqn1)

$$\text{Let } \mathcal{X} = \mathcal{N}$$

(Eqn2)

$$((\mathcal{N}E)^1 \subseteq D_A + P_A)(a = \emptyset^1) \subseteq (\mathcal{N} * a^1 = \emptyset)$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)

$$(\mathcal{N}E)^1 \subseteq D_D$$

(A_{S31})(Quantity Dimension Axiom)(Eqn4)

$$D_D \subseteq D_A$$

(T_{T885})(Eqn5)

$$(\mathcal{N}E)^1 \subseteq D_A$$

(T_{M139})(Hypothetical Syllogism Eqns4&5)(Eqn6)

$$D_A \subseteq D_A + P_A$$

(T_{M082})(Disjunction Introduction)(Eqn7)

$$(\mathcal{N}E)^1 \subseteq D_A + P_A$$

(T_{M139})(Hypothetical Syllogism Eqns6&7)

$$((\mathcal{N}E)^1 \subseteq D_A + P_A) = E$$

(A_{MX16})(Order Axiom)(Eqn8)

$$E(a = \emptyset^1) \subseteq (\mathcal{N} * a^1 = \emptyset)$$

(A_{MX19})(Substitution of Eqn8 into Eqn3)

$$(a = \emptyset^1) \subseteq (\mathcal{N}a^1 = \emptyset)$$

($\mathbf{A}_{MX11}$)(Universal Set Conjunction Axiom)

$$(a = \emptyset) \subseteq (\{\div\} * \mathcal{N}a^1 = \emptyset)$$

($\mathbf{T}_{S799}$)

Proof of ($\mathbf{T}_{S799}$)

$$(\{\div\} * \{0\}^1) = \emptyset$$

($\mathbf{A}_{R54}$)(Division by Zero is Null)(Eqn1)

$$\text{Let } a = \emptyset$$

$$\mathcal{N}a^1 = \{0\}^1$$

($\mathbf{D}_{R23}$)(Definition of Zero)(Eqn2)

$$(\{\div\} * \mathcal{N}a^1) = \emptyset$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$\text{Thus } (a = \emptyset) \subseteq (\{\div\} * \mathcal{N}a^1 = \emptyset)$$

($\mathbf{A}_{MX02}$)(Conditional Introduction)

$$(\{a\}^1 = \{0\}^1) \subseteq (\{b\}^1 \div \{a\}^1 = \emptyset)$$

($\mathbf{T}_{S808}$)

Proof of ($\mathbf{T}_{S808}$)

$$\{b\}^1 \div \{a\}^1 = \{b\}^1 \div \{a\}^1$$

($\mathbf{D}_{M00}$)(Definition of =)(Eqn1)

$$\text{Let } (\{a\}^1 = \{0\}^1)$$

(Eqn2)

$$\{b\}^1 \div \{a\}^1 = \{b\}^1 \div \{0\}^1$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$\{b\}^1 \div \{a\}^1 = (\{\div\} * \{0\}^1) * \{b\}^1$$

($\mathbf{T}_{T474}$)

$$\{b\}^1 \div \{a\}^1 = \emptyset * \{b\}^1$$

($\mathbf{A}_{R54}$)(Division by Zero is Null)

$$\{b\}^1 \div \{a\}^1 = \emptyset$$

($\mathbf{T}_{R072}$)(Null Function Theorem)

$$\text{Thus } (\{a\}^1 = \{0\}^1) \subseteq (\{b\}^1 \div \{a\}^1 = \emptyset)$$

($\mathbf{A}_{MX02}$)(Conditional Introduction)

$$\{a\}^1 \div \{0\}^1 = \emptyset$$

($\mathbf{T}_{S807}$)

Proof of ($\mathbf{T}_{S807}$)

$$\{a\}^1 \div \{0\}^1 = \{a\}^1 \div \{0\}^1$$

($\mathbf{D}_{M00}$)(Definition of =)

$$\{a\}^1 \div \{0\}^1 = (\{\div\} * \{0\}^1) * \{a\}^1$$

($\mathbf{T}_{T474}$)

$$\{a\}^1 \div \{0\}^1 = \emptyset * \{a\}^1$$

($\mathbf{A}_{R54}$)(Division by Zero is Null)

$$\{a\}^1 \div \{0\}^1 = \emptyset$$

($\mathbf{T}_{R072}$)(Null Function Theorem)

$$\{0\}^1 \div \{0\}^1 = \emptyset$$

($\mathbf{T}_{S768}$)

Proof of ($\mathbf{T}_{S768}$)

$$\{a\}^1 \div \{0\}^1 = \emptyset$$

($\mathbf{T}_{S807}$)(Eqn1)

$$\text{Let } \{a\}^1 = \{0\}^1$$

(Eqn2)

$$\{0\}^1 \div \{0\}^1 = \emptyset$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$(X = \{\div\})(a = \emptyset) + (X = \{=\})(a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_i) \subseteq (X * \mathcal{N}a^1 = \emptyset) \quad (\mathbf{T}_{S763})$$

Proof of ($\mathbf{T}_{S763}$)

$$(a^1 = \emptyset) \subseteq (\mathcal{N}a^1 = \emptyset)$$

($\mathbf{T}_{S798}$)(Eqn1)

$$(a = E) \subseteq (\mathcal{N}a^1 = \emptyset)$$

($\mathbf{T}_{S786}$)(Eqn2)

$$(a = \emptyset^1) \subseteq (\mathcal{N}a^1 = \emptyset)$$

($\mathbf{T}_{S809}$)(Eqn3)

$$(a = E^1) \subseteq (\mathcal{N}a^1 = \emptyset)$$

($\mathbf{T}_{S787}$)(Eqn4)

$$(a^1 \subseteq S_i) \subseteq (\mathcal{N}a^1 = \emptyset)$$

($\mathbf{A}_{R18}$)(Quantity of an Infinite Set)(Eqn5)

$$(a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_i) \subseteq (\mathcal{N}a^1 = \emptyset) \quad (\mathbf{T}_{M131b})(\text{Conjunction of Sufficiencies Eqns1,2,3,4\&5})$$

$$(a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_i) \subseteq (\{=\} * \mathcal{N}a^1 = \emptyset)$$

($\mathbf{D}_{A144}$)(Definition of the Reflexive Relationship Set)

(Eqn6)

$$\text{Let } X = \{=\}$$

(Eqn7)

$$(a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_i) \subseteq (X * \mathcal{N}a^1 = \emptyset)$$

($\mathbf{A}_{MX19}$)(Substitution of Eqn7 into Eqn6)

$$\text{Thus } ((X = \{=\}) = E) \subseteq ((a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_i) \subseteq (X * \mathcal{N}a^1 = \emptyset))$$

($\mathbf{A}_{MX02}$)(Conditional Introduction)

$$(X = \{=\})((a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_i)) \subseteq (X * \mathcal{N}a^1 = \emptyset)$$

$(a = \emptyset) \subseteq (\{\div\} * \mathcal{N}a^1 = \emptyset)$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets (Eqn8)
Let $X = \{\div\}$	$(\mathbf{T}_{S799})$ (Eqn9)
$(a = \emptyset) \subseteq (X * \mathcal{N}a^1 = \emptyset)$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn10 into Eqn9)
Thus $((X = \{\div\}) = E) \subseteq ((a = \emptyset) \subseteq (X * \mathcal{N}a^1 = \emptyset))$	$(\mathbf{A}_{MX02})$ (Conditional Introduction)
$(X = \{\div\})(a = \emptyset) \subseteq (X * \mathcal{N}a^1 = \emptyset)$	$(\mathbf{A}_{MX35})$ (Exportation/Importation) for truth sets (Eqn11)
$(X = \{\div\})(a = \emptyset) + (X = \{=\})(a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_1) \subseteq (X * \mathcal{N}a^1 = \emptyset)$	$(\mathbf{T}_{M131b})$ (Conjunction of Sufficiencies Eqns8&11)

$$(X = \{\div\})(a = \emptyset) + (X = \{=\})(a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_1) = (X * \mathcal{N}a^1 = \emptyset) \quad (\mathbf{T}_{S762})$$

Proof of $(\mathbf{T}_{S762})$

$(X = \{\div\})(a = \emptyset) + (X = \{=\})(a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_1) \subseteq (X * \mathcal{N}a^1 = \emptyset)$	$(\mathbf{T}_{S763})$ (Eqn1)
$(X * \mathcal{N}a^1 = \emptyset) \subseteq (X = \{\div\})(a = \emptyset) + (X = \{=\})(a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_1)$	$(\mathbf{A}_{R52})$ (Conditions for Null Quantity)(Eqn2)
$(X = \{\div\})(a = \emptyset) + (X = \{=\})(a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_1) = (X * \mathcal{N}a^1 = \emptyset)$	$(\mathbf{T}_{M430})$ (Biconditional Introduction Eqns1&2)

$$((X \neq \{\div\}) + (a \neq \emptyset)).((X \neq \{=\}) + ((a^1 \neq \emptyset).(a \neq E).(a \neq \emptyset^1).(a \neq E^1).(a^1 \not\subseteq S_1))) = (X * \mathcal{N}a^1 \neq \emptyset)$$

$(\mathbf{T}_{S761})$ Implied Conditions of Quantity

Proof of $(\mathbf{T}_{S761})$

$(X = \{\div\})(a = \emptyset) + (X = \{=\})(a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_1) = (X * \mathcal{N}a^1 = \emptyset)$	$(\mathbf{T}_{S762})$
$\sim((X = \{\div\})(a = \emptyset) + (X = \{=\})(a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_1)) = (X * \mathcal{N}a^1 \neq \emptyset)$	$(\mathbf{T}_{M008})$ (Inverse of an Equality)
$\sim((X = \{\div\})(a = \emptyset)).\sim((X = \{=\})(a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_1)) = (X * \mathcal{N}a^1 \neq \emptyset)$	$(\mathbf{T}_{M026})$ (DeMorgan's Law)
$((X \neq \{\div\}) + (a \neq \emptyset)).((X \neq \{=\}) + \sim((a^1 = \emptyset) + (a = E) + (a = \emptyset^1) + (a = E^1) + (a^1 \subseteq S_1))) = (X * \mathcal{N}a^1 \neq \emptyset)$	$(\mathbf{T}_{M025})$ (DeMorgan's Law)
$((X \neq \{\div\}) + (a \neq \emptyset)).((X \neq \{=\}) + ((a^1 \neq \emptyset).(a \neq E).(a \neq \emptyset^1).(a \neq E^1).(a^1 \not\subseteq S_1))) = (X * \mathcal{N}a^1 \neq \emptyset)$	$(\mathbf{T}_{M026})$ (DeMorgan's Law)

17.2 General Theorems of Quantity

An example of the derivation of theorems of quantity from the more general theorems of dimensions is given below.

$$(a \neq \emptyset) = (\mathcal{N}a \neq \emptyset) \quad (\mathbf{T}_{T275\mathcal{N}})$$

Proof of $(\mathbf{T}_{T275\mathcal{N}})$

$((\mathcal{N}E)^1 \subseteq D_A + P_A) = E \subseteq ((a \neq \emptyset) = (Xa \neq \emptyset))$	$(\mathbf{T}_{T275})$ (Eqn1)
Let $X = \mathcal{N}$	(Eqn2)
$((\mathcal{N}E)^1 \subseteq D_A + P_A) = E \subseteq ((a \neq \emptyset) = (\mathcal{N}a \neq \emptyset))$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn2 into Eqn1)(Eqn3)
$(\mathcal{N}E)^1 \subseteq D_D$	$(\mathbf{A}_{S31})$ (Quantity Dimension Axiom)(Eqn4)
$D_D \subseteq D_A$	$(\mathbf{T}_{T385})$ (Eqn5)
$(\mathcal{N}E)^1 \subseteq D_A$	$(\mathbf{T}_{M139})$ (Hypothetical Syllogism Eqns4&5)
$(\mathcal{N}E)^1 \subseteq D_A + P_A$	$(\mathbf{T}_{M150})$ (Single Premise Syllogism)
$((\mathcal{N}E)^1 \subseteq D_A + P_A) = E$	$(\mathbf{A}_{MX16})$ (Order Axiom)(Eqn6)
$(E = E) \subseteq ((a \neq \emptyset) = (\mathcal{N}a \neq \emptyset))$	$(\mathbf{A}_{MX19})$ (Substitution of Eqn6 into Eqn3)
$E \subseteq ((a \neq \emptyset) = (\mathcal{N}a \neq \emptyset))$	$(\mathbf{T}_{M342})$

$$\begin{aligned} E &= ((a \neq \emptyset) = (\mathcal{N}a \neq \emptyset)) \\ (a \neq \emptyset) &= (\mathcal{N}a \neq \emptyset) \end{aligned}$$

(T_{M275})(Supersets of the Universal Set)
($AMX16$)(Order Axiom)(Eqn7)

17.3 Special Theorems of Quantity and Number

$$((a)^1 \subseteq \sim S_i) \subseteq (a^1 \subseteq S_i) \quad (T_{7316})$$

Proof of (T_{7316})

$$\begin{aligned} (a^1 \subseteq \sim S_i) &\subseteq ((a)^1 \subseteq S_i) & (A_{S98}) & \text{(Complement of a Finite Set)(Eqn1)} \\ \text{Let } a^1 &= (a)^1 & & \text{(Eqn2)} \\ ((a)^1 \subseteq \sim S_i) &\subseteq (a^1 \subseteq S_i) & (A_{MX19}) & \text{(Substitution of Eqn2 into Eqn1)} \end{aligned}$$

$$\emptyset^1 \not\subseteq S_i \quad (T_{R764})$$

Proof of (T_{R764})

$$\begin{aligned} ((a^1 \subseteq S_i) = E) &= ((a^1 \neq \emptyset)(x^1 \subseteq \{ \_ \} * a^1) \subseteq (\{ \_ \} * x^1 . \{ \_ \} * a^1 \neq \emptyset)) & (D_{R28}) & \text{(Definition of an Infinite Set.)(Eqn1)} \\ \text{Let } a^1 &= \emptyset^1 & & \text{(Eqn2)} \\ ((\emptyset^1 \subseteq S_i) = E) &= ((\emptyset^1 \neq \emptyset)(x^1 \subseteq \{ \_ \} * \emptyset^1) \subseteq (\{ \_ \} * x^1 . \{ \_ \} * \emptyset^1 \neq \emptyset)) & (A_{MX19}) & \text{(Substitution of Eqn1 into Eqn1)(Eqn3)} \\ \text{Let } x^1 &\subseteq \{ \_ \} * \emptyset^1 & & \\ x^1 &\subseteq \emptyset & (T_{S576}) & \text{(Null Set Entity Theorem)} \\ x^1 &= \emptyset & (T_{M052}) & \text{(Subsets of the Null Set)} \\ \text{Thus } (x^1 \subseteq \{ \_ \} * \emptyset^1) &= (x^1 = \emptyset) & (A_{MX02}) & \text{(Conditional Introduction)} \\ & & & \text{since this proof is reversible (Eqn4)} \\ ((\emptyset^1 \subseteq S_i) = E) &= ((\emptyset^1 \neq \emptyset)(x^1 = \emptyset) \subseteq (\{ \_ \} * x^1 . \{ \_ \} * \emptyset^1 \neq \emptyset)) & (A_{MX19}) & \text{(Substitution of Eqn4 into Eqn3)} \\ ((\emptyset^1 \subseteq S_i) = E) &= ((\emptyset^1 \neq \emptyset)(x^1 = \emptyset) \subseteq (\{ \_ \} * x^1 . \emptyset \neq \emptyset)) & (T_{S576}) & \text{(Null Set Entity Theorem)} \\ ((\emptyset^1 \subseteq S_i) = E) &= ((\emptyset^1 \neq \emptyset) \subseteq (\{ \_ \} * \emptyset . \emptyset \neq \emptyset)) & (A_{MX19}) & \text{(Substitution)} \\ ((\emptyset^1 \subseteq S_i) = E) &= ((\emptyset^1 \neq \emptyset) \subseteq (\emptyset \neq \emptyset)) & (T_{M050}) & \text{(Null Set Conjunction)} \\ ((\emptyset^1 \subseteq S_i) = E) &= ((\emptyset^1 \neq \emptyset) \subseteq \emptyset) & (T_{M346}) & \\ ((\emptyset^1 \subseteq S_i) = E) &= ((\emptyset^1 \neq \emptyset) = \emptyset) & (T_{M052}) & \text{(Subsets of the Null Set)} \\ ((\emptyset^1 \subseteq S_i) = E) &= (\sim(\emptyset^1 \neq \emptyset) = \sim\emptyset) & (T_{M008}) & \text{(Inverse of an Equality)} \\ ((\emptyset^1 \subseteq S_i) = E) &= (\sim(\emptyset^1 \neq \emptyset) = E) & (T_{M054}) & \text{(Negative of Null Set)} \\ ((\emptyset^1 \subseteq S_i) = \sim(\emptyset^1 \neq \emptyset)) &= E & (T_{M094}) & \text{(Probabilistic Rule)} \\ (\emptyset^1 \subseteq S_i) &= \sim(\emptyset^1 \neq \emptyset) & (A_{MX16}) & \text{(Order Axiom)(Eqn5)} \\ \emptyset^1 \neq \emptyset & & (T_{M780}) & \text{(Existence of } \emptyset^1) \\ (\emptyset^1 \neq \emptyset) &= E & (A_{MX16}) & \text{(Order Axiom)(Eqn6)} \\ (\emptyset^1 \subseteq S_i) &= \sim E & (A_{MX19}) & \text{(Substitution of Eqn6 into Eqn5)} \\ (\emptyset^1 \not\subseteq S_i) &= E & (T_{M008}) & \text{(Inverse of an Equality)} \\ \emptyset^1 \not\subseteq S_i & & (A_{MX16}) & \text{(Order Axiom)} \end{aligned}$$

$$(\mathcal{Y}^1 \subseteq R_N) \subseteq (\mathcal{Y}^1 \subseteq R_P) \quad (T_{428})$$

Proof of (T_{428})

$$\begin{aligned} (\mathcal{X}^1 \subseteq R_N) &\subseteq (\mathcal{X}^1 \subseteq R_P)((Q\mathcal{X})^1 \subseteq D_D)(S\mathcal{X} \subseteq N_R) & (D_{S04}) & \text{(Definition of a Scale)} \\ (\mathcal{X}^1 \subseteq R_N) &\subseteq (\mathcal{X}^1 \subseteq R_P) & (T_{M073}) & \text{(Single Premise Syllogism)} \\ (\mathcal{Y}^1 \subseteq R_N) &\subseteq (\mathcal{Y}^1 \subseteq R_P) & & \text{Resymbolising} \end{aligned}$$

$$\{0\}^1 \neq \emptyset \quad (T_{426})$$

Proof of (T_{426})

$$\begin{aligned} ((X \neq \{ \_ \}) + (a \neq \emptyset)).((X \neq \{ \_ \}) + ((a^1 \neq \emptyset).(a \neq E).(a \neq \emptyset^1).(a \neq E^1).(a^1 \not\subseteq S_i))) &= (X * \mathcal{N}a^1 \neq \emptyset) \\ & & (T_{S761}) & \text{(Implied Conditions of Quantity)(Eqn1)} \\ \text{Let } X &= \{ \_ \} & & \text{(Eqn2)} \end{aligned}$$

$$\begin{aligned}
& ((\{=\} \neq \{\neq\}) + (a \neq \emptyset)).((\{=\} \neq \{=\}) + ((a^1 \neq \emptyset).(a \neq E).(a \neq \emptyset^1).(a \neq E^1).(a^1 \not\subseteq S_i))) = (\{=\} * \mathcal{N}a^1 \neq \emptyset) \\
& \hspace{15em} (AMX19)(\text{Substitution of Eqn2 into Eqn1})(Eqn3) \\
& \{=\} \neq \{\neq\} \hspace{15em} \text{By definition} \\
& (\{=\} \neq \{\neq\}) = E \hspace{15em} (AMX16)(\text{Order Axiom})(Eqn4) \\
& (E + (a \neq \emptyset)).((\{=\} \neq \{=\}) + ((a^1 \neq \emptyset).(a \neq E).(a \neq \emptyset^1).(a \neq E^1).(a^1 \not\subseteq S_i))) = (\{=\} * \mathcal{N}a^1 \neq \emptyset) \\
& \hspace{15em} (AMX19)(\text{Substitution of Eqn4 into Eqn3}) \\
& E.((\{=\} \neq \{=\}) + ((a^1 \neq \emptyset).(a \neq E).(a \neq \emptyset^1).(a \neq E^1).(a^1 \not\subseteq S_i))) = (\{=\} * \mathcal{N}a^1 \neq \emptyset) \\
& \hspace{15em} (TM055)(\text{Universal Set Disjunction}) \\
& (\{=\} \neq \{=\}) + ((a^1 \neq \emptyset).(a \neq E).(a \neq \emptyset^1).(a \neq E^1).(a^1 \not\subseteq S_i)) = (\{=\} * \mathcal{N}a^1 \neq \emptyset) \\
& \hspace{15em} (AMX17)(\text{Universal Set Conjunction Axiom})(Eqn5) \\
& \{=\} = \{=\} \hspace{15em} (DM00)(\text{Definition of } =) \\
& (\{=\} = \{=\}) = E \hspace{15em} (AMX16)(\text{Order Axiom}) \\
& (\{=\} \neq \{=\}) = \sim E \hspace{15em} (TM008)(\text{Inverse of an Equality}) \\
& (\{=\} \neq \{=\}) = \emptyset \hspace{15em} (DM02)(\text{Definition of } \emptyset)(Eqn6) \\
& \emptyset + ((a^1 \neq \emptyset).(a \neq E).(a \neq \emptyset^1).(a \neq E^1).(a^1 \not\subseteq S_i)) = (\{=\} * \mathcal{N}a^1 \neq \emptyset) \\
& \hspace{15em} (AMX19)(\text{Substitution of Eqn6 into Eqn5}) \\
& (a^1 \neq \emptyset).(a \neq E).(a \neq \emptyset^1).(a \neq E^1).(a^1 \not\subseteq S_i) = (\{=\} * \mathcal{N}a^1 \neq \emptyset) \\
& \hspace{15em} (TM668)(\text{Null Set Disjunction}) \\
& (a^1 \neq \emptyset).(a \neq E).(a \neq \emptyset^1).(a \neq E^1).(a^1 \not\subseteq S_i) = (\mathcal{N}a^1 \neq \emptyset) \hspace{2em} (TA080)(Eqn7) \\
& \text{Let } a = \emptyset \hspace{15em} (Eqn8) \\
& (\emptyset^1 \neq \emptyset).(\emptyset \neq E).(\emptyset \neq \emptyset^1).(\emptyset \neq E^1).(\emptyset^1 \not\subseteq S_i) = (\mathcal{N}a^1 \neq \emptyset) \\
& \hspace{15em} (AMX19)(\text{Substitution of Eqn8 into Eqn7})(Eqn9) \\
& \emptyset \neq E \hspace{15em} (TM352) \\
& (\emptyset \neq E) = E \hspace{15em} (AMX16)(\text{Order Axiom})(Eqn10) \\
& \emptyset \neq \emptyset^1 \hspace{15em} (TM780)(\text{Existence of } \emptyset^1) \\
& (\emptyset \neq \emptyset^1) = E \hspace{15em} (AMX16)(\text{Order Axiom})(Eqn11) \\
& \emptyset \neq E^1 \hspace{15em} (TM773)(\text{Existence of } E^1) \\
& (\emptyset \neq E^1) = E \hspace{15em} (AMX16)(\text{Order Axiom})(Eqn12) \\
& \emptyset^1 \not\subseteq S_i \hspace{15em} (TR764) \\
& (\emptyset^1 \not\subseteq S_i) = E \hspace{15em} (AMX16)(\text{Order Axiom})(Eqn13) \\
& E.E.E.E.E = (\mathcal{N}a^1 \neq \emptyset) \hspace{15em} (AMX19)(\text{Substitution of Eqns10to13 into Eqn9}) \\
& E = (\mathcal{N}a^1 \neq \emptyset) \hspace{15em} (TM667)(\text{Conjunction Tautology}) \\
& \mathcal{N}a^1 \neq \emptyset \hspace{15em} (AMX16)(\text{Order Axiom}) \\
& \text{Thus } (a = \emptyset) \subseteq (\mathcal{N}a^1 \neq \emptyset) \hspace{15em} (AMX02)(\text{Conditional Introduction}) \\
& (\mathcal{N}a^1 = \{0\}^1) \subseteq (\mathcal{N}a^1 \neq \emptyset) \hspace{15em} (DR23)(\text{Definition of Zero}) \\
& \{0\}^1 \neq \emptyset \hspace{15em} (AMX19)(\text{Substitution})
\end{aligned}$$

$$N_N = \mathcal{N}E \hspace{15em} (TR740)$$

Proof of (TR740)

$$\begin{aligned}
& Q\mathcal{N} = E \hspace{15em} (TS721\mathcal{N}) \\
& \mathcal{N}Q\mathcal{N} = \mathcal{N}E \hspace{15em} (TR196)(\text{Relational Deduction}) \\
& S\mathcal{N} = \mathcal{N}E \hspace{15em} (TR020a) \\
& N_N = \mathcal{N}E \hspace{15em} (DR70)(\text{Definition of the Quantity Dimension})
\end{aligned}$$

$$C(\mathcal{N}E)^1 = \emptyset \hspace{15em} (T625)$$

Proof of (T625)

$$\begin{aligned}
& CN_N^1 = \emptyset \hspace{15em} (DS05)(\text{Definition of the Upper Bound of Natural Numbers})(Eqn1) \\
& N_N = \mathcal{N}E \hspace{15em} (TR740) \\
& N_N^1 = (\mathcal{N}E)^1 \hspace{15em} (TM479)(Eqn2) \\
& C(\mathcal{N}E)^1 = \emptyset \hspace{15em} (AMX19)(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$\mathcal{N}a^1 \subseteq N_N \hspace{15em} (T500)$$

Proof of (T₇₅₀₀)

$$\begin{aligned} \mathcal{N}\mathbf{E} &= \mathbf{N}_N & (\mathbf{T}_{R740})(\text{Eqn1}) \\ \mathbf{a}^1 &\subseteq \mathbf{E} & (\mathbf{T}_{M744})(\text{Universal Set Theorem}) \\ \mathcal{N}\mathbf{a}^1 &\subseteq \mathcal{N}\mathbf{E} & (\mathbf{T}_{R196})(\text{Relational Deduction})(\text{Eqn2}) \\ \mathcal{N}\mathbf{a}^1 &\subseteq \mathbf{N}_N & (\mathbf{A}_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \end{aligned}$$

$$\{0\}^1 \subseteq \mathbf{N}_N \quad (\mathbf{T}_{7501}) \text{ First Peano Axiom}$$

Proof of (T₇₅₀₁)

$$\begin{aligned} \text{Let } \mathbf{a} &= \emptyset \\ \mathcal{N}\mathbf{a}^1 &= \{0\}^1 & (\mathbf{D}_{R23})(\text{Definition of Zero})(\text{Eqn1}) \\ \mathcal{N}\mathbf{a}^1 &\subseteq \mathbf{N}_N & (\mathbf{T}_{7500})(\text{Eqn2}) \\ \{0\}^1 &\subseteq \mathbf{N}_N & (\mathbf{A}_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \end{aligned}$$

$${}^v\mathcal{N}^*\{0\}^1 = \emptyset \quad (\mathbf{T}_{7452})$$

Proof of (T₇₄₅₂)

$$\begin{aligned} ((\mathbf{c}^1 \subseteq \mathbf{R}_0)(\mathbf{a} \subseteq \mathbf{S}\mathbf{c}) = \mathbf{E}) &\subseteq ((\mathbf{b} = {}^v\mathcal{N}^*\mathbf{a}) = (\mathbf{a} = \mathbf{c}^*\mathbf{b})) & (\mathbf{T}_{S522})(\text{Eqn1}) \\ \text{Let } \mathbf{c} &= \mathcal{N} & (\text{Eqn2}) \\ \text{Let } \mathbf{a} &= \{0\}^1 & (\text{Eqn3}) \\ \text{Let } \mathbf{b} &= \mathbf{a}^1 & (\text{Eqn4}) \\ ((\mathcal{N}^1 \subseteq \mathbf{R}_0)(\{0\}^1 \subseteq \mathbf{S}\mathcal{N}) = \mathbf{E}) &\subseteq ((\mathbf{a}^1 = {}^v\mathcal{N}^*\{0\}^1) = (\{0\}^1 = \mathcal{N}^*\mathbf{a}^1)) & (\mathbf{A}_{MX19})(\text{Substitution of Eqns2,3\&4 into Eqn1})(\text{Eqn5}) \\ \mathcal{N}^1 &\subseteq \mathbf{R}_0 & (\mathbf{T}_{S719}\mathcal{N}) \\ (\mathcal{N}^1 \subseteq \mathbf{R}_0) &= \mathbf{E} & (\mathbf{A}_{MX16})(\text{Order Axiom})(\text{Eqn6}) \\ (\mathbf{E}(\{0\}^1 \subseteq \mathbf{S}\mathcal{N}) = \mathbf{E}) &\subseteq ((\mathbf{a}^1 = {}^v\mathcal{N}^*\{0\}^1) = (\{0\}^1 = \mathcal{N}^*\mathbf{a}^1)) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn6 into Eqn5}) \\ ((\{0\}^1 \subseteq \mathbf{S}\mathcal{N}) = \mathbf{E}) &\subseteq ((\mathbf{a}^1 = {}^v\mathcal{N}^*\{0\}^1) = (\{0\}^1 = \mathcal{N}^*\mathbf{a}^1)) & (\mathbf{A}_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn7}) \\ \mathbf{S}\mathcal{N} &= \mathbf{N}_N & (\mathbf{D}_{R70})(\text{Definition of the Quantity Scale})(\text{Eqn8}) \\ ((\{0\}^1 \subseteq \mathbf{N}_N) = \mathbf{E}) &\subseteq ((\mathbf{a}^1 = {}^v\mathcal{N}^*\{0\}^1) = (\{0\}^1 = \mathcal{N}^*\mathbf{a}^1)) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn8 into Eqn7})(\text{Eqn9}) \\ \{0\}^1 &\subseteq \mathbf{N}_N & (\mathbf{T}_{7501})(\text{First Peano Axiom}) \\ (\{0\}^1 \subseteq \mathbf{N}_N) &= \mathbf{E} & (\mathbf{A}_{MX16})(\text{Order Axiom})(\text{Eqn10}) \\ (\mathbf{E} = \mathbf{E}) &\subseteq ((\mathbf{a}^1 = {}^v\mathcal{N}^*\{0\}^1) = (\{0\}^1 = \mathcal{N}^*\mathbf{a}^1)) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn10 into Eqn9}) \\ \mathbf{E} &\subseteq ((\mathbf{a}^1 = {}^v\mathcal{N}^*\{0\}^1) = (\{0\}^1 = \mathcal{N}^*\mathbf{a}^1)) & (\mathbf{T}_{M342}) \\ \mathbf{E} &= ((\mathbf{a}^1 = {}^v\mathcal{N}^*\{0\}^1) = (\{0\}^1 = \mathcal{N}^*\mathbf{a}^1)) & (\mathbf{T}_{M275})(\text{Supersets of the Universal Set}) \\ (\mathbf{a}^1 = {}^v\mathcal{N}^*\{0\}^1) &= (\{0\}^1 = \mathcal{N}^*\mathbf{a}^1) & (\mathbf{A}_{MX16})(\text{Order Axiom})(\text{Eqn11}) \\ (\mathbf{a} = \emptyset) &= (\mathcal{N}\mathbf{a}^1 = \{0\}^1) & (\mathbf{D}_{R23})(\text{Definition of Zero})(\text{Eqn12}) \\ (\mathbf{a}^1 = {}^v\mathcal{N}^*\{0\}^1) &= (\mathbf{a} = \emptyset) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn12 into Eqn11})(\text{Eqn13}) \\ (\mathbf{a}^1 = \emptyset) &\subseteq (\mathbf{a} = \emptyset) & (\mathbf{A}_{MX50})(\text{Components of a Null Entity})(\text{Eqn14}) \\ (\mathbf{a}^1 = \emptyset) &\subseteq (\mathbf{a}^1 = {}^v\mathcal{N}^*\{0\}^1) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn13 into Eqn14}) \\ \emptyset &= {}^v\mathcal{N}^*\{0\}^1 & (\mathbf{A}_{MX19})(\text{Substitution}) \end{aligned}$$

$$\mathbf{N}_N \subseteq \mathbf{N}_C \quad (\mathbf{T}_{7544})$$

$$\mathbf{N}_Z \subseteq \mathbf{N}_C \quad (\mathbf{T}_{7547})$$

$$\mathbf{N}_Q \subseteq \mathbf{N}_C \quad (\mathbf{T}_{7548})$$

$$\mathbf{N}_A \subseteq \mathbf{N}_C \quad (\mathbf{T}_{7549})$$

$$\mathbf{N}_R \subseteq \mathbf{N}_C \quad (\mathbf{T}_{7550})$$

$$\mathbf{N}_I \subseteq \mathbf{N}_C \quad (\mathbf{T}_{7551})$$

Example Proof of (T₇₅₄₄)

$$\begin{aligned} \mathbf{N}_N + \mathbf{N}_Z + \mathbf{N}_Q + \mathbf{N}_A &\subseteq \mathbf{N}_R & (\mathbf{D}_{R22})(\text{Definition of Natural, Integer, Rational and Algebraic Numbers}) \\ \mathbf{N}_N &\subseteq \mathbf{N}_R & \text{Extraction (Eqn1)} \end{aligned}$$

$\mathbf{N_R} + \mathbf{N_I} \subseteq \mathbf{N_C}$	($\mathbf{D_{R19}}$)(Definition of Real and Imaginary Numbers)
$\mathbf{N_R} \subseteq \mathbf{N_C}$	Extraction (Eqn2)
$\mathbf{N_N} \subseteq \mathbf{N_C}$	($\mathbf{T_{M139}}$)(Hypothetical Syllogism Eqns1&2)

$$\mathbf{N_R^1} \subseteq \mathbf{D_P} \quad (\mathbf{T_{T583}})$$

$$\mathbf{N_I^1} \subseteq \mathbf{D_P} \quad (\mathbf{T_{T584}})$$

Example Proof of ($\mathbf{T_{T583}}$)

$$(\mathbf{X} = \mathbf{N_R}, \mathbf{N_I}) \subseteq (\mathbf{X^1} \subseteq \mathbf{D_P}) \quad (\mathbf{A_{S38}})(\text{Axiom of Numeric Dimensions})$$

$$\mathbf{N_R^1} \subseteq \mathbf{D_P} \quad \text{Extraction}$$

$$\mathbf{N_R^1} \subseteq \mathbf{S_D S_O} \quad (\mathbf{T_{T590}})$$

Proof of ($\mathbf{T_{T590}}$)

$$\mathbf{N_R^1} \subseteq \mathbf{D_P} \quad (\mathbf{T_{T583}})(\text{Eqn1})$$

$$\mathbf{D_P} \subseteq \mathbf{D_D} \quad (\mathbf{A_{R94}})(\text{Point Dimensions are Discrete})(\text{Eqn2})$$

$$\mathbf{N_R^1} \subseteq \mathbf{D_D} \quad (\mathbf{T_{M139}})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3})$$

$$((\mathbf{X_E})^1 \subseteq \mathbf{D_D}) \subseteq ((\mathbf{X_E})^1 \subseteq \mathbf{S_D S_O}) \quad (\mathbf{T_{S723}})$$

$$(\mathbf{D_D} \subseteq \mathbf{S_D S_O}) = \mathbf{E} \quad (\mathbf{T_{M751}})(\text{Entity Mode / Set Mode Rule}) \text{ for } (\mathbf{X_E})^1 \neq \emptyset$$

$$\mathbf{D_D} \subseteq \mathbf{S_D S_O} \quad (\mathbf{A_{MX16}})(\text{Order Axiom})(\text{Eqn4})$$

$$\mathbf{N_R^1} \subseteq \mathbf{S_D S_O} \quad (\mathbf{T_{M139}})(\text{Hypothetical Syllogism Eqns3\&4})$$

$$\mathbf{N_R} \neq \mathbf{N_R^1} \quad (\mathbf{T_{R765}})$$

$$\mathbf{N_R} \neq \emptyset \quad (\mathbf{T_{R766}})$$

Proof of ($\mathbf{T_{R765}}$) and ($\mathbf{T_{R766}}$)

$$(\mathbf{a^1} \subseteq \mathbf{S_O}) \subseteq (\mathbf{a \neq a^1})(\mathbf{a \neq \emptyset}) \quad (\mathbf{T_{S641}})(\text{Eqn1})$$

$$\text{Let } \mathbf{a} = \mathbf{N_R} \quad (\text{Eqn2})$$

$$(\mathbf{N_R^1} \subseteq \mathbf{S_O}) \subseteq (\mathbf{N_R} \neq \mathbf{N_R^1})(\mathbf{N_R} \neq \emptyset) \quad (\mathbf{A_{MX19}})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3})$$

$$\mathbf{N_R^1} \subseteq \mathbf{S_D S_O} \quad (\mathbf{T_{T590}})$$

$$\mathbf{N_R^1} \subseteq \mathbf{S_O} \quad (\mathbf{T_{M073}})(\text{Single Premise Syllogism})$$

$$(\mathbf{N_R^1} \subseteq \mathbf{S_O}) = \mathbf{E} \quad (\mathbf{A_{MX16}})(\text{Order Axiom})(\text{Eqn4})$$

$$\mathbf{E} \subseteq (\mathbf{N_R} \neq \mathbf{N_R^1})(\mathbf{N_R} \neq \emptyset) \quad (\mathbf{A_{MX19}})(\text{Substitution of Eqn4 into Eqn3})$$

$$\mathbf{E} = (\mathbf{N_R} \neq \mathbf{N_R^1})(\mathbf{N_R} \neq \emptyset) \quad (\mathbf{T_{M275}})(\text{Supersets of the Universal Set})$$

$$(\mathbf{N_R} \neq \mathbf{N_R^1})(\mathbf{N_R} \neq \emptyset) \quad (\mathbf{A_{MX16}})(\text{Order Axiom})$$

$$\mathcal{N}\mathbf{a^1} \subset \mathcal{N}\mathbf{E} \quad (\mathbf{T_{T403}})$$

Proof of ($\mathbf{T_{T403}}$)

$$\mathbf{a^1} \subset \mathbf{E} \quad (\mathbf{A_{MX37}})(\text{Plurality Axiom})(\text{Eqn1})$$

$$\mathbf{E} = \mathcal{Q}\mathcal{N} \quad (\mathbf{T_{S721}})(\mathcal{N})(\text{Eqn2})$$

$$\mathbf{E} \subseteq \mathcal{Q}\mathcal{N} \quad (\mathbf{T_{M425}})$$

$$\mathbf{a^1} \subset \mathcal{Q}\mathcal{N} \quad (\mathbf{A_{MX19}})(\text{Substitution of Eqn2 into Eqn1})$$

$$\mathbf{a^1} \subseteq \mathcal{Q}\mathcal{N} \quad (\mathbf{T_{M592}})$$

$$\mathcal{N}\mathbf{a^1} \subset \mathcal{N}\mathbf{E} \quad (\mathbf{A_{R38}})(\text{2nd Relational Deduction Axiom } \mathbf{o} = \{\subset\}) \text{ since } \mathbf{a^1} \subseteq \mathcal{Q}\mathcal{N} \text{ and } \mathbf{E} \subseteq \mathcal{Q}\mathcal{N}$$

$$\mathbf{N_C} \neq \mathbf{N_C^1} \quad (\mathbf{T_{T620}})$$

Proof of ($\mathbf{T_{T620}}$)

$$\mathcal{N}\mathbf{E^1} \subset \mathcal{N}\mathbf{E} \quad (\mathbf{T_{T403}})$$

$$\mathcal{N}\mathbf{E^1} \subset \mathbf{N_N} \quad (\mathbf{T_{R740}})(\text{Eqn1})$$

$$\mathbf{N_N} \subseteq \mathbf{N_C} \quad (\mathbf{T_{T544}})(\text{Eqn2})$$

$$\mathcal{N}\mathbf{E^1} \subset \mathbf{N_C} \quad (\mathbf{T_{M401}})(\text{Eqns1\&2})(\text{Eqn3})$$

$$\mathcal{N}\mathbf{E^1} \neq \emptyset \quad (\mathbf{A_{R18}})$$

$\mathcal{N}\mathcal{E}^1 = (\mathcal{N}\mathcal{E}^1)^1$	($\mathcal{T}_{155}\mathcal{N}$)(Eqn4)
$(\mathcal{N}\mathcal{E}^1)^1 \subset \mathbf{N}_c$	($\mathcal{A}_{MX19}$)(Substitution of Eqn4 into Eqn3)
$((\mathcal{N}\mathcal{E}^1)^1 \subset \mathbf{N}_c) = \mathbf{E}$	($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn5)
$\mathcal{N}\mathcal{E}^1 \neq \emptyset$	($\mathcal{A}_{R18}$)(Eqn6)
$(\mathcal{N}\mathcal{E}^1)^1 \neq \emptyset$	($\mathcal{A}_{MX19}$)(Substitution of Eqn4 into Eqn6)
$((\mathcal{N}\mathcal{E}^1)^1 \neq \emptyset) = \mathbf{E}$	($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn7)
$((\mathcal{N}\mathcal{E}^1)^1 \neq \emptyset)((\mathcal{N}\mathcal{E}^1)^1 \subset \mathbf{N}_c) \subseteq (\mathbf{N}_c \neq \mathbf{N}_c^1)$	($\mathcal{D}_{M19}$)(Definition of a Plural Set)(Eqn8)
$\mathbf{E}\mathbf{E} \subseteq (\mathbf{N}_c \neq \mathbf{N}_c^1)$	($\mathcal{A}_{MX19}$)(Substitution of Eqns5&7 into Eqn8)
$\mathbf{E} \subseteq (\mathbf{N}_c \neq \mathbf{N}_c^1)$	($\mathcal{T}_{M667}$)(Conjunction Tautology)
$\mathbf{E} = (\mathbf{N}_c \neq \mathbf{N}_c^1)$	($\mathcal{T}_{M275}$)(Supersets of the Universal Set)
$\mathbf{N}_c \neq \mathbf{N}_c^1$	($\mathcal{A}_{MX16}$)(Order Axiom)

$$(\mathbf{X} = \{-\})(\mathbf{Y} = \{+\}), (\mathbf{X} = \{\div\})(\mathbf{Y} = \{x\}), (\mathbf{X} = \{\sqrt{\cdot}\})(\mathbf{Y} = \{\wedge\}): (\forall(\mathbf{X}*\{\mathbf{a}\}^1) = \mathbf{Y}*\{\mathbf{a}\}^1) \quad (\mathcal{T}_{5582})$$

Proof of ($\mathcal{T}_{5582}$)

$(\mathbf{X} = \{+\})(\mathbf{Y} = \{-}), (\mathbf{X} = \{x\})(\mathbf{Y} = \{\div\}), (\mathbf{X} = \{\wedge\})(\mathbf{Y} = \{\sqrt{\cdot}\}): (\forall(\mathbf{X}*\{\mathbf{a}\}^1) = \mathbf{Y}*\{\mathbf{a}\}^1)$	($\mathcal{A}_{R19}$)(Axiom of Reverse Arithmetic Operations)
$(\mathbf{X} = \{+\})(\mathbf{Y} = \{-}), (\mathbf{X} = \{x\})(\mathbf{Y} = \{\div\}), (\mathbf{X} = \{\wedge\})(\mathbf{Y} = \{\sqrt{\cdot}\}): (\forall(\mathbf{X}*\{\mathbf{a}\}^1) = \forall(\mathbf{Y}*\{\mathbf{a}\}^1))$	($\mathcal{T}_{R522}$)(relational Deduction)
$(\mathbf{X} = \{+\})(\mathbf{Y} = \{-}), (\mathbf{X} = \{x\})(\mathbf{Y} = \{\div\}), (\mathbf{X} = \{\wedge\})(\mathbf{Y} = \{\sqrt{\cdot}\}): (\mathbf{X}*\{\mathbf{a}\}^1 = \forall(\mathbf{Y}*\{\mathbf{a}\}^1))$	($\mathcal{T}_{R012k}$)(Sub-Relationship Closure Theorem)
$(\mathbf{Y} = \{+\})(\mathbf{X} = \{-}), (\mathbf{Y} = \{x\})(\mathbf{X} = \{\div\}), (\mathbf{Y} = \{\wedge\})(\mathbf{X} = \{\sqrt{\cdot}\}): (\mathbf{Y}*\{\mathbf{a}\}^1 = \forall(\mathbf{X}*\{\mathbf{a}\}^1))$	Resymbolising
$(\mathbf{Y} = \{+\})(\mathbf{X} = \{-}), (\mathbf{Y} = \{x\})(\mathbf{X} = \{\div\}), (\mathbf{Y} = \{\wedge\})(\mathbf{X} = \{\sqrt{\cdot}\}): (\forall(\mathbf{X}*\{\mathbf{a}\}^1) = \mathbf{Y}*\{\mathbf{a}\}^1)$	($\mathcal{T}_{M197}$)(Commutativity of =)
$(\mathbf{X} = \{-})(\mathbf{Y} = \{+\}), (\mathbf{X} = \{\div\})(\mathbf{Y} = \{x\}), (\mathbf{X} = \{\sqrt{\cdot}\})(\mathbf{Y} = \{\wedge\}): (\forall(\mathbf{X}*\{\mathbf{a}\}^1) = \mathbf{Y}*\{\mathbf{a}\}^1)$	($\mathcal{T}_{M404}$)(Commutative .)

$$(\{\mathbf{a}_1\}^{1+++} \{\mathbf{a}_n\}^1 = \mathbf{N}_c) \subseteq (\{\mathbf{a}_1\}^1 \neq \{0\}^1) +++ (\{\mathbf{a}_n\}^1 \neq \{0\}^1) \quad (\mathcal{T}_{542})$$

Proof of ($\mathcal{T}_{542}$)

Let $\{0\}^1 = \mathbf{N}_c$	
$\mathbf{N}_c = \mathbf{N}_c^1$	($\mathcal{T}_{M782}$)
$\emptyset$	($\mathcal{T}_{M024}$)(Non-existence of anti-($\mathcal{T}_{1620}$))
Thus $(\{0\}^1 = \mathbf{N}_c) = \emptyset$	($\mathcal{A}_{MX02}$)(Conditional Introduction)
$(\mathbf{N}_c \neq \{0\}^1) = \sim \emptyset$	($\mathcal{T}_{M008}$)(Inverse of an Equality)
$(\mathbf{N}_c \neq \{0\}^1) = \mathbf{E}$	($\mathcal{T}_{M054}$)(Negative of Null Set)
$\mathbf{N}_c \neq \{0\}^1$	($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn1)
Let $\{\mathbf{a}_1\}^{1+++} \{\mathbf{a}_n\}^1 = \mathbf{N}_c$	(Eqn2)
Where $(\{\mathbf{a}_1\}^1 \neq \emptyset) \dots (\{\mathbf{a}_n\}^1 \neq \emptyset)$	
$(\{\mathbf{a}_1\}^{1+++} \{\mathbf{a}_n\}^1 \neq \{0\}^1)$	($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)
Thus $(\{\mathbf{a}_1\}^{1+++} \{\mathbf{a}_n\}^1 = \mathbf{N}_c) \subseteq (\{\mathbf{a}_1\}^{1+++} \{\mathbf{a}_n\}^1 \neq \{0\}^1)$	($\mathcal{A}_{MX02}$)(Conditional Introduction)
$(\{\mathbf{a}_1\}^{1+++} \{\mathbf{a}_n\}^1 = \mathbf{N}_c) \subseteq \sim(\{\mathbf{a}_1\}^{1+++} \{\mathbf{a}_n\}^1 = \{0\}^1)$	($\mathcal{T}_{M260}$)(Definition of $\neq$)
$(\{\mathbf{a}_1\}^{1+++} \{\mathbf{a}_n\}^1 = \mathbf{N}_c) \subseteq \sim(\{\mathbf{a}_1\}^1 = \{0\}^1) \dots (\{\mathbf{a}_n\}^1 = \{0\}^1)$	($\mathcal{T}_{M589}$)
$(\{\mathbf{a}_1\}^{1+++} \{\mathbf{a}_n\}^1 = \mathbf{N}_c) \subseteq (\{\mathbf{a}_1\}^1 \neq \{0\}^1) +++ (\{\mathbf{a}_n\}^1 \neq \{0\}^1)$	($\mathcal{T}_{M025}$)(DeMorgan's Law)

$$(\mathbf{X} = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\sqrt{\cdot}\}) \subseteq ((\mathbf{X}\mathbf{N}_c)\mathbf{N}_c = \mathbf{N}_c) \quad (\mathcal{T}_{521})$$

Proof of ($\mathcal{T}_{521}$)

Let $\{\mathbf{a}_1\}^{1+++} \{\mathbf{a}_n\}^1 = \mathbf{N}_c$	(Eqn1)
$(\{\mathbf{a}_1\}^{1+++} \{\mathbf{a}_n\}^1 = \mathbf{N}_c) = \mathbf{E}$	($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn2)
$(\{\mathbf{a}_1\}^{1+++} \{\mathbf{a}_n\}^1 = \mathbf{N}_c) \subseteq (\{\mathbf{a}_1\}^1 \neq \{0\}^1) +++ (\{\mathbf{a}_n\}^1 \neq \{0\}^1)$	($\mathcal{T}_{542}$)(Eqn3)
$\mathbf{E} \subseteq (\{\mathbf{a}_1\}^1 \neq \{0\}^1) +++ (\{\mathbf{a}_n\}^1 \neq \{0\}^1)$	($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn3)

$$E = (\{a_1\}^1 \neq \{0\}^1) \text{+++} (\{a_n\}^1 \neq \{0\}^1)$$

(T_{M275})(Supersets of the Universal Set)(Eqn4)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\})(\{a_1\}^1 \neq \{0\}^1) \text{+++} (X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\})(\{a_n\}^1 \neq \{0\}^1) \subseteq ((X\{a_1\}^1)N_c \text{+++} (X\{a_n\}^1)N_c = N_c \text{+++} N_c)$$

(AM_{X27})(Combination of (AS_{43}) 1st Axiom of Arithmetic Operators for $\{a_1\}^1$ to $\{a_n\}^1$)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\})(\{a_1\}^1 \neq \{0\}^1) \text{+++} (\{a_n\}^1 \neq \{0\}^1) \subseteq ((X\{a_1\}^1)N_c \text{+++} (X\{a_n\}^1)N_c = N_c \text{+++} N_c)$$

(T_{M407})(Distribution of . over +)(Eqn5)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\})E \subseteq ((X\{a_1\}^1)N_c \text{+++} (X\{a_n\}^1)N_c = N_c \text{+++} N_c)$$

(AM_{X19})(Substitution of Eqn4 into Eqn5)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((X\{a_1\}^1)N_c \text{+++} (X\{a_n\}^1)N_c = N_c \text{+++} N_c)$$

(AM_{X17})(Universal Set Conjunction Axiom)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((X\{a_1\}^1)N_c \text{+++} (X\{a_n\}^1)N_c = N_c)$$

(AM_{X07})(Disjunction Tautology)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (((X\{a_1\}^1) \text{+++} (X\{a_n\}^1))N_c = N_c)$$

(T_{R002c})(Distribution of + over *)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((X(\{a_1\}^1 \text{+++} \{a_n\}^1))N_c = N_c)$$

(T_{R002b})(Distribution of + over *) (Eqn6)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((XN_c)N_c = N_c)$$

(AM_{X19})(Substitution of Eqn1 into Eqn6)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\forall XE = N_c) \quad (T_{T520})$$

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (X * \sim N_c = \emptyset) \quad (T_{T522})$$

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((XE) * \sim N_c = \emptyset) \quad (T_{T523})$$

Example Proof of (T_{T520})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\forall XE = N_c). (X * \sim N_c = \emptyset). ((XE) * \sim N_c = \emptyset)$$

(AS_{39}) 2nd Axiom of Arithmetic Operators)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\forall XE = N_c)$$

(T_{M073})(Single Premise Syllogism)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((XN_c) * \sim N_c = \emptyset) \quad (T_{S574})$$

Proof of (T_{S574})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((XE) * \sim N_c = \emptyset) \quad (T_{T523})(Eqn1)$$

$$N_c \subseteq E \quad (T_{M744})(Universal Set Theorem)$$

$$XN_c \subseteq XE \quad (T_{R196})(Relational Deduction)$$

$$(XN_c) * \sim N_c \subseteq (XE) * \sim N_c \quad (T_{R197})(Relational Deduction)$$

$$((XN_c) * \sim N_c \subseteq (XE) * \sim N_c) = E \quad (AM_{X16})(Order Axiom)$$

$$((XE) * \sim N_c = \emptyset) \subseteq ((XN_c) * \sim N_c = \emptyset) \quad (T_{M097})(Probabilistic Rule)(Eqn2)$$

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((XN_c) * \sim N_c = \emptyset) \quad (T_{M139})(Hypothetical Syllogism Eqns1\&2)$$

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((X\{a\}^1) * \sim N_c = \emptyset) \quad (T_{S573})$$

Proof of (T_{S574})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((XN_c) * \sim N_c = \emptyset) \quad (T_{S574})(Eqn1)$$

$$\{a\}^1 \subseteq N_c \quad (D_{R18})(Definition of Numbers)$$

$$X\{a\}^1 \subseteq XN_c \quad (T_{R196})(Relational Deduction)$$

$$(X\{a\}^1) * \sim N_c \subseteq (XN_c) * \sim N_c \quad (T_{R197})(Relational Deduction)$$

$$((X\{a\}^1) * \sim N_c \subseteq (XN_c) * \sim N_c) = E \quad (AM_{X16})(Order Axiom)$$

$$((XN_c) * \sim N_c = \emptyset) \subseteq ((X\{a\}^1) * \sim N_c = \emptyset) \quad (T_{M097})(Probabilistic Rule)(Eqn2)$$

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((X\{a\}^1) * \sim N_c = \emptyset) \quad (T_{M139})(Hypothetical Syllogism Eqns1\&2)$$

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((XE)E = N_c) \quad (T_{S587})$$

Proof of (T_{S587})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((XN_c)N_c = N_c) \quad (T_{T521})$$

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((XN_c + \emptyset)N_c = N_c) \quad (T_{M668})(Null Set Disjunction)$$

$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\}\}\} \subseteq ((XN_c + X \sim N_c)N_c = N_c)$	(T_{522})
$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\}\} \subseteq ((X(N_c + \sim N_c))N_c = N_c)$	$(T_{R002b})(\text{Distribution of } * \text{ over } +)$
$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\}\} \subseteq ((X E)N_c = N_c)$	$(T_{M671})(\text{Tertium non Datur})$
$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\}\} \subseteq ((X E)N_c + \emptyset = N_c)$	$(T_{M668})(\text{Null Set Disjunction})$
$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\}\} \subseteq ((X E)N_c + (X E) \sim N_c = N_c)$	(T_{523})
$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\}\} \subseteq ((X E)(N_c + \sim N_c) = N_c)$	$(T_{R002b})(\text{Distribution of } * \text{ over } +)$
$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\}\} \subseteq ((X E)E = N_c)$	$(T_{M671})(\text{Tertium non Datur})$

$(\{-\} * \{a\}^1) = \vee(\{+\} * \{a\}^1)$	(T_{248})
$(\{+\} * \{a\}^1) = \vee(\{-\} * \{a\}^1)$	(T_{246})
$(\{\div\} * \{a\}^1) = \vee(\{x\} * \{a\}^1)$	(T_{244})
$(\{x\} * \{a\}^1) = \vee(\{\div\} * \{a\}^1)$	(T_{242})
$(\{\vee\} * \{a\}^1) = \vee(\{\wedge\} * \{a\}^1)$	(T_{235})
$(\{\wedge\} * \{a\}^1) = \vee(\{\vee\} * \{a\}^1)$	(T_{234})

Example Proof of (T_{248}) & (T_{246})

$(X = \{+\})(Y = \{-}), (X = \{x\})(Y = \{\div\}), (X = \{\wedge\})(Y = \{\vee\}): (\vee(X * \{a\}^1) = Y * \{a\}^1)$	$(A_{R19})(\text{Axiom of Reverse Arithmetic Operations})$
$\vee(\{+\} * \{a\}^1) = (\{-\} * \{a\}^1)$	Extraction
$\vee\vee(\{+\} * \{a\}^1) = \vee(\{-\} * \{a\}^1)$	$(T_{522})(\text{Relational Deduction})$
$(\{+\} * \{a\}^1) = \vee(\{-\} * \{a\}^1)$	$(T_{R012k})(\text{Sub-Relationship Closure Theorem})$

$$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\}\} \subseteq (\vee(X\{a\}^1) * \sim N_c = \emptyset) \quad (T_{577})$$

Proof of (T_{577})

$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\}\} \subseteq ((X E) * \sim N_c = \emptyset)$	$(T_{523})(\text{Eqn1})$
$\{a\}^1 \subseteq E$	$(T_{M744})(\text{Universal Set Theorem})$
$X\{a\}^1 \subseteq X E$	$(T_{R196})(\text{Relational Deduction})$
$(X\{a\}^1) * \sim N_c \subseteq (X E * \sim N_c)$	$(T_{R197})(\text{Relational Deduction})$
$((X\{a\}^1) * \sim N_c \subseteq (X E * \sim N_c)) = E$	$(A_{MX16})(\text{Order Axiom})$
$((X E * \sim N_c) = \emptyset) \subseteq ((X\{a\}^1) * \sim N_c = \emptyset)$	$(T_{M097})(\text{Probabilistic Rule})(\text{Eqn2})$
$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\}\} \subseteq ((X\{a\}^1) * \sim N_c = \emptyset)$	$(T_{M139})(\text{Hypothetical Syllogism Eqns1\&2})(\text{Eqn3})$
$(\{-\} * \{a\}^1) * \sim N_c = \emptyset$	Extraction from Eqn3 (Eqn4)
$(\{-\} * \{a\}^1) = \vee(\{+\} * \{a\}^1)$	$(T_{248})(\text{Eqn5})$
$\vee(\{+\} * \{a\}^1) * \sim N_c = \emptyset$	$(A_{MX19})(\text{Substitution of Eqn5 into Eqn4})(\text{Eqn6})$
$(\{+\} * \{a\}^1) * \sim N_c = \emptyset$	Extraction from Eqn3 (Eqn7)
$(\{+\} * \{a\}^1) = \vee(\{-\} * \{a\}^1)$	$(T_{246})(\text{Eqn8})$
$\vee(\{-\} * \{a\}^1) * \sim N_c = \emptyset$	$(A_{MX19})(\text{Substitution of Eqn8 into Eqn7})(\text{Eqn9})$
$(\{\div\} * \{a\}^1) * \sim N_c = \emptyset$	Extraction from Eqn3 (Eqn10)
$(\{\div\} * \{a\}^1) = \vee(\{x\} * \{a\}^1)$	$(T_{244})(\text{Eqn11})$
$\vee(\{x\} * \{a\}^1) * \sim N_c = \emptyset$	$(A_{MX19})(\text{Substitution of Eqn11 into Eqn10})(\text{Eqn12})$
$(\{x\} * \{a\}^1) * \sim N_c = \emptyset$	Extraction from Eqn3 (Eqn13)
$(\{x\} * \{a\}^1) = \vee(\{\div\} * \{a\}^1)$	$(T_{242})(\text{Eqn14})$
$\vee(\{\div\} * \{a\}^1) * \sim N_c = \emptyset$	$(A_{MX19})(\text{Substitution of Eqn14 into Eqn13})(\text{Eqn15})$
$(\{\vee\} * \{a\}^1) * \sim N_c = \emptyset$	Extraction from Eqn3 (Eqn16)
$(\{\vee\} * \{a\}^1) = \vee(\{\wedge\} * \{a\}^1)$	$(T_{235})(\text{Eqn17})$
$\vee(\{\wedge\} * \{a\}^1) * \sim N_c = \emptyset$	$(A_{MX19})(\text{Substitution of Eqn17 into Eqn16})(\text{Eqn18})$
$(\{\wedge\} * \{a\}^1) * \sim N_c = \emptyset$	Extraction from Eqn3 (Eqn19)
$(\{\wedge\} * \{a\}^1) = \vee(\{\vee\} * \{a\}^1)$	$(T_{234})(\text{Eqn20})$
$\vee(\{\vee\} * \{a\}^1) * \sim N_c = \emptyset$	$(A_{MX19})(\text{Substitution of Eqn20 into Eqn19})(\text{Eqn21})$
$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\}\} \subseteq (\vee(X\{a\}^1) * \sim N_c = \emptyset)$	Reverse Extraction Eqns6,9,12,15,18\&21

$$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\}\} \subseteq (\vee(XN_c) * \sim N_c = \emptyset) \quad (T_{575})$$

Proof of (T_{575})

$$\begin{aligned}
&\text{Let } X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\} \\
&\text{Let } \{a_1\}^1 \text{ +++ } \{a_E\}^1 = N_c \quad (\text{Eqn1}) \\
&X(\{a_1\}^1 \text{ +++ } \{a_E\}^1) = XN_c \quad (\mathcal{T}_{R196})(\text{Relational Deduction}) \\
&\vee(X(\{a_1\}^1 \text{ +++ } \{a_E\}^1)) = \vee(XN_c) \quad (\mathcal{T}_{R196})(\text{Relational Deduction}) \\
&\vee(X(\{a_1\}^1 \text{ +++ } \{a_E\}^1)) * \sim N_c = \vee(XN_c) * \sim N_c \quad (\mathcal{T}_{R197})(\text{Relational Deduction}) \\
&\vee(X\{a_1\}^1 \text{ +++ } X\{a_E\}^1) * \sim N_c = \vee(XN_c) * \sim N_c \quad (\mathcal{T}_{R002b})(\text{Distribution of } * \text{ over } +) \\
&(\vee(X\{a_1\}^1) \text{ +++ } \vee(X\{a_E\}^1)) * \sim N_c = \vee(XN_c) * \sim N_c \quad (\mathcal{T}_{R002b})(\text{Distribution of } * \text{ over } +)(\text{Eqn3}) \\
&\vee(X\{a_1\}^1) * \sim N_c \text{ +++ } \vee(X\{a_E\}^1) * \sim N_c = \vee(XE) * \sim N_c \quad (\mathcal{A}_{S65})(\text{Distribution of } + \text{ over } *) \\
&\emptyset \text{ +++ } \emptyset = \vee(XN_c) * \sim N_c \quad (\mathcal{T}_{S577}) \\
&\emptyset = \vee(XN_c) * \sim N_c \quad (\mathcal{A}_{MX01})(\text{Disjunction Tautology}) \\
&\text{Thus } (X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XN_c) * \sim N_c = \emptyset) \quad (\mathcal{A}_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XE) * \sim N_c = \emptyset) \quad (\mathcal{T}_{S578})$$

Proof of ($\mathcal{T}_{S578}$)

$$\begin{aligned}
(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) &\subseteq (\vee(XN_c) * \sim N_c = \emptyset) \quad (\mathcal{T}_{S575}) \\
(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) &\subseteq (\vee(XN_c + \emptyset) * \sim N_c = \emptyset) \quad (\mathcal{T}_{M668})(\text{Null Set Disjunction}) \\
(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) &\subseteq (\vee(XN_c + X \sim N_c) * \sim N_c = \emptyset) \quad (\mathcal{T}_{T522}) \\
(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) &\subseteq (\vee(X(N_c + \sim N_c)) * \sim N_c = \emptyset) \quad (\mathcal{T}_{R002b})(\text{Distribution of } * \text{ over } +) \\
(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) &\subseteq (\vee(XE) * \sim N_c = \emptyset) \quad (\mathcal{T}_{M671})(\text{Tertium non Datur})
\end{aligned}$$

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\})(\{a\}^1 \neq \{0\}^1) \subseteq (\vee(X\{a\}^1)N_c = N_c) \quad (\mathcal{T}_{S580})$$

Proof of ($\mathcal{T}_{S580}$)

$$\begin{aligned}
(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\})(\{a\}^1 \neq \{0\}^1) &\subseteq ((X\{a\}^1)N_c = N_c) \quad (\mathcal{A}_{S43})(1^{\text{st}} \text{ Axiom of Arithmetic Operators}) \\
&\text{Extraction (Eqn1)} \\
(\{a\}^1 \neq \{0\}^1) &\subseteq (\{-\} * \{a\}^1)N_c = N_c \quad (\mathcal{T}_{T248})(\text{Eqn2}) \\
(\{-\} * \{a\}^1) &= \vee(\{+\} * \{a\}^1) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\
(\{a\}^1 \neq \{0\}^1) &\subseteq \vee(\{+\} * \{a\}^1)N_c = N_c \quad \text{Extraction (Eqn4)} \\
(\{a\}^1 \neq \{0\}^1) &\subseteq (\{+\} * \{a\}^1)N_c = N_c \quad (\mathcal{T}_{T246})(\text{Eqn5}) \\
(\{+\} * \{a\}^1) &= \vee(\{-\} * \{a\}^1) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn5 into Eqn4})(\text{Eqn6}) \\
(\{a\}^1 \neq \{0\}^1) &\subseteq \vee(\{-\} * \{a\}^1)N_c = N_c \quad \text{Extraction (Eqn7)} \\
(\{a\}^1 \neq \{0\}^1) &\subseteq (\{\div\} * \{a\}^1)N_c = N_c \quad (\mathcal{T}_{T244})(\text{Eqn8}) \\
(\{\div\} * \{a\}^1) &= \vee(\{x\} * \{a\}^1) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn8 into Eqn7})(\text{Eqn9}) \\
(\{a\}^1 \neq \{0\}^1) &\subseteq \vee(\{x\} * \{a\}^1)N_c = N_c \quad \text{Extraction (Eqn10)} \\
(\{a\}^1 \neq \{0\}^1) &\subseteq (\{x\} * \{a\}^1)N_c = N_c \quad (\mathcal{T}_{T242})(\text{Eqn11}) \\
(\{x\} * \{a\}^1) &= \vee(\{\div\} * \{a\}^1) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn11 into Eqn10})(\text{Eqn12}) \\
(\{a\}^1 \neq \{0\}^1) &\subseteq \vee(\{\div\} * \{a\}^1)N_c = N_c \quad \text{Extraction (Eqn13)} \\
(\{a\}^1 \neq \{0\}^1) &\subseteq (\{\vee\} * \{a\}^1)N_c = N_c \quad (\mathcal{T}_{T235})(\text{Eqn14}) \\
(\{\vee\} * \{a\}^1) &= \vee(\{\wedge\} * \{a\}^1) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn14 into Eqn13})(\text{Eqn15}) \\
(\{a\}^1 \neq \{0\}^1) &\subseteq \vee(\{\wedge\} * \{a\}^1)N_c = N_c \quad \text{Extraction (Eqn16)} \\
(\{a\}^1 \neq \{0\}^1) &\subseteq (\{\wedge\} * \{a\}^1)N_c = N_c \quad (\mathcal{T}_{T234})(\text{Eqn17}) \\
(\{\wedge\} * \{a\}^1) &= \vee(\{\vee\} * \{a\}^1) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn17 into Eqn16})(\text{Eqn18}) \\
(\{a\}^1 \neq \{0\}^1) &\subseteq \vee(\{\vee\} * \{a\}^1)N_c = N_c \\
(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\})(\{a\}^1 \neq \{0\}^1) &\subseteq (\vee(X\{a\}^1)N_c = N_c) \\
&\text{Reverse Extraction Eqns3,6,9,12,15\&18}
\end{aligned}$$

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XN_c)N_c = N_c) \quad (\mathcal{T}_{S579})$$

Proof of ($\mathcal{T}_{S579}$)

$$\begin{aligned}
&\text{Let } X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\} \\
&\text{Let } \{a_1\}^1 \text{ +++ } \{a_E\}^1 = N_c \quad (\text{Eqn1}) \\
&X(\{a_1\}^1 \text{ +++ } \{a_E\}^1) = XN_c \quad (\mathcal{T}_{R196})(\text{Relational Deduction}) \\
&X(\{a_1\}^1 \text{ +++ } \{a_E\}^1) * N_c = (XN_c) * N_c \quad (\mathcal{T}_{R197})(\text{Relational Deduction}) \\
&(X\{a_1\}^1 \text{ +++ } X\{a_E\}^1) * N_c = (XN_c) * N_c \quad (\mathcal{T}_{R002b})(\text{Distribution of } * \text{ over } +) \\
&X\{a_1\}^1 * N_c \text{ +++ } X\{a_E\}^1 * N_c = (XN_c) * N_c \quad (\mathcal{A}_{S65})(\text{Distribution of } + \text{ over } *) \\
&N_c \text{ +++ } N_c = (XN_c) * N_c \quad (\mathcal{T}_{S580})
\end{aligned}$$

$$N_c = (XN_c) * N_c$$

$$\text{Thus } (X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XN) * N_c = N_c)$$

(A_{MX01})(Disjunction Tautology)

(A_{MX02})(Conditional Introduction)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XE)E = N_c) \quad (T_{S584})$$

Proof of (T_{S584})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XN_c)N_c = N_c)$$

(T_{S579})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XN_c + \emptyset)N_c = N_c)$$

(T_{M668})(Null Set Disjunction)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XN_c + X \sim N_c)N_c = N_c)$$

(T_{S522})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(X(N_c + \sim N_c))N_c = N_c)$$

(T_{R002b})(Distribution of * over +)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XE)N_c = N_c)$$

(T_{M671})(Tertium non Datur)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XE)N_c + \emptyset = N_c)$$

(T_{M668})(Null Set Disjunction)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XE)N_c + \vee(XE)\sim N_c = N_c)$$

(T_{S578})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XE)(N_c + \sim N_c) = N_c)$$

(T_{R002b})(Distribution of * over +)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XE)E = N_c)$$

(T_{M671})(Tertium non Datur)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (QX = N_c).(SSX = N_c).(QSX = N_c) \quad (T_{T519})$$

Proof of (T_{T519})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee XE = N_c)$$

(T_{T520})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (QX = N_c)$$

(T_{R211})(Eqn1)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((XE)E = N_c)$$

(T_{S587})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((SX)E = N_c)$$

(T_{R210})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (SSX = N_c)$$

(T_{R210})(Eqn2)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(XE)E = N_c)$$

(T_{S584})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (\vee(SX)E = N_c)$$

(T_{R210})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (QSX = N_c)$$

(T_{R211})(Eqn3)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (QX = N_c).(SSX = N_c).(QSX = N_c)$$

(T_{M669})(Combination of Eqns1,2&3) &

(T_{M667})(Conjunction Tautology)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (QX = N_c) \quad (T_{T516})$$

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (SSX = N_c) \quad (T_{T517})$$

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (QSX = N_c) \quad (T_{T518})$$

Example Proof of (T_{T516})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (QX = N_c).(SSX = N_c).(QSX = N_c)$$

(T_{T519})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (QX = N_c)$$

(T_{M073})(Single Premise Syllogism)

$$Q\{+\} = N_c \quad (T_{R741})$$

$$SS\{+\} = N_c \quad (T_{T461})$$

$$QS\{+\} = N_c \quad (T_{T509})$$

$$Q\{-\} = N_c \quad (T_{R742})$$

$$SS\{-\} = N_c \quad (T_{T462})$$

$$QS\{-\} = N_c \quad (T_{T510})$$

$$Q\{x\} = N_c \quad (T_{R743})$$

$$SS\{x\} = N_c \quad (T_{T463})$$

$$QS\{x\} = N_c \quad (T_{T511})$$

$$Q\{\div\} = N_c \quad (T_{R744})$$

$$SS\{\div\} = N_c \quad (T_{T464})$$

$$QS\{\div\} = N_c \quad (T_{T512})$$

$$Q\{\wedge\} = N_c \quad (T_{R745})$$

$$SS\{\wedge\} = N_c \quad (T_{T465})$$

$$QS\{\wedge\} = N_c \quad (T_{T513})$$

$$Q\{\vee\} = N_c \quad (T_{R869})$$

$$SS\{\vee\} = N_c \quad (T_{T466})$$

$$QS\{\vee\} = N_c \quad (T_{T514})$$

Example Proof of (T_{R741})

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (QX = N_c).(SSX = N_c).(QSX = N_c)$$

(T_{T519})(Axiom of Arithmetic Operators)

$$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (QX = N_c)$$

(T_{M073})(Single Premise Syllogism)

$$Q\{+\} = N_c$$

(A_{MX19})(Substitution)

$$\{-\} * \{0\}^1 = \{=\} \quad (T_{R752}) \text{ Identity of Subtraction}$$

$$\{\div\} * \{1\}^1 = \{=\} \quad (T_{R753}) \text{ Identity of Division}$$

$$\{\sqrt{}\} * \{1\}^1 = \{=\} \quad (T_{R754}) \text{ Identity of Root}$$

Example Proof of (T_{R752})

$$\begin{aligned} \{+\} * \{0\}^1 &= \{=\} & (A_{R69}) & \text{(Identity of Addition)} \\ \forall (\{+\} * \{0\}^1) &= \forall \{=\} & (T_{R196}) & \text{(Relational Deduction)} \\ \forall (\{+\} * \{0\}^1) &= \{=\} & (T_{A485}) & \\ \{-\} * \{0\}^1 &= \{=\} & (T_{T248}) & \end{aligned}$$

$$\{+\} * \{0\}^1 * \{a\}^1 = \{a\}^1 \quad (T_{R755})$$

$$\{-\} * \{0\}^1 * \{a\}^1 = \{a\}^1 \quad (T_{T467})$$

$$\{x\} * \{1\}^1 * \{a\}^1 = \{a\}^1 \quad (T_{T468})$$

$$\{\div\} * \{1\}^1 * \{a\}^1 = \{a\}^1 \quad (T_{R805})$$

$$\{\wedge\} * \{1\}^1 * \{a\}^1 = \{a\}^1 \quad (T_{T469})$$

$$\{\sqrt{}\} * \{1\}^1 * \{a\}^1 = \{a\}^1 \quad (T_{T470})$$

Example Proof of (T_{R755})

$$\begin{aligned} \{a\}^1 &= \{a\}^1 & (D_{M00}) & \text{(Definition of } =) \\ \{=\} * \{a\}^1 &= \{a\}^1 & (D_{A144}) & \text{(Definition of the Reflexive Relationship Set)} \\ (\{+\} * \{0\}^1) * \{a\}^1 &= \{a\}^1 & (A_{R69}) & \text{(Identity of Addition)} \end{aligned}$$

$$(\{a\}^1 \subseteq N_N).(\{a\}^1 = \{b\}^1) \subseteq (\{b\}^1 \subseteq N_N) \quad (T_{R750}) \text{ Fifth Peano Axiom}$$

Proof of (T_{R750})

$$\begin{aligned} \text{Let } \{a\}^1 &\subseteq N_N & (\text{Eqn1}) \\ \text{Let } \{a\}^1 &= \{b\}^1 & (\text{Eqn2}) \\ \{b\}^1 &\subseteq N_N & (A_{MX19}) \text{(Substitution of Eqn2 into Eqn1)} \\ \text{Thus } (\{a\}^1 &\subseteq N_N).(\{a\}^1 = \{b\}^1) \subseteq (\{b\}^1 \subseteq N_N) & (A_{MX02}) \text{(Conditional Introduction)} \end{aligned}$$

$$(\{a\}^1 \subseteq N_C)(\{b\}^1 \subseteq N_C)(X = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\sqrt{}\}) \subseteq ((X * \{b\}^1) * \{a\}^1 \subseteq N_C) \quad (T_{T504})$$

Proof of (T_{T504})

$$\begin{aligned} \text{Let } (X &= \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\sqrt{}\}) \\ \text{Let } \{a\}^1 &\subseteq N_C \\ (\{a\}^1 &\subseteq N_C) = E & (A_{MX16}) \text{(Order Axiom)(Eqn1)} \\ \text{Let } \{b\}^1 &\subseteq N_C \\ X * \{b\}^1 &\subseteq X * N_C & (T_{R196}) \text{(Relational Deduction)} \\ (X * \{b\}^1 &\subseteq X * N_C) = E & (A_{MX16}) \text{(Order Axiom)(Eqn2)} \\ (\{a\}^1 &\subseteq N_C)(X * \{b\}^1 \subseteq X * N_C) \subseteq ((X * \{b\}^1) * \{a\}^1 \subseteq (X * N_C) * N_C) & (A_{R08}) \text{(1st Relational Deduction Axiom)(Eqn3)} \\ EE &\subseteq ((X * \{b\}^1) * \{a\}^1 \subseteq (X * N_C) * N_C) & (A_{MX19}) \text{(Substitution of Eqns1\&2 into Eqn3)} \\ E &\subseteq ((X * \{b\}^1) * \{a\}^1 \subseteq (X * N_C) * N_C) & (A_{MX11}) \text{(Universal Set Conjunction Axiom)} \\ E &= ((X * \{b\}^1) * \{a\}^1 \subseteq (X * N_C) * N_C) & (T_{M275}) \text{(Supersets of the Universal Set)} \\ (X * \{b\}^1) * \{a\}^1 &\subseteq (X * N_C) * N_C & (A_{MX16}) \text{(Order Axiom)(Eqn4)} \\ ((X * N_C) * N_C &= N_C) & (T_{T521}) \text{(Eqn5)} \\ (X * \{b\}^1) * \{a\}^1 &\subseteq N_C & (A_{MX19}) \text{(Substitution of Eqn5 into Eqn4)} \\ \text{Thus } (\{a\}^1 &\subseteq N_C)(\{b\}^1 \subseteq N_C)(X = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\sqrt{}\}) \subseteq ((X * \{b\}^1) * \{a\}^1 \subseteq N_C) & (A_{MX02}) \text{(Conditional Introduction)} \end{aligned}$$

$$((X * \{b\}^1) * \{a\}^1 \not\subseteq N_C) \subseteq (\{a\}^1 \not\subseteq N_C) + (\{b\}^1 \not\subseteq N_C) + (X \neq \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\sqrt{}\}) \quad (T_{T505})$$

Proof of (T_{T505})

$$\begin{aligned} (\{a\}^1 &\subseteq N_C)(\{b\}^1 \subseteq N_C)(X = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\sqrt{}\}) \subseteq ((X * \{b\}^1) * \{a\}^1 \subseteq N_C) & (T_{T504}) \\ \sim((X * \{b\}^1) * \{a\}^1 &\subseteq N_C) \subseteq \sim((\{a\}^1 \subseteq N_C)(\{b\}^1 \subseteq N_C)(X = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\sqrt{}\})) & (T_{M012}) \text{(Transposition)} \end{aligned}$$

$$\begin{aligned} \sim((X * \{b\}^1) * \{a\}^1 \subseteq N_c) &\subseteq \sim(\{a\}^1 \subseteq N_c) + \sim(\{b\}^1 \subseteq N_c) + \sim(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) \\ &\quad (T_{M025})(\text{DeMorgan's Law}) \\ ((X * \{b\}^1) * \{a\}^1 \not\subseteq N_c) &\subseteq (\{a\}^1 \not\subseteq N_c) + (\{b\}^1 \not\subseteq N_c) + \sim(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) \\ &\quad (T_{M259})(\text{Definition of } \not\subseteq) \\ ((X * \{b\}^1) * \{a\}^1 \not\subseteq N_c) &\subseteq (\{a\}^1 \not\subseteq N_c) + (\{b\}^1 \not\subseteq N_c) + (X \neq \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) \\ &\quad (T_{M260})(\text{Definition of } \neq) \end{aligned}$$

$$(\{a\}^1 \subseteq N_c) \subseteq (\{a\} = \{a\}^1) \quad (T_{T531})$$

Proof of (T₅₃₁)

$$\begin{aligned} (\{a\} \neq \{a\}^1) &\subseteq (\{a\}^1 \not\subseteq N_c) & (A_{S40})(\text{Axiom of Collections of Numbers}) \\ (\{a\}^1 \subseteq N_c) &\subseteq (\{a\} = \{a\}^1) & (T_{M012})(\text{Transposition}) \end{aligned}$$

$$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) (\{a\}^1 \neq \{0\}^1) \subseteq (Q(X\{a\}^1) = N_c) \quad (T_{T552})$$

Proof of (T₅₅₂)

$$\begin{aligned} (X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) (\{a\}^1 \neq \{0\}^1) &\subseteq (\vee(X\{a\}^1)N_c = N_c) & (T_{S580}) \\ (X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) (\{a\}^1 \neq \{0\}^1) &\subseteq (\vee(X\{a\}^1)N_c + \emptyset = N_c) & (T_{M668})(\text{Null Set Disjunction})(\text{Eqn1}) \\ (X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) &\subseteq (\vee(XE) * \sim N_c = \emptyset) & (T_{S578})(\text{Eqn2}) \\ \{a\}^1 \subseteq E & & (T_{M744})(\text{Universal Set Theorem}) \\ X\{a\}^1 \subseteq XE & & (T_{R196})(\text{Relational Deduction}) \\ \vee(X\{a\}^1) \subseteq \vee(XE) & & (T_{R196})(\text{Relational Deduction}) \\ \vee(X\{a\}^1) \sim N_c \subseteq \vee(XE) \sim N_c & & (T_{R197})(\text{Relational Deduction}) \\ (\vee(X\{a\}^1) \sim N_c \subseteq \vee(XE) \sim N_c) = E & & (A_{MX16})(\text{Order Axiom}) \\ (\vee(XE) \sim N_c = \emptyset) \subseteq (\vee(X\{a\}^1) \sim N_c = \emptyset) & & (T_{M097})(\text{Probabilistic Rule})(\text{Eqn3}) \\ (X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) &\subseteq (\vee(X\{a\}^1) \sim N_c = \emptyset) & (T_{M139})(\text{Hypothetical Syllogism}) \\ & & \text{Eqns2\&3}(\text{Eqn4}) \\ (X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) (\{a\}^1 \neq \{0\}^1) &\subseteq (\vee(X\{a\}^1)N_c + \vee(X\{a\}^1) \sim N_c = N_c) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn1}) \\ (X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) (\{a\}^1 \neq \{0\}^1) &\subseteq (\vee(X\{a\}^1)(N_c + \sim N_c) = N_c) & (T_{R002b})(\text{Distribution of } * \text{ over } +) \\ (X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) (\{a\}^1 \neq \{0\}^1) &\subseteq (\vee(X\{a\}^1)E = N_c) & (T_{M671})(\text{Tertium non Datur}) \\ (X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) (\{a\}^1 \neq \{0\}^1) &\subseteq (S^V(X\{a\}^1) = N_c) & (T_{R210}) \\ (X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) (\{a\}^1 \neq \{0\}^1) &\subseteq (Q(X\{a\}^1) = N_c) & (T_{R016})(\text{Reversibility of Matching Entities}) \end{aligned}$$

$$(X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) (\{b\}^1 \neq \{0\}^1) \subseteq (\{a\}^1 \subseteq Q(X * \{b\}^1)) \quad (T_{T479})$$

Proof of (T₄₇₉)

$$\begin{aligned} (X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) (\{b\}^1 \neq \{0\}^1) &\subseteq (Q(X\{b\}^1) = N_c) & (T_{T552})(\text{Eqn1}) \\ \text{Let } \{a\}^1 \subseteq N_c & & (\text{Eqn2}) \\ (X = \{+, \{-, \{x, \{\div, \{\wedge, \{\vee\}\} \}) (\{b\}^1 \neq \{0\}^1) &\subseteq (\{a\}^1 \subseteq Q(X * \{b\}^1)) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \end{aligned}$$

$$(X = \{+, \{-, \{x, \{\div\}\}) \subseteq ((\vee(X * \{b\}^1))^1 \subseteq R_o) \quad (T_{T478})$$

Proof of (T₄₇₈)

$$\begin{aligned} (X = (\{+, \{-, \{x, \{\div\}\})) &\subseteq ((SX)^1 \subseteq R_p) & (A_{R72})(\text{Axiom of Arithmetic Operation})(\text{Eqn1}) \\ ((SX)^1 \subseteq R_p) &= ((SX)^1 \subseteq R_o) \cdot ((\vee(SX))^1 \subseteq R_o) & (D_{R31})(\text{Definition of a Pairing Relationship Set}) \\ ((SX)^1 \subseteq R_p) &\subseteq ((\vee(SX))^1 \subseteq R_o) & (T_{M073})(\text{Single Premise Syllogism})(\text{Eqn2}) \\ (X = (\{+, \{-, \{x, \{\div\}\})) &\subseteq ((\vee(SX))^1 \subseteq R_o) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\ (X = (\{+, \{-, \{x, \{\div\}\})) &\subseteq ((\vee(XE))^1 \subseteq R_o) & (T_{R210})(\text{Eqn3}) \\ \{b\}^1 \subseteq E & & (T_{M744})(\text{Universal Set Theorem}) \\ X * \{b\}^1 \subseteq XE & & (T_{R196})(\text{Relational Deduction}) \\ \vee(X * \{b\}^1) \subseteq \vee(XE) & & (T_{R196})(\text{Relational Deduction}) \\ (\vee(XE) \subseteq R_o) &\subseteq (\vee(X * \{b\}^1) \subseteq R_o) & (T_{M364})(\text{Deductive Rule}) \\ ((\vee(XE))^1 \subseteq R_o) &\subseteq ((\vee(X * \{b\}^1))^1 \subseteq R_o) & (A_{S41})(\text{Axiom of Relationship Types})(\text{Eqn4}) \\ (X = (\{+, \{-, \{x, \{\div\}\})) &\subseteq ((\vee(X * \{b\}^1))^1 \subseteq R_o) & (T_{M139})(\text{Hypothetical Syllogism Eqns3\&4}) \end{aligned}$$

$$(X = (\{+, \{-, \{x, \{\div\}\})) \subseteq (((X * \{b\}^1)^1 \subseteq R_o) \quad (T_{T413})$$

Proof of (T₇₄₁₃)

$(a \subseteq b) \subseteq (Xa \subseteq Xb)$	(T _{R196})(Relational Deduction)
$((a \subseteq b) \subseteq (Xa \subseteq Xb)) = E$	(A _{MX16})(Order Axiom)
$((a \subseteq b) = E) \subseteq ((Xa \subseteq Xb) = E)$	(T _{M094})(Probabilistic Rule)(Eqn1)
$((Xa \subseteq Xb) = E) \subseteq ((Xb \subseteq Ro) \subseteq (Xa \subseteq Ro))$	(T _{M364})(Deductive Rule)(Eqn2)
$((a \subseteq b) = E) \subseteq ((Xb \subseteq Ro) \subseteq (Xa \subseteq Ro))$	(T _{M139})(Hypothetical Syllogism Eqns1&2)(Eqn3)
$(c \subseteq Rx) = (c^1 \subseteq Rx)$	(A _{S41})(Axiom of Relationship Types)(Eqn4)
Let $R_x = R_o$	(Eqn5)
$(c \subseteq Ro) = (c^1 \subseteq Ro)$	(A _{MX19})(Substitution of Eqn5 into Eqn4)
$(Xa \subseteq Ro) = ((Xa)^1 \subseteq Ro)$	Resymbolising (Eqn6)
$(Xb \subseteq Ro) = ((Xb)^1 \subseteq Ro)$	Resymbolising (Eqn7)
$((a \subseteq b) = E) \subseteq (((Xb)^1 \subseteq Ro) \subseteq ((Xa)^1 \subseteq Ro))$	(A _{MX19})(Substitution of Eqns6&7 into Eqn3)(Eqn8)
Let $a = \{b\}^1$	(Eqn9)
Let $b = E$	(Eqn10)
$((\{b\}^1 \subseteq E) = E) \subseteq (((XE)^1 \subseteq Ro) \subseteq ((X\{b\}^1)^1 \subseteq Ro))$	(A _{MX19})(Substitution of Eqns9&10 into Eqn8)(Eqn8)
$(E = E) \subseteq (((XE)^1 \subseteq Ro) \subseteq ((X\{b\}^1)^1 \subseteq Ro))$	(T _{M001})(Universality of (T _{M744}) Universal Set Theorem)
$E \subseteq (((XE)^1 \subseteq Ro) \subseteq ((X\{b\}^1)^1 \subseteq Ro))$	(T _{M342})
$E = (((XE)^1 \subseteq Ro) \subseteq ((X\{b\}^1)^1 \subseteq Ro))$	(T _{M275})(Supersets of the Universal Set)
$((XE)^1 \subseteq Ro) \subseteq (((X\{b\}^1)^1 \subseteq Ro)$	(A _{MX16})(Order Axiom)(Eqn11)
$(X = \{+, -, \{x\}, \{\div\}) \subseteq ((SX)^1 \subseteq R_p)$	(A _{R72})(Axiom of Arithmetic Operation)
$(X = \{+, -, \{x\}, \{\div\}) \subseteq ((XE)^1 \subseteq R_p)$	(T _{R210})(Eqn12)
$R_p \subseteq R_o$	(T _{R523})
$(R_p \subseteq R_o) = E$	(A _{MX16})(Order Axiom)
$((XE)^1 \subseteq R_p) \subseteq ((XE)^1 \subseteq Ro)$	(T _{M272})(Deductive Rule)(Eqn13)
$(X = \{+, -, \{x\}, \{\div\}) \subseteq ((XE)^1 \subseteq Ro)$	(T _{M139})(Hypothetical Syllogism Eqns12&13)(Eqn14)
$(X = \{+, -, \{x\}, \{\div\}) \subseteq ((X*\{b\}^1)^1 \subseteq Ro)$	(T _{M139})(Hypothetical Syllogism Eqns14&11)

$$(X = \{+, -, \{x\}, \{\div\}) \subseteq ((X*\{b\}^1)*\{a\}^1 = ((X*\{b\}^1)*\{a\}^1)^1) \quad (T_{508})$$

Proof of (T₅₀₈)

$((X*\{b\}^1)^1 \subseteq Ro) (\{a\}^1 \subseteq Q(X*\{b\}^1)) (\{a\}^1 \neq \emptyset) \subseteq ((X*\{b\}^1)*\{a\}^1 = ((X*\{b\}^1)*\{a\}^1)^1)$	(T _{R533})(Definition of a One-to-One relationship set)(Eqn1)
$\{a\}^1 \neq \emptyset$	(A _{R59})(Axiom of the Existence of Numbers)
$(\{a\}^1 \neq \emptyset) = E$	(A _{MX16})(Order Axiom)(Eqn2)
$((X*\{b\}^1)^1 \subseteq Ro) (\{a\}^1 \subseteq Q(X*\{b\}^1)) E \subseteq ((X*\{b\}^1)*\{a\}^1 = ((X*\{b\}^1)*\{a\}^1)^1)$	(A _{MX19})(Substitution of Eqn2 into Eqn1)
$((X*\{b\}^1)^1 \subseteq Ro) (\{a\}^1 \subseteq Q(X*\{b\}^1)) \subseteq ((X*\{b\}^1)*\{a\}^1 = ((X*\{b\}^1)*\{a\}^1)^1)$	(A _{MX11})(Universal Set Conjunction Axiom)(Eqn3)
$(X = \{+, -, \{x\}, \{\div\}) \subseteq ((X*\{b\}^1)^1 \subseteq Ro)$	(T ₇₄₁₃)
$(X = \{+, -, \{x\}, \{\div\}) (\{a\}^1 \subseteq Q(X*\{b\}^1)) \subseteq (((X*\{b\}^1)^1 \subseteq Ro) (\{a\}^1 \subseteq Q(X*\{b\}^1)))$	(T _{M029})(Deduction)(Eqn4)
$(X = \{+, -, \{x\}, \{\div\}) (\{a\}^1 \subseteq Q(X*\{b\}^1)) \subseteq ((X*\{b\}^1)*\{a\}^1 = ((X*\{b\}^1)*\{a\}^1)^1)$	(T _{M139})(Hypothetical Syllogism Eqns4&3)(Eqn5)
$(X = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) (\{b\}^1 \neq \{0\}^1) \subseteq (\{a\}^1 \subseteq Q(X*\{b\}^1))$	(T ₇₄₇₉)(Eqn6)
$(X = \{+, -, \{x\}, \{\div\}) \subseteq (X = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\vee\})$	(T _{M753})
$(X = \{+, -, \{x\}, \{\div\}) (\{b\}^1 \neq \{0\}^1) \subseteq (X = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) (\{b\}^1 \neq \{0\}^1)$	(T _{M029})(Deduction)(Eqn7)
$(X = \{+, -, \{x\}, \{\div\}) (\{b\}^1 \neq \{0\}^1) \subseteq (\{a\}^1 \subseteq Q(X*\{b\}^1))$	(T _{M139})(Hypothetical Syllogism Eqns7&6)
$(X = \{+, -, \{x\}, \{\div\}) (\{b\}^1 \neq \{0\}^1) \subseteq (X = \{+, -, \{x\}, \{\div\}) (\{a\}^1 \subseteq Q(X*\{b\}^1))$	(T _{M256})(Conditional Subsets Rule)(Eqn8)
$(X = \{+, -, \{x\}, \{\div\}) (\{b\}^1 \neq \{0\}^1) \subseteq ((X*\{b\}^1)*\{a\}^1 = ((X*\{b\}^1)*\{a\}^1)^1)$	(T _{M139})(Hypothetical Syllogism Eqns8&5)

$$(o = \{+, -, \{x\}, \{\div\}) \subseteq ((\{a\}^1 \langle o \rangle \{b\}^1)^1 = \{a\}^1 \langle o \rangle \{b\}^1) \quad (T_{533})$$

Proof of (T₇₅₃₃)

$$\begin{aligned}
 (\mathbf{o} = \{+, \{-\}, \{x\}, \{\div\}) &\subseteq (((\mathbf{o} * \{b\}^1) * \{a\}^1) = ((\mathbf{o} * \{b\}^1) * \{a\}^1)^1) \\
 &\quad (\mathbf{T}_{7508})(\text{Eqn1}) \\
 (\mathbf{o} = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) &\subseteq (\{a\}^1 \langle \mathbf{o} \rangle \{b\}^1 = (\mathbf{o} * \{b\}^1) * \{a\}^1) \\
 &\quad (\mathbf{D}_{R20})(\text{Definition of an Arithmetical Operator})(\text{Eqn2}) \\
 (\mathbf{o} = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) &\subseteq ((\{a\}^1 \langle \mathbf{o} \rangle \{b\}^1)^1 = ((\mathbf{o} * \{b\}^1) * \{a\}^1)^1) \\
 &\quad (\mathbf{T}_{M479})(\text{Eqn3}) \\
 (\mathbf{o} = \{+, \{-\}, \{x\}, \{\div\}) &\subseteq (\{a\}^1 \langle \mathbf{o} \rangle \{b\}^1 = (\{a\}^1 \langle \mathbf{o} \rangle \{b\}^1)^1) \\
 &\quad (\mathbf{A}_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})
 \end{aligned}$$

$$\{a\}^1 + \{0\}^1 = \{a\}^1 \quad (\mathbf{T}_{7492})$$

$$\{a\}^1 - \{0\}^1 = \{a\}^1 \quad (\mathbf{T}_{7493})$$

$$\{a\}^1 \times \{1\}^1 = \{a\}^1 \quad (\mathbf{T}_{7494})$$

$$\{a\}^1 \div \{1\}^1 = \{a\}^1 \quad (\mathbf{T}_{7495})$$

$$\{a\}^1 \wedge \{1\}^1 = \{a\}^1 \quad (\mathbf{T}_{7496})$$

$$\{a\}^1 \vee \{1\}^1 = \{a\}^1 \quad (\mathbf{T}_{7497})$$

Example Proof of (T₇₄₉₂)

$$(\{+\} * \{0\}^1) * \{a\}^1 = \{a\}^1 \quad (\mathbf{T}_{R755})$$

$$\{a\}^1 + \{0\}^1 = \{a\}^1 \quad (\mathbf{T}_{7471})$$

$$(\{a\}^1 \subseteq \mathbf{N}_N) \subseteq ((\{+\} * \{1\}^1) * \{a\}^1 \subseteq \mathbf{N}_N)$$

(T₇₅₀₂) Sixth Peano Axiom

Proof of (T₇₅₀₂)

Let $\mathbf{a}, \mathbf{b}^1 = \emptyset$

$$(\mathbf{a}, \mathbf{b}^1 = \emptyset) = \mathbf{E}$$

(A_{MX16})(Order Axiom)(Eqn1)

$$(\mathbf{a}\mathbf{b}^1 = \emptyset) = (\mathcal{N}(\mathbf{a} + \mathbf{b}^1)^1 = \mathcal{N}\mathbf{a}^1 + \mathcal{N}\mathbf{b}^1)$$

(A_{S37})(Distribution of $\mathcal{N}$ over Disjoint Sets)(Eqn2)

$$\mathbf{E} = (\mathcal{N}(\mathbf{a} + \mathbf{b}^1)^1 = \mathcal{N}\mathbf{a}^1 + \mathcal{N}\mathbf{b}^1)$$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

$$\mathcal{N}(\mathbf{a} + \mathbf{b}^1)^1 = \mathcal{N}\mathbf{a}^1 + \mathcal{N}\mathbf{b}^1$$

(A_{MX16})(Order Axiom)(Eqn3)

$$\mathcal{N}(\mathbf{a} + \mathbf{b}^1)^1 \subseteq \mathbf{N}_N$$

(T₇₅₀₀)(Eqn4)

$$\mathcal{N}\mathbf{a}^1 + \mathcal{N}\mathbf{b}^1 \subseteq \mathbf{N}_N$$

(A_{MX19})(Substitution of Eqn3 into Eqn4)(Eqn5)

$$\text{Let } \{a\}^1 \subseteq \mathbf{N}_N$$

(Eqn6)

$$\text{Let } \mathcal{N}\mathbf{a}^1 = \{a\}^1$$

(Eqn7)

$$\mathcal{N}\mathbf{a}^1 \subseteq \mathbf{N}_N$$

(A_{MX19})(Substitution of Eqn7 into Eqn6)

$$\text{Let } \mathcal{N}\mathbf{b}^1 = \{1\}^1$$

(Eqn8)

$$\{a\}^1 + \{1\}^1 \subseteq \mathbf{N}_N$$

(A_{MX19})(Substitution of Eqns7\&8 into Eqn5)

$$(\{+\} * \{1\}^1) * \{a\}^1 \subseteq \mathbf{N}_N$$

(T₇₄₇₁)

$$\text{Thus } (\{a\}^1 \subseteq \mathbf{N}_N) \subseteq ((\{+\} * \{1\}^1) * \{a\}^1 \subseteq \mathbf{N}_N)$$

(A_{MX02})(Conditional Introduction)

$$\mathbf{Q}(\{+\} * \{a\}^1) = \mathbf{N}_C \quad (\mathbf{T}_{7536})$$

$$\mathbf{Q}(\{-\} * \{a\}^1) = \mathbf{N}_C \quad (\mathbf{T}_{7538})$$

Example Proof of (T₇₅₃₆)

$$(\mathbf{X} = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) (\{a\}^1 \neq \{0\}^1) \subseteq ((\mathbf{X}\{a\}^1) \mathbf{N}_C = \mathbf{N}_C)$$

(A_{S43})(1st Axiom of Arithmetic Operators)

$$(\{a\}^1 \neq \{0\}^1) \subseteq ((\{-\} * \{a\}^1) * \mathbf{N}_C = \mathbf{N}_C)$$

Extraction (Eqn1)

$$\{-\} * \{0\}^1 = \{=\}$$

(T₇₅₂)(Identity of Subtraction)

$$(\{-\} * \{0\}^1) * \mathbf{N}_C = \{=\} * \mathbf{N}_C$$

(T_{R197})(Relational Deduction)

$$(\{-\} * \{0\}^1) * \mathbf{N}_C = \mathbf{N}_C$$

(T_{A080})(Eqn2)

$$\text{Let } \{a\}^1 = \{0\}^1$$

(Eqn3)

$$(\{-\} * \{a\}^1) \mathbf{N}_C = \mathbf{N}_C$$

(A_{MX19})(Substitution of Eqn3 into Eqn2)

$$\text{Thus } (\{a\}^1 = \{0\}^1) \subseteq ((\{-\} * \{a\}^1) * \mathbf{N}_C = \mathbf{N}_C)$$

(A_{MX02})(Conditional Introduction)(Eqn4)

$$(\{a\}^1 = \{0\}^1) + (\{a\}^1 \neq \{0\}^1) \subseteq ((\{-\} * \{a\}^1) * \mathbf{N}_C = \mathbf{N}_C) + ((\{-\} * \{a\}^1) * \mathbf{N}_C = \mathbf{N}_C)$$

(A_{MX21})(Combination of Eqns1\&4)

$$(\{a\}^1 = \{0\}^1) + (\{a\}^1 \neq \{0\}^1) \subseteq ((\{-\} * \{a\}^1) * \mathbf{N}_C = \mathbf{N}_C)$$

(A_{MX01})(Disjunction Tautology)

$E \subseteq ((\{-\} * \{a\}^1) * N_c = N_c)$	(T_{M671}) (Tertium non Datur)
$E = ((\{-\} * \{a\}^1) * N_c = N_c)$	(T_{M275}) (Supersets of the Universal Set)
$(\{-\} * \{a\}^1) * N_c = N_c$	(A_{MX16}) (Order Axiom)(Eqn5)
$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((XE) * \sim N_c = \emptyset)$	(T_{T523})
$(\{-\} * E) * \sim N_c = \emptyset$	Extraction
$((\{-\} * E) * \sim N_c = \emptyset) = E$	(A_{MX16}) (Order Axiom)(Eqn6)
$\{a\}^1 \subseteq E$	(T_{M744}) (Universal Set Theorem)
$\{-\} * \{a\}^1 \subseteq \{-\} * E$	(T_{R196}) (Relational Deduction)
$(\{-\} * \{a\}^1) * \sim N_c \subseteq (\{-\} * E) * \sim N_c$	(T_{R197}) (Relational Deduction)
$((\{-\} * \{a\}^1) * \sim N_c \subseteq (\{-\} * E) * \sim N_c) = E$	(A_{MX16}) (Order Axiom)
$((\{-\} * E) * \sim N_c = \emptyset) \subseteq ((\{-\} * \{a\}^1) * \sim N_c = \emptyset)$	(T_{M097}) (Probabilistic Rule)(Eqn7)
$E \subseteq ((\{-\} * \{a\}^1) * \sim N_c = \emptyset)$	(A_{MX19}) (Substitution of Eqn6 into Eqn7)
$E = ((\{-\} * \{a\}^1) * \sim N_c = \emptyset)$	(T_{M275}) (Supersets of the Universal Set)
$(\{-\} * \{a\}^1) * \sim N_c = \emptyset$	(A_{MX16}) (Order Axiom)(Eqn8)
$(\{-\} * \{a\}^1) * N_c + (\{-\} * \{a\}^1) * \sim N_c = N_c + \emptyset$	(A_{MX21}) (Combination of Eqns5&8)
$(\{-\} * \{a\}^1) * N_c + (\{-\} * \{a\}^1) * \sim N_c = N_c$	(T_{M668}) (Null Set Disjunction)
$(\{-\} * \{a\}^1) * (N_c + \sim N_c) = N_c$	(T_{R002b}) (Distribution of * over +)
$(\{-\} * \{a\}^1) * E = N_c$	(T_{M671}) (Tertium non Datur)
$\vee(\{+\} * \{a\}^1) * E = N_c$	(T_{T248})
$Q(\{+\} * \{a\}^1) = N_c$	(T_{R211})

$$(\{a\}^1 \neq \{0\}^1) \subseteq (Q(\{x\} * \{a\}^1) = N_c) \quad (T_{T537})$$

$$(\{a\}^1 \neq \{0\}^1) \subseteq (Q(\{\div\} * \{a\}^1) = N_c) \quad (T_{T539})$$

$$(\{a\}^1 \neq \{0\}^1) \subseteq (Q(\{\wedge\} * \{a\}^1) = N_c) \quad (T_{T540})$$

$$(\{a\}^1 \neq \{0\}^1) \subseteq (Q(\{\vee\} * \{a\}^1) = N_c) \quad (T_{T541})$$

Example Proof of (T_{T537})

$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\})(\{a\}^1 \neq \{0\}^1) \subseteq ((X\{a\}^1)N_c = N_c)$	(A_{S43}) (1 st Axiom of Arithmetic Operators)
$(\{a\}^1 \neq \{0\}^1) \subseteq ((\{\div\} * \{a\}^1) * N_c = N_c)$	Extraction (Eqn1)
$(X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq ((XE) * \sim N_c = \emptyset)$	(T_{T523})
$(\{\div\} * E) * \sim N_c = \emptyset$	Extraction
$((\{\div\} * E) * \sim N_c = \emptyset) = E$	(A_{MX16}) (Order Axiom)(Eqn2)
$\{a\}^1 \subseteq E$	(T_{M744}) (Universal Set Theorem)
$\{\div\} * \{a\}^1 \subseteq \{\div\} * E$	(T_{R196}) (Relational Deduction)
$(\{\div\} * \{a\}^1) * \sim N_c \subseteq (\{\div\} * E) * \sim N_c$	(T_{R197}) (Relational Deduction)
$((\{\div\} * \{a\}^1) * \sim N_c \subseteq (\{\div\} * E) * \sim N_c) = E$	(A_{MX16}) (Order Axiom)
$((\{\div\} * E) * \sim N_c = \emptyset) \subseteq ((\{\div\} * \{a\}^1) * \sim N_c = \emptyset)$	(T_{M097}) (Probabilistic Rule)(Eqn3)
$E \subseteq ((\{\div\} * \{a\}^1) * \sim N_c = \emptyset)$	(A_{MX19}) (Substitution of Eqn2 into Eqn3)
$E = ((\{\div\} * \{a\}^1) * \sim N_c = \emptyset)$	(T_{M275}) (Supersets of the Universal Set)
$(\{\div\} * \{a\}^1) * \sim N_c = \emptyset$	(A_{MX16}) (Order Axiom)(Eqn4)
$(\{a\}^1 \neq \{0\}^1) \subseteq ((\{\div\} * \{a\}^1) * N_c + (\{\div\} * \{a\}^1) * \sim N_c = N_c + \emptyset)$	(A_{MX21}) (Combination of Eqns1&4)
$(\{a\}^1 \neq \{0\}^1) \subseteq ((\{\div\} * \{a\}^1) * N_c + (\{\div\} * \{a\}^1) * \sim N_c = N_c)$	(T_{M668}) (Null Set Disjunction)
$(\{a\}^1 \neq \{0\}^1) \subseteq ((\{\div\} * \{a\}^1) * (N_c + \sim N_c) = N_c)$	(T_{R002b}) (Distribution of * over +)
$(\{a\}^1 \neq \{0\}^1) \subseteq ((\{\div\} * \{a\}^1) * E = N_c)$	(T_{M671}) (Tertium non Datur)
$(\{a\}^1 \neq \{0\}^1) \subseteq (\vee(\{x\} * \{a\}^1) * E = N_c)$	(T_{T244})
$(\{a\}^1 \neq \{0\}^1) \subseteq (Q(\{x\} * \{a\}^1) = N_c)$	(T_{R211})

$$\{a\}^1 - \{a\}^1 = \{0\}^1 \quad (T_{T498})$$

Proof of (T_{T498})

$\{a\}^1 + \{0\}^1 = \{a\}^1$	(T_{T492})
$(\{+\} * \{a\}^1) * \{0\}^1 = \{a\}^1$	(T_{T471})
$(\{-\} * \{a\}^1) * (\{+\} * \{a\}^1) * \{0\}^1 = (\{-\} * \{a\}^1) * \{a\}^1$	(T_{R522}) (Relational Deduction)
$\vee(\{+\} * \{a\}^1) * (\{+\} * \{a\}^1) * \{0\}^1 = (\{-\} * \{a\}^1) * \{a\}^1$	(T_{T248}) (Eqn1)
$((\vee(\{+\} * \{a\}^1))^1 \subseteq R_o)(\{0\}^1 \subseteq Q(\{+\} * \{a\}^1)) \subseteq (\{0\}^1 = \vee(\{+\} * \{a\}^1) * (\{+\} * \{a\}^1) * \{0\}^1)$	(T_{S529}) (Eqn2)

$((\forall(\{+\}\ast\{a\}^1))^1 \subseteq R_o)(\{0\}^1 \subseteq Q(\{+\}\ast\{a\}^1)) \subseteq (\{0\}^1 = (\{-\}\ast\{a\}^1)\ast\{a\}^1)$	$(A_{MX19})(\text{Substitution of Eqn1 into Eqn2})(\text{Eqn3})$
$(X = \{+\}, \{-\}, \{x\}, \{\div\}) \subseteq ((\forall(X\ast\{a\}^1))^1 \subseteq R_o)$	(T_{T478})
$(\forall(\{+\}\ast\{a\}^1))^1 \subseteq R_o$	Extraction
$((\forall(\{+\}\ast\{a\}^1))^1 \subseteq R_o) = E$	$(A_{MX16})(\text{Order Axiom})(\text{Eqn4})$
$E(\{0\}^1 \subseteq Q(\{+\}\ast\{a\}^1)) \subseteq (\{0\}^1 = (\{-\}\ast\{a\}^1)\ast\{a\}^1)$	$(A_{MX19})(\text{Substitution of Eqn4 into Eqn3})$
$(\{0\}^1 \subseteq Q(\{+\}\ast\{a\}^1)) \subseteq (\{0\}^1 = (\{-\}\ast\{a\}^1)\ast\{a\}^1)$	$(A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn5})$
$Q(\{+\}\ast\{a\}^1) = N_c$	$(T_{T536})(\text{Eqn6})$
$\{0\}^1 \subseteq N_N$	$(T_{T501})(\text{First Peano Axiom})$
$\{0\}^1 \subseteq N_N + N_Z + N_Q + N_A$	$(T_{M150})(\text{Single Premise Syllogism})(\text{Eqn7})$
$N_N + N_Z + N_Q + N_A \subseteq N_R$	$(D_{R22})(\text{Definition of Natural, Integer, Rational and Algebraic Numbers})(\text{Eqn8})$
$\{0\}^1 \subseteq N_R$	$(T_{M139})(\text{Hypothetical Syllogism Eqns7\&8})$
$\{0\}^1 \subseteq N_R + N_i$	$(T_{M150})(\text{Single Premise Syllogism})(\text{Eqn9})$
$N_R + N_i \subseteq N_c$	$(D_{R19})(\text{Definition of Real and Imaginary Numbers})(\text{Eqn10})$
$\{0\}^1 \subseteq N_c$	$(T_{M139})(\text{Hypothetical Syllogism Eqns9\&10})(\text{Eqn11})$
$\{0\}^1 \subseteq Q(\{+\}\ast\{a\}^1)$	$(A_{MX19})(\text{Substitution of Eqn6 into Eqn11})$
$(\{0\}^1 \subseteq Q(\{+\}\ast\{a\}^1)) = E$	$(A_{MX16})(\text{Order Axiom})(\text{Eqn12})$
$E \subseteq (\{0\}^1 = (\{-\}\ast\{a\}^1)\ast\{a\}^1)$	$(A_{MX19})(\text{Substitution of Eqn12 into Eqn5})$
$E = (\{0\}^1 = (\{-\}\ast\{a\}^1)\ast\{a\}^1)$	$(T_{M275})(\text{Supersets of the Universal Set})$
$\{0\}^1 = (\{-\}\ast\{a\}^1)\ast\{a\}^1$	$(A_{MX16})(\text{Order Axiom})$
$\{0\}^1 = \{a\}^1 - \{a\}^1$	(T_{T472})

Alternative Example Proof of (T_{T498})

$((\forall(\{+\}\ast\{a\}^1))^1 \subseteq R_o)(\{0\}^1 \subseteq Q(\{+\}\ast\{a\}^1)) \subseteq (\{0\}^1 = \forall(\{+\}\ast\{a\}^1)\ast(\{+\}\ast\{a\}^1)\ast\{0\}^1)$	$(T_{S529})(\text{Eqn1})$
$(X = (\{+\}, \{-\}, \{x\}, \{\div\})) \subseteq ((S_X)^1 \subseteq R_P)$	$(A_{R72})(\text{Axiom of Arithmetic Operation})$
$(S_{\{-\}})^1 \subseteq R_P$	Extraction
$(\{-\}E)^1 \subseteq R_P$	(T_{R210})
$\{-\}E \subseteq R_P$	$(A_{S41})(\text{Axiom of Relationship Types})(\text{Eqn2})$
$\{a\}^1 \subseteq E$	$(T_{M744})(\text{Universal Set Theorem})$
$\{-\}\ast\{a\}^1 \subseteq \{-\}\ast E$	$(T_{R196})(\text{Relational Deduction})(\text{Eqn3})$
$\{-\}\ast\{a\}^1 \subseteq R_P$	$(T_{M139})(\text{Hypothetical Syllogism Eqns2\&3})(\text{Eqn4})$
$R_P \subseteq R_o$	$(T_{R523})(\text{Eqn5})$
$(\{-\}\ast\{a\}^1) \subseteq R_o$	$(T_{M139})(\text{Hypothetical Syllogism Eqns4\&5})$
$\forall(\{+\}\ast\{a\}^1) \subseteq R_o$	(T_{T248})
$(\forall(\{+\}\ast\{a\}^1))^1 \subseteq R_o$	$(A_{S41})(\text{Axiom of Relationship Types})$
$((\forall(\{+\}\ast\{a\}^1))^1 \subseteq R_o) = E$	$(A_{MX16})(\text{Order Axiom})(\text{Eqn6})$
$Q(\{+\}\ast\{a\}^1) = N_c$	$(T_{T536})(\text{Eqn7})$
$\{0\}^1 \subseteq N_N$	$(T_{T501})(\text{First Peano Axiom})$
$\{0\}^1 \subseteq N_N + N_Z + N_Q + N_A$	$(T_{M150})(\text{Single Premise Syllogism})(\text{Eqn8})$
$N_N + N_Z + N_Q + N_A \subseteq N_R$	$(D_{R22})(\text{Definition of Natural, Integer, Rational and Algebraic Numbers})(\text{Eqn9})$
$\{0\}^1 \subseteq N_R$	$(T_{M139})(\text{Hypothetical Syllogism Eqns8\&9})$
$\{0\}^1 \subseteq N_R + N_i$	$(T_{M150})(\text{Single Premise Syllogism})(\text{Eqn10})$
$N_R + N_i \subseteq N_c$	$(D_{R19})(\text{Definition of Real and Imaginary Numbers})(\text{Eqn11})$
$\{0\}^1 \subseteq N_c$	$(T_{M139})(\text{Hypothetical Syllogism Eqns10\&11})(\text{Eqn12})$
$\{0\}^1 \subseteq Q(\{+\}\ast\{a\}^1)$	$(A_{MX19})(\text{Substitution of Eqn7 into Eqn12})$
$(\{0\}^1 \subseteq Q(\{+\}\ast\{a\}^1)) = E$	$(A_{MX16})(\text{Order Axiom})(\text{Eqn13})$
$EE \subseteq (\{0\}^1 = (\{-\}\ast\{a\}^1)\ast\{a\}^1)$	$(A_{MX19})(\text{Substitution of Eqns6\&13 into Eqn1})$
$E \subseteq (\{0\}^1 = (\{-\}\ast\{a\}^1)\ast\{a\}^1)$	$(T_{M667})(\text{Conjunction Tautology})$
$E = (\{0\}^1 = (\{-\}\ast\{a\}^1)\ast\{a\}^1)$	$(T_{M275})(\text{Supersets of the Universal Set})$
$\{0\}^1 = (\{-\}\ast\{a\}^1)\ast\{a\}^1$	$(A_{MX16})(\text{Order Axiom})$
$\{0\}^1 = \{a\}^1 - \{a\}^1$	(T_{T472})

$$(\{a\}^1 = \{b\}^1) = ((\{+\}\ast\{c\}^1)\ast\{a\}^1 = (\{+\}\ast\{c\}^1)\ast\{b\}^1) \quad (T_{R857})$$

$$(\{a\}^1 = \{b\}^1) = (\{a\}^1 + \{c\}^1 = \{b\}^1 + \{c\}^1) \quad (T_{R858})$$

$$(\{a\}^1 = \{b\}^1) = ((\{-\}\ast\{c\}^1)\ast\{a\}^1 = (\{-\}\ast\{c\}^1)\ast\{b\}^1) \quad (T_{R859})$$

$$(\{a\}^1 = \{b\}^1) = (\{a\}^1 - \{c\}^1 = \{b\}^1 - \{c\}^1)$$

($\mathcal{T}_{R860}$)

Example Proof of ($\mathcal{T}_{R857}$) & ($\mathcal{T}_{R858}$)

Let $\{a\}^1 = \{b\}^1$

$$(\{+\}\{c\}^1) * \{a\}^1 = (\{+\}\{c\}^1) * \{b\}^1$$

($\mathcal{T}_{R522}$)(Relational Deduction)

$$\text{Thus } (\{a\}^1 = \{b\}^1) \subseteq ((\{+\}\{c\}^1) * \{a\}^1 = (\{+\}\{c\}^1) * \{b\}^1)$$

($\mathcal{A}_{MX02}$)(Conditional Introduction)(Eqn1)

$$\text{Let } (\{+\}\{c\}^1) * \{a\}^1 = (\{+\}\{c\}^1) * \{b\}^1$$

$$\forall (\{+\}\{c\}^1) * (\{+\}\{c\}^1) * \{a\}^1 = \forall (\{+\}\{c\}^1) * (\{+\}\{c\}^1) * \{b\}^1$$

($\mathcal{T}_{R522}$)(Relational Deduction)(Eqn2)

$$((\forall (\{+\}\{c\}^1))^1 \subseteq \mathcal{R}_0)(\{b\}^1 \subseteq \mathcal{Q}(\{+\}\{c\}^1)) \subseteq (\{b\}^1 = \forall (\{+\}\{c\}^1) * (\{+\}\{c\}^1) * \{b\}^1)$$

($\mathcal{T}_{S529}$)(Eqn3)

$$(X = \{+, \{-\} \{x\}, \{\div\}) \subseteq ((\forall (X * \{c\}^1))^1 \subseteq \mathcal{R}_0)$$

($\mathcal{T}_{T478}$)

$$(\forall (\{+\}\{c\}^1))^1 \subseteq \mathcal{R}_0$$

Extraction

$$((\forall (\{+\}\{c\}^1))^1 \subseteq \mathcal{R}_0) = E$$

($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn4)

$$\{b\}^1 \subseteq \mathcal{N}_c$$

($\mathcal{D}_{R18}$)(Definition of Numbers)(Eqn5)

$$\mathcal{Q}(\{+\}\{c\}^1) = \mathcal{N}_c$$

($\mathcal{T}_{T536}$)(Eqn6)

$$\{b\}^1 \subseteq \mathcal{Q}(\{+\}\{c\}^1)$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn6 into Eqn5)

$$(\{b\}^1 \subseteq \mathcal{Q}(\{+\}\{c\}^1)) = E$$

($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn7)

$$EE \subseteq (\{b\}^1 = \forall (\{+\}\{c\}^1) * (\{+\}\{c\}^1) * \{b\}^1)$$

($\mathcal{A}_{MX19}$)(Substitution of Eqns4&7 into Eqn3)

$$E \subseteq (\{b\}^1 = \forall (\{+\}\{c\}^1) * (\{+\}\{c\}^1) * \{b\}^1)$$

($\mathcal{T}_{M667}$)(Conjunction Tautology)

$$E = (\{b\}^1 = \forall (\{+\}\{c\}^1) * (\{+\}\{c\}^1) * \{b\}^1)$$

($\mathcal{T}_{M275}$)(Supersets of the Universal Set)

$$\{b\}^1 = \forall (\{+\}\{c\}^1) * (\{+\}\{c\}^1) * \{b\}^1$$

($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn8)

$$\{a\}^1 = \forall (\{+\}\{c\}^1) * (\{+\}\{c\}^1) * \{a\}^1$$

Resymbolising (Eqn9)

$$\{a\}^1 = \{b\}^1$$

($\mathcal{A}_{MX19}$)(Substitution of Eqns8&9 into Eqn2)

$$\text{Thus } ((\{+\}\{c\}^1) * \{a\}^1 = (\{+\}\{c\}^1) * \{b\}^1) \subseteq (\{a\}^1 = \{b\}^1)$$

($\mathcal{A}_{MX02}$)(Conditional Introduction)(Eqn10)

$$(\{a\}^1 = \{b\}^1) = ((\{+\}\{c\}^1) * \{a\}^1 = (\{+\}\{c\}^1) * \{b\}^1)$$

($\mathcal{T}_{M430}$)(Biconditional Introduction Eqns1&10)

$$(\{a\}^1 = \{b\}^1) = (\{a\}^1 + \{c\}^1 = \{b\}^1 + \{c\}^1)$$

($\mathcal{T}_{T471}$)

$$((\{+\}\{1\}^1) * \{a\}^1 = (\{+\}\{1\}^1) * \{b\}^1) = (\{a\}^1 = \{b\}^1)$$

($\mathcal{T}_{R751}$) Eighth Peano Axiom

Proof of ($\mathcal{T}_{R751}$)

$$(\{a\}^1 = \{b\}^1) = ((\{+\}\{c\}^1) * \{a\}^1 = (\{+\}\{c\}^1) * \{b\}^1)$$

($\mathcal{T}_{R857}$)(Eqn1)

$$\text{Let } \{c\}^1 = \{1\}^1$$

(Eqn2)

$$(\{a\}^1 = \{b\}^1) = ((\{+\}\{1\}^1) * \{a\}^1 = (\{+\}\{1\}^1) * \{b\}^1)$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$(\{a\}^1 = \{b\}^1) = ((\{x\} * \{c\}^1) * \{a\}^1 = (\{x\} * \{c\}^1) * \{b\}^1) \quad (\mathcal{T}_{R861})$$

$$(\{a\}^1 = \{b\}^1) = (\{a\}^1 \times \{c\}^1 = \{b\}^1 \times \{c\}^1) \quad (\mathcal{T}_{R862})$$

$$(\{a\}^1 = \{b\}^1) = ((\{\div\} * \{c\}^1) * \{a\}^1 = (\{\div\} * \{c\}^1) * \{b\}^1) \quad (\mathcal{T}_{R863})$$

$$(\{a\}^1 = \{b\}^1) = (\{a\}^1 \div \{c\}^1 = \{b\}^1 \div \{c\}^1) \quad (\mathcal{T}_{R864})$$

Example Proof of ($\mathcal{T}_{R861}$) & ($\mathcal{T}_{R862}$)

Let $\{a\}^1 = \{b\}^1$

$$(\{x\} * \{c\}^1) * \{a\}^1 = (\{x\} * \{c\}^1) * \{b\}^1$$

($\mathcal{T}_{R522}$)(Relational Deduction)

$$\text{Thus } (\{a\}^1 = \{b\}^1) \subseteq ((\{x\} * \{c\}^1) * \{a\}^1 = (\{x\} * \{c\}^1) * \{b\}^1)$$

($\mathcal{A}_{MX02}$)(Conditional Introduction)(Eqn1)

$$\text{Let } (\{x\} * \{c\}^1) * \{a\}^1 = (\{x\} * \{c\}^1) * \{b\}^1$$

$$\forall (\{x\} * \{c\}^1) * (\{x\} * \{c\}^1) * \{a\}^1 = \forall (\{x\} * \{c\}^1) * (\{x\} * \{c\}^1) * \{b\}^1$$

($\mathcal{T}_{R522}$)(Relational Deduction)(Eqn2)

$$((\forall (\{x\} * \{c\}^1))^1 \subseteq \mathcal{R}_0)(\{b\}^1 \subseteq \mathcal{Q}(\{x\} * \{c\}^1)) \subseteq (\{b\}^1 = \forall (\{x\} * \{c\}^1) * (\{x\} * \{c\}^1) * \{b\}^1)$$

($\mathcal{T}_{S529}$)(Eqn3)

$$(X = \{+, \{-\} \{x\}, \{\div\}) \subseteq ((\forall (X * \{c\}^1))^1 \subseteq \mathcal{R}_0)$$

($\mathcal{T}_{T478}$)

$$(\forall (\{x\} * \{c\}^1))^1 \subseteq \mathcal{R}_0$$

($\mathcal{A}_{MX19}$)(Substitution)

$$((\forall (\{x\} * \{c\}^1))^1 \subseteq \mathcal{R}_0) = E$$

($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn4)

$$\{b\}^1 \subseteq \mathcal{N}_c$$

($\mathcal{D}_{R18}$)(Definition of Numbers)(Eqn5)

$$\mathcal{Q}(\{x\} * \{c\}^1) = \mathcal{N}_c$$

($\mathcal{T}_{T537}$)(Eqn6)

$$\{b\}^1 \subseteq \mathcal{Q}(\{x\} * \{c\}^1)$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn6 into Eqn5)

$$(\{b\}^1 \subseteq \mathcal{Q}(\{x\} * \{c\}^1)) = E$$

($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn7)

$$EE \subseteq (\{b\}^1 = \forall (\{x\} * \{c\}^1) * (\{x\} * \{c\}^1) * \{b\}^1)$$

($\mathcal{A}_{MX19}$)(Substitution of Eqns4&7 into Eqn3)

$$E \subseteq (\{b\}^1 = \forall (\{x\} * \{c\}^1) * (\{x\} * \{c\}^1) * \{b\}^1)$$

($\mathcal{T}_{M667}$)(Conjunction Tautology)

$$E = (\{b\}^1 = \forall (\{x\} * \{c\}^1) * (\{x\} * \{c\}^1) * \{b\}^1)$$

($\mathcal{T}_{M275}$)(Supersets of the Universal Set)

$$\begin{aligned}
\{b\}^1 &= \vee(\{x\} * \{c\}^1) * (\{x\} * \{c\}^1) * \{b\}^1 & (\mathbf{A}_{MX16})(\text{Order Axiom})(\text{Eqn8}) \\
\{a\}^1 &= \vee(\{x\} * \{c\}^1) * (\{x\} * \{c\}^1) * \{a\}^1 & \text{Resymbolising (Eqn9)} \\
\{a\}^1 &= \{b\}^1 & (\mathbf{A}_{MX19})(\text{Substitution of Eqns8\&9 into Eqn2}) \\
\text{Thus } ((\{x\} * \{c\}^1) * \{a\}^1 &= (\{x\} * \{c\}^1) * \{b\}^1) \subseteq (\{a\}^1 = \{b\}^1) & (\mathbf{A}_{MX02})(\text{Conditional Introduction})(\text{Eqn10}) \\
(\{a\}^1 = \{b\}^1) &= ((\{x\} * \{c\}^1) * \{a\}^1 = (\{x\} * \{c\}^1) * \{b\}^1) & (\mathbf{T}_{M430})(\text{Biconditional Introduction Eqns1\&10}) \\
(\{a\}^1 = \{b\}^1) &= (\{a\}^1 \times \{c\}^1 = \{b\}^1 \times \{c\}^1) & (\mathbf{T}_{T471})
\end{aligned}$$

$$(\{a\}^1 = \{b\}^1) \subseteq (((\wedge) * \{c\}^1) * \{a\}^1 = ((\wedge) * \{c\}^1) * \{b\}^1) \quad (\mathbf{T}_{R865})$$

$$(\{a\}^1 = \{b\}^1) \subseteq (\{a\}^1 \wedge \{c\}^1 = \{b\}^1 \wedge \{c\}^1) \quad (\mathbf{T}_{R866})$$

$$(\{a\}^1 = \{b\}^1) \subseteq ((\vee) * \{c\}^1) * \{a\}^1 = ((\vee) * \{c\}^1) * \{b\}^1 \quad (\mathbf{T}_{R867})$$

$$(\{a\}^1 = \{b\}^1) \subseteq (\{a\}^1 \vee \{c\}^1 = \{b\}^1 \vee \{c\}^1) \quad (\mathbf{T}_{R868})$$

Example Proof of ($\mathbf{T}_{R865}$) & ($\mathbf{T}_{R866}$)

$$\begin{aligned}
\text{Let } \{a\}^1 &= \{b\}^1 \\
(\wedge) * \{c\}^1 &= (\wedge) * \{c\}^1 & (\mathbf{T}_{R522})(\text{Relational Deduction}) \\
\text{Thus } (\{a\}^1 = \{b\}^1) &\subseteq (((\wedge) * \{c\}^1) * \{a\}^1 = ((\wedge) * \{c\}^1) * \{b\}^1) & (\mathbf{A}_{MX02})(\text{Conditional Introduction}) \\
(\{a\}^1 = \{b\}^1) &\subseteq (\{a\}^1 \wedge \{c\}^1 = \{b\}^1 \wedge \{c\}^1) & (\mathbf{T}_{T475})
\end{aligned}$$

$$(\{+\} * ((\{+\} * \{c\}^1) * \{b\}^1)) * \{a\}^1 = (\{+\} * \{c\}^1) * ((\{+\} * \{b\}^1) * \{a\}^1) \quad (\mathbf{T}_{R851}) \text{ Associativity of Algebraic +}$$

$$\{a\}^1 + (\{b\}^1 + \{c\}^1) = (\{a\}^1 + \{b\}^1) + \{c\}^1 \quad (\mathbf{T}_{T480}) \text{ Associativity of Algebraic +}$$

$$(\{x\} * ((\{x\} * \{c\}^1) * \{b\}^1)) * \{a\}^1 = (\{x\} * \{c\}^1) * ((\{x\} * \{b\}^1) * \{a\}^1) \quad (\mathbf{T}_{R852}) \text{ Associativity of Algebraic x}$$

$$\{a\}^1 \times (\{b\}^1 \times \{c\}^1) = (\{a\}^1 \times \{b\}^1) \times \{c\}^1 \quad (\mathbf{T}_{T481}) \text{ Associativity of Algebraic x}$$

Example Proof of ($\mathbf{T}_{R851}$) & ($\mathbf{T}_{T480}$)

$$\begin{aligned}
(\{+\}^1 \subseteq \mathbf{Rv}) &= ((\{a\}^1 + \{b\}^1) + \{c\}^1 = \{a\}^1 + (\{b\}^1 + \{c\}^1)) & (\mathbf{D}_{R39})(\text{Definition of an Associative Operator})(\text{Eqn1}) \\
\{+\}^1 + \{x\}^1 &\subseteq \mathbf{RvRc} & (\mathbf{A}_{R81})(\text{Arithmetical Associativity and Commutativity Axiom}) \\
\{+\}^1 &\subseteq \mathbf{Rv} & (\mathbf{T}_{M072})(\text{Single Premise Syllogism}) \\
(\{+\}^1 \subseteq \mathbf{Rv}) &= \mathbf{E} & (\mathbf{A}_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\
\mathbf{E} &= ((\{a\}^1 + \{b\}^1) + \{c\}^1 = \{a\}^1 + (\{b\}^1 + \{c\}^1)) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
(\{a\}^1 + \{b\}^1) + \{c\}^1 &= \{a\}^1 + (\{b\}^1 + \{c\}^1) & (\mathbf{A}_{MX16})(\text{Order Axiom}) \\
(\{+\} * \{c\}^1) * (\{a\}^1 + \{b\}^1) &= (\{+\} * (\{b\}^1 + \{c\}^1)) * \{a\}^1 & (\mathbf{T}_{T471}) \\
(\{+\} * \{c\}^1) * ((\{+\} * \{b\}^1) * \{a\}^1) &= (\{+\} * ((\{+\} * \{c\}^1) * \{b\}^1)) * \{a\}^1 & (\mathbf{T}_{T471})
\end{aligned}$$

$$(\{+\} * \{b\}^1) * \{a\}^1 = (\{+\} * \{a\}^1) * \{b\}^1 \quad (\mathbf{T}_{R853}) \text{ Commutativity of Algebraic +}$$

$$\{a\}^1 + \{b\}^1 = \{b\}^1 + \{a\}^1 \quad (\mathbf{T}_{T482}) \text{ Commutativity of Algebraic +}$$

$$(\{x\} * \{b\}^1) * \{a\}^1 = (\{x\} * \{a\}^1) * \{b\}^1 \quad (\mathbf{T}_{R854}) \text{ Commutativity of Algebraic x}$$

$$\{a\}^1 \times \{b\}^1 = \{b\}^1 \times \{a\}^1 \quad (\mathbf{T}_{T483}) \text{ Commutativity of Algebraic x}$$

Example Proof of ($\mathbf{T}_{R853}$) & ($\mathbf{T}_{T482}$)

$$\begin{aligned}
(\{+\}^1 \subseteq \mathbf{Rc}) &= (\{a\}^1 + \{b\}^1 = \{b\}^1 + \{a\}^1) & (\mathbf{D}_{R40})(\text{Definition of a Commutative Operator})(\text{Eqn1}) \\
\{+\}^1 + \{x\}^1 &\subseteq \mathbf{RvRc} & (\mathbf{A}_{R81})(\text{Arithmetical Associativity and Commutativity Axiom}) \\
\{+\}^1 &\subseteq \mathbf{Rc} & (\mathbf{T}_{M072})(\text{Single Premise Syllogism}) \\
(\{+\}^1 \subseteq \mathbf{Rc}) &= \mathbf{E} & (\mathbf{A}_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\
\mathbf{E} &= (\{a\}^1 + \{b\}^1 = \{b\}^1 + \{a\}^1) & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
\{a\}^1 + \{b\}^1 &= \{b\}^1 + \{a\}^1 & (\mathbf{A}_{MX16})(\text{Order Axiom}) \\
(\{+\} * \{b\}^1) * \{a\}^1 &= (\{+\} * \{a\}^1) * \{b\}^1 & (\mathbf{T}_{T471})
\end{aligned}$$

$$\{a\}^1 \div \{a\}^1 = \{1\}^1 \quad (\mathbf{T}_{T499})$$

Proof of ($\mathbf{T}_{T499}$)

$$\begin{aligned}
\{a\}^1 \times \{1\}^1 &= \{a\}^1 & (\mathbf{T}_{T494}) \\
(\{x\} * \{1\}^1) * \{a\}^1 &= \{a\}^1 & (\mathbf{T}_{T473}) \\
(\{x\} * \{a\}^1) * \{1\}^1 &= \{a\}^1 & (\mathbf{T}_{R854})(\text{Commutativity of Algebraic x})
\end{aligned}$$

$$\begin{aligned}
& (\{ \div \} * \{ a \}^1) * (\{ x \} * \{ a \}^1) * \{ 1 \}^1 = (\{ \div \} * \{ a \}^1) * \{ a \}^1 & (TR522)(\text{Relational Deduction}) \\
& \vee (\{ x \} * \{ a \}^1) * (\{ x \} * \{ a \}^1) * \{ 1 \}^1 = (\{ \div \} * \{ a \}^1) * \{ a \}^1 & (T742)(\text{Eqn1}) \\
& ((\vee (\{ x \} * \{ a \}^1))^1 \subseteq R_o) (\{ 1 \}^1 \subseteq Q(\{ x \} * \{ a \}^1)) \subseteq (\{ 1 \}^1 = \vee (\{ x \} * \{ a \}^1) * (\{ x \} * \{ a \}^1) * \{ 1 \}^1) & (TS529)(\text{Eqn2}) \\
& ((\vee (\{ x \} * \{ a \}^1))^1 \subseteq R_o) (\{ 1 \}^1 \subseteq Q(\{ x \} * \{ a \}^1)) \subseteq (\{ 1 \}^1 = (\{ \div \} * \{ a \}^1) * \{ a \}^1) & (AMX19)(\text{Substitution of Eqn1 into Eqn2})(\text{Eqn3}) \\
& (X = \{ + \}, \{ - \}, \{ x \}, \{ \div \}) \subseteq ((\vee (X * \{ a \}^1))^1 \subseteq R_o) & (T478) \\
& (\vee (\{ x \} * \{ a \}^1))^1 \subseteq R_o & \text{Extraction} \\
& ((\vee (\{ x \} * \{ a \}^1))^1 \subseteq R_o) = E & (AMX16)(\text{Order Axiom})(\text{Eqn4}) \\
& E(\{ 1 \}^1 \subseteq Q(\{ x \} * \{ a \}^1)) \subseteq (\{ 1 \}^1 = (\{ \div \} * \{ a \}^1) * \{ a \}^1) & (AMX19)(\text{Substitution of Eqn4 into Eqn3}) \\
& (\{ 1 \}^1 \subseteq Q(\{ x \} * \{ a \}^1)) \subseteq (\{ 1 \}^1 = (\{ \div \} * \{ a \}^1) * \{ a \}^1) & (AMX11)(\text{Universal Set Conjunction Axiom})(\text{Eqn5}) \\
& (X = \{ + \}, \{ - \}, \{ x \}, \{ \div \}, \{ \wedge \}, \{ \vee \}) (\{ a \}^1 \neq \{ 0 \}^1) \subseteq (Q(X \{ a \}^1) = N_c) & (T552) \\
& (\{ a \}^1 \neq \{ 0 \}^1) \subseteq (Q(\{ x \} * \{ a \}^1) = N_c) & \text{Extraction (Eqn6)} \\
& (\{ a \}^1 \subseteq N_N) \subseteq ((\{ + \} * \{ 1 \}^1) * \{ a \}^1 \subseteq N_N) & (T502)(\text{Sixth Peano Axiom})(\text{Eqn7}) \\
& \text{Let } \{ a \}^1 = \{ 0 \}^1 & (\text{Eqn8}) \\
& (\{ 0 \}^1 \subseteq N_N) \subseteq ((\{ + \} * \{ 1 \}^1) * \{ 0 \}^1 \subseteq N_N) & (AMX19)(\text{Substitution of Eqn8 into Eqn7})(\text{Eqn9}) \\
& \{ 0 \}^1 \subseteq N_N & (T501)(\text{First Peano Axiom}) \\
& (\{ 0 \}^1 \subseteq N_N) = E & (AMX16)(\text{Order Axiom})(\text{Eqn10}) \\
& E \subseteq ((\{ + \} * \{ 1 \}^1) * \{ 0 \}^1 \subseteq N_N) & (AMX19)(\text{Substitution of Eqn10 into Eqn9}) \\
& E = ((\{ + \} * \{ 1 \}^1) * \{ 0 \}^1 \subseteq N_N) & (TM275)(\text{Supersets of the Universal Set}) \\
& (\{ + \} * \{ 1 \}^1) * \{ 0 \}^1 \subseteq N_N & (AMX16)(\text{Order Axiom}) \\
& (\{ + \} * \{ 0 \}^1) * \{ 1 \}^1 \subseteq N_N & (TR853)(\text{Commutativity of Algebraic } +) \\
& \{ 1 \}^1 \subseteq N_N & (TR755) \\
& \{ 1 \}^1 \subseteq N_N + N_Z + N_Q + N_A & (TM150)(\text{Single Premise Syllogism})(\text{Eqn11}) \\
& N_N + N_Z + N_Q + N_A \subseteq N_R & (DR22)(\text{Definition of Natural, Integer, Rational and Algebraic Numbers})(\text{Eqn12}) \\
& \{ 1 \}^1 \subseteq N_R & (TM139)(\text{Hypothetical Syllogism Eqns11\&12}) \\
& \{ 1 \}^1 \subseteq N_R + N_I & (TM150)(\text{Single Premise Syllogism})(\text{Eqn13}) \\
& N_R + N_I \subseteq N_C & (DR19)(\text{Definition of Real and Imaginary Numbers})(\text{Eqn14}) \\
& \{ 1 \}^1 \subseteq N_C & (TM139)(\text{Hypothetical Syllogism Eqns13\&14})(\text{Eqn15}) \\
& (\{ a \}^1 \neq \{ 0 \}^1) \subseteq (\{ 1 \}^1 \subseteq Q(\{ x \} * \{ a \}^1)) & (AMX19)(\text{Substitution of Eqn6 into Eqn15}) \\
& (\{ a \}^1 \neq \{ 0 \}^1) \subseteq ((\{ 1 \}^1 \subseteq Q(\{ x \} * \{ a \}^1)) = E) & (AMX16)(\text{Order Axiom}) \\
& ((\{ a \}^1 \neq \{ 0 \}^1) = E) \subseteq ((\{ 1 \}^1 \subseteq Q(\{ x \} * \{ a \}^1) = E) & (TM094)(\text{Probabilistic Rule})(\text{Eqn16}) \\
& ((\{ a \}^1 \neq \{ 0 \}^1) = E) \subseteq (E \subseteq (\{ 1 \}^1 = (\{ \div \} * \{ a \}^1) * \{ a \}^1)) & (AMX19)(\text{Substitution of Eqn16 into Eqn5}) \\
& ((\{ a \}^1 \neq \{ 0 \}^1) = E) \subseteq (E = (\{ 1 \}^1 = (\{ \div \} * \{ a \}^1) * \{ a \}^1)) & (TM275)(\text{Supersets of the Universal Set}) \\
& ((\{ a \}^1 \neq \{ 0 \}^1) \subseteq (\{ 1 \}^1 = (\{ \div \} * \{ a \}^1) * \{ a \}^1)) = E & (TM094)(\text{Probabilistic Rule}) \\
& (\{ a \}^1 \neq \{ 0 \}^1) \subseteq (\{ 1 \}^1 = (\{ \div \} * \{ a \}^1) * \{ a \}^1) & (AMX16)(\text{Order Axiom}) \\
& (\{ a \}^1 \neq \{ 0 \}^1) \subseteq (\{ 1 \}^1 = \{ a \}^1 \div \{ a \}^1) & (T474) \\
& \{ 1 \}^1 = \{ a \}^1 \div \{ a \}^1 & (TS761)(\text{Implied Conditions of Quantity})
\end{aligned}$$

$$\mathcal{N}c^1 \div \mathcal{N}c^1 = \{ 1 \}^1 \quad (TR806)$$

Proof of (TR806)

$$\begin{aligned}
& \{ a \}^1 \div \{ a \}^1 = \{ 1 \}^1 & (T499)(\text{Eqn1}) \\
& \text{Let } \mathcal{N}c^1 = \{ a \}^1 & (\text{Eqn2}) \\
& \mathcal{N}c^1 \div \mathcal{N}c^1 = \{ 1 \}^1 & (AMX19)(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$\begin{aligned}
& (\{ x \} * \{ c \}^1) * (\{ a \}^1 + \{ b \}^1) = (\{ x \} * \{ c \}^1) * \{ a \}^1 + (\{ x \} * \{ c \}^1) * \{ b \}^1 & (TR788) \text{ Distribution of } x \text{ over } + \\
& (\{ a \}^1 + \{ b \}^1) \times \{ c \}^1 = (\{ a \}^1 \times \{ c \}^1) + (\{ b \}^1 \times \{ c \}^1) & (T484) \text{ Distribution of } x \text{ over } + \\
& (\{ x \} * \{ c \}^1) * (\{ a \}^1 - \{ b \}^1) = (\{ x \} * \{ c \}^1) * \{ a \}^1 - (\{ x \} * \{ c \}^1) * \{ b \}^1 & (TR789) \text{ Distribution of } x \text{ over } - \\
& (\{ a \}^1 - \{ b \}^1) \times \{ c \}^1 = (\{ a \}^1 \times \{ c \}^1) - (\{ b \}^1 \times \{ c \}^1) & (T485) \text{ Distribution of } x \text{ over } - \\
& (\{ \div \} * \{ c \}^1) * (\{ a \}^1 + \{ b \}^1) = (\{ \div \} * \{ c \}^1) * \{ a \}^1 + (\{ \div \} * \{ c \}^1) * \{ b \}^1 & (TR790) \text{ Distribution of } \div \text{ over } +
\end{aligned}$$

$$(\{a\}^1 + \{b\}^1) \div \{c\}^1 = (\{a\}^1 \div \{c\}^1) + (\{b\}^1 \div \{c\}^1) \quad (T_{7486}) \text{ Distribution of } \div \text{ over } +$$

$$(\{ \div \} * \{c\}^1) * (\{a\}^1 - \{b\}^1) = (\{ \div \} * \{c\}^1) * \{a\}^1 - (\{ \div \} * \{c\}^1) * \{b\}^1 \quad (T_{7791}) \text{ Distribution of } \div \text{ over } -$$

$$(\{a\}^1 - \{b\}^1) \div \{c\}^1 = (\{a\}^1 \div \{c\}^1) - (\{b\}^1 \div \{c\}^1) \quad (T_{7487}) \text{ Distribution of } \div \text{ over } -$$

Example Proof of (T₇₇₈₈) & (T₇₄₈₄)

$$\begin{aligned} (\{x\}^1 \subseteq R_D * \{+\}^1) &= ((\{a\}^1 + \{b\}^1) \times \{c\}^1 = (\{a\}^1 \times \{c\}^1) + (\{b\}^1 \times \{c\}^1)) \\ &\quad (D_{R41}) (\text{Definition of a Distributive Operator}) (\text{Eqn1}) \\ (X = \{x\}, \{ \div \}) (Y = \{+\}, \{-\}) &\subseteq (X^1 \subseteq R_D * Y^1) \\ &\quad (A_{R74}) (\text{Arithmetical Distribution Axiom}) \\ \{x\}^1 &\subseteq R_D * \{+\}^1 \\ (\{x\}^1 \subseteq R_D * \{+\}^1) &= E \\ E &= ((\{a\}^1 + \{b\}^1) \times \{c\}^1 = (\{a\}^1 \times \{c\}^1) + (\{b\}^1 \times \{c\}^1)) \\ &\quad (A_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\ (\{a\}^1 + \{b\}^1) \times \{c\}^1 &= (\{a\}^1 \times \{c\}^1) + (\{b\}^1 \times \{c\}^1) \\ &\quad (A_{MX16}) (\text{Order Axiom}) \\ (\{x\} * \{c\}^1) * (\{a\}^1 + \{b\}^1) &= (\{x\} * \{c\}^1) * \{a\}^1 + (\{x\} * \{c\}^1) * \{b\}^1 \\ &\quad (T_{7473}) \end{aligned}$$

$$(\{x\} * \mathcal{N}c^1) * (\mathcal{N}a^1 + \mathcal{N}b^1) = (\{x\} * \mathcal{N}c^1) * \mathcal{N}a^1 + (\{x\} * \mathcal{N}c^1) * \mathcal{N}b^1 \quad (T_{7792}) \text{ Distribution of } x \text{ over } +$$

$$(\{x\} * \mathcal{N}c^1) * (\mathcal{N}a^1 - \mathcal{N}b^1) = (\{x\} * \mathcal{N}c^1) * \mathcal{N}a^1 - (\{x\} * \mathcal{N}c^1) * \mathcal{N}b^1 \quad (T_{7793}) \text{ Distribution of } x \text{ over } -$$

$$(\{ \div \} * \mathcal{N}c^1) * (\mathcal{N}a^1 + \mathcal{N}b^1) = (\{ \div \} * \mathcal{N}c^1) * \mathcal{N}a^1 + (\{ \div \} * \mathcal{N}c^1) * \mathcal{N}b^1 \quad (T_{7794}) \text{ Distribution of } \div \text{ over } +$$

$$(\{ \div \} * \mathcal{N}c^1) * (\mathcal{N}a^1 - \mathcal{N}b^1) = (\{ \div \} * \mathcal{N}c^1) * \mathcal{N}a^1 - (\{ \div \} * \mathcal{N}c^1) * \mathcal{N}b^1 \quad (T_{7795}) \text{ Distribution of } \div \text{ over } -$$

Example Proof of (T₇₇₉₂)

$$\begin{aligned} (\{x\} * \{c\}^1) * (\{a\}^1 + \{b\}^1) &= (\{x\} * \{c\}^1) * \{a\}^1 + (\{x\} * \{c\}^1) * \{b\}^1 \quad (T_{7788}) (\text{Distribution of } x \text{ over } +) (\text{Eqn1}) \\ \text{Let } \{a\}^1 &= \mathcal{N}a^1 \quad (\text{Eqn2}) \\ \text{Let } \{b\}^1 &= \mathcal{N}b^1 \quad (\text{Eqn3}) \\ \text{Let } \{c\}^1 &= \mathcal{N}c^1 \quad (\text{Eqn4}) \\ (\{x\} * \mathcal{N}c^1) * (\mathcal{N}a^1 + \mathcal{N}b^1) &= (\{x\} * \mathcal{N}c^1) * \mathcal{N}a^1 + (\{x\} * \mathcal{N}c^1) * \mathcal{N}b^1 \\ &\quad (A_{MX19}) (\text{Substitution of Eqns2,3\&4 into Eqn1}) \end{aligned}$$

$$(\{a\}^1 = \{b\}^1) \subseteq (X * \{a\}^1 = X * \{b\}^1) \quad (T_{855})$$

Proof of (T₈₅₅)

$$\begin{aligned} \text{Let } \{a\}^1 &= \{b\}^1 \\ X * \{a\}^1 &= X * \{b\}^1 \quad (T_{852}) (\text{Relational Deduction}) \\ \text{Thus } (\{a\}^1 = \{b\}^1) &\subseteq (X * \{a\}^1 = X * \{b\}^1) \quad (A_{MX02}) (\text{Conditional Introduction}) \end{aligned}$$

$$(ab = \emptyset) = ((\{+\} * \mathcal{N}b^1) * \mathcal{N}a^1 = \mathcal{N}(a + b)^1) \quad (T_{800})$$

Proof of (T₈₀₀)

$$\begin{aligned} (ab = \emptyset) &= (\mathcal{N}(a + b)^1 = \mathcal{N}a^1 + \mathcal{N}b^1) \quad (A_{S37}) (\text{Distribution of } \mathcal{N} \text{ over Disjoint Sets}) \\ (ab = \emptyset) &= (\mathcal{N}(a + b)^1 = (\{+\} * \mathcal{N}b^1) * \mathcal{N}a^1) \quad (T_{7471}) \end{aligned}$$

$$\mathcal{N}a^1 = \mathcal{N}(ab)^1 + \mathcal{N}(\underline{ab})^1 \quad (T_{7491})$$

$$\mathcal{N}b^1 = \mathcal{N}(ab)^1 + \mathcal{N}(\underline{ab})^1 \quad (T_{7796})$$

Example Proof of (T₇₄₉₁)

$$\begin{aligned} a &= a \quad (D_{M00}) (\text{Definition of } =) \\ a &= a.E \quad (A_{MX11}) (\text{Universal Set Conjunction Axiom}) \\ a &= a.(b + \underline{b}) \quad (T_{M671}) (\text{Tertium non Datur}) \\ a &= ab + \underline{ab} \quad (T_{M407}) (\text{Distribution of } . \text{ over } +) \\ a^1 &= (ab + \underline{ab})^1 \quad (T_{M479}) \\ \mathcal{N}a^1 &= \mathcal{N}(ab + \underline{ab})^1 \quad (T_{R196}) (\text{Relational Deduction}) \\ \mathcal{N}a^1 &= \mathcal{N}(ab)^1 + \mathcal{N}(\underline{ab})^1 \quad (A_{S37}) (\text{Distribution of } \mathcal{N} \text{ over Disjoint Sets}) \\ \text{Thus } (a^1 \neq \emptyset) &\subseteq (\mathcal{N}a^1 = \mathcal{N}(ab)^1 + \mathcal{N}(\underline{ab})^1) \quad (A_{MX02}) (\text{Conditional Introduction}) \end{aligned}$$

$$\mathcal{N}(\underline{ab})^1 = \mathcal{N}a^1 - \mathcal{N}(\underline{ab})^1 \quad (\mathcal{T}_{R797})$$

Proof of ($\mathcal{T}_{R797}$)

$$\begin{aligned} \mathcal{N}a^1 &= \mathcal{N}(\underline{ab})^1 + \mathcal{N}(\underline{ab})^1 & (\mathcal{T}_{T491}) \\ (\{-\}\mathcal{N}(\underline{ab})^1) * \mathcal{N}a^1 &= (\{-\}\mathcal{N}(\underline{ab})^1) * (\mathcal{N}(\underline{ab})^1 + \mathcal{N}(\underline{ab})^1) & (\mathcal{T}_{R196})(\text{Relational Deduction}) \\ \mathcal{N}a^1 - \mathcal{N}(\underline{ab})^1 &= \mathcal{N}(\underline{ab})^1 + \mathcal{N}(\underline{ab})^1 - \mathcal{N}(\underline{ab})^1 & (\mathcal{T}_{T472}) \\ \mathcal{N}a^1 - \mathcal{N}(\underline{ab})^1 &= \mathcal{N}(\underline{ab})^1 + \{0\}^1 & (\mathcal{T}_{T498}) \\ \mathcal{N}a^1 - \mathcal{N}(\underline{ab})^1 &= \mathcal{N}(\underline{ab})^1 & (\mathcal{T}_{T492}) \end{aligned}$$

$$((o = \{+\}, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) = E) \subseteq ((\{a\}^1 = \{b\}^1).(\{c\}^1 = \{d\}^1)) \subseteq (\{a\}^1 \langle o \rangle \{c\}^1 = \{b\}^1 \langle o \rangle \{d\}^1))$$

($\mathcal{T}_{T553}$) Arithmetic Combination

Proof of ($\mathcal{T}_{T553}$)

$$\begin{aligned} \text{Let } o &= \{+\}, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\} \\ \text{Let } \{a\}^1 &= \{b\}^1 & (\text{Eqn1}) \\ \text{Let } \{c\}^1 &= \{d\}^1 & (\text{Eqn2}) \\ \{a\}^1 \langle o \rangle \{c\}^1 &= \{a\}^1 \langle o \rangle \{c\}^1 & (\mathcal{D}_{M00})(\text{Definition of } =)(\text{Eqn3}) \\ \{a\}^1 \langle o \rangle \{c\}^1 &= \{b\}^1 \langle o \rangle \{d\}^1 & (\mathcal{A}_{MX19})(\text{Substitution of Eqns1\&2 into Eqn3}) \\ \text{Thus } (o = \{+\}, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}).(\{a\}^1 = \{b\}^1).(\{c\}^1 = \{d\}^1) &\subseteq (\{a\}^1 \langle o \rangle \{c\}^1 = \{b\}^1 \langle o \rangle \{d\}^1) & (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\ ((o = \{+\}, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) = E) &\subseteq ((\{a\}^1 = \{b\}^1).(\{c\}^1 = \{d\}^1)) \subseteq (\{a\}^1 \langle o \rangle \{c\}^1 = \{b\}^1 \langle o \rangle \{d\}^1) & (\mathcal{A}_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \end{aligned}$$

$$\mathcal{N}(a + b)^1 = (\{-\}\mathcal{N}(\underline{ab})^1) * (\{+\}\mathcal{N}b^1) * \mathcal{N}a^1 \quad (\mathcal{T}_{R747})$$

$$\mathcal{N}(a + b)^1 = \mathcal{N}a^1 + \mathcal{N}b^1 - \mathcal{N}(a.b)^1 \quad (\mathcal{T}_{R798})$$

Proof of ($\mathcal{T}_{R747}$) & ($\mathcal{T}_{R798}$)

$$\begin{aligned} a &= a & (\mathcal{D}_{M00})(\text{Definition of } =) \\ a &= a.E & (\mathcal{A}_{MX17})(\text{Universal Set Conjunction Axiom}) \\ a &= a.(b + \underline{b}) & (\mathcal{T}_{M671})(\text{Tertium non Datur}) \\ a &= ab + \underline{ab} & (\mathcal{T}_{M407})(\text{Distribution of } . \text{ over } +)(\text{Eqn1}) \\ b &= b & (\mathcal{D}_{M00})(\text{Definition of } =) \\ b &= b.E & (\mathcal{A}_{MX17})(\text{Universal Set Conjunction Axiom}) \\ b &= b.(a + \underline{a}) & (\mathcal{T}_{M671})(\text{Tertium non Datur}) \\ b &= ab + \underline{ab} & (\mathcal{T}_{M407})(\text{Distribution of } . \text{ over } +)(\text{Eqn2}) \\ a + b &= ab + \underline{ab} + ab + \underline{ab} & (\mathcal{A}_{MX21})(\text{Combination of Eqns1\&2}) \\ a + b &= ab + \underline{ab} + \underline{ab} & (\mathcal{A}_{MX01})(\text{Disjunction Tautology}) \\ (a + b)^1 &= (ab + \underline{ab} + \underline{ab})^1 & (\mathcal{T}_{M479}) \\ \mathcal{N}(a + b)^1 &= \mathcal{N}(ab + \underline{ab} + \underline{ab})^1 & (\mathcal{T}_{R196})(\text{Relational Deduction}) \\ \mathcal{N}(a + b)^1 &= \mathcal{N}(\underline{ab})^1 + \mathcal{N}(\underline{ab})^1 + \mathcal{N}(\underline{ab})^1 & (\mathcal{A}_{S37})(\text{Distribution of } \mathcal{N} \text{ over Disjoint Sets})(\text{Eqn3}) \\ \mathcal{N}a^1 &= \mathcal{N}(\underline{ab})^1 + \mathcal{N}(\underline{ab})^1 & (\mathcal{T}_{T491})(\text{Eqn4}) \\ \mathcal{N}b^1 &= \mathcal{N}(\underline{ab})^1 + \mathcal{N}(\underline{ab})^1 & (\mathcal{T}_{T491})(\text{Eqn5}) \\ \mathcal{N}a^1 + \mathcal{N}b^1 &= \mathcal{N}(\underline{ab})^1 + \mathcal{N}(\underline{ab})^1 + \mathcal{N}(\underline{ab})^1 + \mathcal{N}(\underline{ab})^1 & (\mathcal{T}_{T553})(\text{Arithmetic Combination } o = \{+\})(\text{Eqn6}) \\ \mathcal{N}a^1 + \mathcal{N}b^1 &= \mathcal{N}(\underline{ab})^1 + \mathcal{N}(a + b)^1 & (\mathcal{A}_{MX19})(\text{Substitution of Eqn3 into Eqn6}) \\ (\{-\} * \mathcal{N}(\underline{ab})^1) * (\mathcal{N}a^1 + \mathcal{N}b^1) &= (\{-\} * \mathcal{N}(\underline{ab})^1) * (\mathcal{N}(\underline{ab})^1 + \mathcal{N}(a + b)^1) & (\mathcal{T}_{R196})(\text{Relational Deduction}) \\ \mathcal{N}a^1 + \mathcal{N}b^1 - \mathcal{N}(\underline{ab})^1 &= \mathcal{N}(\underline{ab})^1 + \mathcal{N}(a + b)^1 - \mathcal{N}(\underline{ab})^1 & (\mathcal{T}_{T472}) \\ \mathcal{N}a^1 + \mathcal{N}b^1 - \mathcal{N}(\underline{ab})^1 &= \mathcal{N}(a + b)^1 + \{0\}^1 & (\mathcal{T}_{T498}) \\ \mathcal{N}a^1 + \mathcal{N}b^1 - \mathcal{N}(\underline{ab})^1 &= (\{+\} * \{0\}^1) * \mathcal{N}(a + b)^1 & (\mathcal{T}_{T471}) \\ \mathcal{N}a^1 + \mathcal{N}b^1 - \mathcal{N}(\underline{ab})^1 &= \{=\} * \mathcal{N}(a + b)^1 & (\mathcal{A}_{R69})(\text{Identity of Addition}) \\ \mathcal{N}a^1 + \mathcal{N}b^1 - \mathcal{N}(\underline{ab})^1 &= \mathcal{N}(a + b)^1 & (\mathcal{T}_{A080}) \\ \mathcal{N}(a + b)^1 &= (\{-\}\mathcal{N}(\underline{ab})^1) * (\{+\}\mathcal{N}b^1) * \mathcal{N}a^1 & (\mathcal{T}_{T471}) \& (\mathcal{T}_{T472}) \end{aligned}$$

$$\{1\}^1 = (\{+\} * \{1\}^1) * \{0\}^1 \quad (\mathcal{T}_{T557})$$

Proof of (T₇₅₅₇)

$$\begin{aligned} \{+\} * \{0\}^1 &= \{=\} & (\mathbf{A}_{R69}) & (\text{Identity of Addition}) \\ (\{+\} * \{0\}^1) * \{1\}^1 &= \{=\} * \{1\}^1 & (\mathbf{T}_{R196}) & (\text{Relational Deduction}) \\ (\{+\} * \{0\}^1) * \{1\}^1 &= \{1\}^1 & (\mathbf{D}_{A144}) & (\text{Definition of the Reflexive Relationship Set}) \\ (\{+\} * \{1\}^1) * \{0\}^1 &= \{1\}^1 & (\mathbf{T}_{R853}) & (\text{Commutativity of Algebraic } +) \end{aligned}$$

$$\{1\}^1 = (\{+\} * \{0\}^1) * \{10\}^1 \quad (\mathbf{T}_{7559})$$

Proof of (T₇₅₅₉)

$$\begin{aligned} \{1\}^1 &= (\{+\} * \{0\}^1) * \{a\}^1 & (\mathbf{D}_{R45}) & (\text{Definition of the Zero Index})(\text{Eqn1}) \\ \text{Let } \{a\}^1 &= \{10\}^1 & & (\text{Eqn2}) \\ \{1\}^1 &= (\{+\} * \{0\}^1) * \{10\}^1 & (\mathbf{A}_{MX19}) & (\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$\{10\}^1 = (\{+\} * \{1\}^1) * \{10\}^1 \quad (\mathbf{T}_{7560})$$

Proof of (T₇₅₆₀)

$$\begin{aligned} \{+\} * \{1\}^1 &= \{=\} & (\mathbf{A}_{R71}) & (\text{Identity of Power}) \\ (\{+\} * \{1\}^1) * \{10\}^1 &= \{=\} * \{10\}^1 & (\mathbf{T}_{R196}) & (\text{Relational Deduction}) \\ (\{+\} * \{1\}^1) * \{10\}^1 &= \{10\}^1 & (\mathbf{D}_{A144}) & (\text{Definition of the Reflexive Relationship Set}) \end{aligned}$$

$$\{+\} * \{>\} * \{0\}^1 = \{>\} \quad (\mathbf{T}_{7564})$$

$$\{+\} * \{<\} * \{0\}^1 = \{<\} \quad (\mathbf{T}_{7566})$$

$$\{+\} * \{=\} * \{0\}^1 = \{=\} \quad (\mathbf{T}_{7568})$$

Example Proof of (T₇₅₆₄)

$$\begin{aligned} (\mathbf{X} = \{>\}, \{<\}, \{=\}) &\subseteq (\{+\} * \mathbf{X} * \{0\}^1 = \mathbf{X}) & (\mathbf{A}_{S44}) & (\text{Axiom of Numerical Order}) \\ \{+\} * \{>\} * \{0\}^1 &= \{>\} & & (\text{Extraction}) \end{aligned}$$

$$\forall (\{+\} * \{>\} * \{0\}^1) = \{<\} \quad (\mathbf{T}_{7565})$$

$$\forall (\{+\} * \{<\} * \{0\}^1) = \{>\} \quad (\mathbf{T}_{7567})$$

Example Proof of (T₇₅₆₅)

$$\begin{aligned} \{+\} * \{>\} * \{0\}^1 &= \{>\} & (\mathbf{T}_{7564}) \\ \forall (\{+\} * \{>\} * \{0\}^1) &= \forall \{>\} & (\mathbf{T}_{R522}) & (\text{Relational Deduction}) \\ \forall (\{+\} * \{>\} * \{0\}^1) &= \{<\} & (\mathbf{D}_{A143}) & (\text{Symmetrical Opposite of an Asymmetrical Relationship Set}) \end{aligned}$$

$$(\{+\} * \{>\} * \{0\}^1) * \{a\}^1 = \{>\} * \{a\}^1 \quad (\mathbf{T}_{7569})$$

Proof of (T₇₅₆₉)

$$\begin{aligned} \{+\} * \{>\} * \{0\}^1 &= \{>\} & (\mathbf{T}_{7564}) \\ (\{+\} * \{>\} * \{0\}^1) * \{a\}^1 &= \{>\} * \{a\}^1 & (\mathbf{T}_{R522b}) & (\text{Relational Deduction}) \end{aligned}$$

$$(\{b\}^1 \subseteq \{>\} * \{0\}^1) \subseteq ((\{+\} * \{b\}^1) * \{a\}^1 \subseteq \{>\} * \{a\}^1) \quad (\mathbf{T}_{7570})$$

Proof of (T₇₅₇₀)

$$\begin{aligned} \text{Let } \{b\}^1 &\subseteq \{>\} * \{0\}^1 \\ \{+\} * \{b\}^1 &\subseteq \{+\} * \{>\} * \{0\}^1 & (\mathbf{T}_{R196}) & (\text{Relational Deduction}) \\ \{+\} * \{b\}^1 &\subseteq \{>\} & (\mathbf{T}_{7564}) \\ (\{+\} * \{b\}^1) * \{a\}^1 &\subseteq \{>\} * \{a\}^1 & (\mathbf{T}_{R197}) & (\text{Relational Deduction}) \\ \text{Thus } (\{b\}^1 &\subseteq \{>\} * \{0\}^1) \subseteq ((\{+\} * \{b\}^1) * \{a\}^1 \subseteq \{>\} * \{a\}^1) & (\mathbf{A}_{MX02}) & (\text{Conditional Introduction}) \end{aligned}$$

$$\{<\} * \{0\}^1 = (\{-\} * \{>\} * \{0\}^1) * \{0\}^1 \quad (\mathbf{T}_{7577})$$

Proof of (T₇₅₇₇)

$$\begin{aligned} \forall (\{+\} * \{>\} * \{0\}^1) &= \{<\} & (\mathbf{T}_{7565}) \\ (\{-\} * \{>\} * \{0\}^1) &= \{<\} & (\mathbf{T}_{7248}) \\ (\{-\} * \{>\} * \{0\}^1) * \{0\}^1 &= \{<\} * \{0\}^1 & (\mathbf{T}_{R522b}) & (\text{Relational Deduction}) \end{aligned}$$

$$\mathcal{N}a^1 \subseteq \{>\} * \{0\}^1 \quad (T_{581})$$

Proof of (T_{581})

$$\mathbf{N}_N \subseteq \{>\} * \{0\}^1$$

(A_{549})(Axiom of Natural Numbers)(Eqn1)

$$\mathcal{N}a^1 \subseteq \mathbf{N}_N$$

(T_{500})(Eqn2)

$$\mathcal{N}a^1 \subseteq \{>\} * \{0\}^1$$

(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \{>\} * \{0\}^1)$$

(T_{611})

Proof of (T_{611})

$$\mathcal{N}a^1 \subseteq \{>\} * \{0\}^1$$

(T_{581})

$$(\mathcal{N}a^1 \subseteq \{>\} * \{0\}^1) = E$$

(A_{MX16})(Order Axiom)(Eqn1)

$$(a = \emptyset) = (\mathcal{N}a^1 = \{0\}^1)$$

(D_{R23})(Definition of Zero)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \neq \{0\}^1)$$

(T_{M008})(Inverse of an Equality)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \sim\{0\}^1)$$

(T_{M090})(Inverse Unit Subset Rule) since $\mathcal{N}a^1 = (\mathcal{N}a^1)^1$

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \{ \neq \} * \{0\}^1)$$

(A_{084}) since $\{0\}^1 \neq \emptyset$

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \{ \neq \} * \{0\}^1) E$$

(A_{MX17})(Universal Set Conjunction Axiom)(Eqn2)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \{ \neq \} * \{0\}^1) (\mathcal{N}a^1 \subseteq \{>\} * \{0\}^1)$$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq (\{ \neq \} * \{0\}^1) (\{>\} * \{0\}^1))$$

(T_{M132})(Composition)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq (\{ \neq \} . \{>\}) * \{0\}^1)$$

(R_{239})(Distribution of . over * for Entities)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq (\{ \neq \} . \{>\} + \{=\}) * \{0\}^1)$$

(D_{A141})(Disjunction with $\{=\}$)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq (\{ \neq \} . \{>\} + \{ \neq \} . \{=\}) * \{0\}^1)$$

(T_{M407})(Distribution of . over +)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq (\{ \neq \} . \{>\} + \emptyset) * \{0\}^1)$$

(T_{M621})(Complement Conjunction)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq (\{ \neq \} . \{>\}) * \{0\}^1)$$

(T_{M668})(Null Set Disjunction)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \{>\} * \{0\}^1)$$

(T_{589})

$$\{1\}^1 \subseteq \{>\} * \{0\}^1$$

(T_{582})

Proof of (T_{582})

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \{>\} * \{0\}^1)$$

(T_{611})(Eqn1)

$$(a^1 \neq \emptyset^1) . (a^1 \neq \emptyset) \subseteq (a \neq \emptyset)$$

(T_{M501})(Eqn2)

$$(a^1 \neq \emptyset^1) . (a^1 \neq \emptyset) \subseteq (\mathcal{N}a^1 \subseteq \{>\} * \{0\}^1)$$

(A_{MX19})(Substitution of Eqn1 into Eqn2)

$$E . (a^1 \neq \emptyset^1) . (a^1 \neq \emptyset) \subseteq (\mathcal{N}a^1 \subseteq \{>\} * \{0\}^1)$$

(A_{MX17})(Universal Set Conjunction Axiom)(Eqn3)

Let $a = a^1$

$$(a = a^1) = E$$

(A_{MX16})(Order Axiom)(Eqn4)

$$(a = a^1) . (a^1 \neq \emptyset^1) . (a^1 \neq \emptyset) \subseteq (\mathcal{N}a^1 \subseteq \{>\} * \{0\}^1)$$

(A_{MX19})(Substitution of Eqn4 into Eqn3)(Eqn5)

$$(a = a^1) (a^1 \neq \emptyset^1) (a^1 \neq \emptyset) = (\{1\}^1 = \mathcal{N}a^1)$$

(D_{R10})(Definition of One)(Eqn6)

$$(\{1\}^1 = \mathcal{N}a^1) \subseteq (\mathcal{N}a^1 \subseteq \{>\} * \{0\}^1)$$

(A_{MX19})(Substitution of Eqn6 into Eqn5)

$$\{1\}^1 \subseteq \{>\} * \{0\}^1$$

(A_{MX19})(Substitution)

$$(\{+\} * \{1\}^1) * \mathbf{N}_R \subseteq \{>\} * \mathbf{N}_R$$

(T_{622})

Proof of (T_{622})

$$\{1\}^1 \subseteq \{>\} * \{0\}^1$$

(T_{582})

$$\{+\} * \{1\}^1 \subseteq \{+\} * \{>\} * \{0\}^1$$

(T_{R196})(Relational Deduction)

$$\{+\} * \{1\}^1 \subseteq \{>\}$$

(T_{564})

$$(\{+\} * \{1\}^1) * \mathbf{N}_R \subseteq \{>\} * \mathbf{N}_R$$

(T_{R197})(Relational Deduction)

$$(\{+\} * \{1\}^1) * \mathbf{N}_N \subseteq \{>\} * \mathbf{N}_N$$

(T_{607})

Proof of (T_{607})

$$\{1\}^1 \subseteq \{>\} * \{0\}^1$$

(T_{582})

$$\{+\} * \{1\}^1 \subseteq \{+\} * \{>\} * \{0\}^1$$

(T_{R196})(Relational Deduction)

$$\begin{aligned} \{+\} * \{1\}^1 &\subseteq \{>\} & (T_{564}) \\ (\{+\} * \{1\}^1) * N_N &\subseteq \{>\} * N_N & (T_{197})(\text{Relational Deduction}) \end{aligned}$$

$$(\{+\} * \{1\}^1) * \{a\}^1 \subseteq \{>\} * \{a\}^1 \quad (T_{571})$$

Proof of (T_{571})

$$\begin{aligned} \{b\}^1 &\subseteq \{>\} * \{0\}^1 \subseteq ((\{+\} * \{b\}^1) * \{a\}^1 \subseteq \{>\} * \{a\}^1) & (T_{570})(\text{Eqn1}) \\ \text{Let } \{b\}^1 &= \{1\}^1 & (\text{Eqn2}) \\ (\{1\}^1 &\subseteq \{>\} * \{0\}^1) \subseteq ((\{+\} * \{1\}^1) * \{a\}^1 \subseteq \{>\} * \{a\}^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\ \{1\}^1 &\subseteq \{>\} * \{0\}^1 & (T_{582}) \\ (\{1\}^1 &\subseteq \{>\} * \{0\}^1) = E & (A_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\ E &\subseteq ((\{+\} * \{1\}^1) * \{a\}^1 \subseteq \{>\} * \{a\}^1) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\ E &= ((\{+\} * \{1\}^1) * \{a\}^1 \subseteq \{>\} * \{a\}^1) & (T_{M275})(\text{Supersets of the Universal Set}) \\ (\{+\} * \{1\}^1) * \{a\}^1 &\subseteq \{>\} * \{a\}^1 & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$N_N \neq \emptyset \quad (T_{588})$$

Proof of (T_{588})

$$\begin{aligned} \{a\}^1 &\neq \emptyset & (A_{R59})(\text{Axiom of the Existence of Numbers})(\text{Eqn1}) \\ \text{Let } \{a\}^1 &= N_N^1 & (\text{Eqn2}) \\ N_N^1 &\neq \emptyset & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (N_N^1 &\neq \emptyset) = E & (A_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\ N_N^1 &\subseteq N_N & (T_{500}) \\ (N_N^1 &\subseteq N_N) = E & (A_{MX16})(\text{Order Axiom}) \\ (N_N^1 &\neq \emptyset) \subseteq (N_N \neq \emptyset) & (T_{M096})(\text{Probabilistic Rule})(\text{Eqn4}) \\ E &\subseteq (N_N \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqn3 into Eqn4}) \\ E &= (N_N \neq \emptyset) & (T_{M275})(\text{Supersets of the Universal Set}) \\ N_N &\neq \emptyset & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$N_C \neq \emptyset \quad (T_{619})$$

Proof of (T_{619})

$$\begin{aligned} N_N &\subseteq N_C & (T_{544}) \\ (N_N &\subseteq N_C) = E & (A_{MX16})(\text{Order Axiom}) \\ (N_N &\neq \emptyset) \subseteq (N_C \neq \emptyset) & (T_{M096})(\text{Probabilistic Rule}) \\ E &\subseteq (N_C \neq \emptyset) & (T_{M001})(\text{Universality of } (T_{588})) \\ E &= (N_C \neq \emptyset) & (T_{M275})(\text{Supersets of the Universal Set}) \\ N_C &\neq \emptyset & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$N_N \neq N_N^1 \quad (T_{589})$$

Proof of (T_{589})

$$\begin{aligned} N_N^1 &\subset N_N & (T_{403}) \\ N_N^1 &\subset N_N & (T_{R740})(\text{Eqn1}) \\ E^1 &\neq \emptyset & (T_{M773})(\text{Existence of } E^1) \\ N_N^1 &= (N_N^1)^1 & (T_{155N})(\text{Eqn2}) \\ (N_N^1)^1 &\subset N_N & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ ((N_N^1)^1 &\subset N_N) = E & (A_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\ E^1 &\neq \emptyset & (T_{M773})(\text{Existence of } E^1) \\ N_N^1 &\neq \emptyset & (T_{275N})(\text{Eqn4}) \\ (N_N^1)^1 &\neq \emptyset & (A_{MX19})(\text{Substitution of Eqn2 into Eqn4}) \\ ((N_N^1)^1 &\neq \emptyset) = E & (A_{MX16})(\text{Order Axiom})(\text{Eqn5}) \\ ((N_N^1)^1 &\neq \emptyset)((N_N^1)^1 \subset N_N) \subseteq (N_N \neq N_N^1) & (D_{M19})(\text{Definition of a Plural Set})(\text{Eqn6}) \\ EE &\subseteq (N_N \neq N_N^1) & (A_{MX19})(\text{Substitution of Eqns3\&5 into Eqn6}) \end{aligned}$$

$E \subseteq (N_N \neq N_N^1)$	(T_{M667})(Conjunction Tautology)
$E = (N_N \neq N_N^1)$	(T_{M275})(Supersets of the Universal Set)
$N_N \neq N_N^1$	($AMX16$)(Order Axiom)

$N_N \subseteq N_R$	(T_{T591})
$N_Z \subseteq N_R$	(T_{T592})
$N_Q \subseteq N_R$	(T_{T593})
$N_A \subseteq N_R$	(T_{T594})

Example Proof of (T_{T591})

$N_N + N_Z + N_Q + N_A \subseteq N_R$	(D_{R22})(Definition of Natural, Integer, Rational and Algebraic Numbers)
$N_N \subseteq N_R$	(T_{M072})(Single Premise Syllogism)

$N_N^1 \subseteq S_D S_O$	(T_{T585})

Proof of (T_{T585})

$(N_R^1 \subseteq S_D S_O)(N_N \subseteq N_R)(N_N \neq \emptyset)(N_N \neq N_N^1) \subseteq (N_N^1 \subseteq S_D S_O)$	(T_{T957})(Eqn1)
$N_N \neq \emptyset$	(T_{T588})
$(N_N \neq \emptyset) = E$	($AMX16$)(Order Axiom)(Eqn2)
$N_N \neq N_N^1$	(T_{T589})
$(N_N \neq N_N^1) = E$	($AMX16$)(Order Axiom) (Eqn3)
$N_R^1 \subseteq S_D S_O$	(T_{T590})
$(N_R^1 \subseteq S_D S_O) = E$	($AMX16$)(Order Axiom)(Eqn4)
$N_N \subseteq N_R$	(T_{T591})
$(N_N \subseteq N_R) = E$	($AMX16$)(Order Axiom)(Eqn5)
$EEEE \subseteq (N_N^1 \subseteq S_D S_O)$	($AMX19$)(Substitution of Eqns2,3,4&5 into Eqn1)
$E \subseteq (N_N^1 \subseteq S_D S_O)$	(T_{M667})(Conjunction Tautology)
$E = (N_N^1 \subseteq S_D S_O)$	(T_{M275})(Supersets of the Universal Set)
$N_N^1 \subseteq S_D S_O$	($AMX16$)(Order Axiom)

$N_N.(\{>\}_\neq N_N) = \{0\}^1$	(T_{T572})

Proof of (T_{T572})

$BN_N^1 = \{0\}^1$	(D_{R87})(Definition of Lower Bound of Natural Numbers)(Eqn1)
$N_N^1 \subseteq S_D S_O$	(T_{T585})
$BN_N^1 = N_N.(\{>\}_\neq N_N)$	(T_{S818})(Eqn2)
$\{0\}^1 = N_N.(\{>\}_\neq N_N)$	($AMX19$)(Substitution of Eqn1 into Eqn2)

$N_N = N_W + \{0\}^1$	(T_{T556})

Proof of (T_{T556})

$N_N = N_N$	(D_{M00})(Definition of =)
$N_N = N_N.E$	($AMX11$)(Universal Set Conjunction Axiom)
$N_N = N_N.(\sim\{0\}^1 + \{0\}^1)$	(T_{M671})(Tertium non Datur)
$N_N = N_N.\sim\{0\}^1 + N_N.\{0\}^1$	(T_{M407})(Distribution of . over +)(Eqn1)
$\{0\}^1 \subseteq N_N$	(T_{T501})(First Peano Axiom)
$\{0\}^1 = N_N.\{0\}^1$	($AMX44$)($\subseteq$ Equality Form)(Eqn2)
$N_N = N_N.\sim\{0\}^1 + \{0\}^1$	($AMX19$)(Substitution of Eqn2 into Eqn1)
$N_N = N_W + \{0\}^1$	(D_{R90})(Definition of N_W)

$N_W \subseteq N_N$	(T_{T599})

Proof of (T_{T599})

$N_N = N_W + \{0\}^1$	(T_{T556})

$$N_W \subseteq N_N$$

(T_{M072b})(Single Premise Syllogism)

$$N_W \subseteq N_R$$

(T_{T598})

Proof of (T_{T598})

$$N_W \subseteq N_N$$

(T_{T599})(Eqn1)

$$N_N \subseteq N_R$$

(T_{T591})(Eqn2)

$$N_W \subseteq N_R$$

(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq N_W)$$

(T_{T601})

Proof of (T_{T601})

$$\mathcal{N}a^1 \subseteq N_N$$

(T_{T500})(Eqn1)

$$N_N = N_W + \{0\}^1$$

(T_{T556})(Eqn2)

$$\mathcal{N}a^1 \subseteq N_W + \{0\}^1$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(\mathcal{N}a^1 \subseteq N_W + \{0\}^1) = E$$

(A_{MX16})(Order Axiom)(Eqn3)

$$(a = \emptyset) = (\mathcal{N}a^1 = \{0\}^1)$$

(D_{R23})(Definition of Zero)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \neq \{0\}^1)$$

(T_{M008})(Inverse of an Equality)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \sim\{0\}^1)$$

(T_{M090})(Inverse Unit Subset Rule)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \sim\{0\}^1)E$$

(A_{MX11})(Universal Set Conjunction Axiom)(Eqn4)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \sim\{0\}^1)(\mathcal{N}a^1 \subseteq N_W + \{0\}^1)$$

(A_{MX19})(Substitution of Eqn3 into Eqn4)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \sim\{0\}^1 \cdot (N_W + \{0\}^1))$$

(T_{M132})(Composition)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \sim\{0\}^1 \cdot N_W + \sim\{0\}^1 \cdot \{0\}^1)$$

(T_{M407})(Distribution of $\cdot$ over $+$)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \sim\{0\}^1 \cdot N_W + \emptyset)$$

(T_{M621})(Complement Conjunction)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq \sim\{0\}^1 \cdot N_W)$$

(T_{M668})(Null Set Disjunction)(Eqn5)

$$N_W = N_N \cdot \sim\{0\}^1$$

(D_{R90})(Definition of N_W)

$$N_W \subseteq \sim\{0\}^1$$

(T_{M073b})(Single Premise Syllogism)

$$N_W = N_W \cdot \sim\{0\}^1$$

(A_{MX44})($\subseteq$ Equality Form)(Eqn6)

$$(a \neq \emptyset) = (\mathcal{N}a^1 \subseteq N_W)$$

(A_{MX19})(Substitution of Eqn6 into Eqn5)

$$\{1\}^1 \neq \emptyset$$

(T_{T606})

Proof of (T_{T606})

$$(a = a^1)(a^1 \neq \emptyset)(a^1 \neq \emptyset^1) = (\{1\}^1 = \mathcal{N}a^1)$$

(D_{R10})(Definition of One)(Eqn1)

$$(a^1 \neq \emptyset) = (\mathcal{N}a^1 \neq \emptyset)$$

(T_{T275N})(Eqn2)

$$(a = a^1)(a^1 \neq \emptyset)(a^1 \neq \emptyset^1) \subseteq (a^1 \neq \emptyset)$$

(T_{M083})(Conjunction Elimination)(Eqn3)

$$(a = a^1)(a^1 \neq \emptyset)(a^1 \neq \emptyset^1) \subseteq (\mathcal{N}a^1 \neq \emptyset)$$

(A_{MX19})(Substitution of Eqn2 into Eqn3)(Eqn4)

$$(\{1\}^1 = \mathcal{N}a^1) \subseteq (\mathcal{N}a^1 \neq \emptyset)$$

(A_{MX19})(Substitution of Eqn1 into Eqn4)

$$\{1\}^1 \neq \emptyset$$

(A_{MX19})(Substitution)

$$N_W \neq N_W^1$$

(T_{T597})

Proof of (T_{T597})

$$(a^1 \neq \emptyset)(a^1 \subset b) \subseteq (b \neq b^1)$$

(D_{M19})(Definition of a Plural Set)(Eqn1)

$$\text{Let } a^1 = \{1\}^1$$

(Eqn2)

$$\text{Let } b = N_W$$

(Eqn3)

$$(\{1\}^1 \neq \emptyset)(\{1\}^1 \subset N_W) \subseteq (N_W \neq N_W^1)$$

(A_{MX19})(Substitution of Eqns2&3 into Eqn1)(Eqn4)

$$\{1\}^1 \neq \emptyset$$

(T_{T606})

$$(\{1\}^1 \neq \emptyset) = E$$

(A_{MX16})(Order Axiom)(Eqn5)

$$\{1\}^1 \subset N_W$$

(A_{R16})(One is a Whole Number)

$$(\{1\}^1 \subset N_W) = E$$

(A_{MX16})(Order Axiom)(Eqn6)

$$EE \subseteq (N_W \neq N_W^1)$$

(A_{MX19})(Substitution of Eqns5&6 into Eqn4)

$$E \subseteq (N_W \neq N_W^1)$$

(T_{M667})(Conjunction Tautology)

$$E = (N_W \neq N_W^1)$$

(T_{M275})(Supersets of the Universal Set)

$$N_W \neq N_W^1$$

(A_{MX16})(Order Axiom)

$$N_W \neq \emptyset \quad (T_{596})$$

Proof of (T₅₉₆)

$$\begin{aligned} \{1\}^1 &\subseteq N_W & (A_{R16})(\text{One is a Whole Number}) \\ \{1\}^1 &\subseteq N_W & (T_{592}) \\ (\{1\}^1 \subseteq N_W) &= E & (A_{MX16})(\text{Order Axiom}) \\ (\{1\}^1 \neq \emptyset) &\subseteq (N_W \neq \emptyset) & (T_{M096})(\text{Probabilistic Rule})(\text{Eqn1}) \\ \{1\}^1 &\neq \emptyset & (T_{606}) \\ (\{1\}^1 \neq \emptyset) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\ E &\subseteq (N_W \neq \emptyset) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ E &= (N_W \neq \emptyset) & (T_{M275})(\text{Supersets of the Universal Set}) \\ N_W &\neq \emptyset & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$N_W^1 \subseteq S_D S_O \quad (T_{586})$$

Proof of (T₅₈₆)

$$\begin{aligned} (N_W \neq \emptyset)(N_W \neq N_W^1)(N_W \subseteq N_R)(N_R^1 \subseteq S_D S_O) &\subseteq (N_W^1 \subseteq S_D S_O) & (T_{597})(\text{Eqn1}) \\ N_W &\neq \emptyset & (T_{596}) \\ (N_W \neq \emptyset) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\ N_W &\neq N_W^1 & (T_{597}) \\ (N_W \neq N_W^1) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn3}) \\ N_W &\subseteq N_R & (T_{598}) \\ (N_W \subseteq N_R) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\ N_R^1 &\subseteq S_D S_O & (T_{590}) \\ (N_R^1 \subseteq S_D S_O) &= E & (A_{MX16})(\text{Order Axiom})(\text{Eqn5}) \\ E &\subseteq (N_W^1 \subseteq S_D S_O) & (A_{MX19})(\text{Substitution of Eqns2,3,4\&5 into Eqn1}) \\ E &\subseteq (N_W^1 \subseteq S_D S_O) & (T_{M667})(\text{Conjunction Tautology}) \\ E &= (N_W^1 \subseteq S_D S_O) & (T_{M275})(\text{Supersets of the Universal Set}) \\ N_W^1 &\subseteq S_D S_O & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$N_W.(\{>\}_* N_W) = \{1\}^1 \quad (T_{573})$$

Proof of (T₅₇₃)

$$\begin{aligned} B N_W^1 &= \{1\}^1 & (D_{R89})(\text{Definition of Lower Bound of Whole Numbers})(\text{Eqn1}) \\ N_W^1 &\subseteq S_D S_O & (T_{586}) \\ B N_W^1 &= N_W.(\{>\}_* N_W) & (T_{5818})(\text{Eqn2}) \\ \{1\}^1 &= N_W.(\{>\}_* N_W) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \end{aligned}$$

$$(\{a\}^1 \subseteq N_N) \subseteq ((\{+\}_* \{1\}^1) * \{a\}^1 \subseteq \sim \{0\}^1) \quad (T_{578})$$

Proof of (T₅₇₈)

$$\begin{aligned} (\{+\}_* \{1\}^1) * \{a\}^1 &\subseteq \{>\}_* \{a\}^1 & (T_{571})(\text{Eqn1}) \\ \{0\}^1 &= N_N.(\{>\}_* N_N) & (T_{572}) \\ \{0\}^1 &\subseteq \{>\}_* N_N & (T_{M073b})(\text{Single Premise Syllogism})(\text{Eqn2}) \\ \text{Let } \{a\}^1 &\subseteq N_N \\ \{>\}_* \{a\}^1 &\subseteq \{>\}_* N_N & (T_{R196})(\text{Relational Deduction}) \\ \{>\}_* N_N &\subseteq \{>\}_* \{a\}^1 & (T_{M012})(\text{Transposition})(\text{Eqn3}) \\ \{0\}^1 &\subseteq \{>\}_* \{a\}^1 & (T_{M139})(\text{Hypothetical Syllogism Eqns2\&3}) \\ \{>\}_* \{a\}^1 &\subseteq \sim \{0\}^1 & (T_{M012})(\text{Transposition})(\text{Eqn4}) \\ (\{+\}_* \{1\}^1) * \{a\}^1 &\subseteq \sim \{0\}^1 & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&4}) \\ \text{Thus } (\{a\}^1 \subseteq N_N) &\subseteq ((\{+\}_* \{1\}^1) * \{a\}^1 \subseteq \sim \{0\}^1) & (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(\{+\}_* \{1\}^1) * \{a\}^1 \neq \emptyset \quad (T_{555})$$

Proof of (T₅₅₅)

$$\begin{aligned} (X = \{+, \{-\}, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) &= ((X * \{b\}^1) * \{a\}^1 \neq \emptyset) \\ & \quad (A_{547})(2^{\text{nd}} \text{ Axiom of the Existence of Numbers}) \\ (\{+\} * \{b\}^1) * \{a\}^1 &\neq \emptyset \\ \text{Let } \{b\}^1 &= \{1\}^1 \\ (\{+\} * \{1\}^1) * \{a\}^1 &\neq \emptyset \end{aligned}$$

(Eqn2)
(A_{MX19})(Substitution of Eqn2 into Eqn1)(Eqn3)

$$(\{a\}^1 \subseteq N_N) \subseteq ((\{+\} * \{1\}^1) * \{a\}^1 \neq \{0\}^1)$$

(T₅₀₃) Seventh Peano Axiom

Proof of (T₅₀₃)

$$\begin{aligned} \text{Let } \{a\}^1 &\subseteq N_N \\ (\{+\} * \{1\}^1) * \{a\}^1 &\subseteq \sim\{0\}^1 \\ (\{+\} * \{1\}^1) * \{a\}^1 &\not\subseteq \{0\}^1 \\ ((\{+\} * \{1\}^1) * \{a\}^1 \not\subseteq \{0\}^1) &= E \\ ((\{+\} * \{1\}^1) * \{a\}^1 \sim \{0\}^1) &((\{+\} * \{1\}^1) * \{a\}^1 \neq \{0\}^1) = E \\ ((\{+\} * \{1\}^1) * \{a\}^1 \neq \{0\}^1) &= E \\ (\{+\} * \{1\}^1) * \{a\}^1 &\neq \{0\}^1 \\ \text{Thus } (\{a\}^1 \subseteq N_N) &\subseteq ((\{+\} * \{1\}^1) * \{a\}^1 \neq \{0\}^1) \end{aligned}$$

(T₅₇₈)
(T_{M090})(Inverse Unit Subset Rule)
Since $(\{+\} * \{1\}^1) * \{a\}^1 \neq \emptyset$ (T₅₅₅)
(A_{MX16})(Order Axiom)
(T_{M198})
(T_{M084})
(A_{MX16})(Order Axiom)
(A_{MX02})(Conditional Introduction)

$$(\{+\} * \{1\}^1) * N_N \subseteq \sim\{0\}^1$$

(T₆₀₈)

Proof of (T₆₀₈)

$$\begin{aligned} (\{a_1\}^1 \subseteq N_N) &\subseteq ((\{+\} * \{1\}^1) * \{a_1\}^1 \neq \{0\}^1) \\ (\{a_1\}^1 \subseteq N_N) \dots (\{a_n\}^1 \subseteq N_N) &\subseteq ((\{+\} * \{1\}^1) * \{a_1\}^1 \neq \{0\}^1) \dots ((\{+\} * \{1\}^1) * \{a_n\}^1 \neq \{0\}^1) \\ (\{a_1\}^1 \dots \{a_n\}^1 \subseteq N_N) &\subseteq ((\{+\} * \{1\}^1) * \{a_1\}^1 \neq \{0\}^1) \dots ((\{+\} * \{1\}^1) * \{a_n\}^1 \neq \{0\}^1) \\ (\{a_1\}^1 \dots \{a_n\}^1 \subseteq N_N) &\subseteq \sim((\{+\} * \{1\}^1) * \{a_1\}^1 = \{0\}^1) \dots \sim((\{+\} * \{1\}^1) * \{a_n\}^1 = \{0\}^1) \\ (\{a_1\}^1 \dots \{a_n\}^1 \subseteq N_N) &\subseteq \sim(\{0\}^1 \subseteq (\{+\} * \{1\}^1) * \{a_1\}^1 \dots (\{+\} * \{1\}^1) * \{a_n\}^1) \\ (\{a_1\}^1 \dots \{a_n\}^1 \subseteq N_N) &\subseteq \sim(\{0\}^1 \subseteq (\{+\} * \{1\}^1) * (\{a_1\}^1 \dots \{a_n\}^1)) \\ \text{Let } \{a_1\}^1 \dots \{a_n\}^1 &= N_N \\ (N_N \subseteq N_N) &\subseteq \sim(\{0\}^1 \subseteq (\{+\} * \{1\}^1) * N_N) \\ (N_N \subseteq N_N) + (N_N \subseteq N_N) &\subseteq \sim(\{0\}^1 \subseteq (\{+\} * \{1\}^1) * N_N) \\ (N_N \subseteq N_N) + E &\subseteq \sim(\{0\}^1 \subseteq (\{+\} * \{1\}^1) * N_N) \\ E &\subseteq \sim(\{0\}^1 \subseteq (\{+\} * \{1\}^1) * N_N) \\ E &= \sim(\{0\}^1 \subseteq (\{+\} * \{1\}^1) * N_N) \\ \sim(\{0\}^1 \subseteq (\{+\} * \{1\}^1) * N_N) & \\ \{0\}^1 &\not\subseteq (\{+\} * \{1\}^1) * N_N \\ \{0\}^1 &\subseteq (\{+\} * \{1\}^1) * N_N \\ (\{+\} * \{1\}^1) * N_N &\subseteq \sim\{0\}^1 \end{aligned}$$

(T₅₀₃)(Seventh Peano Axiom)(Eqn1)
(A_{MX21})(Combination of Eqn1 for r = 1 to n)
(T_{M192})(Conjunction of Sufficiencies for a Series)
(T_{M188})(DeMorgan's Law for Series)
(T_{M526})
(T_{S282})(Distribution of * over +)(Eqn2)
(Eqn3)
(A_{MX19})(Substitution of Eqn3 into Eqn2)
(T_{M266})(Definition of $\subseteq$)
(T_{M001})(Universality of (D_{M00}) Definition of =)
(T_{M055})(Universal Set Disjunction)
(T_{M275})(Supersets of the Universal Set)
(A_{MX16})(Order Axiom)
(T_{M259})(Definition of $\not\subseteq$)
(T_{M604})
(T_{M012})(Transposition)

$$N_W = N_N.(\{>\} * \{0\}^1)$$

(T₅₇₄)

Proof of (T₅₇₄)

$$\begin{aligned} N_N &\subseteq \{>\} * \{0\}^1 \\ N_N &= N_N.(\{>\} * \{0\}^1) \\ N_W &= N_N. \sim\{0\}^1 \\ N_W &= N_N.(\{>\} * \{0\}^1). \sim\{0\}^1 \\ N_W &= N_N.(\{>\} * \{0\}^1). (\{>\} * \{0\}^1) \\ N_W &= N_N.(\{>\}. \{>\}) * \{0\}^1 \\ N_W &= N_N.(((\{>\} + \{=\}) . \{>\}) * \{0\}^1) \\ N_W &= N_N.(((\{>\}. \{>\} + \{=\}. \{>\})) * \{0\}^1) \\ N_W &= N_N.(((\{>\}. \{>\} + \emptyset) * \{0\}^1) \\ N_W &= N_N.(\{>\}. \{>\}) * \{0\}^1 \end{aligned}$$

(A_{S49})(Axiom of Natural Numbers)
(A_{MX44})($\subseteq$ Equality Form)(Eqn1)
(D_{R90})(Definition of N_W)(Eqn2)
(A_{MX19})(Substitution of Eqn1 into Eqn2)
(T_{A084}) since $\{0\}^1 \neq \emptyset$
(T_{R239})(Distribution of . over * for Entities)
(D_{A141})(Disjunction with $\{=\}$)
(T_{M407})(Distribution of . over +)
(T_{M621})(Complement Conjunction)
(T_{M668})(Null Set Disjunction)

$$\mathbf{N}_W = \mathbf{N}_N.(\{>\}*\{0\}^1) \quad (\mathcal{T}_{S689})$$

$$\{-a\}^1 = (\{-\}*\{a\}^1)*\{0\}^1 \quad (\mathcal{T}_{T685})$$

$$\{-a\}^1 = \{0\}^1 - \{a\}^1 \quad (\mathcal{T}_{T684})$$

Proof of ($\mathcal{T}_{T685}$) & ($\mathcal{T}_{T684}$)

$$(\{+\}*\{-a\}^1) = \vee(\{+\}*\{a\}^1) \quad (\mathcal{A}_{S58})(\text{Axiom of Negative Numbers})$$

$$(\{+\}*\{-a\}^1)*\{0\}^1 = \vee(\{+\}*\{a\}^1)*\{0\}^1 \quad (\mathcal{T}_{R196})(\text{Relational Deduction})$$

$$\{0\}^1 + \{-a\}^1 = \vee(\{+\}*\{a\}^1)*\{0\}^1 \quad (\mathcal{T}_{T471})$$

$$\{-a\}^1 = \vee(\{+\}*\{a\}^1)*\{0\}^1 \quad (\mathcal{T}_{T492})$$

$$\{-a\}^1 = (\{-\}*\{a\}^1)*\{0\}^1 \quad (\mathcal{T}_{T248})$$

$$\{-a\}^1 = \{0\}^1 - \{a\}^1 \quad (\mathcal{T}_{T472})$$

$$\{-1\}^1 = (\{-\}*\{1\}^1)*\{0\}^1 \quad (\mathcal{T}_{T576})$$

Proof of ($\mathcal{T}_{T576}$)

$$\{-a\}^1 = (\{-\}*\{a\}^1)*\{0\}^1 \quad (\mathcal{T}_{T685})(\text{Eqn1})$$

$$\text{Let } \{a\}^1 = \{1\}^1 \quad (\text{Eqn2})$$

$$\{-1\}^1 = (\{-\}*\{1\}^1)*\{0\}^1 \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3})$$

$$(\{b\}^1 \subseteq \{>\}*\{0\}^1) \subseteq ((\{+\}*\{b\}^1)*\{a\}^1 \subseteq \{>\}*\{a\}^1) \quad (\mathcal{T}_{T618})$$

Proof of ($\mathcal{T}_{T618}$)

$$(\{b\}^1 \subseteq \{>\}*\{0\}^1) \subseteq ((\{+\}*\{b\}^1)*\{a\}^1 \subseteq \{>\}*\{a\}^1) \quad (\mathcal{T}_{T570})(\text{Eqn1})$$

$$(\{+\}*\{0\}^1)*\{a\}^1 = \{a\}^1 \quad (\mathcal{T}_{R755})(\text{Eqn2})$$

$$\text{Let } \{b\}^1 = \{0\}^1 \quad (\text{Eqn3})$$

$$(\{+\}*\{b\}^1)*\{a\}^1 = \{a\}^1 \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn3 into Eqn2})$$

$$\text{Thus } (\{b\}^1 = \{0\}^1) \subseteq ((\{+\}*\{b\}^1)*\{a\}^1 = \{a\}^1) \quad (\mathcal{A}_{MX02})(\text{Conditional Introduction})$$

$$(\{b\}^1 \subseteq \{=\}*\{0\}^1) \subseteq ((\{+\}*\{b\}^1)*\{a\}^1 \subseteq \{=\}*\{a\}^1) \quad (\mathcal{T}_{S411})(\text{Definition of an Ordering Relation})(\text{Eqn4})$$

$$(\{b\}^1 \subseteq \{>\}*\{0\}^1) + (\{b\}^1 \subseteq \{=\}*\{0\}^1) \subseteq ((\{+\}*\{b\}^1)*\{a\}^1 \subseteq \{>\}*\{a\}^1) + ((\{+\}*\{b\}^1)*\{a\}^1 \subseteq \{=\}*\{a\}^1)$$

$$(\mathcal{A}_{MX21})(\text{Combination of Eqns1\&4})$$

$$(\{b\}^1 \subseteq \{>\}*\{0\}^1 + \{=\}*\{0\}^1) \subseteq ((\{+\}*\{b\}^1)*\{a\}^1 \subseteq \{>\}*\{a\}^1 + \{=\}*\{a\}^1)$$

$$(\mathcal{T}_{M526})$$

$$(\{b\}^1 \subseteq (\{>\} + \{=\})*\{0\}^1) \subseteq ((\{+\}*\{b\}^1)*\{a\}^1 \subseteq (\{>\} + \{=\})*\{a\}^1)$$

$$(\mathcal{T}_{R002c})(\text{Distribution of } + \text{ over } *)$$

$$(\{b\}^1 \subseteq \{>\}*\{0\}^1) \subseteq ((\{+\}*\{b\}^1)*\{a\}^1 \subseteq \{>\}*\{a\}^1) \quad (\mathcal{D}_{A141})(\text{Disjunction with } \{=\})$$

$$\mathcal{N}a^1 + \mathcal{N}b^1 \subseteq \{>\}\mathcal{N}a^1 \quad (\mathcal{T}_{T617})$$

Proof of ($\mathcal{T}_{T617}$)

$$(\{b\}^1 \subseteq \{>\}*\{0\}^1) \subseteq ((\{+\}*\{b\}^1)*\{a\}^1 \subseteq \{>\}*\{a\}^1) \quad (\mathcal{T}_{T618})(\text{Eqn1})$$

$$\text{Let } \{a\}^1 = \mathcal{N}a^1 \quad (\text{Eqn2})$$

$$\text{Let } \{b\}^1 = \mathcal{N}b^1 \quad (\text{Eqn3})$$

$$(\mathcal{N}b^1 \subseteq \{>\}*\{0\}^1) \subseteq ((\{+\}*\mathcal{N}b^1)*\mathcal{N}a^1 \subseteq \{>\}*\mathcal{N}a^1) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})(\text{Eqn4})$$

$$\mathcal{N}b^1 \subseteq \{>\}*\{0\}^1 \quad (\mathcal{T}_{T581})$$

$$(\mathcal{N}b^1 \subseteq \{>\}*\{0\}^1) = \mathbf{E} \quad (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn5})$$

$$\mathbf{E} \subseteq ((\{+\}*\mathcal{N}b^1)*\mathcal{N}a^1 \subseteq \{>\}*\mathcal{N}a^1) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn5 into Eqn4})$$

$$\mathbf{E} = ((\{+\}*\mathcal{N}b^1)*\mathcal{N}a^1 \subseteq \{>\}*\mathcal{N}a^1) \quad (\mathcal{T}_{M275})(\text{Supersets of the Universal Set})$$

$$(\{+\}*\mathcal{N}b^1)*\mathcal{N}a^1 \subseteq \{>\}*\mathcal{N}a^1 \quad (\mathcal{A}_{MX16})(\text{Order Axiom})$$

$$\mathcal{N}a^1 + \mathcal{N}b^1 \subseteq \{>\}\mathcal{N}a^1 \quad (\mathcal{T}_{T471})$$

$$(\{+\}*\mathcal{N}b^1)*\mathcal{N}a^1 \subseteq \{>\}\mathcal{N}(a + b)^1 \quad (\mathcal{T}_{R746})$$

Proof of ($\mathcal{T}_{R746}$)

$$a = a \quad (\mathcal{D}_{M00})(\text{Definition of } =)$$

$$a = a \wedge b = b \quad (\mathcal{A}_{MX11})(\text{Universal Set Conjunction Axiom})$$

$a = a(b + \underline{b})$	(T_{M671})(Tertium non Datur)
$a = ab + \underline{ab}$	(T_{M407})(Distribution of . over +)(Eqn1)
$\mathcal{N}a^1 = \mathcal{N}(ab + \underline{ab})^1$	(T_{R522})(Relational Deduction)
$\mathcal{N}a^1 = \mathcal{N}(ab)^1 + \mathcal{N}(\underline{ab})^1$	(A_{S37})(Distribution of $\mathcal{N}$ over Disjoint Sets)(Eqn2)
$b = b$	(D_{M00})(Definition of =)
$b = bE$	(A_{MX11})(Universal Set Conjunction Axiom)
$b = b(a + \underline{a})$	(T_{M671})(Tertium non Datur)
$b = ab + \underline{ab}$	(T_{M407})(Distribution of . over +)(Eqn3)
$\mathcal{N}b^1 = \mathcal{N}(ab + \underline{ab})^1$	(T_{R522})(Relational Deduction)
$\mathcal{N}b^1 = \mathcal{N}(ab)^1 + \mathcal{N}(\underline{ab})^1$	(A_{S37})(Distribution of $\mathcal{N}$ over Disjoint Sets)(Eqn4)
$a + b = ab + \underline{ab} + ab + \underline{ab}$	(A_{MX21})(Combination of Eqns1&3)
$a + b = ab + \underline{ab} + \underline{ab}$	(A_{MX01})(Disjunction Tautology)
$\mathcal{N}(a + b)^1 = \mathcal{N}(ab + \underline{ab} + \underline{ab})^1$	(T_{R522})(Relational Deduction)
$\mathcal{N}(a + b)^1 = \mathcal{N}(ab)^1 + \mathcal{N}(\underline{ab})^1 + \mathcal{N}(\underline{ab})^1$	(A_{S37})(Distribution of $\mathcal{N}$ over Disjoint Sets)(Eqn5)
$\mathcal{N}(ab)^1 + \mathcal{N}(\underline{ab})^1 + \mathcal{N}(ab)^1 + \mathcal{N}(\underline{ab})^1 \subseteq \{\geq\}(\mathcal{N}(ab)^1 + \mathcal{N}(\underline{ab})^1)$	(T_{T617})(Eqn6)
$\mathcal{N}a^1 + \mathcal{N}b^1 \subseteq \{\geq\} \mathcal{N}(a + b)^1$	(A_{MX19})(Substitution of Eqns2,4&5 into Eqn6)
$(\{+\} * \mathcal{N}b^1) * \mathcal{N}a^1 \subseteq \{\geq\} \mathcal{N}(a + b)^1$	(T_{T471})
Thus $((a^1 \neq \emptyset)(b^1 \neq \emptyset) = E) \subseteq ((\{+\} * \mathcal{N}b^1) * \mathcal{N}a^1 \subseteq \{\geq\} \mathcal{N}(a + b)^1)$	(A_{MX02})(Conditional Introduction)

$$(\{x\} * \{a\}^1) * \{0\}^1 = \{0\}^1 \quad (T_{T654})$$

Proof of (T_{T654})

$\{a\}^1 - \{a\}^1 = \{0\}^1$	(T_{T498})
$\{a\}^1 \times \{1\}^1 - \{a\}^1 \times \{1\}^1 = \{0\}^1$	(T_{T494})
$(\{x\} * \{a\}^1) * \{1\}^1 - (\{x\} * \{a\}^1) * \{1\}^1 = \{0\}^1$	(T_{T473})
$(\{x\} * \{a\}^1) * (\{1\}^1 - \{1\}^1) = \{0\}^1$	(T_{R789})(Distribution of x over -)
$(\{x\} * \{a\}^1) * \{0\}^1 = \{0\}^1$	(T_{T498})

$$(\{a\}^1 \neq \{0\}^1) \subseteq (\{0\}^1 = (\{\div\} * \{a\}^1) * \{0\}^1) \quad (T_{T655})$$

Proof of (T_{T655})

$(X = (\{+\}, \{-\}, \{x\}, \{\div\})) \subseteq ((SX)^1 \subseteq R_P)$	(A_{R72})(Axiom of Arithmetic Operation)
$(S\{x\})^1 \subseteq R_P$	Extraction
$(\{x\} * E)^1 \subseteq R_P$	(T_{R210})
$(\forall (\{x\} * E))^1 \subseteq R_P$	(T_{S517})
$((\{x\} * E)^1 \subseteq R_P) = E$	(A_{MX16})(Order Axiom)(Eqn1)
$\{a\}^1 \subseteq E$	(T_{M744})(Universal Set Theorem)
$\{x\} * \{a\}^1 \subseteq \{x\} * E$	(T_{R196})(Relational Deduction)
$(\{x\} * \{a\}^1 \subseteq \{x\} * E) = E$	(A_{MX16})(Order Axiom)(Eqn2)
$(\{x\} * \{a\}^1 \subseteq \{x\} * E)((\{x\} * E)^1 \subseteq R_P) \subseteq ((\{x\} * \{a\}^1)^1 \subseteq R_P)$	(T_{T534})(Eqn3)
$EE \subseteq ((\{x\} * \{a\}^1)^1 \subseteq R_P)$	(A_{MX19})(Substitution of Eqn1&2 into Eqn3)
$E \subseteq ((\{x\} * \{a\}^1)^1 \subseteq R_P)$	(T_{M667})(Conjunction Tautology)
$E = ((\{x\} * \{a\}^1)^1 \subseteq R_P)$	(T_{M275})(Supersets of the Universal Set)
$(\{x\} * \{a\}^1)^1 \subseteq R_P$	(A_{MX16})(Order Axiom)
$\forall (\{x\} * \{a\}^1)^1 \subseteq R_P$	(T_{S517})(Eqn4)
$R_P \subseteq R_O$	(T_{R523})(Eqn5)
$\forall (\{x\} * \{a\}^1)^1 \subseteq R_O$	(T_{M139})(Hypothetical Syllogism Eqns4&5)
$(\forall (\{x\} * \{a\}^1)^1 \subseteq R_O) = E$	(A_{MX16})(Order Axiom)(Eqn6)
$(\{a\}^1 \neq \{0\}^1) \subseteq (Q(\{x\} * \{a\}^1) = N_C)$	(T_{T537})(Eqn7)
$\{0\}^1 \subseteq N_N$	(T_{T501})(First Peano Axiom)(Eqn8)
$N_N \subseteq N_C$	(T_{T544})(Eqn9)
$\{0\}^1 \subseteq N_C$	(T_{M139})(Hypothetical Syllogism Eqns8&9)(Eqn10)
$(\{a\}^1 \neq \{0\}^1) \subseteq (\{0\}^1 \subseteq Q(\{x\} * \{a\}^1))$	(A_{MX19})(Substitution of Eqn7 into Eqn10)
$(\forall (\{x\} * \{a\}^1)^1 \subseteq R_O)(\{a\}^1 \neq \{0\}^1) \subseteq (\forall (\{x\} * \{a\}^1)^1 \subseteq R_O)(\{0\}^1 \subseteq Q(\{x\} * \{a\}^1))$	(T_{M029})(Deduction)(Eqn11)

$$\begin{aligned}
& (\vee(\{x\} * \{a\}^1) \subseteq \mathbf{R}_0)(\{0\}^1 \subseteq \mathbf{Q}(\{x\} * \{a\}^1)) \subseteq (\{0\}^1 = \vee(\{x\} * \{a\}^1) * (\{x\} * \{a\}^1) * \{0\}^1) \\
& \quad (\mathcal{T}_{S519})(\text{Eqn12}) \\
& (\vee(\{x\} * \{a\}^1) \subseteq \mathbf{R}_0)(\{a\}^1 \neq \{0\}^1) \subseteq (\{0\}^1 = \vee(\{x\} * \{a\}^1) * (\{x\} * \{a\}^1) * \{0\}^1) \\
& \quad (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns11\&12})(\text{Eqn13}) \\
& \mathbf{E}(\{a\}^1 \neq \{0\}^1) \subseteq (\{0\}^1 = \vee(\{x\} * \{a\}^1) * (\{x\} * \{a\}^1) * \{0\}^1) \quad (\mathbf{A}_{MX19})(\text{Substitution of Eqn6 into Eqn13}) \\
& (\{a\}^1 \neq \{0\}^1) \subseteq (\{0\}^1 = \vee(\{x\} * \{a\}^1) * (\{x\} * \{a\}^1) * \{0\}^1) \quad (\mathbf{A}_{MX17})(\text{Universal Set Conjunction Axiom})(\text{Eqn14}) \\
& (\{x\} * \{a\}^1) * \{0\}^1 = \{0\}^1 \quad (\mathcal{T}_{T654}) \\
& \vee(\{x\} * \{a\}^1) * (\{x\} * \{a\}^1) * \{0\}^1 = \vee(\{x\} * \{a\}^1) * \{0\}^1 \quad (\mathcal{T}_{R522})(\text{Relational Deduction})(\text{Eqn15}) \\
& (\{a\}^1 \neq \{0\}^1) \subseteq (\{0\}^1 = \vee(\{x\} * \{a\}^1) * \{0\}^1) \quad (\mathbf{A}_{MX19})(\text{Substitution of Eqn15 into Eqn14}) \\
& (\{a\}^1 \neq \{0\}^1) \subseteq (\{0\}^1 = (\{x\} * \{a\}^1) * \{0\}^1) \quad (\mathcal{T}_{T244})
\end{aligned}$$

$$\{0\}^1 \times \{a\}^1 = \{0\}^1 \quad (\mathcal{T}_{T656})$$

$$(\{a\}^1 \neq \{0\}^1) \subseteq (\{0\}^1 \div \{a\}^1 = \{0\}^1) \quad (\mathcal{T}_{T657})$$

Example Proof of ($\mathcal{T}_{T656}$)

$$(\{x\} * \{a\}^1) * \{0\}^1 = \{0\}^1 \quad (\mathcal{T}_{T654})$$

$$\{0\}^1 \times \{a\}^1 = \{0\}^1 \quad (\mathcal{T}_{T473})$$

$$\begin{aligned}
& (\{\wedge\} * \{c\}^1) * (\{a\}^1 \times \{b\}^1) = (\{\wedge\} * \{c\}^1) * \{a\}^1 \times (\{\wedge\} * \{c\}^1) * \{b\}^1 \quad (\mathcal{T}_{T664}) \text{ Distribution of } \wedge \text{ over } \times \\
& (\{\wedge\} * \{c\}^1) * (\{a\}^1 \div \{b\}^1) = (\{\wedge\} * \{c\}^1) * \{a\}^1 \div (\{\wedge\} * \{c\}^1) * \{b\}^1 \quad (\mathcal{T}_{T665}) \text{ Distribution of } \wedge \text{ over } \div \\
& (\{\vee\} * \{c\}^1) * (\{a\}^1 \times \{b\}^1) = (\{\vee\} * \{c\}^1) * \{a\}^1 \times (\{\vee\} * \{c\}^1) * \{b\}^1 \quad (\mathcal{T}_{T666}) \text{ Distribution of } \vee \text{ over } \times \\
& (\{\vee\} * \{c\}^1) * (\{a\}^1 \div \{b\}^1) = (\{\vee\} * \{c\}^1) * \{a\}^1 \div (\{\vee\} * \{c\}^1) * \{b\}^1 \quad (\mathcal{T}_{T667}) \text{ Distribution of } \vee \text{ over } \div \\
& (\{a\}^1 \times \{b\}^1) \wedge \{c\}^1 = (\{a\}^1 \wedge \{c\}^1) \times (\{b\}^1 \wedge \{c\}^1) \quad (\mathcal{T}_{T668}) \text{ Distribution of } \wedge \text{ over } \times \\
& (\{a\}^1 \div \{b\}^1) \wedge \{c\}^1 = (\{a\}^1 \wedge \{c\}^1) \div (\{b\}^1 \wedge \{c\}^1) \quad (\mathcal{T}_{T669}) \text{ Distribution of } \wedge \text{ over } \div \\
& (\{a\}^1 \times \{b\}^1) \vee \{c\}^1 = (\{a\}^1 \vee \{c\}^1) \times (\{b\}^1 \vee \{c\}^1) \quad (\mathcal{T}_{T670}) \text{ Distribution of } \vee \text{ over } \times \\
& (\{a\}^1 \div \{b\}^1) \vee \{c\}^1 = (\{a\}^1 \vee \{c\}^1) \div (\{b\}^1 \vee \{c\}^1) \quad (\mathcal{T}_{T671}) \text{ Distribution of } \vee \text{ over } \div
\end{aligned}$$

Example Proof of ($\mathcal{T}_{T668}$) & ($\mathcal{T}_{T664}$)

$$\begin{aligned}
& (\{\wedge\}^1 \subseteq \mathbf{R}_0 * \{x\}^1) = ((\{a\}^1 \times \{b\}^1) \wedge \{c\}^1 = (\{a\}^1 \wedge \{c\}^1) \times (\{b\}^1 \wedge \{c\}^1)) \\
& \quad (\mathbf{D}_{R41})(\text{Definition of a Distributive Operator})(\text{Eqn1}) \\
& (\mathbf{X} = \{\wedge\}, \{\vee\})(\mathbf{Y} = \{x\}, \{\div\}) \subseteq (\mathbf{X}^1 \subseteq \mathbf{R}_0 * \mathbf{Y}^1) \quad (\mathbf{A}_{S56})(2^{\text{nd}} \text{ Arithmetical Distribution Axiom}) \\
& \{\wedge\}^1 \subseteq \mathbf{R}_0 * \{x\}^1 \quad \text{Extraction} \\
& (\{\wedge\}^1 \subseteq \mathbf{R}_0 * \{x\}^1) = \mathbf{E} \quad (\mathbf{A}_{MX16})(\text{Order Axiom})(\text{Eqn2}) \\
& \mathbf{E} = ((\{a\}^1 \times \{b\}^1) \wedge \{c\}^1 = (\{a\}^1 \wedge \{c\}^1) \times (\{b\}^1 \wedge \{c\}^1)) \quad (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
& (\{a\}^1 \times \{b\}^1) \wedge \{c\}^1 = (\{a\}^1 \wedge \{c\}^1) \times (\{b\}^1 \wedge \{c\}^1) \quad (\mathbf{A}_{MX16})(\text{Order Axiom}) \\
& (\{\wedge\} * \{c\}^1) * (\{a\}^1 \times \{b\}^1) = (\{\wedge\} * \{c\}^1) * \{a\}^1 \times (\{\wedge\} * \{c\}^1) * \{b\}^1 \\
& \quad (\mathcal{T}_{T473})
\end{aligned}$$

$$\{a\}^1 \times \{b\}^1 \wedge \{-1\}^1 = \{a\}^1 \div \{b\}^1 \quad (\mathcal{T}_{T661})$$

Proof of ($\mathcal{T}_{T661}$)

$$\begin{aligned}
& (\{\wedge\} * \{-1\}^1) * \{a\}^1 = \vee(\{x\} * ((\{\wedge\} * \{b\}^1) * \{a\}^1)) * \{1\}^1 \quad (\mathbf{D}_{S02})(\text{Definition of a Negative Index})(\text{Eqn1}) \\
& \text{Let } \{b\}^1 = \{1\}^1 \quad (\text{Eqn2}) \\
& \text{Let } \{a\}^1 = \{c\}^1 \quad (\text{Eqn3}) \\
& (\{\wedge\} * \{-1\}^1) * \{c\}^1 = \vee(\{x\} * ((\{\wedge\} * \{1\}^1) * \{c\}^1)) * \{1\}^1 \quad (\mathbf{A}_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\
& (\{\wedge\} * \{-1\}^1) * \{c\}^1 = \vee(\{x\} * \{c\}^1) * \{1\}^1 \quad (\mathcal{T}_{T469}) \\
& (\{x\} * \{c\}^1) * (\{\wedge\} * \{-1\}^1) * \{c\}^1 = (\{x\} * \{c\}^1) * \vee(\{x\} * \{c\}^1) * \{1\}^1 \quad (\mathcal{T}_{R196})(\text{Relational Deduction}) \\
& (\{x\} * \{c\}^1) * (\{\wedge\} * \{-1\}^1) * \{c\}^1 = \{1\}^1 \quad (\mathcal{T}_{S519}) \\
& \text{Since } (\{x\} * \{c\}^1)^1 \subseteq \mathbf{R}_0 \text{ and } \{1\}^1 \subseteq \mathbf{Q}^{\vee}(\{x\} * \{c\}^1) \\
& ((\{\wedge\} * \{-1\}^1) * \{c\}^1) \times \{c\}^1 = \{1\}^1 \quad (\mathcal{T}_{T473}) \\
& \{c\}^1 \wedge \{-1\}^1 \times \{c\}^1 = \{1\}^1 \quad (\mathcal{T}_{T475}) \\
& \{a\}^1 \times \{c\}^1 \times \{c\}^1 \wedge \{-1\}^1 = \{a\}^1 \times \{1\}^1 \quad (\mathcal{T}_{R862}) \\
& \{a\}^1 \times \{c\}^1 \times \{c\}^1 \wedge \{-1\}^1 = \{a\}^1 \quad (\mathcal{T}_{T494}) \\
& \{a\}^1 \times \{c\}^1 \times \{c\}^1 \wedge \{-1\}^1 \div \{c\}^1 = \{a\}^1 \div \{c\}^1 \quad (\mathcal{T}_{R864})
\end{aligned}$$

$$\begin{aligned} \{a\}^1 \times \{1\}^1 \times \{c\}^1 \wedge \{-1\}^1 &= \{a\}^1 \div \{c\}^1 & (T_{7499}) \\ \{a\}^1 \times \{c\}^1 \wedge \{-1\}^1 &= \{a\}^1 \div \{c\}^1 & (T_{7494})(\text{Eqn4}) \\ \{a\}^1 \times \{b\}^1 \wedge \{-1\}^1 &= \{a\}^1 \div \{b\}^1 & \text{Resymbolising} \end{aligned}$$

$$\{a\}^1 \times (\{b\}^1 \div \{c\}^1) = (\{a\}^1 \times \{b\}^1) \div \{c\}^1 \quad (T_{7660})$$

Proof of (T₇₆₆₀)

$$\begin{aligned} \{a\}^1 \times (\{b\}^1 \times \{c\}^1) &= (\{a\}^1 \times \{b\}^1) \times \{c\}^1 & (T_{7481})(\text{Associativity of Algebraic } x)(\text{Eqn1}) \\ \text{Let } \{c\}^1 &= (\{c\}^1 \wedge \{-1\}^1) & (\text{Eqn2}) \\ \{a\}^1 \times (\{b\}^1 \times (\{c\}^1 \wedge \{-1\}^1)) &= (\{a\}^1 \times \{b\}^1) \times (\{c\}^1 \wedge \{-1\}^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \{a\}^1 \times (\{b\}^1 \div \{c\}^1) &= (\{a\}^1 \times \{b\}^1) \div \{c\}^1 & (T_{7661}) \end{aligned}$$

$$(\{a\}^1 \div \{b\}^1) \div \{c\}^1 = \{a\}^1 \div (\{b\}^1 \times \{c\}^1) \quad (T_{7662})$$

Proof of (T₇₆₆₂)

$$\begin{aligned} (\{a\}^1 \times \{b\}^1) \times \{c\}^1 &= \{a\}^1 \times (\{b\}^1 \times \{c\}^1) & (T_{7481})(\text{Associativity of Algebraic } x)(\text{Eqn1}) \\ \text{Let } \{b\}^1 &= (\{b\}^1 \wedge \{-1\}^1) & (\text{Eqn2}) \\ \text{Let } \{c\}^1 &= (\{c\}^1 \wedge \{-1\}^1) & (\text{Eqn3}) \\ (\{a\}^1 \times \{b\}^1 \wedge \{-1\}^1) \times (\{c\}^1 \wedge \{-1\}^1) &= \{a\}^1 \times ((\{b\}^1 \wedge \{-1\}^1) \times (\{c\}^1 \wedge \{-1\}^1)) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\ (\{a\}^1 \times \{b\}^1 \wedge \{-1\}^1) \times (\{c\}^1 \wedge \{-1\}^1) &= \{a\}^1 \times ((\{b\}^1 \times \{c\}^1) \wedge \{-1\}^1) & (T_{7668})(\text{Distribution of } \wedge \text{ over } x) \\ (\{a\}^1 \div \{b\}^1) \div \{c\}^1 &= \{a\}^1 \div (\{b\}^1 \times \{c\}^1) & (T_{7661}) \end{aligned}$$

$$(\{a\}^1 \times \{b\}^1) \div (\{c\}^1 \times \{d\}^1) = (\{a\}^1 \div \{c\}^1) \times (\{b\}^1 \div \{d\}^1) \quad (T_{7663})$$

Proof of (T₇₆₆₃)

$$\begin{aligned} (\{a\}^1 \times \{b\}^1) \times \{c\}^1 &= \{a\}^1 \times (\{b\}^1 \times \{c\}^1) & (T_{7481})(\text{Associativity of Algebraic } x)(\text{Eqn1}) \\ \text{Let } \{c\}^1 &= (\{c\}^1 \times \{d\}^1) & (\text{Eqn2}) \\ (\{a\}^1 \times \{b\}^1) \times (\{c\}^1 \times \{d\}^1) &= \{a\}^1 \times (\{b\}^1 \times (\{c\}^1 \times \{d\}^1)) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (\{a\}^1 \times \{b\}^1) \times (\{c\}^1 \times \{d\}^1) &= \{a\}^1 \times ((\{c\}^1 \times \{d\}^1) \times \{b\}^1) & (T_{7483})(\text{Commutativity of Algebraic } x) \\ (\{a\}^1 \times \{b\}^1) \times (\{c\}^1 \times \{d\}^1) &= \{a\}^1 \times (\{c\}^1 \times (\{d\}^1 \times \{b\}^1)) & (T_{7481})(\text{Associativity of Algebraic } x) \\ (\{a\}^1 \times \{b\}^1) \times (\{c\}^1 \times \{d\}^1) &= \{a\}^1 \times (\{c\}^1 \times (\{b\}^1 \times \{d\}^1)) & (T_{7483})(\text{Commutativity of Algebraic } x) \\ (\{a\}^1 \times \{b\}^1) \times (\{c\}^1 \times \{d\}^1) &= (\{a\}^1 \times \{c\}^1) \times (\{b\}^1 \times \{d\}^1) & (T_{7481})(\text{Associativity of Algebraic } x)(\text{Eqn3}) \\ \text{Let } \{c\}^1 &= \{c\}^1 \wedge \{-1\}^1 & (\text{Eqn4}) \\ \text{Let } \{d\}^1 &= \{d\}^1 \wedge \{-1\}^1 & (\text{Eqn5}) \\ (\{a\}^1 \times \{b\}^1) \times ((\{c\}^1 \wedge \{-1\}^1) \times (\{d\}^1 \wedge \{-1\}^1)) &= (\{a\}^1 \times (\{c\}^1 \wedge \{-1\}^1)) \times (\{b\}^1 \times (\{d\}^1 \wedge \{-1\}^1)) & (A_{MX19})(\text{Substitution of Eqns4\&5 into Eqn3}) \\ (\{a\}^1 \times \{b\}^1) \times ((\{c\}^1 \times \{d\}^1) \wedge \{-1\}^1) &= (\{a\}^1 \times (\{c\}^1 \wedge \{-1\}^1)) \times (\{b\}^1 \times (\{d\}^1 \wedge \{-1\}^1)) & (T_{7668})(\text{Distribution of } \wedge \text{ over } x) \\ (\{a\}^1 \times \{b\}^1) \div (\{c\}^1 \times \{d\}^1) &= (\{a\}^1 \div \{c\}^1) \times (\{b\}^1 \div \{d\}^1) & (T_{7661}) \end{aligned}$$

$$\{1\}^1 \div (\{a\}^1 \times \{b\}^1) = (\{1\}^1 \div \{a\}^1) \times (\{1\}^1 \div \{b\}^1) \quad (T_{7675})$$

Proof of (T₇₆₇₅)

$$\begin{aligned} (\{a\}^1 \times \{b\}^1) \div (\{c\}^1 \times \{d\}^1) &= (\{a\}^1 \div \{c\}^1) \times (\{b\}^1 \div \{d\}^1) & (T_{7663})(\text{Eqn1}) \\ \text{Let } \{a\}^1 &= \{1\}^1 & (\text{Eqn2}) \\ \text{Let } \{b\}^1 &= \{1\}^1 & (\text{Eqn3}) \\ (\{1\}^1 \times \{1\}^1) \div (\{c\}^1 \times \{d\}^1) &= (\{1\}^1 \div \{c\}^1) \times (\{1\}^1 \div \{d\}^1) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\ \{1\}^1 \div (\{c\}^1 \times \{d\}^1) &= (\{1\}^1 \div \{c\}^1) \times (\{1\}^1 \div \{d\}^1) & (T_{7494}) \\ \{1\}^1 \div (\{a\}^1 \times \{b\}^1) &= (\{1\}^1 \div \{a\}^1) \times (\{1\}^1 \div \{b\}^1) & \text{Resymbolising} \end{aligned}$$

$$\{a\}^1 \wedge (\{1\}^1 \div \{b\}^1) = \{a\}^1 \sqrt{\{b\}^1} \quad (T_{7674})$$

Proof of (T₇₆₇₄)

$$\begin{aligned} (\sqrt{\{a\}^1} \times \{b\}^1) \times \{a\}^1 &= (\{a\}^1 \times (\{b\}^1 \times \{a\}^1)) \times \{a\}^1 & (D_{S01})(\text{Definition of Root}) \\ (\sqrt{\{a\}^1} \times \{b\}^1) \times \{a\}^1 &= (\{a\}^1 \times ((\{b\}^1 \times \{a\}^1) \times \{a\}^1)) \times \{a\}^1 & (T_{7244}) \\ \{a\}^1 \sqrt{\{b\}^1} &= \{a\}^1 \wedge (\{1\}^1 \div \{b\}^1) & (T_{7476}), (T_{7475}) \& (T_{7474}) \end{aligned}$$

$$(\{a\}^1 \wedge \{b\}^1) \wedge \{c\}^1 = \{a\}^1 \wedge (\{b\}^1 \times \{c\}^1) \quad (\mathcal{T}_{7672})$$

Proof of ($\mathcal{T}_{7672}$)

$$\begin{aligned} (\{a\}^1 \times \{c\}^1) \times (\{b\}^1 \times \{a\}^1) &= (\{a\}^1 \times (\{b\}^1 \times \{c\}^1)) \times \{a\}^1 & (\mathcal{A}_{557}) (\text{Axiom of Index Multiplication}) \\ (\{a\}^1 \wedge \{b\}^1) \wedge \{c\}^1 &= \{a\}^1 \wedge (\{b\}^1 \times \{c\}^1) & (\mathcal{T}_{7475}) \& (\mathcal{T}_{7473}) \end{aligned}$$

$$(\{a\}^1 \vee \{b\}^1) \vee \{c\}^1 = \{a\}^1 \vee (\{b\}^1 \times \{c\}^1) \quad (\mathcal{T}_{7673})$$

Proof of ($\mathcal{T}_{7673}$)

$$\begin{aligned} (\{a\}^1 \wedge \{b\}^1) \wedge \{c\}^1 &= \{a\}^1 \wedge (\{b\}^1 \times \{c\}^1) & (\mathcal{T}_{7672}) (\text{Eqn1}) \\ \text{Let } \{b\}^1 &= (\{1\}^1 \div \{b\}^1) & (\text{Eqn2}) \\ \text{Let } \{c\}^1 &= (\{1\}^1 \div \{c\}^1) & (\text{Eqn3}) \\ (\{a\}^1 \wedge (\{1\}^1 \div \{b\}^1)) \wedge (\{1\}^1 \div \{c\}^1) &= \{a\}^1 \wedge ((\{1\}^1 \div \{b\}^1) \times (\{1\}^1 \div \{c\}^1)) & (\mathcal{A}_{MX19}) (\text{Substitution of Eqns2\&3 into Eqn1}) \\ (\{a\}^1 \wedge (\{1\}^1 \div \{b\}^1)) \wedge (\{1\}^1 \div \{c\}^1) &= \{a\}^1 \wedge (\{1\}^1 \div (\{b\}^1 \times \{c\}^1)) & (\mathcal{T}_{7675}) \\ (\{a\}^1 \vee \{b\}^1) \vee \{c\}^1 &= \{a\}^1 \vee (\{b\}^1 \times \{c\}^1) & (\mathcal{T}_{7674}) \end{aligned}$$

$$\{a\}^1 + \{-b\}^1 = \{a\}^1 - \{b\}^1 \quad (\mathcal{T}_{7686})$$

Proof of ($\mathcal{T}_{7686}$)

$$\begin{aligned} (\{+\} \times \{-b\}^1) &= \vee (\{+\} \times \{b\}^1) & (\mathcal{A}_{558}) (\text{Axiom of Negative Numbers}) \\ (\{+\} \times \{-b\}^1) \times \{a\}^1 &= \vee (\{+\} \times \{b\}^1) \times \{a\}^1 & (\mathcal{T}_{7196}) (\text{Relational Deduction}) \\ (\{+\} \times \{-b\}^1) \times \{a\}^1 &= (\{-\} \times \{b\}^1) \times \{a\}^1 & (\mathcal{T}_{7248}) \\ \{a\}^1 + \{-b\}^1 &= \{a\}^1 - \{b\}^1 & (\mathcal{T}_{7471}) \& (\mathcal{T}_{7472}) \end{aligned}$$

$$(\{a\}^1 + \{b\}^1) - \{c\}^1 = \{a\}^1 + (\{b\}^1 - \{c\}^1) \quad (\mathcal{T}_{7683})$$

Proof of ($\mathcal{T}_{7683}$)

$$\begin{aligned} (\{a\}^1 + \{b\}^1) + \{c\}^1 &= \{a\}^1 + (\{b\}^1 + \{c\}^1) & (\mathcal{T}_{7480}) (\text{Associativity of Algebraic } +) (\text{Eqn1}) \\ \text{Let } \{c\}^1 &= \{-c\}^1 & (\text{Eqn2}) \\ (\{a\}^1 + \{b\}^1) + \{-c\}^1 &= \{a\}^1 + (\{b\}^1 + \{-c\}^1) & (\mathcal{A}_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\ (\{a\}^1 + \{b\}^1) - \{c\}^1 &= \{a\}^1 + (\{b\}^1 - \{c\}^1) & (\mathcal{T}_{7686}) \end{aligned}$$

$$\{a\}^1 \times (\{b\}^1 \div \{c\}^1) = \{b\}^1 \times (\{a\}^1 \div \{c\}^1) \quad (\mathcal{T}_{7679})$$

Proof of ($\mathcal{T}_{7679}$)

$$\begin{aligned} \{a\}^1 \times (\{b\}^1 \div \{c\}^1) &= (\{a\}^1 \times \{b\}^1) \div \{c\}^1 & (\mathcal{T}_{7660}) \\ \{a\}^1 \times (\{b\}^1 \div \{c\}^1) &= (\{b\}^1 \times \{a\}^1) \div \{c\}^1 & (\mathcal{T}_{7483}) (\text{Commutativity of Algebraic } x) (\text{Eqn1}) \\ \{b\}^1 \times (\{a\}^1 \div \{c\}^1) &= (\{b\}^1 \times \{a\}^1) \div \{c\}^1 & (\mathcal{T}_{7660}) (\text{Eqn2}) \\ \{b\}^1 \times (\{a\}^1 \div \{c\}^1) &= \{a\}^1 \times (\{b\}^1 \div \{c\}^1) & (\mathcal{A}_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(\{a\}^1 = \{b\}^1 + \{c\}^1) = (\{a\}^1 - \{c\}^1 = \{b\}^1) \quad (\mathcal{T}_{7681})$$

Proof of ($\mathcal{T}_{7681}$)

$$\begin{aligned} \text{Let } \{a\}^1 &= \{b\}^1 + \{c\}^1 \\ \{a\}^1 - \{c\}^1 &= (\{b\}^1 + \{c\}^1) - \{c\}^1 & (\mathcal{T}_{7660}) \\ \{a\}^1 - \{c\}^1 &= \{b\}^1 + (\{c\}^1 - \{c\}^1) & (\mathcal{T}_{7683}) \\ \{a\}^1 - \{c\}^1 &= \{b\}^1 + \{0\}^1 & (\mathcal{T}_{7498}) \\ \{a\}^1 - \{c\}^1 &= \{b\}^1 & (\mathcal{T}_{7492}) \\ \text{Thus } (\{a\}^1 = \{b\}^1 + \{c\}^1) &= (\{a\}^1 - \{c\}^1 = \{b\}^1) & (\mathcal{A}_{MX02}) (\text{Conditional Introduction}) \text{ since this proof is reversible} \end{aligned}$$

$$(\{a\}^1 = \{b\}^1 \times \{c\}^1) = (\{a\}^1 \div \{c\}^1 = \{b\}^1) \quad (\mathcal{T}_{7680})$$

Proof of ($\mathcal{T}_{7680}$)

$$\begin{aligned} \text{Let } \{a\}^1 \div \{c\}^1 &= \{b\}^1 & (\text{Eqn1}) \\ (\{a\}^1 \div \{c\}^1) \times \{c\}^1 &= \{b\}^1 \times \{c\}^1 & (\mathcal{T}_{7662}) \\ \{c\}^1 \times (\{a\}^1 \div \{c\}^1) &= \{b\}^1 \times \{c\}^1 & (\mathcal{T}_{7483}) (\text{Commutativity of Algebraic } x) \\ \{a\}^1 \times (\{c\}^1 \div \{c\}^1) &= \{b\}^1 \times \{c\}^1 & (\mathcal{T}_{7679}) \end{aligned}$$

$$\begin{aligned}
\{a\}^1 \times \{1\}^1 &= \{b\}^1 \times \{c\}^1 & (T_{7499}) \\
\{a\}^1 &= \{b\}^1 \times \{c\}^1 & (T_{7494})(\text{Eqn2}) \\
\text{Thus } (\{a\}^1 = \{b\}^1 \times \{c\}^1) &= (\{a\}^1 \div \{c\}^1 = \{b\}^1) & (A_{MX02})(\text{Conditional Introduction}) \text{ since this proof is reversible}
\end{aligned}$$

$$(\{b\}^1 \neq \{0\}^1) \subseteq (\{a\}^1 \subseteq (\{a\}^1 \wedge \{b\}^1) \vee \{b\}^1) \quad (T_{7687})$$

Proof of (T₇₆₈₇)

$$\begin{aligned}
(\{b\}^1 \neq \{0\}^1) &\subseteq (Q(\{\wedge\} * \{b\}^1) = N_c) & (T_{7540})(\text{Eqn1}) \\
\{a\}^1 &\subseteq N_c & (D_{R18})(\text{Definition of Numbers})(\text{Eqn2}) \\
(\{b\}^1 \neq \{0\}^1) &\subseteq (\{a\}^1 \subseteq Q(\{\wedge\} * \{b\}^1)) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})(\text{Eqn3}) \\
(\{a\}^1 \subseteq Q(\{\wedge\} * \{b\}^1)) &\subseteq (\{a\}^1 \subseteq \vee(\{\wedge\} * \{b\}^1) * (\{\wedge\} * \{b\}^1) * \{a\}^1) & (T_{R526})(\text{Eqn4}) \\
(\{b\}^1 \neq \{0\}^1) &\subseteq (\{a\}^1 \subseteq \vee(\{\wedge\} * \{b\}^1) * (\{\wedge\} * \{b\}^1) * \{a\}^1) & (T_{M139})(\text{Hypothetical Syllogism Eqns3\&4}) \\
(\{b\}^1 \neq \{0\}^1) &\subseteq (\{a\}^1 \subseteq (\{\vee\} * \{b\}^1) * (\{\wedge\} * \{b\}^1) * \{a\}^1) & (T_{7235}) \\
(\{b\}^1 \neq \{0\}^1) &\subseteq (\{a\}^1 \subseteq (\{a\}^1 \wedge \{b\}^1) \vee \{b\}^1) & (T_{7475}) \& (T_{7476})
\end{aligned}$$

$$(\{c\}^1 \neq \{0\}^1) = E \subseteq ((\{a\}^1 = \{b\}^1 \wedge \{c\}^1) \subseteq (\{b\}^1 \subseteq \{a\}^1 \vee \{c\}^1)) \quad (T_{7682})$$

Proof of (T₇₆₈₂)

$$\begin{aligned}
\text{Let } \{a\}^1 &= \{b\}^1 \wedge \{c\}^1 \\
\{a\}^1 \vee \{c\}^1 &= (\{b\}^1 \wedge \{c\}^1) \vee \{c\}^1 & (T_{R868})(\text{Eqn1}) \\
(\{c\}^1 \neq \{0\}^1) &\subseteq (\{b\}^1 \subseteq (\{b\}^1 \wedge \{c\}^1) \vee \{c\}^1) & (T_{7687})(\text{Eqn2}) \\
(\{c\}^1 \neq \{0\}^1) &\subseteq (\{b\}^1 \subseteq \{a\}^1 \vee \{c\}^1) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\
\text{Thus } ((\{a\}^1 = \{b\}^1 \wedge \{c\}^1) = E) &\subseteq ((\{c\}^1 \neq \{0\}^1) \subseteq (\{b\}^1 \subseteq \{a\}^1 \vee \{c\}^1)) & (A_{MX02})(\text{Conditional Introduction}) \\
(\{a\}^1 = \{b\}^1 \wedge \{c\}^1)(\{c\}^1 \neq \{0\}^1) &\subseteq (\{b\}^1 \subseteq \{a\}^1 \vee \{c\}^1) & (A_{MX35})(\text{Exportation/Importation}) \text{ For truth sets} \\
((\{c\}^1 \neq \{0\}^1) = E) &\subseteq ((\{a\}^1 = \{b\}^1 \wedge \{c\}^1) \subseteq (\{b\}^1 \subseteq \{a\}^1 \vee \{c\}^1)) & (A_{MX35})(\text{Exportation/Importation}) \text{ For truth sets}
\end{aligned}$$

$$(\{a\}^1 \div \{c\}^1 = \{b\}^1 \div \{d\}^1) \subseteq ((\{a\}^1 + \{b\}^1) \div (\{c\}^1 + \{d\}^1) = \{a\}^1 \div \{c\}^1) \quad (T_{7677})$$

Proof of (T₇₆₇₇)

$$\begin{aligned}
\text{Let } \{a\}^1 \div \{c\}^1 &= \{n\}^1 & (\text{Eqn1}) \\
\{a\}^1 &= \{n\}^1 \times \{c\}^1 & (T_{7680})(\text{Eqn2}) \\
\text{Let } \{b\}^1 \div \{d\}^1 &= \{n\}^1 & (\text{Eqn3}) \\
\{b\}^1 &= \{n\}^1 \times \{d\}^1 & (T_{7680})(\text{Eqn4}) \\
\{a\}^1 + \{b\}^1 &= (\{n\}^1 \times \{c\}^1) + (\{n\}^1 \times \{d\}^1) & (T_{7553})(\text{Arithmetical Combination of Eqns 2\&4}) \\
\{a\}^1 + \{b\}^1 &= \{n\}^1 \times (\{c\}^1 + \{d\}^1) & (T_{7484})(\text{Distribution of } x \text{ over } +) \\
(\{a\}^1 + \{b\}^1) \div (\{c\}^1 + \{d\}^1) &= (\{n\}^1 \times (\{c\}^1 + \{d\}^1)) \div (\{c\}^1 + \{d\}^1) & (T_{R864}) \\
(\{a\}^1 + \{b\}^1) \div (\{c\}^1 + \{d\}^1) &= \{n\}^1 \times ((\{c\}^1 + \{d\}^1) \div (\{c\}^1 + \{d\}^1)) & (T_{7679}) \\
(\{a\}^1 + \{b\}^1) \div (\{c\}^1 + \{d\}^1) &= \{n\}^1 \times \{1\}^1 & (T_{7499}) \\
(\{a\}^1 + \{b\}^1) \div (\{c\}^1 + \{d\}^1) &= \{n\}^1 & (T_{7494}) \\
\text{Thus } (\{a\}^1 \div \{c\}^1 = \{n\}^1)(\{b\}^1 \div \{d\}^1 = \{n\}^1) &\subseteq ((\{a\}^1 + \{b\}^1) \div (\{c\}^1 + \{d\}^1) = \{n\}^1) & (A_{MX02})(\text{Conditional Introduction})(\text{Eqn5}) \\
(\{a\}^1 \div \{c\}^1 = \{a\}^1 \div \{c\}^1)(\{b\}^1 \div \{d\}^1 = \{a\}^1 \div \{c\}^1) &\subseteq ((\{a\}^1 + \{b\}^1) \div (\{c\}^1 + \{d\}^1) = \{a\}^1 \div \{c\}^1) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn5}) \\
E(\{b\}^1 \div \{d\}^1 = \{a\}^1 \div \{c\}^1) &\subseteq ((\{a\}^1 + \{b\}^1) \div (\{c\}^1 + \{d\}^1) = \{a\}^1 \div \{c\}^1) & (T_{M001})(\text{Universality of } (D_{M00}) \text{ Definition of } =) \\
(\{b\}^1 \div \{d\}^1 = \{a\}^1 \div \{c\}^1) &\subseteq ((\{a\}^1 + \{b\}^1) \div (\{c\}^1 + \{d\}^1) = \{a\}^1 \div \{c\}^1) & (A_{MX11})(\text{Universal Set Conjunction Axiom})
\end{aligned}$$

$$(a = b) \subseteq (a^1 =_{\mathcal{N}} b^1) \quad (T_{7424})$$

$$(a \subset b) \subseteq (a^1 <_{\mathcal{N}} b^1) \quad (T_{7425})$$

$$(a \supset b) \subseteq (a^1 >_{\mathcal{N}} b^1) \quad (T_{7427})$$

$$(a \subseteq b) \subseteq (a^1 \leq_{\mathcal{N}} b^1) \quad (T_{7430})$$

$$(a \supseteq b) \subseteq (a^1 \geq_{\mathcal{N}} b^1) \quad (\mathcal{T}_{7431})$$

Example Proof of ($\mathcal{T}_{7425}$)

$$(((\mathcal{X}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{\subset\}b^1) \subseteq (\mathcal{X}a^1 \subseteq \{\subset\}\mathcal{X}b^1))$$

($\mathcal{A}_{S33}$)(Axiom of Physical Proper Sub-entities for Discrete Dimensions)(Eqn1)

$$\text{Let } \mathcal{X} = \mathcal{N}$$

(Eqn2)

$$(((\mathcal{N}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}) \subseteq ((a^1 \subseteq \{\subset\}b^1) \subseteq (\mathcal{N}a^1 \subseteq \{\subset\}\mathcal{N}b^1)) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3})$$

$$(\mathcal{N}\mathcal{E})^1 \subseteq \mathcal{D}_0$$

($\mathcal{A}_{S31}$)(Quantity Dimension Axiom)

$$((\mathcal{N}\mathcal{E})^1 \subseteq \mathcal{D}_0) = \mathcal{E}$$

($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn4)

$$(\mathcal{E} = \mathcal{E}) \subseteq ((a^1 \subseteq \{\subset\}b^1) \subseteq (\mathcal{N}a^1 \subseteq \{\subset\}\mathcal{N}b^1))$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn4 into Eqn3)

$$\mathcal{E} \subseteq ((a^1 \subseteq \{\subset\}b^1) \subseteq (\mathcal{N}a^1 \subseteq \{\subset\}\mathcal{N}b^1))$$

($\mathcal{T}_{M342}$)

$$\mathcal{E} = ((a^1 \subseteq \{\subset\}b^1) \subseteq (\mathcal{N}a^1 \subseteq \{\subset\}\mathcal{N}b^1))$$

($\mathcal{T}_{M275}$)(Supersets of the Universal Set)

$$(a^1 \subseteq \{\subset\}b^1) \subseteq (\mathcal{N}a^1 \subseteq \{\subset\}\mathcal{N}b^1)$$

($\mathcal{A}_{MX16}$)(Order Axiom)

$$(a^1 \subseteq \{\subset\}b^1) \subseteq (a^1 <_{\mathcal{N}} b^1)$$

($\mathcal{T}_{7459}\mathcal{N}$)(Eqn5)

$$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = \mathcal{E}) \subseteq ((a \subset b) = (a^1 \subseteq \{\subset\}b^1))$$

($\mathcal{T}_{R653}$)(Definition of a Relationship)(Eqn6)

$$((a^1 \neq \emptyset)(b^1 \neq \emptyset) = \mathcal{E}) \subseteq ((a \subset b) \subseteq (a^1 <_{\mathcal{N}} b^1))$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn6 into Eqn5)

$$(a \subset b) \subseteq (a^1 <_{\mathcal{N}} b^1)$$

($\mathcal{T}_{S761}$)(Implied Conditions of Quantity)

$$\mathcal{N}(@_{\mathcal{T}^1})a^1 \times \mathcal{N}(@_{\mathcal{T}^1})b^1 = \mathcal{N}(\mathcal{V}\mathcal{S}(@_{\mathcal{T}^1})a.\mathcal{V}\mathcal{Q}(@_{\mathcal{T}^1})b)^1 \quad (\mathcal{T}_{7414})$$

Proof of ($\mathcal{T}_{7414}$)

$$(\{x\} * (\mathcal{N}(@_{\mathcal{T}^1})b^1) * \mathcal{N}(@_{\mathcal{T}^1})a^1 = \mathcal{N}(\mathcal{V}\mathcal{S}(@_{\mathcal{T}^1})a.\mathcal{V}\mathcal{Q}(@_{\mathcal{T}^1})b)^1$$

($\mathcal{D}_{S03}$)(Definition of Integer Multiplication)

$$\mathcal{N}(@_{\mathcal{T}^1})a^1 \times \mathcal{N}(@_{\mathcal{T}^1})b^1 = \mathcal{N}(\mathcal{V}\mathcal{S}(@_{\mathcal{T}^1})a.\mathcal{V}\mathcal{Q}(@_{\mathcal{T}^1})b)^1$$

($\mathcal{T}_{7473}$)

$$(\{x\} * \mathcal{N}b^1) * \mathcal{N}a^1 = \mathcal{N}(\mathcal{V}\mathcal{S}a.\mathcal{V}\mathcal{Q}b)^1$$

($\mathcal{T}_{7415}$)

$$\mathcal{N}a^1 \times \mathcal{N}b^1 = \mathcal{N}(\mathcal{V}\mathcal{S}a.\mathcal{V}\mathcal{Q}b)^1$$

($\mathcal{T}_{7416}$)

Proof of ($\mathcal{T}_{7415}$) & ($\mathcal{T}_{7416}$)

$$(\{x\} * \mathcal{N}(@_{\mathcal{T}^1})b^1) * \mathcal{N}(@_{\mathcal{T}^1})a^1 = \mathcal{N}(\mathcal{V}\mathcal{S}(@_{\mathcal{T}^1})a.\mathcal{V}\mathcal{Q}(@_{\mathcal{T}^1})b)^1$$

($\mathcal{D}_{S03}$)(Definition of Integer Multiplication)(Eqn1)

$$\text{Let } (@_{\mathcal{T}^1}) = \{=\}$$

(Eqn2)

$$(\{x\} * \mathcal{N}\{=\}b^1) * \mathcal{N}\{=\}a^1 = \mathcal{N}(\mathcal{V}\mathcal{S}\{=\}a.\mathcal{V}\mathcal{Q}\{=\}b)^1$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$(\{x\} * \mathcal{N}b^1) * \mathcal{N}a^1 = \mathcal{N}(\mathcal{V}\mathcal{S}a.\mathcal{V}\mathcal{Q}b)^1$$

($\mathcal{T}_{A080}$)

$$\mathcal{N}a^1 \times \mathcal{N}b^1 = \mathcal{N}(\mathcal{V}\mathcal{S}a.\mathcal{V}\mathcal{Q}b)^1$$

($\mathcal{T}_{7473}$)

$$(\{+\} * \{b\}) * \{a\} = (\{+\} * \{a\}) * \{b\}$$

($\mathcal{T}_{7262}$) Commutativity of Algebraic +

$$(\{x\} * \{b\}) * \{a\} = (\{x\} * \{a\}) * \{b\}$$

($\mathcal{T}_{7263}$) Commutativity of Algebraic x

Example Proof of ($\mathcal{T}_{7263}$)

$$(\mathcal{o}^1 \subseteq \mathcal{R}_c) = (\{a\} \langle \mathcal{o} \rangle \{b\} = \{b\} \langle \mathcal{o} \rangle \{a\})$$

($\mathcal{D}_{R40}$)(Definition of a Commutative Operator)(Eqn1)

$$\text{Let } \mathcal{o} = \{+\}$$

(Eqn2)

$$(\{+\}^1 \subseteq \mathcal{R}_c) = (\{a\} \langle \{+\} \rangle \{b\} = \{b\} \langle \{+\} \rangle \{a\})$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn2 into Eqn1)

$$(\{+\}^1 \subseteq \mathcal{R}_c) = (\{a\} + \{b\} = \{b\} + \{a\})$$

($\mathcal{D}_{A229}$)(Definition of $\langle \rangle$)

$$\{+\}^1 + \{x\}^1 \subseteq \mathcal{R}_c$$

($\mathcal{A}_{R81}$)(Arithmetical Associativity and Commutativity Axiom)

$$\{+\}^1 \subseteq \mathcal{R}_c$$

($\mathcal{T}_{M072}$)(Single Premise Syllogism)

$$(\{+\}^1 \subseteq \mathcal{R}_c) = \mathcal{E}$$

($\mathcal{A}_{MX16}$)(Order Axiom)(Eqn4)

$$\mathcal{E} = (\{a\} + \{b\} = \{b\} + \{a\})$$

($\mathcal{A}_{MX19}$)(Substitution of Eqn4 into Eqn3)

$$\{a\} + \{b\} = \{b\} + \{a\}$$

($\mathcal{A}_{MX16}$)(Order Axiom)

$$(\{+\} * \{b\}) * \{a\} = (\{+\} * \{a\}) * \{b\}$$

($\mathcal{T}_{7471}$)

$$(\{x\} * \{1\}^1) * \{a\} = \{a\} \quad (\mathcal{T}_{S845})$$

$$(\{x\} * \{a\}) * \{1\}^1 = \{a\} \quad (\mathcal{T}_{S839})$$

Proof of ($\mathcal{T}_{S845}$) & ($\mathcal{T}_{S839}$)

$$\begin{aligned} \text{Let } \{a\} &= \{a_1\}^{1+++} \{a_n\}^1 & (\text{Eqn1}) \\ (\{x\} * \{1\}^1) * \{a\} &= (\{x\} * \{1\}^1) * (\{a_1\}^{1+++} \{a_n\}^1) & (\mathcal{R}_{522})(\text{Relational Deduction}) \\ (\{x\} * \{1\}^1) * \{a\} &= (\{x\} * \{1\}^1) * \{a_1\}^{1+++} (\{x\} * \{1\}^1) * \{a_n\}^1 & (\mathcal{A}_{S64})(\text{Distribution of } * \text{ over } +) \\ (\{x\} * \{1\}^1) * \{a\} &= (\{1\}^1 \times \{a_1\}^1) +++ (\{1\}^1 \times \{a_n\}^1) & (\mathcal{D}_{R20})(\text{Definition of an Arithmetical Operator}) \\ (\{x\} * \{1\}^1) * \{a\} &= \{a_1\}^{1+++} \{a_n\}^1 & (\mathcal{T}_{494})(\text{Eqn2}) \\ (\{x\} * \{1\}^1) * \{a\} &= \{a\} & (\mathcal{A}_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\ (\{x\} * \{a\}) * \{1\}^1 &= \{a\} & (\mathcal{T}_{263})(\text{Commutativity of Algebraic } x) \end{aligned}$$

$$(\{ \div \} * \{1\}^1) * \{a\} = \{a\} \quad (\mathcal{T}_{S846})$$

Proof of ($\mathcal{T}_{S846}$)

$$\begin{aligned} \text{Let } \{a\} &= \{a_1\}^{1+++} \{a_n\}^1 & (\text{Eqn1}) \\ (\{ \div \} * \{1\}^1) * \{a\} &= (\{ \div \} * \{1\}^1) * (\{a_1\}^{1+++} \{a_n\}^1) & (\mathcal{R}_{522})(\text{Relational Deduction}) \\ (\{ \div \} * \{1\}^1) * \{a\} &= (\{ \div \} * \{1\}^1) * \{a_1\}^{1+++} (\{ \div \} * \{1\}^1) * \{a_n\}^1 & (\mathcal{A}_{S64})(\text{Distribution of } * \text{ over } +) \\ (\{ \div \} * \{1\}^1) * \{a\} &= (\{a_1\}^{1 \div} \{1\}^1) +++ (\{a_n\}^{1 \div} \{1\}^1) & (\mathcal{D}_{R20})(\text{Definition of an Arithmetical Operator}) \\ (\{ \div \} * \{1\}^1) * \{a\} &= \{a_1\}^1 +++ \{a_n\}^1 & (\mathcal{T}_{495})(\text{Eqn2}) \\ (\{ \div \} * \{1\}^1) * \{a\} &= \{a\} & (\mathcal{A}_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \end{aligned}$$

$$(\{-\} * \{a\}^1) * \{a\}^1 = \{0\}^1 \quad (\mathcal{T}_{S838})$$

Proof of ($\mathcal{T}_{S838}$)

$$\begin{aligned} \{a\}^1 - \{a\}^1 &= \{0\}^1 & (\mathcal{T}_{498}) \\ (\{-\} * \{a\}^1) * \{a\}^1 &= \{0\}^1 & (\mathcal{D}_{R20})(\text{Definition of an Arithmetical Operator}) \end{aligned}$$

$$(\{x\} * \{0\}^1) * \{a\} = \{0\}^1 \quad (\mathcal{T}_{S836})$$

$$(\{x\} * \{a\}) * \{0\}^1 = \{0\}^1 \quad (\mathcal{T}_{S837})$$

Proof of ($\mathcal{T}_{S836}$) & ($\mathcal{T}_{S837}$)

$$\begin{aligned} \text{Let } \{a\} &= \{a_1\}^{1+++} \{a_n\}^1 \\ (\{x\} * \{0\}^1) * \{a\} &= (\{x\} * \{0\}^1) * (\{a_1\}^{1+++} \{a_n\}^1) & (\mathcal{R}_{522})(\text{Relational Deduction}) \\ (\{x\} * \{0\}^1) * \{a\} &= (\{x\} * \{0\}^1) * \{a_1\}^{1+++} (\{x\} * \{0\}^1) * \{a_n\}^1 & (\mathcal{A}_{S64})(\text{Distribution of } * \text{ over } +) \\ (\{x\} * \{0\}^1) * \{a\} &= (\{x\} * \{a_1\}^1) * \{0\}^{1+++} (\{x\} * \{a_n\}^1) * \{0\}^1 & (\mathcal{T}_{263})(\text{Commutativity of Algebraic } x) \\ (\{x\} * \{0\}^1) * \{a\} &= (\{0\}^1 \times \{a_1\}^1) +++ (\{0\}^1 \times \{a_n\}^1) & (\mathcal{D}_{R20})(\text{Definition of an Arithmetical Operator}) \\ (\{x\} * \{0\}^1) * \{a\} &= \{0\}^1 +++ \{0\}^1 & (\mathcal{T}_{656}) \\ (\{x\} * \{0\}^1) * \{a\} &= \{0\}^1 & (\mathcal{A}_{MX01})(\text{Disjunction Tautology}) \\ (\{x\} * \{a\}) * \{0\}^1 &= \{0\}^1 & (\mathcal{T}_{263})(\text{Commutativity of Algebraic } x) \end{aligned}$$

$$(\{+\} * \{0\}^1) * \{a\} = \{a\} \quad (\mathcal{T}_{260})$$

$$(\{+\} * \{a\}) * \{0\}^1 = \{a\} \quad (\mathcal{T}_{266})$$

Proof of ($\mathcal{T}_{260}$) & ($\mathcal{T}_{266}$)

$$\begin{aligned} \text{Let } \{a\} &= \{a_1\}^{1+++} \{a_n\}^1 & (\text{Eqn1}) \\ (\{+\} * \{0\}^1) * \{a\} &= (\{+\} * \{0\}^1) * (\{a_1\}^{1+++} \{a_n\}^1) & (\mathcal{R}_{522})(\text{Relational Deduction}) \\ (\{+\} * \{0\}^1) * \{a\} &= (\{+\} * \{0\}^1) * \{a_1\}^{1+++} (\{+\} * \{0\}^1) * \{a_n\}^1 & (\mathcal{A}_{S64})(\text{Distribution of } * \text{ over } +) \\ (\{+\} * \{0\}^1) * \{a\} &= \{a_1\}^{1+++} \{a_n\}^1 & (\mathcal{T}_{755})(\text{Eqn2}) \\ (\{+\} * \{0\}^1) * \{a\} &= \{a\} & (\mathcal{A}_{MX19})(\text{Substitution of Eqn1 into Eqn2}) \\ (\{+\} * \{a\}) * \{0\}^1 &= \{a\} & (\mathcal{T}_{262})(\text{Commutativity of Algebraic } +) \end{aligned}$$

$$\{>\} * \{0\}^1 = (\{+\} * \{>\} * \{0\}^1) * \{0\}^1 \quad (\mathcal{T}_{258})$$

Proof of ($\mathcal{T}_{258}$)

$$\begin{aligned} \forall (\{+\} * \{>\} * \{0\}^1) &= \{<\} & (\mathcal{T}_{565}) \\ \forall \forall (\{+\} * \{>\} * \{0\}^1) &= \forall \{<\} & (\mathcal{R}_{522})(\text{Relational Deduction}) \\ (\{+\} * \{>\} * \{0\}^1) &= \forall \{<\} & (\mathcal{R}_{012k})(\text{Sub-Relationship Closure Theorem}) \\ (\{+\} * \{>\} * \{0\}^1) &= \{>\} & (\mathcal{D}_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set}) \end{aligned}$$

$$(\{+\} * \{>\} * \{0\}^1) * \{0\}^1 = \{>\} * \{0\}^1 \quad (\mathcal{T}_{R522b})(\text{Relational Deduction})$$

$$\forall (\{+\} * \{a\}) = \{-\} * \{a\} \quad (\mathcal{T}_{T201})$$

$$\forall (\{-\} * \{a\}) = \{+\} * \{a\} \quad (\mathcal{T}_{T198})$$

$$\forall (\{x\} * \{a\}) = \{\div\} * \{a\} \quad (\mathcal{T}_{T156})$$

$$\forall (\{\div\} * \{a\}) = \{x\} * \{a\} \quad (\mathcal{T}_{T154})$$

Example Proof of ($\mathcal{T}_{T198}$)

$$\begin{aligned} \text{Let } \{a\} &= \{a_1\}^1 +++ \{a_n\}^1 & (\text{Eqn1}) \\ (\{-\} * \{a_r\}^1) &= \forall (\{+\} * \{a_r\}^1) & (\mathcal{T}_{T248})(\text{Eqn2}) \\ (\{-\} * \{a_r\}^1) +++ (\{-\} * \{a_r\}^1) &= \forall (\{+\} * \{a_r\}^1) +++ \forall (\{+\} * \{a_r\}^1) & (\mathcal{T}_{M230})(\text{Combination of Eqn2 for } r=1 \text{ to } n) \\ \{-\} * \{a_r\}^1 +++ \{a_r\}^1 &= \forall (\{+\} * \{a_r\}^1 +++ \{+\} * \{a_r\}^1) & (\mathcal{A}_{S64})(\text{Distribution of } * \text{ over } +) \\ \{-\} * \{a_r\}^1 +++ \{a_r\}^1 &= \forall (\{+\} * (\{a_r\}^1 +++ \{a_r\}^1)) & (\mathcal{A}_{S64})(\text{Distribution of } * \text{ over } +)(\text{Eqn3}) \\ \{-\} * \{a\}^1 &= \forall (\{+\} * \{a\}^1) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn1 into Eqn3}) \end{aligned}$$

$$\forall (\{+\} * \{a\}) = \{-\} * \{a\} \quad (\mathcal{T}_{T152})$$

$$\forall (\{-\} * \{a\}) = \{+\} * \{a\} \quad (\mathcal{T}_{T151})$$

$$\forall (\{x\} * \{a\}) = \{\div\} * \{a\} \quad (\mathcal{T}_{T157})$$

$$\forall (\{\div\} * \{a\}) = \{x\} * \{a\} \quad (\mathcal{T}_{T158})$$

Example Proof of ($\mathcal{T}_{T157}$)

$$\begin{aligned} \forall (\{x\} * \{a\}) &= \{\div\} * \{a\} & (\mathcal{T}_{T156})(\text{Eqn1}) \\ \text{Let } \{a\} &= \forall \{a\} & (\text{Eqn2}) \\ \forall (\{x\} * \forall \{a\}) &= \{\div\} * \forall \{a\} & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$\forall \{a\}^1 = (\{-\} * \{a\}^1) * \{0\}^1 \quad (\mathcal{T}_{T225})$$

Proof of ($\mathcal{T}_{T225}$)

$$\begin{aligned} \{-a\}^1 &= (\{-\} * \{a\}^1) * \{0\}^1 & (\mathcal{T}_{T685}) \\ \forall \{a\}^1 &= (\{-\} * \{a\}^1) * \{0\}^1 & (\mathcal{D}_{S12})(\text{Definition of } \forall \{a\}) \end{aligned}$$

$$\forall \{a\} = (\{-\} * \{a\}) * \{0\}^1 \quad (\mathcal{T}_{T224})$$

Proof of ($\mathcal{T}_{T224}$)

$$\begin{aligned} \text{Let } \{a\} &= \{a_1\}^1 +++ \{a_n\}^1 & (\text{Eqn1}) \\ \forall \{a_r\}^1 &= (\{-\} * \{a_r\}^1) * \{0\}^1 & (\mathcal{T}_{T225})(\text{Eqn2}) \\ \forall \{a_1\}^1 +++ \forall \{a_n\}^1 &= (\{-\} * \{a_1\}^1) * \{0\}^1 +++ (\{-\} * \{a_n\}^1) * \{0\}^1 & (\mathcal{T}_{M230})(\text{Combination of Eqn2 for } r=1 \text{ to } n) \\ \forall (\{a_1\}^1 +++ \forall \{a_n\}^1) &= (\{-\} * \{a_1\}^1) * \{0\}^1 +++ (\{-\} * \{a_n\}^1) * \{0\}^1 & (\mathcal{A}_{S64})(\text{Distribution of } * \text{ over } +)(\text{Eqn3}) \\ \forall \{a\}^1 &= (\{-\} * \{a\}^1) * \{0\}^1 +++ (\{-\} * \{a_n\}^1) * \{0\}^1 & (\mathcal{A}_{MX19})(\text{Substitution of Eqn1 into Eqn3}) \\ \forall \{a\}^1 &= ((\{-\} * \{a_1\}^1) +++ (\{-\} * \{a_n\}^1)) * \{0\}^1 & (\mathcal{A}_{S65})(\text{Distribution of } + \text{ over } *) \\ \forall \{a\}^1 &= (\{-\} * (\{a_1\}^1 +++ \{a_n\}^1)) * \{0\}^1 & (\mathcal{A}_{S64})(\text{Distribution of } * \text{ over } +)(\text{Eqn4}) \\ \forall \{a\}^1 &= (\{-\} * \{a\}^1) * \{0\}^1 & (\mathcal{A}_{MX19})(\text{Substitution of Eqn1 into Eqn4}) \end{aligned}$$

$$\forall (\{>\} * \forall \{a\}^1) = \{<\} * \{a\}^1 \quad (\mathcal{T}_{T223})$$

Proof of ($\mathcal{T}_{T223}$)

$$\begin{aligned} \forall (\{>\} * \{a\}^1) &= \{<\} * \forall \{a\}^1 & (\mathcal{A}_{R55})(\text{Axiom of Number Symmetry})(\text{Eqn1}) \\ \text{Let } \{a\}^1 &= \forall \{a\}^1 & (\text{Eqn2}) \\ \forall \{a\}^1 &= \forall \forall \{a\}^1 & (\mathcal{T}_{R522})(\text{Relational Deduction}) \\ \forall \{a\}^1 &= \{a\}^1 & (\mathcal{T}_{R012k})(\text{Sub-Relationship Closure Theorem})(\text{Eqn3}) \\ \forall (\{>\} * \forall \{a\}^1) &= \{<\} * \{a\}^1 & (\mathcal{A}_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \end{aligned}$$

$$\forall (\{<\} * \{a\}^1) = \{>\} * \{a\}^1 \quad (\mathcal{T}_{T217})$$

$$\forall (\{<\} * \forall \{a\}^1) = \{>\} * \{a\}^1 \quad (\mathcal{T}_{T216})$$

Example Proof of ($\mathcal{T}_{217}$)

$$\begin{aligned} \forall(\{>\} * \{a\}^1) &= \{<\} * \forall\{a\}^1 && (\mathcal{A}_{R55})(\text{Axiom of Number Symmetry}) \\ \forall\forall(\{>\} * \{a\}^1) &= \forall(\{<\} * \forall\{a\}^1) && (\mathcal{T}_{R522})(\text{Relational Deduction}) \\ \{>\} * \{a\}^1 &= \forall(\{<\} * \forall\{a\}^1) && (\mathcal{T}_{R012k})(\text{Sub-Relationship Closure Theorem}) \end{aligned}$$

$$\forall(\{>\} * \{0\}^1) = \{<\} * \{0\}^1 \quad (\mathcal{T}_{215})$$

$$\forall(\{<\} * \{0\}^1) = \{>\} * \{0\}^1 \quad (\mathcal{T}_{214})$$

Example Proof of ($\mathcal{T}_{215}$)

$$\begin{aligned} \forall(\{>\} * \{a\}^1) &= \{<\} * \forall\{a\}^1 && (\mathcal{A}_{R55})(\text{Axiom of Number Symmetry})(\text{Eqn1}) \\ \text{Let } \{a\}^1 &= \{0\}^1 && (\text{Eqn2}) \\ \forall(\{>\} * \{0\}^1) &= \{<\} * \forall\{0\}^1 && (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ \forall(\{>\} * \{0\}^1) &= \{<\} * \{0\}^1 && (\mathcal{A}_{R56})(\text{Reverse of Zero}) \end{aligned}$$

$$\forall(\{>\} * \{1\}^1) = \{<\} * \forall\{1\}^1 \quad (\mathcal{T}_{213})$$

$$\forall(\{>\} * \forall\{1\}^1) = \{<\} * \{1\}^1 \quad (\mathcal{T}_{212})$$

$$\forall(\{<\} * \{1\}^1) = \{>\} * \forall\{1\}^1 \quad (\mathcal{T}_{211})$$

$$\forall(\{<\} * \forall\{1\}^1) = \{>\} * \{1\}^1 \quad (\mathcal{T}_{202})$$

Example Proof of ($\mathcal{T}_{213}$)

$$\begin{aligned} \forall(\{>\} * \{a\}^1) &= \{<\} * \forall\{a\}^1 && (\mathcal{A}_{R55})(\text{Axiom of Number Symmetry})(\text{Eqn1}) \\ \text{Let } \{a\}^1 &= \{1\}^1 && (\text{Eqn2}) \\ \forall(\{>\} * \{1\}^1) &= \{<\} * \forall\{1\}^1 && (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$\{+\} * \{>\} * \{0\}^1 = \forall(\{-\} * \{>\} * \{0\}^1) \quad (\mathcal{T}_{197})$$

$$\{+\} * \{<\} * \{0\}^1 = \forall(\{-\} * \{<\} * \{0\}^1) \quad (\mathcal{T}_{196})$$

$$\{-\} * \{>\} * \{0\}^1 = \forall(\{+\} * \{>\} * \{0\}^1) \quad (\mathcal{T}_{195})$$

$$\{-\} * \{<\} * \{0\}^1 = \forall(\{+\} * \{<\} * \{0\}^1) \quad (\mathcal{T}_{194})$$

Example Proof of ($\mathcal{T}_{197}$)

$$\begin{aligned} \forall(\{-\} * \{a\}) &= \{+\} * \{a\} && (\mathcal{T}_{198})(\text{Eqn1}) \\ \text{Let } \{a\} &= \{>\} * \{0\}^1 && (\text{Eqn2}) \\ \forall(\{-\} * \{>\} * \{0\}^1) &= \{+\} * \{>\} * \{0\}^1 && (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$\{-\} * \{a\} = \{+\} * \forall\{a\} \quad (\mathcal{T}_{193})$$

Proof of ($\mathcal{T}_{193}$)

$$\begin{aligned} (\{+\} * \forall\{a\}) &= \forall(\{+\} * \{a\}) && (\mathcal{A}_{S58})(\text{Axiom of Negative Numbers})(\text{Eqn1}) \\ \forall(\{+\} * \{a\}) &= \{-\} * \{a\} && (\mathcal{T}_{201})(\text{Eqn2}) \\ \{+\} * \forall\{a\} &= \{-\} * \{a\} && (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$\{+\} * \{a\} = \{-\} * \forall\{a\} \quad (\mathcal{T}_{S934})$$

Proof of ($\mathcal{T}_{S934}$)

$$\begin{aligned} \{-\} * \{a\} &= \{+\} * \forall\{a\} && (\mathcal{T}_{193}) \\ \forall(\{+\} * \{a\}) &= \forall(\{-\} * \forall\{a\}) && (\mathcal{T}_{201}) \text{ \& } (\mathcal{T}_{198}) \\ \forall\forall(\{+\} * \{a\}) &= \forall\forall(\{-\} * \forall\{a\}) && (\mathcal{T}_{R522})(\text{Relational Deduction}) \\ \{+\} * \{a\} &= \{-\} * \forall\{a\} && (\mathcal{T}_{R012k})(\text{Sub-relationship Closure Theorem}) \end{aligned}$$

$$\{+\} * \{>\} * \{0\}^1 = \{-\} * \forall(\{>\} * \{0\}^1) \quad (\mathcal{T}_{179})$$

$$\{+\} * \{<\} * \{0\}^1 = \{-\} * \forall(\{<\} * \{0\}^1) \quad (\mathcal{T}_{177})$$

$$\{-\} * \{>\} * \{0\}^1 = \{+\} * \forall(\{>\} * \{0\}^1) \quad (\mathcal{T}_{175})$$

$$\{-\} * \{<\} * \{0\}^1 = \{+\} * \forall(\{<\} * \{0\}^1) \quad (\mathcal{T}_{168})$$

$$\{+\}\{>\}\{0\}^1 = \{-\}\{<\}\{0\}^1 \quad (\mathcal{T}_{165})$$

$$\{+\}\{<\}\{0\}^1 = \{-\}\{>\}\{0\}^1 \quad (\mathcal{T}_{163})$$

$$\{-\}\{>\}\{0\}^1 = \{+\}\{<\}\{0\}^1 \quad (\mathcal{T}_{162})$$

$$\{-\}\{<\}\{0\}^1 = \{+\}\{>\}\{0\}^1 \quad (\mathcal{T}_{160})$$

Example Proof of ($\mathcal{T}_{179}$) & ($\mathcal{T}_{165}$)

$$\{+\}\{a\} = \{-\}\{a\}^v \quad (\mathcal{T}_{934})(\text{Eqn1})$$

$$\text{Let } \{a\} = \{>\}\{0\}^1 \quad (\text{Eqn2})$$

$$\{+\}\{>\}\{0\}^1 = \{-\}\{>\}\{0\}^1 \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$\{+\}\{>\}\{0\}^1 = \{-\}\{<\}\{0\}^1 \quad (\mathcal{T}_{215})$$

$$\{x\}\{>\}\{1\}^1 = \{ \div \}\{>\}\{1\}^1 \quad (\mathcal{T}_{150})$$

$$\{x\}\{<\}\{1\}^1 = \{ \div \}\{<\}\{1\}^1 \quad (\mathcal{T}_{149})$$

$$\{ \div \}\{>\}\{1\}^1 = \{x\}\{>\}\{1\}^1 \quad (\mathcal{T}_{148})$$

$$\{ \div \}\{<\}\{1\}^1 = \{x\}\{<\}\{1\}^1 \quad (\mathcal{T}_{147})$$

Example Proof of ($\mathcal{T}_{150}$)

$$\{ \div \}\{a\} = \{x\}\{a\} \quad (\mathcal{T}_{154})(\text{Eqn1})$$

$$\text{Let } \{a\} = \{>\}\{1\}^1 \quad (\text{Eqn2})$$

$$\{ \div \}\{>\}\{1\}^1 = \{x\}\{>\}\{1\}^1 \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$\{x\}\{>\}\{1\}^1 = \{ \div \}\{>\}\{1\}^1 \quad (\mathcal{T}_{146})$$

$$\{x\}\{<\}\{1\}^1 = \{ \div \}\{<\}\{1\}^1 \quad (\mathcal{T}_{145})$$

$$\{ \div \}\{>\}\{1\}^1 = \{x\}\{>\}\{1\}^1 \quad (\mathcal{T}_{144})$$

$$\{ \div \}\{<\}\{1\}^1 = \{x\}\{<\}\{1\}^1 \quad (\mathcal{T}_{142})$$

Example Proof of ($\mathcal{T}_{146}$)

$$\{ \div \}\{a\} = \{x\}\{a\}^v \quad (\mathcal{T}_{158})(\text{Eqn1})$$

$$\text{Let } \{a\} = \{>\}\{1\}^1 \quad (\text{Eqn2})$$

$$\{ \div \}\{>\}\{1\}^1 = \{x\}\{>\}\{1\}^1 \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$\{x\}\{>\}\{1\}^1 = \{ \div \}\{>\}\{1\}^1 \quad (\mathcal{T}_{141})$$

$$\{x\}\{<\}\{1\}^1 = \{ \div \}\{<\}\{1\}^1 \quad (\mathcal{T}_{137})$$

$$\{ \div \}\{>\}\{1\}^1 = \{x\}\{>\}\{1\}^1 \quad (\mathcal{T}_{135})$$

$$\{ \div \}\{<\}\{1\}^1 = \{x\}\{<\}\{1\}^1 \quad (\mathcal{T}_{134})$$

Example Proof of ($\mathcal{T}_{135}$)

$$\{ \div \}\{a\} = \{x\}\{a\} \quad (\mathcal{T}_{154})(\text{Eqn1})$$

$$\text{Let } \{a\} = \{>\}\{1\}^1 \quad (\text{Eqn2})$$

$$\{ \div \}\{>\}\{1\}^1 = \{x\}\{>\}\{1\}^1 \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$\{x\}\{>\}\{1\}^1 = \{ \div \}\{>\}\{1\}^1 \quad (\mathcal{T}_{117})$$

$$\{x\}\{<\}\{1\}^1 = \{ \div \}\{<\}\{1\}^1 \quad (\mathcal{T}_{104})$$

$$\{ \div \}\{>\}\{1\}^1 = \{x\}\{>\}\{1\}^1 \quad (\mathcal{T}_{103})$$

$$\{ \div \}\{<\}\{1\}^1 = \{x\}\{<\}\{1\}^1 \quad (\mathcal{T}_{096})$$

Example Proof of ($\mathcal{T}_{117}$)

$$\{ \div \}\{a\} = \{x\}\{a\} \quad (\mathcal{T}_{154})(\text{Eqn1})$$

$$\text{Let } \{a\} = \{>\}\{1\}^1 \quad (\text{Eqn2})$$

$$\{ \div \}\{>\}\{1\}^1 = \{x\}\{>\}\{1\}^1 \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(\mathbf{X} = \{x\}, \{ \div \}) \subseteq ((\mathbf{X} \ast \{1\}^1) \ast \{a\} = \{a\}) \quad (\mathcal{T}_{932})$$

Proof of (T_{S932})

$$\begin{aligned}
 (X = \{x\}, \{\div\})(Y = V, \{=\})(Z = V, \{=\}) &\subseteq ((X * Y * \{a\}) * Z * \{b\} = Y * Z * (X * \{a\}) * \{b\}) \\
 &\quad (A_{R47})(\text{Positive/Negative Axiom})(\text{Eqn1}) \\
 \text{Let } \{a\} &= \{1\}^1 && (\text{Eqn2}) \\
 \text{Let } \{b\} &= \{a\} && (\text{Eqn3}) \\
 (X = \{x\}, \{\div\})(Y = V, \{=\})(Z = V, \{=\}) &\subseteq ((X * Y * \{1\}^1) * Z * \{a\} = Y * Z * (X * \{1\}^1) * \{a\}) \\
 &\quad (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\
 (X = \{x\}, \{\div\}) &\subseteq ((X * V * \{1\}^1) * \{=\} * \{a\} = V * \{=\} * (X * \{1\}^1) * \{a\}) \\
 &\quad \text{Extraction} \\
 (X = \{x\}, \{\div\}) &\subseteq ((X * V * \{1\}^1) * \{a\} = V * (X * \{1\}^1) * \{a\}) \\
 &\quad (T_{A080}) \\
 (X = \{x\}, \{\div\}) &\subseteq ((X *^V \{1\}^1) * \{a\} =^V (X * \{1\}^1) * \{a\}) \\
 &\quad (D_{R27})(\text{Simplification of the Relationship Symmetry Function}) \\
 (X = \{x\}, \{\div\}) &\subseteq ((X *^V \{1\}^1) * \{a\} =^V (X * \{1\}^1) * \{a\}) \\
 &\quad (T_{M131b})(\text{Conjunction of Sufficiencies}) \\
 (X = \{x\}, \{\div\}) &\subseteq ((X *^V \{1\}^1) * \{a\} =^V \{a\}) && (T_{S845}) \& (T_{S846})
 \end{aligned}$$

$$(\{x\} *^V \{1\}^1) * \{a\} =^V \{a\} \quad (T_{7095})$$

$$(\{\div\} *^V \{1\}^1) * \{a\} =^V \{a\} \quad (T_{7093})$$

Example Proof of (T₇₀₉₅)

$$\begin{aligned}
 (X = \{x\}, \{\div\}) &\subseteq ((X *^V \{1\}^1) * \{a\} =^V \{a\}) && (T_{S932}) \\
 (\{x\} *^V \{1\}^1) * \{a\} &=^V \{a\} && \text{Extraction}
 \end{aligned}$$

$$(\{x\} *^V \{1\}^1) *^V \{a\} = \{a\} \quad (T_{7092})$$

$$(\{\div\} *^V \{1\}^1) *^V \{a\} = \{a\} \quad (T_{7086})$$

Example Proof of (T₇₀₉₂)

$$\begin{aligned}
 (\{x\} *^V \{1\}^1) * \{a\} &=^V \{a\} && (T_{7095})(\text{Eqn1}) \\
 \text{Let } \{a\} &=^V \{a\} && (\text{Eqn2}) \\
 (\{x\} *^V \{1\}^1) *^V \{a\} &=^V \{a\} && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
 (\{x\} *^V \{1\}^1) * \{a\} &= \{a\} && (T_{R012k})(\text{Sub-Relationship Closure Theorem})
 \end{aligned}$$

$$(\{x\} * \{1\}^1) *^V \{a\} =^V \{a\} \quad (T_{7085})$$

$$(\{\div\} * \{1\}^1) *^V \{a\} =^V \{a\} \quad (T_{7084})$$

Example Proof of (T₇₀₈₅)

$$\begin{aligned}
 (\{x\} * \{1\}^1) * \{a\} &= \{a\} && (T_{S845})(\text{Eqn1}) \\
 \text{Let } \{a\} &=^V \{a\} && (\text{Eqn2}) \\
 (\{x\} * \{1\}^1) *^V \{a\} &=^V \{a\} && (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
 \end{aligned}$$

$$(\{a\} + \{b\} \subseteq \{>\} * \{0\}^1)(X = \{x\}, \{\div\})(Y = V, \{=\})(Z = V, \{=\}) \subseteq ((X * Y * \{a\}) * Z * \{b\} \subseteq Y * Z * (\{>\} * \{0\}^1)) \quad (T_{S933})$$

Proof of (T_{S933})

$$\begin{aligned}
 (X = \{x\}, \{\div\})(Y = V, \{=\})(Z = V, \{=\}) &\subseteq ((X * Y * \{a\}) * Z * \{b\} = Y * Z * (X * \{a\}) * \{b\}) \\
 &\quad (A_{R47})(\text{Positive/Negative Axiom})(\text{Eqn1}) \\
 (\{a\} + \{b\} \subseteq \{>\} * \{0\}^1)(X = \{x\}, \{\div\}) &\subseteq ((X * \{a\}) * \{b\} \subseteq \{>\} * \{0\}^1) \\
 &\quad (A_{R45})(\text{Axiom of Positive Number Operations}) \\
 (\{a\} + \{b\} \subseteq \{>\} * \{0\}^1)(X = \{x\}, \{\div\})(Y = V, \{=\})(Z = V, \{=\}) &\subseteq (Y * Z * (X * \{a\}) * \{b\} \subseteq Y * Z * (\{>\} * \{0\}^1)) \\
 &\quad (T_{R196})(\text{Relational Deduction})(\text{Eqn2}) \\
 (\{a\} + \{b\} \subseteq \{>\} * \{0\}^1)(X = \{x\}, \{\div\})(Y = V, \{=\})(Z = V, \{=\}) &\subseteq ((X * Y * \{a\}) * Z * \{b\} \subseteq Y * Z * (\{>\} * \{0\}^1)) \\
 &\quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})
 \end{aligned}$$

$$(\{a\} \subseteq \{>\} * \{0\}^1).(\{b\} \subseteq \{>\} * \{0\}^1) \subseteq ((\{x\} * \{a\}) * \{b\} \subseteq \{>\} * \{0\}^1) \quad (T_{7081})$$

$$(\{a\} \subseteq \{<\} * \{0\}^1).(\{b\} \subseteq \{<\} * \{0\}^1) \subseteq ((\{x\} * \{a\}) * \{b\} \subseteq \{>\} * \{0\}^1) \quad (T_{7080})$$

$$(\{a\} \subseteq \{>\} * \{0\}^1).(\{b\} \subseteq \{<\} * \{0\}^1) \subseteq ((\{x\} * \{a\}) * \{b\} \subseteq \{<\} * \{0\}^1) \quad (T_{7079})$$

$$(\{a\} \subseteq \{\leftarrow\} * \{0\}^1).(\{b\} \subseteq \{\rightarrow\} * \{0\}^1) \subseteq ((\{x\} * \{a\}) * \{b\} \subseteq \{\leftarrow\} * \{0\}^1) \quad (T_{7074})$$

$$(\{a\} \subseteq \{\rightarrow\} * \{0\}^1).(\{b\} \subseteq \{\rightarrow\} * \{0\}^1) \subseteq ((\{\div\} * \{a\}) * \{b\} \subseteq \{\rightarrow\} * \{0\}^1) \quad (T_{7073})$$

$$(\{a\} \subseteq \{\leftarrow\} * \{0\}^1).(\{b\} \subseteq \{\leftarrow\} * \{0\}^1) \subseteq ((\{\div\} * \{a\}) * \{b\} \subseteq \{\rightarrow\} * \{0\}^1) \quad (T_{7060})$$

$$(\{a\} \subseteq \{\rightarrow\} * \{0\}^1).(\{b\} \subseteq \{\leftarrow\} * \{0\}^1) \subseteq ((\{\div\} * \{a\}) * \{b\} \subseteq \{\leftarrow\} * \{0\}^1) \quad (T_{7057})$$

$$(\{a\} \subseteq \{\leftarrow\} * \{0\}^1).(\{b\} \subseteq \{\rightarrow\} * \{0\}^1) \subseteq ((\{\div\} * \{a\}) * \{b\} \subseteq \{\leftarrow\} * \{0\}^1) \quad (T_{7055})$$

Example Proof of (T₇₀₇₄)

$$(\{a\} + \{b\} \subseteq \{\rightarrow\} * \{0\}^1)(X = \{x\}, \{\div\})(Y = V, \{=\})(Z = V, \{=\}) \subseteq ((X * Y * \{a\}) * Z * \{b\} \subseteq Y * Z * (\{\rightarrow\} * \{0\}^1)) \quad (T_{5933})$$

$$(\{a\} + \{b\} \subseteq \{\rightarrow\} * \{0\}^1) \subseteq ((\{x\} * V * \{a\}) * \{=\} * \{b\} \subseteq V * \{=\} * (\{\rightarrow\} * \{0\}^1))$$

Extraction

$$(\{a\} + \{b\} \subseteq \{\rightarrow\} * \{0\}^1) \subseteq ((\{x\} * V * \{a\}) * \{b\} \subseteq V * (\{\rightarrow\} * \{0\}^1)) \quad (T_{A080})$$

$$(\{a\} + \{b\} \subseteq \{\rightarrow\} * \{0\}^1) \subseteq ((\{x\} * V * \{a\}) * \{b\} \subseteq V * (\{\rightarrow\} * \{0\}^1))$$

(D_{R27})(Simplification of the Relationship Symmetry Function)

$$(\{a\} \subseteq \{\rightarrow\} * \{0\}^1).(\{b\} \subseteq \{\rightarrow\} * \{0\}^1) \subseteq ((\{x\} * V * \{a\}) * \{b\} \subseteq V * (\{\rightarrow\} * \{0\}^1))$$

(T_{M131b})(Conjunction of Sufficiencies)(Eqn1)

$$\text{Let } V * \{a\} = \{a\}$$

(Eqn2)

$$(V * \{a\} \subseteq \{\rightarrow\} * \{0\}^1).(\{b\} \subseteq \{\rightarrow\} * \{0\}^1) \subseteq ((\{x\} * \{a\}) * \{b\} \subseteq V * (\{\rightarrow\} * \{0\}^1))$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(V * \{a\} \subseteq V * (\{\rightarrow\} * \{0\}^1)).(\{b\} \subseteq \{\rightarrow\} * \{0\}^1) \subseteq ((\{x\} * \{a\}) * \{b\} \subseteq V * (\{\rightarrow\} * \{0\}^1))$$

(T_{R196})(Relational Deduction)

$$(\{a\} \subseteq V * (\{\rightarrow\} * \{0\}^1)).(\{b\} \subseteq \{\rightarrow\} * \{0\}^1) \subseteq ((\{x\} * \{a\}) * \{b\} \subseteq V * (\{\rightarrow\} * \{0\}^1))$$

(T_{R012k})(Sub-Relationship Closure Theorem)

$$(\{a\} \subseteq \{\leftarrow\} * \{0\}^1).(\{b\} \subseteq \{\rightarrow\} * \{0\}^1) \subseteq ((\{x\} * \{a\}) * \{b\} \subseteq \{\leftarrow\} * \{0\}^1)$$

(T_{T215})

$$(\{x\} * \{\rightarrow\} * \{0\}^1) * \{\rightarrow\} * \{0\}^1 = \{\rightarrow\} * \{0\}^1 \quad (T_{7046})$$

$$(\{\div\} * \{\rightarrow\} * \{0\}^1) * \{\rightarrow\} * \{0\}^1 = \{\rightarrow\} * \{0\}^1 \quad (T_{7035})$$

$$(\{x\} * \{\leftarrow\} * \{0\}^1) * \{\leftarrow\} * \{0\}^1 = \{\rightarrow\} * \{0\}^1 \quad (T_{7033})$$

$$(\{\div\} * \{\leftarrow\} * \{0\}^1) * \{\leftarrow\} * \{0\}^1 = \{\rightarrow\} * \{0\}^1 \quad (T_{7031})$$

$$(\{x\} * \{\rightarrow\} * \{0\}^1) * \{\leftarrow\} * \{0\}^1 = \{\leftarrow\} * \{0\}^1 \quad (T_{7029})$$

$$(\{\div\} * \{\rightarrow\} * \{0\}^1) * \{\leftarrow\} * \{0\}^1 = \{\leftarrow\} * \{0\}^1 \quad (T_{7028})$$

$$(\{x\} * \{\leftarrow\} * \{0\}^1) * \{\rightarrow\} * \{0\}^1 = \{\leftarrow\} * \{0\}^1 \quad (T_{7027})$$

$$(\{\div\} * \{\leftarrow\} * \{0\}^1) * \{\rightarrow\} * \{0\}^1 = \{\leftarrow\} * \{0\}^1 \quad (T_{7026})$$

Example Proof of (T₇₀₂₇)

$$(\{a\} \subseteq \{\leftarrow\} * \{0\}^1).(\{b\} \subseteq \{\rightarrow\} * \{0\}^1) \subseteq ((\{x\} * \{a\}) * \{b\} \subseteq \{\leftarrow\} * \{0\}^1)$$

(T₇₀₇₄)(Eqn1)

$$\text{Let } \{a\} = \{\leftarrow\} * \{0\}^1$$

(Eqn2)

$$\text{Let } \{b\} = \{\rightarrow\} * \{0\}^1$$

(Eqn3)

$$(\{x\} * \{\leftarrow\} * \{0\}^1) * \{\rightarrow\} * \{0\}^1 \subseteq \{\leftarrow\} * \{0\}^1$$

(A_{MX19})(Substitution of Eqns2&3 into Eqn1)(Eqn4)

$$\{1\}^1 \subseteq \{\rightarrow\} * \{0\}^1$$

(T₅₈₂)

$$V * \{1\}^1 \subseteq V * (\{\rightarrow\} * \{0\}^1)$$

(T_{R196})(Relational Deduction)

$$V * \{1\}^1 \subseteq \{\leftarrow\} * \{0\}^1$$

(T_{T215})

$$\{x\} * V * \{1\}^1 \subseteq \{x\} * \{\leftarrow\} * \{0\}^1$$

(T_{R196})(Relational Deduction)

$$(\{x\} * V * \{1\}^1) * \{\rightarrow\} * \{0\}^1 \subseteq (\{x\} * \{\leftarrow\} * \{0\}^1) * \{\rightarrow\} * \{0\}^1$$

(T_{R197})(Relational Deduction)

$$V * (\{\rightarrow\} * \{0\}^1) \subseteq (\{x\} * \{\leftarrow\} * \{0\}^1) * \{\rightarrow\} * \{0\}^1$$

(T₇₀₉₅)

$$\{\leftarrow\} * \{0\}^1 \subseteq (\{x\} * \{\leftarrow\} * \{0\}^1) * \{\rightarrow\} * \{0\}^1$$

(T_{T215})(Eqn5)

$$(\{x\} * \{\leftarrow\} * \{0\}^1) * \{\rightarrow\} * \{0\}^1 = \{\leftarrow\} * \{0\}^1$$

(T_{M430})(Biconditional Introduction Eqns4&5)

$$(\{x\} * \{\geq\} * \{0\}^1) * \{\geq\} * \{0\}^1 = \{\geq\} * \{0\}^1 \quad (T_{7021})$$

$$(\{x\} * \{\leq\} * \{0\}^1) * \{\geq\} * \{0\}^1 = \{\leq\} * \{0\}^1 \quad (T_{7019})$$

$$(\{x\} * \{\geq\} * \{0\}^1) * \{\leq\} * \{0\}^1 = \{\leq\} * \{0\}^1 \quad (T_{7272})$$

$$(\{x\} * \{\leq\} * \{0\}^1) * \{\leq\} * \{0\}^1 = \{\geq\} * \{0\}^1 \quad (T_{7005})$$

Example Proof of (T₇₀₂₁)

$$\begin{aligned} & (\{x\} * \{\>\} * \{0\}^1) * \{\>\} * \{0\}^1 = \{\>\} * \{0\}^1 \quad (T_{7046})(\text{Eqn1}) \\ & (\{x\} * \{\>\} * \{0\}^1) * \{0\}^1 = \{0\}^1 \quad (T_{5837}) \\ & (\{x\} * \{\>\} * \{0\}^1) * \{=\} * \{0\}^1 = \{=\} * \{0\}^1 \quad (D_{A144})(\text{Definition of the Reflexive Relationship Set})(\text{Eqn2}) \\ & (\{x\} * \{\geq\} * \{0\}^1) * \{0\}^1 = \{0\}^1 \quad (T_{5837}) \\ & (\{x\} * \{\geq\} * \{0\}^1) * \{=\} * \{0\}^1 = \{=\} * \{0\}^1 \quad (D_{A144})(\text{Definition of the Reflexive Relationship Set})(\text{Eqn3}) \\ & (\{x\} * \{\>\} * \{0\}^1) * \{\>\} * \{0\}^1 + (\{x\} * \{\>\} * \{0\}^1) * \{=\} * \{0\}^1 = \{\>\} * \{0\}^1 + \{=\} * \{0\}^1 \\ & \quad (A_{MX21})(\text{Combination of Eqns1\&2}) \\ & (\{x\} * \{\>\} * \{0\}^1) * (\{\>\} * \{0\}^1 + \{=\} * \{0\}^1) = \{\>\} * \{0\}^1 + \{=\} * \{0\}^1 \\ & \quad (T_{R002b})(\text{Distribution of } * \text{ over } +) \\ & (\{x\} * \{\>\} * \{0\}^1) * (\{\>\} + \{=\}) * \{0\}^1 = (\{\>\} + \{=\}) * \{0\}^1 \\ & \quad (T_{R002c})(\text{Distribution of } + \text{ over } *) \\ & (\{x\} * \{\>\} * \{0\}^1) * \{\geq\} * \{0\}^1 = \{\geq\} * \{0\}^1 \quad (D_{A141})(\text{Disjunction with } \{=\}) \\ & (\{x\} * \{\geq\} * \{0\}^1) * \{\>\} * \{0\}^1 = \{\geq\} * \{0\}^1 \quad (T_{7263})(\text{Commutativity of Algebraic } x)(\text{Eqn4}) \\ & (\{x\} * \{\geq\} * \{0\}^1) * \{\>\} * \{0\}^1 + (\{x\} * \{\geq\} * \{0\}^1) * \{=\} * \{0\}^1 = \{\geq\} * \{0\}^1 + \{=\} * \{0\}^1 \\ & \quad (A_{MX21})(\text{Combination of Eqns4\&3}) \\ & (\{x\} * \{\geq\} * \{0\}^1) * (\{\>\} * \{0\}^1 + \{=\} * \{0\}^1) = \{\geq\} * \{0\}^1 + \{=\} * \{0\}^1 \\ & \quad (T_{R002b})(\text{Distribution of } * \text{ over } +) \\ & (\{x\} * \{\geq\} * \{0\}^1) * (\{\>\} + \{=\}) * \{0\}^1 = (\{\geq\} + \{=\}) * \{0\}^1 \\ & \quad (T_{R002c})(\text{Distribution of } + \text{ over } *) \\ & (\{x\} * \{\geq\} * \{0\}^1) * (\{\>\} + \{=\}) * \{0\}^1 = (\{\>\} + \{=\} + \{=\}) * \{0\}^1 \\ & \quad (D_{A141})(\text{Disjunction with } \{=\}) \\ & (\{x\} * \{\geq\} * \{0\}^1) * (\{\>\} + \{=\}) * \{0\}^1 = (\{\>\} + \{=\}) * \{0\}^1 \\ & \quad (A_{MX01})(\text{Disjunction Tautology}) \\ & (\{x\} * \{\geq\} * \{0\}^1) * \{\geq\} * \{0\}^1 = \{\geq\} * \{0\}^1 \quad (D_{A141})(\text{Disjunction with } \{=\}) \end{aligned}$$

$$(\{\div\} * \{\>\} * \{1\}^1) * \{1\}^1 = (\{\<\} * \{1\}^1)(\{\>\} * \{0\}^1) \quad (T_{5842})$$

Proof of (T₅₈₄₂)

$$\begin{aligned} & (\{a\} \subseteq \{\>\} * \{b\}) (\{b\} \subseteq \{\>\} * \{0\}^1) \subseteq ((\{\div\} * \{a\}) * \{b\} \subseteq (\{\<\} * \{1\}^1)(\{\>\} * \{0\}^1)) \\ & \quad (A_{R46})(\text{Axiom of Fractions})(\text{Eqn1}) \\ & \text{Let } \{b\} = \{1\}^1 \quad (\text{Eqn2}) \\ & \text{Let } \{a\} = \{\>\} * \{1\}^1 \quad (\text{Eqn3}) \\ & (\{\>\} * \{1\}^1 \subseteq \{\>\} * \{1\}^1) (\{1\}^1 \subseteq \{\>\} * \{0\}^1) \subseteq ((\{\div\} * (\{\>\} * \{1\}^1)) * \{1\}^1 \subseteq (\{\<\} * \{1\}^1)(\{\>\} * \{0\}^1)) \\ & \quad (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\ & (\{\>\} * \{1\}^1 \subseteq \{\>\} * \{1\}^1) (\{1\}^1 \subseteq \{\>\} * \{0\}^1) \subseteq ((\{\div\} * (\{\>\} * \{1\}^1)) * \{1\}^1 \subseteq (\{\<\} * \{1\}^1)(\{\>\} * \{0\}^1)) \\ & ((\{\>\} * \{1\}^1 \subseteq \{\>\} * \{1\}^1) + (\{\>\} * \{1\}^1 = \{\>\} * \{1\}^1)) (\{1\}^1 \subseteq \{\>\} * \{0\}^1) \subseteq \\ & ((\{\div\} * (\{\>\} * \{1\}^1)) * \{1\}^1 \subseteq (\{\<\} * \{1\}^1)(\{\>\} * \{0\}^1)) \quad (T_{M266})(\text{Definition of } \subseteq) \\ & ((\{\>\} * \{1\}^1 \subseteq \{\>\} * \{1\}^1) + E) (\{1\}^1 \subseteq \{\>\} * \{0\}^1) \subseteq ((\{\div\} * (\{\>\} * \{1\}^1)) * \{1\}^1 \subseteq (\{\<\} * \{1\}^1)(\{\>\} * \{0\}^1)) \\ & \quad (T_{M001})(\text{Universality of } (D_{M00}) \text{ Definition of } =) \\ & E(\{1\}^1 \subseteq \{\>\} * \{0\}^1) \subseteq ((\{\div\} * (\{\>\} * \{1\}^1)) * \{1\}^1 \subseteq (\{\<\} * \{1\}^1)(\{\>\} * \{0\}^1)) \\ & \quad (T_{M055})(\text{Universal Set Disjunction}) \\ & EE \subseteq ((\{\div\} * (\{\>\} * \{1\}^1)) * \{1\}^1 \subseteq (\{\<\} * \{1\}^1)(\{\>\} * \{0\}^1)) \\ & \quad (T_{M001})(\text{Universality of } (T_{582})) \\ & E \subseteq ((\{\div\} * (\{\>\} * \{1\}^1)) * \{1\}^1 \subseteq (\{\<\} * \{1\}^1)(\{\>\} * \{0\}^1)) \\ & \quad (T_{M667})(\text{Conjunction Tautology}) \\ & E = ((\{\div\} * (\{\>\} * \{1\}^1)) * \{1\}^1 \subseteq (\{\<\} * \{1\}^1)(\{\>\} * \{0\}^1)) \\ & \quad (T_{M275})(\text{Supersets of the Universal Set}) \\ & (\{\div\} * (\{\>\} * \{1\}^1)) * \{1\}^1 \subseteq (\{\<\} * \{1\}^1)(\{\>\} * \{0\}^1) \quad (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$(\mathbf{o} = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (((\mathbf{o} * \{a\}) * \{c\}^1 \subseteq (\mathbf{o} * \{b\}) * \{c\}^1) \subseteq (\mathbf{o} * \{a\}) \subseteq (\mathbf{o} * \{b\})) \quad (T_{5840})$$

Proof of (T₅₈₄₀)

$$\begin{aligned} & (\mathbf{o} = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (((\mathbf{o} * \{a\}) * \{c\}^1 \subseteq (\mathbf{o} * \{b\}) * \{c\}^1) \subseteq (\{a\} \subseteq \{b\})) \\ & \quad (A_{R43})(\text{Axiom of Arithmetic Induction})(\text{Eqn1}) \\ & (\{a\} \subseteq \{b\}) \subseteq ((\mathbf{o} * \{a\}) \subseteq (\mathbf{o} * \{b\})) \\ & \quad (T_{R196})(\text{Relational Deduction})(\text{Eqn2}) \\ & (\mathbf{o} = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (((\mathbf{o} * \{a\}) * \{c\}^1 \subseteq (\mathbf{o} * \{b\}) * \{c\}^1) \subseteq (\mathbf{o} * \{a\}) \subseteq (\mathbf{o} * \{b\})) \\ & \quad (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \end{aligned}$$

$$\{\div\} * \{\>\} * \{1\}^1 = \{x\} * ((\{\<\} * \{1\}^1)(\{\>\} * \{0\}^1)) \quad (T_{5982})$$

Proof of (T_{S982})

$$\begin{aligned}
 (\div) * \{>\} * \{1\}^1 &= (\{<\} * \{1\}^1) (\{>\} * \{0\}^1) & (T_{S842}) \\
 (\div) * \{>\} * \{1\}^1 &= (\{x\} * \{1\}^1) * (\{<\} * \{1\}^1) (\{>\} * \{0\}^1) & (T_{S845}) \\
 (\div) * \{>\} * \{1\}^1 &= (\{x\} * ((\{<\} * \{1\}^1) (\{>\} * \{0\}^1))) * \{1\}^1 & (T_{T263}) \text{(Commutativity of Algebraic x)} \\
 \{>\} * \{>\} * \{1\}^1 &= \{x\} * ((\{<\} * \{1\}^1) (\{>\} * \{0\}^1)) & (T_{S840})
 \end{aligned}$$

$$\{x\} * \{>\} * \{1\}^1 = \{>\} * ((\{<\} * \{1\}^1) (\{>\} * \{0\}^1)) \quad (T_{S984})$$

Proof of (T_{S984})

$$\begin{aligned}
 \{>\} * \{>\} * \{1\}^1 &= \{x\} * ((\{<\} * \{1\}^1) (\{>\} * \{0\}^1)) & (T_{S982}) \\
 \vee (\{>\} * \{>\} * \{1\}^1) &= \vee (\{x\} * ((\{<\} * \{1\}^1) (\{>\} * \{0\}^1))) & (T_{R196}) \text{(Relational Deduction)} \\
 \{x\} * \{>\} * \{1\}^1 &= \vee (\{x\} * ((\{<\} * \{1\}^1) (\{>\} * \{0\}^1))) & (T_{T154}) \\
 \{x\} * \{>\} * \{1\}^1 &= \{>\} * ((\{<\} * \{1\}^1) (\{>\} * \{0\}^1)) & (T_{T156})
 \end{aligned}$$

$$(\{x\} * \{>\} * \{1\}^1) * \{1\}^1 = (\{>\} * ((\{<\} * \{1\}^1) (\{>\} * \{0\}^1))) * \{1\}^1 \quad (T_{S936})$$

$$(\{>\} * \{>\} * \{1\}^1) * \{1\}^1 = (\{x\} * ((\{<\} * \{1\}^1) (\{>\} * \{0\}^1))) * \{1\}^1 \quad (T_{S931})$$

Example Proof of (T_{S936})

$$\begin{aligned}
 \{x\} * \{>\} * \{1\}^1 &= \{>\} * ((\{<\} * \{1\}^1) (\{>\} * \{0\}^1)) & (T_{S984}) \\
 (\{x\} * \{>\} * \{1\}^1) * \{1\}^1 &= (\{>\} * ((\{<\} * \{1\}^1) (\{>\} * \{0\}^1))) * \{1\}^1 & (T_{R197}) \text{(Relational Deduction)}
 \end{aligned}$$

$$(\mathbf{o} = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (((\mathbf{o} * \{a\}) * \{1\}^1 \subseteq (\mathbf{o} * \{b\}) * \{1\}^1) \subseteq ((\mathbf{o} * \{a\}) \subseteq (\mathbf{o} * \{b\}))) \quad (T_{S930})$$

Proof of (T_{S930})

$$(\mathbf{o} = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (((\mathbf{o} * \{a\}) * \{c\}^1 \subseteq (\mathbf{o} * \{b\}) * \{c\}^1) \subseteq ((\mathbf{o} * \{a\}) \subseteq (\mathbf{o} * \{b\}))) \quad (T_{S840}) \text{(Eqn1)}$$

$$\text{Let } \{c\}^1 = \{1\}^1 \quad \text{(Eqn2)}$$

$$(\mathbf{o} = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) \subseteq (((\mathbf{o} * \{a\}) * \{1\}^1 \subseteq (\mathbf{o} * \{b\}) * \{1\}^1) \subseteq ((\mathbf{o} * \{a\}) \subseteq (\mathbf{o} * \{b\}))) \quad (\mathbf{A}_{MX19}) \text{(Substitution of Eqn1 into Eqn2)}$$

$$((\{+\} * \{a\}) * \{1\}^1 \subseteq (\{+\} * \{b\}) * \{1\}^1) \subseteq ((\{+\} * \{a\}) \subseteq (\{+\} * \{b\})) \quad (T_{S920})$$

$$((\{-\} * \{a\}) * \{1\}^1 \subseteq (\{-\} * \{b\}) * \{1\}^1) \subseteq ((\{-\} * \{a\}) \subseteq (\{-\} * \{b\})) \quad (T_{S915})$$

$$((\{x\} * \{a\}) * \{1\}^1 \subseteq (\{x\} * \{b\}) * \{1\}^1) \subseteq ((\{x\} * \{a\}) \subseteq (\{x\} * \{b\})) \quad (T_{S909})$$

$$((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\div\} * \{b\}) * \{1\}^1) \subseteq ((\{\div\} * \{a\}) \subseteq (\{\div\} * \{b\})) \quad (T_{S908})$$

Example Proof of (T_{S908})

$$\begin{aligned}
 (\mathbf{o} = \{+, -, \{x\}, \{\div\}, \{\wedge\}, \{\vee\}) &\subseteq (((\mathbf{o} * \{a\}) * \{1\}^1 \subseteq (\mathbf{o} * \{b\}) * \{1\}^1) \subseteq ((\mathbf{o} * \{a\}) \subseteq (\mathbf{o} * \{b\}))) & (T_{S930}) \\
 ((\{-\} * \{a\}) * \{1\}^1 \subseteq (\{-\} * \{b\}) * \{1\}^1) &\subseteq ((\{\div\} * \{a\}) \subseteq (\{\div\} * \{b\})) & \text{Extraction}
 \end{aligned}$$

$$(\{a\} \subseteq \{b\}) = (((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\div\} * \{b\}) * \{1\}^1)) \quad (T_{S905})$$

Proof of (T_{S905})

Let $\{a\} \subseteq \{b\}$

$$\{\div\} * \{a\} \subseteq \{\div\} * \{b\} \quad (T_{R196}) \text{(Relational Deduction)}$$

$$(\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\div\} * \{b\}) * \{1\}^1 \quad (T_{R197}) \text{(Relational Deduction)}$$

$$\text{Thus } (\{a\} \subseteq \{b\}) \subseteq (((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\div\} * \{b\}) * \{1\}^1) \subseteq ((\{\div\} * \{a\}) \subseteq (\{\div\} * \{b\}))) \quad (\mathbf{A}_{MX02}) \text{(Conditional Introduction)(Eqn1)}$$

$$\text{Let } (\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\div\} * \{b\}) * \{1\}^1$$

$$(\{\div\} * \{a\}) \subseteq (\{\div\} * \{b\}) \quad (T_{S908})$$

$$\vee (\{\div\} * \{a\}) \subseteq \vee (\{\div\} * \{b\}) \quad (T_{R196}) \text{(Relational Deduction)}$$

$$(\{x\} * \{a\}) \subseteq (\{x\} * \{b\}) \quad (T_{T154})$$

$$(\{x\} * \{a\}) * \{1\}^1 \subseteq (\{x\} * \{b\}) * \{1\}^1 \quad (T_{R197}) \text{(Relational Deduction)}$$

$$(\{x\} * \{1\}^1) * \{a\} \subseteq (\{x\} * \{1\}^1) * \{b\} \quad (T_{T263}) \text{(Commutativity of Algebraic x)}$$

$$\{a\} \subseteq \{b\} \quad (T_{S845})$$

$$\text{Thus } (((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\div\} * \{b\}) * \{1\}^1) \subseteq (\{a\} \subseteq \{b\})) \quad (\mathbf{A}_{MX02}) \text{(Conditional Introduction)(Eqn2)}$$

$$(\{a\} \subseteq \{b\}) = (((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\div\} * \{b\}) * \{1\}^1)) \quad (T_{M430}) \text{(Biconditional Introduction Eqn1\&2)}$$

$$(\{a\} \subseteq \{>\} * \{1\}^1) = ((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) \quad (\mathcal{T}_{S903})$$

Proof of ($\mathcal{T}_{S903}$)

$$\begin{aligned} (\{\div\} * \{>\} * \{1\}^1) * \{1\}^1 &= (\{x\} * ((\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1))) * \{1\}^1 & (\mathcal{T}_{S931})(\text{Eqn1}) \\ \text{Let } \{a\} \subseteq \{>\} * \{1\}^1 & & (\text{Eqn2}) \\ \{\div\} * \{a\} \subseteq \{\div\} * \{>\} * \{1\}^1 & & (\mathcal{T}_{R196})(\text{Relational Deduction}) \\ (\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\div\} * \{>\} * \{1\}^1) * \{1\}^1 & & (\mathcal{T}_{R197})(\text{Relational Deduction})(\text{Eqn3}) \\ (\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{x\} * ((\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1))) * \{1\}^1 & & (\mathcal{A}_{MX19})(\text{Substitution of Eqn1 into Eqn3}) \\ (\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{x\} * \{1\}^1) * ((\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) & & (\mathcal{T}_{T263})(\text{Commutativity of Algebraic x}) \\ (\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1) & & (\mathcal{T}_{S845}) \\ \text{Thus } (\{a\} \subseteq \{>\} * \{1\}^1) \subseteq ((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) & & (\mathcal{A}_{MX02})(\text{Conditional Introduction})(\text{Eqn3}) \\ (\{\div\} * \{>\} * \{1\}^1) * \{1\}^1 &= (\{x\} * ((\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1))) * \{1\}^1 & (\mathcal{T}_{S931})(\text{Eqn1}) \\ \text{Let } (\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1) & & (\text{Eqn4}) \\ \{x\} * ((\{\div\} * \{a\}) * \{1\}^1) \subseteq \{x\} * ((\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) & & (\mathcal{T}_{R196})(\text{Relational Deduction}) \\ (\{x\} * ((\{\div\} * \{a\}) * \{1\}^1)) * \{1\}^1 \subseteq (\{x\} * ((\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1))) * \{1\}^1 & & (\mathcal{T}_{R197})(\text{Relational Deduction})(\text{Eqn5}) \\ (\{x\} * ((\{\div\} * \{a\}) * \{1\}^1)) * \{1\}^1 \subseteq (\{\div\} * \{>\} * \{1\}^1) * \{1\}^1 & & (\mathcal{A}_{MX19})(\text{Substitution of Eqn1 into Eqn5}) \\ (\{x\} * \{1\}^1) * ((\{\div\} * \{a\}) * \{1\}^1) \subseteq (\{\div\} * \{>\} * \{1\}^1) * \{1\}^1 & & (\mathcal{T}_{T263})(\text{Commutativity of Algebraic x}) \\ ((\{\div\} * \{a\}) * \{1\}^1) \subseteq (\{\div\} * \{>\} * \{1\}^1) * \{1\}^1 & & (\mathcal{T}_{S845}) \\ \vee((\{\div\} * \{a\}) * \{1\}^1) \subseteq \vee((\{\div\} * \{>\} * \{1\}^1) * \{1\}^1) & & (\mathcal{T}_{R196})(\text{Relational Deduction}) \\ \{a\} \subseteq \{>\} * \{1\}^1 & & (\mathcal{T}_{S905}) \\ \text{Thus } ((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) \subseteq (\{a\} \subseteq \{>\} * \{1\}^1) & & (\mathcal{A}_{MX02})(\text{Conditional Introduction})(\text{Eqn6}) \\ (\{a\} \subseteq \{>\} * \{1\}^1) = ((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) & & (\mathcal{T}_{M430})(\text{Biconditional Introduction}) \\ & & \text{Eqns3\&6)} \end{aligned}$$

$$(\{a\} \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) = ((\{\div\} * \{a\}) * \{1\}^1 \subseteq \{>\} * \{1\}^1) \quad (\mathcal{T}_{S985})$$

Proof of ($\mathcal{T}_{S985}$)

$$\begin{aligned} (\{a\} \subseteq \{>\} * \{1\}^1) &= ((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) & (\mathcal{T}_{S903})(\text{Eqn1}) \\ \text{Let } \{a\} = (\{\div\} * \{a\}) * \{1\}^1 & & (\text{Eqn2}) \\ ((\{\div\} * \{a\}) * \{1\}^1 \subseteq \{>\} * \{1\}^1) &= (\{a\} \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(\{a\} \subseteq \{>\} * \{b\}^1)(\{b\}^1 \subseteq \{>\} * \{0\}^1) \subseteq ((\{\div\} * \{a\}) * \{b\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) \quad (\mathcal{T}_{S953}) \text{ Theorem of Fractions}$$

Proof of ($\mathcal{T}_{S953}$)

$$\begin{aligned} (\{a\} \subseteq \{>\} \diamond \{b\})(\{b\} \subseteq \{>\} * \{0\}^1) &\subseteq ((\{\div\} * \{a\}) * \{b\} \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) & (\mathcal{A}_{R46})(\text{Axiom of Fractions})(\text{Eqn1}) \\ \text{Let } \{b\} = \{b\}^1 & & (\text{Eqn2}) \\ (\{a\} \subseteq \{>\} \diamond \{b\})(\{b\}^1 \subseteq \{>\} * \{0\}^1) &\subseteq ((\{\div\} * \{a\}) * \{b\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(\{a\} \subseteq \{>\} * \{1\}^1) \subseteq ((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) \quad (\mathcal{T}_{T261}) \text{ Theorem of Fractions}$$

Proof of ($\mathcal{T}_{T261}$)

$$\begin{aligned} (\{a\} \subseteq \{>\} \diamond \{b\})(\{b\}^1 \subseteq \{>\} * \{0\}^1) &\subseteq ((\{\div\} * \{a\}) * \{b\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) & (\mathcal{T}_{S953})(\text{Theorem of Fractions})(\text{Eqn1}) \\ \text{Let } \{b\}^1 = \{1\}^1 & & (\text{Eqn2}) \\ (\{a\} \subseteq \{>\} \diamond \{1\}^1)(\{1\}^1 \subseteq \{>\} * \{0\}^1) &\subseteq ((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3}) \\ \{1\}^1 \subseteq \{>\} * \{0\}^1 & & (\mathcal{T}_{T582}) \\ (\{1\}^1 \subseteq \{>\} * \{0\}^1) = E & & (\mathcal{A}_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\ (\{a\} \subseteq \{>\} \diamond \{1\}^1)E \subseteq ((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) & & (\mathcal{A}_{MX19})(\text{Substitution of Eqn4 into Eqn3}) \\ (\{a\} \subseteq \{>\} \diamond \{1\}^1) \subseteq ((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) & & (\mathcal{A}_{MX17})(\text{Universal Set Conjunction Axiom}) \end{aligned}$$

$$(\{a\} \subseteq \{>\} * \{1\}^1)(\{b\} = (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) \subseteq ((\{\div\} * \{a\}) * \vee(\{\div\} * \{a\}) * \{b\} \subseteq (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) \quad (\mathcal{T}_{S879})$$

$$(\{a\} \subseteq \{>\} * \{1\}^1)(\{b\} = (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1)) \subseteq ((\{\div\} * \{b\}) * \vee(\{\div\} * \{b\}) * \{a\} \subseteq \{>\} * \{1\}^1) \quad (\mathcal{T}_{S935})$$

Example Proof of ($\mathcal{T}_{S879}$)

$$\begin{aligned} \text{Let } \{a\} \subseteq \{>\} * \{1\}^1 & & (\text{Eqn1}) \\ \text{Let } \{b\} = (\{\leq\} * \{1\}^1)(\{>\} * \{0\}^1) & & (\text{Eqn2}) \end{aligned}$$

$$\begin{aligned}
& \{b\} \subseteq (\{<\} * \{1\}^1) & (T_{M073b})(\text{Single Premise Syllogism})(\text{Eqn3}) \\
& \{>\} \diamond (\{<\} * \{1\}^1) \subseteq \{>\} \diamond \{b\} & (T_{R262})(\text{Relational Deduction})(\text{Eqn4}) \text{ since } \{b\} \neq \emptyset \\
& \{>\} * \{a\}^1 \subseteq \{>\} \diamond (\{<\} * \{a\}^1) & (T_{S897})(\text{Eqn5}) \\
& \text{Let } \{a\}^1 = \{1\}^1 & (\text{Eqn6}) \\
& \{>\} * \{1\}^1 \subseteq \{>\} \diamond (\{<\} * \{1\}^1) & (A_{MX19})(\text{Substitution of Eqn6 into Eqn5})(\text{Eqn7}) \\
& \{a\} \subseteq \{>\} \diamond (\{<\} * \{1\}^1) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&7})(\text{Eqn8}) \\
& \{a\} \subseteq \{>\} \diamond \{b\} & (T_{M139})(\text{Hypothetical Syllogism Eqns8\&4})(\text{Eqn9}) \\
& (\{>\} * \{a\}) * \{b\} \subseteq (\{<\} * \{1\}^1) (\{>\} * \{0\}^1) & (A_{R46})(\text{Axiom of Fractions}) \\
& & \text{since } \{a\} \subseteq \{>\} \diamond \{b\} \text{ and } \{b\} = (\{>\} * \{0\}^1) \text{ (Eqn10)} \\
& \text{Let } \{b\} = {}^v(\{>\} * \{a\}) * \{b\} & (\text{Eqn11}) \\
& (\{>\} * \{a\}) * {}^v(\{>\} * \{a\}) * \{b\} \subseteq (\{<\} * \{1\}^1) (\{>\} * \{0\}^1) & (A_{MX19})(\text{Substitution of Eqn11 into Eqn10})(\text{Eqn7}) \\
& \text{Thus } (\{a\} \subseteq \{>\} * \{1\}^1) (\{b\} = (\{<\} * \{1\}^1) (\{>\} * \{0\}^1)) \subseteq ((\{>\} * \{a\}) * {}^v(\{>\} * \{a\}) * \{b\} \subseteq (\{<\} * \{1\}^1) (\{>\} * \{0\}^1)) & (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$(\{>\} * ((\{>\} * \{a\}) * \{b\})) * \{1\}^1 = (\{>\} * \{b\}) * \{a\} \quad (T_{S880})$$

Proof of (T_{S880})

$$\begin{aligned}
& (\{>\} * ((\{>\} * \{a\}) * \{b\})) = (\{x\} * (\{>\} * \{b\}) * \{a\}) & (A_{R26})(\text{Axiom of Double Division}) \\
& (\{>\} * ((\{>\} * \{a\}) * \{b\})) * \{1\}^1 = (\{x\} * (\{>\} * \{b\}) * \{a\}) * \{1\}^1 & (T_{R197})(\text{Relational Deduction}) \\
& (\{>\} * ((\{>\} * \{a\}) * \{b\})) * \{1\}^1 = (\{x\} * \{1\}^1) * ((\{>\} * \{b\}) * \{a\}) & (T_{T263})(\text{Commutativity of Algebraic } x) \\
& (\{>\} * ((\{>\} * \{a\}) * \{b\})) * \{1\}^1 = (\{>\} * \{b\}) * \{a\} & (T_{S845})
\end{aligned}$$

$$(\{a\} \subseteq \{>\} \diamond \{b\}) (\{b\} \subseteq \{>\} * \{0\}^1) \subseteq ((\{>\} * \{b\}) * \{a\} \subseteq \{>\} * \{1\}^1) \quad (T_{S892})$$

Proof of (T_{S892})

$$\begin{aligned}
& (\{a\} \subseteq \{>\} \diamond \{b\}) (\{b\} \subseteq \{>\} * \{0\}^1) \subseteq ((\{>\} * \{a\}) * \{b\} \subseteq (\{<\} * \{1\}^1) (\{>\} * \{0\}^1)) & (A_{R46})(\text{Axiom of Fractions})(\text{Eqn1}) \\
& & (T_{S985})(\text{Eqn2}) \\
& (\{a\} \subseteq (\{<\} * \{1\}^1) (\{>\} * \{0\}^1)) = ((\{>\} * \{a\}) * \{1\}^1 \subseteq \{>\} * \{1\}^1) & (T_{S985})(\text{Eqn2}) \\
& \text{Let } \{a\} = (\{>\} * \{a\}) * \{b\} & (\text{Eqn3}) \\
& ((\{>\} * \{a\}) * \{b\} \subseteq (\{<\} * \{1\}^1) (\{>\} * \{0\}^1)) = ((\{>\} * ((\{>\} * \{a\}) * \{b\})) * \{1\}^1 \subseteq \{>\} * \{1\}^1) & (A_{MX19})(\text{Substitution of Eqn3 into Eqn2})(\text{Eqn4}) \\
& & (A_{MX19})(\text{Substitution of Eqn4 into Eqn1})(\text{Eqn5}) \\
& (\{a\} \subseteq \{>\} \diamond \{b\}) (\{b\} \subseteq \{>\} * \{0\}^1) \subseteq ((\{>\} * ((\{>\} * \{a\}) * \{b\})) * \{1\}^1 \subseteq \{>\} * \{1\}^1) & (A_{MX19})(\text{Substitution of Eqn4 into Eqn1})(\text{Eqn5}) \\
& & (T_{S880})(\text{Eqn6}) \\
& (\{>\} * ((\{>\} * \{a\}) * \{b\})) * \{1\}^1 = (\{>\} * \{b\}) * \{a\} & (T_{S880})(\text{Eqn6}) \\
& (\{a\} \subseteq \{>\} \diamond \{b\}) (\{b\} \subseteq \{>\} * \{0\}^1) \subseteq ((\{>\} * \{b\}) * \{a\} \subseteq \{>\} * \{1\}^1) & (A_{MX19})(\text{Substitution of Eqn6 into Eqn5})
\end{aligned}$$

$$(\{a\} \subseteq \{>\} * \{b\}^1) (\{b\}^1 \subseteq \{>\} * \{0\}^1) \subseteq ((\{>\} * \{b\}^1) * \{a\} \subseteq \{>\} * \{1\}^1) \quad (T_{S884})$$

Proof of (T_{S884})

$$\begin{aligned}
& (\{a\} \subseteq \{>\} \diamond \{b\}) (\{b\} \subseteq \{>\} * \{0\}^1) \subseteq ((\{>\} * \{b\}) * \{a\} \subseteq \{>\} * \{1\}^1) & (T_{S892})(\text{Eqn1}) \\
& \text{Let } \{b\} = \{b\}^1 & (\text{Eqn2}) \\
& (\{a\} \subseteq \{>\} * \{b\}^1) (\{b\}^1 \subseteq \{>\} * \{0\}^1) \subseteq ((\{>\} * \{b\}^1) * \{a\} \subseteq \{>\} * \{1\}^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$(\{a\} \subseteq \{>\} * \{1\}^1) \subseteq ((\{>\} * \{1\}^1) * \{a\} \subseteq \{>\} * \{1\}^1) \quad (T_{S881})$$

Proof of (T_{S881})

$$\begin{aligned}
& (\{a\} \subseteq \{>\} * \{b\}^1) (\{b\}^1 \subseteq \{>\} * \{0\}^1) \subseteq ((\{>\} * \{b\}^1) * \{a\} \subseteq \{>\} * \{1\}^1) & (T_{S884})(\text{Eqn1}) \\
& \text{Let } \{b\}^1 = \{1\}^1 & (\text{Eqn2}) \\
& (\{a\} \subseteq \{>\} * \{1\}^1) (\{1\}^1 \subseteq \{>\} * \{0\}^1) \subseteq ((\{>\} * \{1\}^1) * \{a\} \subseteq \{>\} * \{1\}^1) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
& (\{a\} \subseteq \{>\} * \{1\}^1) \subseteq ((\{>\} * \{1\}^1) * \{a\} \subseteq \{>\} * \{1\}^1) & (T_{M007})(\text{Universality of } (T_{S882})) \\
& (\{a\} \subseteq \{>\} * \{1\}^1) \subseteq ((\{>\} * \{1\}^1) * \{a\} \subseteq \{>\} * \{1\}^1) & (A_{MX11})(\text{Universal Set Conjunction Axiom})
\end{aligned}$$

$${}^v((\{>\} * \{a\}) * \{1\}^1) = (\{>\} * {}^v\{a\}) * \{1\}^1 \quad (T_{S870})$$

$${}^v((\{>\} * \{a\}) * \{1\}^1) = (\{>\} * \{a\}) * {}^v\{1\}^1 \quad (T_{S872})$$

Proof of (T_{S870})

$$\begin{aligned}
& (X = \{x\}, \{>\})(Y = V, \{=\})(Z = V, \{=\}) \subseteq ((X * Y * \{a\}) * Z * \{b\} = Y * Z * (X * \{a\}) * \{b\}) & (A_{R47})(\text{Positive/Negative Axiom})
\end{aligned}$$

$$(\{\div\} * \mathbf{V} * \{a\}) * \{=\} * \{b\} = \mathbf{V} * \{=\} * (\{\div\} * \{a\}) * \{b\}$$

Extraction

$$(\{\div\} * \mathbf{V} * \{a\}) * \{b\} = \mathbf{V} * (\{\div\} * \{a\}) * \{b\}$$

(T_{A080})

$$(\{\div\} * \mathbf{V} * \{a\}) * \{b\} = \mathbf{V} * (\{\div\} * \{a\}) * \{b\}$$

(D_{R27})(Simplification of the Relationship Symmetry Function)(Eqn1)

$$\text{Let } \{b\} = \{1\}^1$$

(Eqn2)

$$(\{\div\} * \mathbf{V} * \{a\}) * \{1\}^1 = \mathbf{V} * (\{\div\} * \{a\}) * \{1\}^1$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(\{>\} * \{1\}^1) \subseteq (\{>\} * \{0\}^1)$$

(T_{S861})

$$(\{>\} * \{1\}^1)(\{>\} * \{0\}^1) = (\{>\} * \{1\}^1)$$

(T_{S864})Proof of (T_{S861}) & (T_{S864})

$$\{1\}^1 \subseteq \{>\} * \{0\}^1$$

(T_{T582})

$$\{>\} * \{1\}^1 \subseteq \{>\} * \{>\} * \{0\}^1$$

(T_{R196})(Relational Deduction)(Eqn1)

$$\{>\} * \{>\} * \{0\}^1 \subseteq \{>\} * \{0\}^1$$

(T_{S414})(Eqn2)

$$(\{>\} * \{1\}^1) \subseteq (\{>\} * \{0\}^1)$$

(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$(\{>\} * \{1\}^1) = (\{>\} * \{1\}^1)(\{>\} * \{0\}^1)$$

(A_{MX44})($\subseteq$ Equality Form)

$$(\{<\} * \mathbf{V} * \{1\}^1) \subseteq (\{<\} * \{0\}^1)$$

(T_{S862})

$$(\{<\} * \mathbf{V} * \{1\}^1)(\{<\} * \{0\}^1) = (\{<\} * \mathbf{V} * \{1\}^1)$$

(T_{S865})Proof of (T_{S862}) & (T_{S865})

$$\{1\}^1 \subseteq \{>\} * \{0\}^1$$

(T_{T582})

$$\mathbf{V} * \{1\}^1 \subseteq \mathbf{V} * (\{>\} * \{0\}^1)$$

(T_{R196})(Relational Deduction)

$$\mathbf{V} * \{1\}^1 \subseteq \{<\} * \{0\}^1$$

(T_{T215})

$$\{<\} * \mathbf{V} * \{1\}^1 \subseteq \{<\} * \{<\} * \{0\}^1$$

(T_{R196})(Relational Deduction)(Eqn1)

$$\{<\} * \{<\} * \{0\}^1 \subseteq \{<\} * \{0\}^1$$

(T_{S414})(Eqn2)

$$\{<\} * \mathbf{V} * \{1\}^1 \subseteq \{<\} * \{0\}^1$$

(T_{M139})(Hypothetical Syllogism Eqns1&2)

$$\{<\} * \mathbf{V} * \{1\}^1 = (\{<\} * \mathbf{V} * \{1\}^1)(\{<\} * \{0\}^1)$$

(A_{MX44})($\subseteq$ Equality Form)

$$(\{a\} \subseteq \{<\} * \mathbf{V} * \{1\}^1) = ((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{>\} * \mathbf{V} * \{1\}^1)(\{<\} * \{0\}^1))$$

(T_{S980})Proof of (T_{S980})

$$(\{a\} \subseteq \{>\} * \{1\}^1) = ((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(T_{S903})

$$(\mathbf{V} * \{a\} \subseteq \mathbf{V} * (\{>\} * \{1\}^1)) = ((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(T_{R022d})(Relationship Opposite of a Proposition)

$$(\mathbf{V} * \{a\} \subseteq \{<\} * \mathbf{V} * \{1\}^1) = ((\{\div\} * \{a\}) * \{1\}^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(T_{T213})(Eqn1)

$$(\mathbf{V} * \{a\} \subseteq \{<\} * \mathbf{V} * \{1\}^1) = (\mathbf{V} * ((\{\div\} * \{a\}) * \{1\}^1) \subseteq \mathbf{V} * ((\{<\} * \{1\}^1)(\{>\} * \{0\}^1)))$$

(T_{R022d})(Relationship Opposite of a Proposition)

$$(\mathbf{V} * \{a\} \subseteq \{<\} * \mathbf{V} * \{1\}^1) = (\mathbf{V} * ((\{\div\} * \{a\}) * \{1\}^1) \subseteq \mathbf{V} * (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(T_{R012a})(Distribution of $\mathbf{V}^*$ over $\cdot$)

$$(\mathbf{V} * \{a\} \subseteq \{<\} * \mathbf{V} * \{1\}^1) = (\mathbf{V} * ((\{\div\} * \{a\}) * \{1\}^1) \subseteq (\{>\} * \mathbf{V} * \{1\}^1)(\{<\} * \{0\}^1))$$

(T_{T211}) & (T_{T215})

$$(\mathbf{V} * \{a\} \subseteq \{<\} * \mathbf{V} * \{1\}^1) = (((\{\div\} * \mathbf{V} * \{a\}) * \{1\}^1) \subseteq (\{>\} * \mathbf{V} * \{1\}^1)(\{<\} * \{0\}^1))$$

(T_{S870})

$$(\{a\} \subseteq \{<\} * \mathbf{V} * \{1\}^1) = (((\{\div\} * \{a\}) * \{1\}^1) \subseteq (\{>\} * \mathbf{V} * \{1\}^1)(\{<\} * \{0\}^1))$$

Resymbolising

$$(\{a\} \subseteq (\{>\} * \mathbf{V} * \{1\}^1)(\{<\} * \{0\}^1)) = ((\{\div\} * \{a\}) * \{1\}^1 \subseteq \{<\} * \mathbf{V} * \{1\}^1)$$

(T_{S977})Proof of (T_{S977})

$$(\{a\} \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) = ((\{\div\} * \{a\}) * \{1\}^1 \subseteq \{>\} * \{1\}^1)$$

(T_{S985})

$$(\mathbf{V} * \{a\} \subseteq \mathbf{V} * ((\{<\} * \{1\}^1)(\{>\} * \{0\}^1))) = ((\{\div\} * \{a\}) * \{1\}^1 \subseteq \{>\} * \{1\}^1)$$

(T_{R022d})(Relationship Opposite of a Proposition)

$$(\mathbf{V} * \{a\} \subseteq \mathbf{V} * ((\{<\} * \{1\}^1)(\{>\} * \{0\}^1))) = ((\{\div\} * \{a\}) * \{1\}^1 \subseteq \{>\} * \{1\}^1)$$

(T_{R012a})(Distribution of $\mathbf{V}^*$ over $\cdot$)

$$(\mathbf{V} * \{a\} \subseteq (\{>\} * \mathbf{V} * \{1\}^1)(\{<\} * \{0\}^1)) = ((\{\div\} * \{a\}) * \{1\}^1 \subseteq \{>\} * \{1\}^1)$$

(T_{T211}) & (T_{T215})

$$(\mathbf{V} * \{a\} \subseteq (\{>\} * \mathbf{V} * \{1\}^1)(\{<\} * \{0\}^1)) = (\mathbf{V} * ((\{\div\} * \{a\}) * \{1\}^1) \subseteq \mathbf{V} * (\{>\} * \{1\}^1))$$

(T_{R022d})(Relationship Opposite of a Proposition)

$$(\mathbf{V} * \{a\} \subseteq (\{>\} * \mathbf{V} * \{1\}^1)(\{<\} * \{0\}^1)) = (\mathbf{V} * ((\{\div\} * \{a\}) * \{1\}^1) \subseteq \{<\} * \mathbf{V} * \{1\}^1)$$

(T_{T213})

$$(\mathbf{V} * \{a\} \subseteq (\{>\} * \mathbf{V} * \{1\}^1)(\{<\} * \{0\}^1)) = (((\{\div\} * \mathbf{V} * \{a\}) * \{1\}^1) \subseteq \{<\} * \mathbf{V} * \{1\}^1)$$

$$(\{a\} \subseteq \{>\}^V\{1\}^1)(\{<\}^V\{0\}^1) = ((\{ \div \}^V\{a\}^1)\{1\}^1 \subseteq \{<\}^V\{1\}^1) \text{ Resymbolising} \quad (\mathcal{T}_{S870})$$

$$(\{ \div \}^V\{<\}^V\{1\}^1)\{1\}^1 = (\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1) \quad (\mathcal{T}_{S860})$$

Proof of ($\mathcal{T}_{S860}$)

$$\begin{aligned} (\{ \div \}^V\{>\}^V\{1\}^1)\{1\}^1 &= (\{<\}^V\{1\}^1)(\{>\}^V\{0\}^1) & (\mathcal{T}_{S842}) \\ \vee((\{ \div \}^V\{>\}^V\{1\}^1)\{1\}^1) &= \vee((\{<\}^V\{1\}^1)(\{>\}^V\{0\}^1)) & (\mathcal{T}_{R022})(\text{Relationship Opposite of a Proposition}) \\ & & (\text{Eqn1}) \\ \vee((\{ \div \}^V\{a\}^1)\{1\}^1) &= (\{ \div \}^V\{a\}^1)\{1\}^1 & (\mathcal{T}_{S870})(\text{Eqn2}) \\ \text{Let } \{a\} &= \{>\}^V\{1\}^1 & (\text{Eqn3}) \\ \vee((\{ \div \}^V\{>\}^V\{1\}^1)\{1\}^1) &= (\{ \div \}^V\{>\}^V\{1\}^1)\{1\}^1 & (\mathcal{A}_{MX19})(\text{Substitution of Eqn3 into Eqn2})(\text{Eqn4}) \\ (\{ \div \}^V\{>\}^V\{1\}^1)\{1\}^1 &= \vee((\{<\}^V\{1\}^1)(\{>\}^V\{0\}^1)) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn4 into Eqn1}) \\ (\{ \div \}^V\{<\}^V\{1\}^1)\{1\}^1 &= \vee((\{<\}^V\{1\}^1)(\{>\}^V\{0\}^1)) & (\mathcal{T}_{T213}) \\ (\{ \div \}^V\{<\}^V\{1\}^1)\{1\}^1 &= \vee(\{<\}^V\{1\}^1)\vee(\{>\}^V\{0\}^1) & (\mathcal{T}_{R012a})(\text{Distribution of } \mathbf{V}^* \text{ over } \cdot) \\ (\{ \div \}^V\{<\}^V\{1\}^1)\{1\}^1 &= (\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1) & (\mathcal{T}_{T211}) \text{ \& } (\mathcal{T}_{T215}) \end{aligned}$$

$$\{ \div \}^V\{<\}^V\{1\}^1 = \{x\}^*((\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1)) \quad (\mathcal{T}_{S973})$$

Proof of ($\mathcal{T}_{S973}$)

$$\begin{aligned} (\{ \div \}^V\{<\}^V\{1\}^1)\{1\}^1 &= (\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1) & (\mathcal{T}_{S860})(\text{Eqn1}) \\ (\{x\}^*\{a\}^1)\{1\}^1 &= \{a\} & (\mathcal{T}_{S839})(\text{Eqn2}) \\ \text{Let } \{a\} &= (\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1) & (\text{Eqn3}) \\ (\{x\}^*((\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1)))\{1\}^1 &= (\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn3 into Eqn2})(\text{Eqn4}) \\ (\{ \div \}^V\{<\}^V\{1\}^1)\{1\}^1 &= (\{x\}^*((\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1)))\{1\}^1 & (\mathcal{A}_{MX19})(\text{Substitution of Eqn4 into Eqn1}) \\ \{ \div \}^V\{<\}^V\{1\}^1 &= \{x\}^*((\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1)) & (\mathcal{T}_{S840}) \end{aligned}$$

$$\{x\}^*\{<\}^V\{1\}^1 = \{ \div \}^*((\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1)) \quad (\mathcal{T}_{S976})$$

Proof of ($\mathcal{T}_{S976}$)

$$\begin{aligned} \{ \div \}^V\{<\}^V\{1\}^1 &= \{x\}^*((\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1)) & (\mathcal{T}_{S973}) \\ \vee(\{ \div \}^V\{<\}^V\{1\}^1) &= \vee(\{x\}^*((\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1))) & (\mathcal{T}_{R022})(\text{Relationship Opposite of a Proposition}) \\ \{x\}^*\{<\}^V\{1\}^1 &= \{ \div \}^*((\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1)) & (\mathcal{T}_{T154}) \text{ \& } (\mathcal{T}_{T156}) \end{aligned}$$

$$(\{ \div \}^V\{a\}^1)\{1\}^1 = (\{ \div \}^V\{a\}^1)\{1\}^1 \quad (\mathcal{T}_{S859})$$

Proof of ($\mathcal{T}_{S859}$)

$$\begin{aligned} \vee((\{ \div \}^V\{a\}^1)\{1\}^1) &= (\{ \div \}^V\{a\}^1)\{1\}^1 & (\mathcal{T}_{S870})(\text{Eqn1}) \\ \vee((\{ \div \}^V\{a\}^1)\{1\}^1) &= (\{ \div \}^V\{a\}^1)\{1\}^1 & (\mathcal{T}_{S872})(\text{Eqn2}) \\ (\{ \div \}^V\{a\}^1)\{1\}^1 &= (\{ \div \}^V\{a\}^1)\{1\}^1 & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(\{ \div \}^V\{a\}^1)\{1\}^1 = (\{ \div \}^V\{a\}^1)\{1\}^1 \quad (\mathcal{T}_{S858})$$

Proof of ($\mathcal{T}_{S858}$)

$$\begin{aligned} (\{ \div \}^V\{a\}^1)\{1\}^1 &= (\{ \div \}^V\{a\}^1)\{1\}^1 & (\mathcal{T}_{S859})(\text{Eqn1}) \\ \text{Let } \{a\} &= \{a\} & (\text{Eqn2}) \\ (\{ \div \}^V\{a\}^1)\{1\}^1 &= (\{ \div \}^V\{a\}^1)\{1\}^1 & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \end{aligned}$$

$$(\{a\} \subseteq \{>\}^V\{1\}^1) = ((\{ \div \}^V\{a\}^1)\{1\}^1 \subseteq (\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1)) \quad (\mathcal{T}_{S972})$$

Proof of ($\mathcal{T}_{S972}$)

$$\begin{aligned} (\{a\} \subseteq \{>\}^V\{1\}^1) &= ((\{ \div \}^V\{a\}^1)\{1\}^1 \subseteq (\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1)) & (\mathcal{T}_{S980}) \\ (\vee\{a\} \subseteq \vee(\{<\}^V\{1\}^1)) &= ((\{ \div \}^V\{a\}^1)\{1\}^1 \subseteq (\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1)) & (\mathcal{T}_{R022d})(\text{Relationship Opposite of a Proposition}) \\ (\vee\{a\} \subseteq \{>\}^V\{1\}^1) &= ((\{ \div \}^V\{a\}^1)\{1\}^1 \subseteq (\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1)) & (\mathcal{T}_{T202}) \\ (\{a\} \subseteq \{>\}^V\{1\}^1) &= ((\{ \div \}^V\{a\}^1)\{1\}^1 \subseteq (\{>\}^V\{1\}^1)(\{<\}^V\{0\}^1)) \text{ Resymbolising} \end{aligned}$$

$$(\{a\} \subseteq \{>\} * \{1\}^1) = (((\div) * \{a\}^1) * \{1\}^1 \subseteq (\{>\} * \{1\}^1)(\{<\} * \{0\}^1)) \quad (T_{S859})$$

$$(\{a\} \subseteq \{<\} * \{1\}^1) = (((\div) * \{a\}^1) * \{1\}^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \quad (T_{S960})$$

Proof of (T_{S960})

$$(\{a\} \subseteq \{<\} * \{1\}^1) = (((\div) * \{a\}^1) * \{1\}^1 \subseteq (\{>\} * \{1\}^1)(\{<\} * \{0\}^1)) \quad (T_{S980})$$

$$(\{a\} \subseteq \{<\} * \{1\}^1) = (\vee(((\div) * \{a\}^1) * \{1\}^1) \subseteq \vee((\{>\} * \{1\}^1)(\{<\} * \{0\}^1)))$$

(T_{R022a})(Relationship Opposite of a Proposition)

$$(\{a\} \subseteq \{<\} * \{1\}^1) = (\vee(((\div) * \{a\}^1) * \{1\}^1) \subseteq \vee(\{>\} * \{1\}^1) \vee (\{<\} * \{0\}^1))$$

(T_{R012a})(Distribution of $\vee$ over $\cdot$)

$$(\{a\} \subseteq \{<\} * \{1\}^1) = (\vee(((\div) * \{a\}^1) * \{1\}^1) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(T_{T212}) & (T_{T214})

$$(\{a\} \subseteq \{<\} * \{1\}^1) = (((\div) * \{a\}^1) * \{1\}^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(T_{S872})

$$(\{x\} * ((\{<\} * \{1\}^1) \cdot (\{>\} * \{0\}^1))) * ((\{<\} * \{1\}^1) \cdot (\{>\} * \{0\}^1)) = (\{<\} * \{1\}^1) \cdot (\{>\} * \{0\}^1) \quad (T_{T270})$$

Proof of (T_{T270})

$$(\{a\} \subseteq \{>\} \diamond \{b\})(\{b\} \subseteq \{>\} * \{0\}^1) \subseteq (((\div) * \{a\}) * \{b\} \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(A_{R46})(Axiom of Fractions)(Eqn1)

$$\text{Let } \{a\} = \{>\} * \{1\}^1$$

(Eqn2)

$$\text{Let } \{b\} = (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)$$

(Eqn3)

$$(\{>\} * \{1\}^1 \subseteq \{>\} \diamond ((\{<\} * \{1\}^1)(\{>\} * \{0\}^1)))((\{<\} * \{1\}^1)(\{>\} * \{0\}^1) \subseteq \{>\} * \{0\}^1) \subseteq$$

$$(((\div) * \{>\} * \{1\}^1) * ((\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(A_{MX19})(Substitution of Eqns2&3 into Eqn1)(Eqn4)

$$\{>\} * \{1\}^1 \subseteq \{>\} \diamond (\{<\} * \{1\}^1)$$

(T_{S897})(Eqn5)

$$(\{>\} * \{1\}^1) \subseteq (\{>\} * \{0\}^1)$$

(T_{S861})(Eqn6)

$$\{>\} * \{1\}^1 \subseteq \{>\} \diamond ((\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(T_{M132})(Composition of Eqns5&6)

$$(\{>\} * \{1\}^1 \subseteq \{>\} \diamond ((\{<\} * \{1\}^1)(\{>\} * \{0\}^1))) = E$$

(A_{MX16})(Order Axiom)(Eqn7)

$$(\{<\} * \{1\}^1)(\{>\} * \{0\}^1) \subseteq \{>\} * \{0\}^1$$

(T_{M083})(Conjunction Elimination)

$$(((\{<\} * \{1\}^1)(\{>\} * \{0\}^1) \subseteq \{>\} * \{0\}^1) = E$$

(A_{MX16})(Order Axiom)(Eqn8)

$$EE \subseteq (((\div) * \{>\} * \{1\}^1) * ((\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(A_{MX19})(Substitution of Eqns7&8 into Eqn4)

$$E \subseteq (((\div) * \{>\} * \{1\}^1) * ((\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(T_{M667})(Conjunction Tautology)

$$E = (((\div) * \{>\} * \{1\}^1) * ((\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(T_{M275})(Supersets of the Universal Set)

$$(((\div) * \{>\} * \{1\}^1) * ((\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(A_{MX16})(Order Axiom)

$$(\{x\} * ((\{<\} * \{1\}^1)(\{>\} * \{0\}^1))) * ((\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(T_{S982})

$$(\{a\}^1 \subseteq \{>\} * \{0\}^1) \subseteq (((\div) * \{>\} * \{a\}^1) * \{a\}^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \quad (T_{S955})$$

Proof of (T_{S955})

$$(\{a\} \subseteq \{>\} \diamond \{b\})(\{b\} \subseteq \{>\} * \{0\}^1) \subseteq (((\div) * \{a\}) * \{b\} \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(A_{R46})(Axiom of Fractions)(Eqn1)

$$\text{Let } \{a\} = \{>\} * \{a\}^1$$

(Eqn2)

$$\text{Let } \{b\} = \{a\}^1$$

(Eqn3)

$$(\{>\} * \{a\}^1 \subseteq \{>\} * \{a\}^1)(\{a\}^1 \subseteq \{>\} * \{0\}^1) \subseteq (((\div) * \{>\} * \{a\}^1) * \{a\}^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(A_{MX19})(Substitution of Eqns2&3 into Eqn1)

$$(\{>\} * \{a\}^1 \subseteq \{>\} * \{a\}^1)(\{a\}^1 \subseteq \{>\} * \{0\}^1) \subseteq (((\div) * \{>\} * \{a\}^1) * \{a\}^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

$$(((\div) * \{>\} * \{a\}^1) * \{a\}^1 \subseteq \{>\} * \{a\}^1) + (\{>\} * \{a\}^1 \subseteq \{>\} * \{a\}^1)(\{a\}^1 \subseteq \{>\} * \{0\}^1) \subseteq$$

$$(((\div) * \{>\} * \{a\}^1) * \{a\}^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(T_{M266})(Definition of $\subseteq$)

$$(((\div) * \{>\} * \{a\}^1) * \{a\}^1 \subseteq \{>\} * \{a\}^1) + E)(\{a\}^1 \subseteq \{>\} * \{0\}^1) \subseteq (((\div) * \{>\} * \{a\}^1) * \{a\}^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(T_{M001})(Universality of (D_{M00}) Definition of $=$)

$$E(\{a\}^1 \subseteq \{>\} * \{0\}^1) \subseteq (((\div) * \{>\} * \{a\}^1) * \{a\}^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

(T_{M055})(Universal Set Disjunction)

$$(\{a\}^1 \subseteq \{>\} * \{0\}^1) \subseteq (((\div) * \{>\} * \{a\}^1) * \{a\}^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1))$$

($\mathbf{A}_{MX17}$)(Universal Set Conjunction Axiom)

$$(\{x\} * \{c\}) * (\{a\} + \{b\}) = (\{x\} * \{c\}) * \{a\} + (\{x\} * \{c\}) * \{b\} \quad (\mathcal{T}_{S855})$$

Proof of ($\mathcal{T}_{S855}$)

$$(\{x\} * \{c\}) * (\{a\} + \{b\}) = (\{x\} * \{c\}) * (\{a\} + \{b\}) \quad (\mathcal{D}_{M00})(\text{Definition of } =)$$

$$(\{x\} * \{c\}) * (\{a\} + \{b\}) = (\{x\} * \{c\}) * \{a\} + (\{x\} * \{c\}) * \{b\} \quad (\mathcal{T}_{R002b})(\text{Distribution of } * \text{ over } +)$$

$$(\{x\} * (\{c\} + \{d\})) * (\{a\} + \{b\}) = (\{x\} * \{c\}) * \{a\} + (\{x\} * \{c\}) * \{b\} + (\{x\} * \{d\}) * \{a\} + (\{x\} * \{d\}) * \{b\} \quad (\mathcal{T}_{S854})$$

Proof of ($\mathcal{T}_{S854}$)

$$(\{x\} * (\{c\} + \{d\})) * (\{a\} + \{b\}) = (\{x\} * (\{c\} + \{d\})) * (\{a\} + \{b\}) \quad (\mathcal{D}_{M00})(\text{Definition of } =)$$

$$(\{x\} * (\{c\} + \{d\})) * (\{a\} + \{b\}) = (\{x\} * (\{c\} + \{d\})) * \{a\} + (\{x\} * (\{c\} + \{d\})) * \{b\} \quad (\mathcal{T}_{R002b})(\text{Distribution of } * \text{ over } +)$$

$$(\{x\} * (\{c\} + \{d\})) * (\{a\} + \{b\}) = (\{x\} * \{a\}) * (\{c\} + \{d\}) + (\{x\} * \{b\}) * (\{c\} + \{d\}) \quad (\mathcal{T}_{T263})(\text{Commutativity of Algebraic } x)$$

$$(\{x\} * (\{c\} + \{d\})) * (\{a\} + \{b\}) = (\{x\} * \{a\}) * \{c\} + (\{x\} * \{a\}) * \{d\} + (\{x\} * \{b\}) * \{c\} + (\{x\} * \{b\}) * \{d\} \quad (\mathcal{T}_{R002b})(\text{Distribution of } * \text{ over } +)$$

$$(\{x\} * (\{c\} + \{d\})) * (\{a\} + \{b\}) = (\{x\} * \{c\}) * \{a\} + (\{x\} * \{d\}) * \{a\} + (\{x\} * \{c\}) * \{b\} + (\{x\} * \{d\}) * \{b\} \quad (\mathcal{T}_{T263})(\text{Commutativity of Algebraic } x)$$

$$(\{x\} * (\{a\} + \{b\})) * (\{a\} + \{b\}) = (\{x\} * \{a\}) * \{a\} + (\{x\} * \{a\}) * \{b\} + (\{x\} * \{b\}) * \{a\} \quad (\mathcal{T}_{S853})$$

Proof of ($\mathcal{T}_{S853}$)

$$(\{x\} * (\{c\} + \{d\})) * (\{a\} + \{b\}) = (\{x\} * \{c\}) * \{a\} + (\{x\} * \{c\}) * \{b\} + (\{x\} * \{d\}) * \{a\} + (\{x\} * \{d\}) * \{b\} \quad (\mathcal{T}_{S854})(\text{Eqn1})$$

$$\text{Let } \{c\} = \{a\} \quad (\text{Eqn2})$$

$$\text{Let } \{d\} = \{b\} \quad (\text{Eqn3})$$

$$(\{x\} * (\{a\} + \{b\})) * (\{a\} + \{b\}) = (\{x\} * \{a\}) * \{a\} + (\{x\} * \{a\}) * \{b\} + (\{x\} * \{b\}) * \{a\} + (\{x\} * \{b\}) * \{b\} \quad (\mathbf{A}_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})$$

$$(\{x\} * (\{a\} + \{b\})) * (\{a\} + \{b\}) = (\{x\} * \{a\}) * \{a\} + (\{x\} * \{a\}) * \{b\} + (\{x\} * \{a\}) * \{b\} + (\{x\} * \{b\}) * \{b\} \quad (\mathcal{T}_{T263})(\text{Commutativity of Algebraic } x)$$

$$(\{x\} * (\{a\} + \{b\})) * (\{a\} + \{b\}) = (\{x\} * \{a\}) * \{a\} + (\{x\} * \{a\}) * \{b\} + (\{x\} * \{b\}) * \{b\} \quad (\mathbf{A}_{MX01})(\text{Disjunction Tautology})$$

$$(\{x\} * (\{a\} + \{1\}^1)) * (\{a\} + \{1\}^1) = (\{x\} * \{a\}) * \{a\} + \{a\} + \{1\}^1 \quad (\mathcal{T}_{S852})$$

Proof of ($\mathcal{T}_{S852}$)

$$(\{x\} * (\{a\} + \{b\})) * (\{a\} + \{b\}) = (\{x\} * \{a\}) * \{a\} + (\{x\} * \{a\}) * \{b\} + (\{x\} * \{b\}) * \{b\} \quad (\mathcal{T}_{S853})(\text{Eqn1})$$

$$\text{Let } \{b\} = \{1\}^1 \quad (\text{Eqn2})$$

$$(\{x\} * (\{a\} + \{1\}^1)) * (\{a\} + \{1\}^1) = (\{x\} * \{a\}) * \{a\} + (\{x\} * \{a\}) * \{1\}^1 + (\{x\} * \{1\}^1) * \{1\}^1 \quad (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3})$$

$$(\{x\} * \{1\}^1) * \{a\} = \{a\} \quad (\mathcal{T}_{S845})$$

$$(\{x\} * \{a\}) * \{1\}^1 = \{a\} \quad (\mathcal{T}_{T263})(\text{Commutativity of Algebraic } x)(\text{Eqn4})$$

$$(\{x\} * (\{a\} + \{1\}^1)) * (\{a\} + \{1\}^1) = (\{x\} * \{a\}) * \{a\} + \{a\} + \{1\}^1 \quad (\mathbf{A}_{MX19})(\text{Substitution of Eqn4 into Eqn3})$$

$$(\{x\} * (\{a\} + \{0\}^1)) * (\{a\} + \{0\}^1) = (\{x\} * \{a\}) * \{a\} + \{0\}^1 \quad (\mathcal{T}_{S851})$$

Proof of ($\mathcal{T}_{S851}$)

$$(\{x\} * (\{a\} + \{b\})) * (\{a\} + \{b\}) = (\{x\} * \{a\}) * \{a\} + (\{x\} * \{a\}) * \{b\} + (\{x\} * \{b\}) * \{b\} \quad (\mathcal{T}_{S853})(\text{Eqn1})$$

$$\text{Let } \{b\} = \{0\}^1 \quad (\text{Eqn2})$$

$$(\{x\} * (\{a\} + \{0\}^1)) * (\{a\} + \{0\}^1) = (\{x\} * \{a\}) * \{a\} + (\{x\} * \{a\}) * \{0\}^1 + (\{x\} * \{0\}^1) * \{0\}^1 \quad (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})(\text{Eqn3})$$

$$(\{x\} * \{a\}) * \{0\}^1 = \{0\}^1 \quad (\mathcal{T}_{S837})(\text{Eqn4})$$

$$(\{x\} * (\{a\} + \{0\}^1)) * (\{a\} + \{0\}^1) = (\{x\} * \{a\}) * \{a\} + \{0\}^1 + \{0\}^1 \quad (\mathbf{A}_{MX19})(\text{Substitution of Eqn4 into Eqn3})$$

$$(\{x\} * (\{a\} + \{0\}^1)) * (\{a\} + \{0\}^1) = (\{x\} * \{a\}) * \{a\} + \{0\}^1 \quad (\mathbf{A}_{MX01})(\text{Disjunction Tautology})$$

$$(\{x\} * (\{a\} + \{1\}^1 + \{0\}^1)) * (\{a\} + \{1\}^1 + \{0\}^1) = (\{x\} * \{a\}) * \{a\} + \{a\} + \{1\}^1 + \{0\}^1 \quad (T_{S850})$$

Proof of (T_{S850})

$$(\{x\} * (\{a\} + \{b\})) * (\{a\} + \{b\}) = (\{x\} * \{a\}) * \{a\} + (\{x\} * \{a\}) * \{b\} + (\{x\} * \{b\}) * \{b\} \quad (T_{S853})(\text{Eqn1})$$

$$\text{Let } \{b\} = \{1\}^1 + \{0\}^1 \quad (\text{Eqn2})$$

$$\begin{aligned} & (\{x\} * (\{a\} + \{1\}^1 + \{0\}^1)) * (\{a\} + \{1\}^1 + \{0\}^1) = \\ & (\{x\} * \{a\}) * \{a\} + (\{x\} * \{a\}) * (\{1\}^1 + \{0\}^1) + (\{x\} * (\{1\}^1 + \{0\}^1)) * (\{1\}^1 + \{0\}^1) \end{aligned}$$

($AMX19$)(Substitution of Eqn2 into Eqn1)

$$\begin{aligned} & (\{x\} * (\{a\} + \{1\}^1 + \{0\}^1)) * (\{a\} + \{1\}^1 + \{0\}^1) = \\ & (\{x\} * \{a\}) * \{a\} + (\{x\} * \{a\}) * \{1\}^1 + (\{x\} * \{a\}) * \{0\}^1 + (\{x\} * (\{1\}^1 + \{0\}^1)) * (\{1\}^1 + \{0\}^1) \end{aligned}$$

(T_{R002b})(Distribution of $*$ over $+$)

$$\begin{aligned} & (\{x\} * (\{a\} + \{1\}^1 + \{0\}^1)) * (\{a\} + \{1\}^1 + \{0\}^1) = \\ & (\{x\} * \{a\}) * \{a\} + (\{x\} * \{a\}) * \{1\}^1 + (\{x\} * \{a\}) * \{0\}^1 + (\{x\} * \{1\}^1) * \{1\}^1 + (\{x\} * \{1\}^1) * \{0\}^1 + (\{x\} * \{0\}^1) * \{0\}^1 \end{aligned}$$

(T_{S853})

$$\begin{aligned} & (\{x\} * (\{a\} + \{1\}^1 + \{0\}^1)) * (\{a\} + \{1\}^1 + \{0\}^1) = \\ & (\{x\} * \{a\}) * \{a\} + \{a\} + (\{x\} * \{a\}) * \{0\}^1 + \{1\}^1 + (\{x\} * \{1\}^1) * \{0\}^1 + (\{x\} * \{0\}^1) * \{0\}^1 \end{aligned}$$

(T_{S839})

$$(\{x\} * (\{a\} + \{1\}^1 + \{0\}^1)) * (\{a\} + \{1\}^1 + \{0\}^1) = (\{x\} * \{a\}) * \{a\} + \{a\} + \{0\}^1 + \{1\}^1 + \{0\}^1 + \{0\}^1$$

(T_{S837})

$$(\{x\} * (\{a\} + \{1\}^1 + \{0\}^1)) * (\{a\} + \{1\}^1 + \{0\}^1) = (\{x\} * \{a\}) * \{a\} + \{a\} + \{1\}^1 + \{0\}^1$$

($AMX01$)(Disjunction Tautology)

$$(\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1 = (\{<=\} * \{1\}^1).(\{>=\} * \{0\}^1) \quad (T_{S849})$$

Proof of (T_{S849})

$$(\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1 = (\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1$$

($DM00$)(Definition of $=$)(Eqn1)

$$\{1\}^1 \subseteq \{>\} * \{0\}^1 \quad (T_{S82})$$

$$\{1\}^1 = \{1\}^1.(\{>\} * \{0\}^1) \quad (AMX44)(\subseteq \text{Equality Form})(\text{Eqn2})$$

$$(\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1 = (\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1.(\{>\} * \{0\}^1) + \{0\}^1$$

($AMX19$)(Substitution of Eqn2 into Eqn1)

$$(\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1 = ((\{<\} * \{1\}^1) + \{1\}^1).(\{>\} * \{0\}^1) + \{0\}^1$$

($TM407$)(Distribution of $.$ over $+$)

$$(\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1 = ((\{<\} * \{1\}^1) + \{=\} * \{1\}^1).(\{>\} * \{0\}^1) + \{0\}^1$$

($DA144$)(Definition of the Reflexive Relationship Set)

$$(\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1 = ((\{<\} + \{=\}) * \{1\}^1).(\{>\} * \{0\}^1) + \{0\}^1$$

($AS65$)(Distribution of $+$ over $*$)

$$(\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1 = (\{<=\} * \{1\}^1).(\{>\} * \{0\}^1) + \{0\}^1$$

($DA141$)(Disjunction with $\{=\}$)(Eqn3)

$$\{1\}^1 \subseteq \{>\} * \{0\}^1 \quad (T_{S82})$$

$$\{0\}^1 \subseteq \{>\} * \{1\}^1 \quad (T_{R519})(\text{Active/Passive Theorem})$$

$$\{0\}^1 \subseteq \{<\} * \{1\}^1 \quad (DA143)(\text{Symmetrical Opposite of an Asymmetrical})$$

$$\{0\}^1 \subseteq \{<\} * \{1\}^1 + \{1\}^1 \quad (TM150)$$

$$\{0\}^1 \subseteq \{<\} * \{1\}^1 + \{=\} * \{1\}^1 \quad (DA144)(\text{Definition of the Reflexive Relationship Set})$$

$$\{0\}^1 \subseteq (\{<\} + \{=\}) * \{1\}^1 \quad (AS65)(\text{Distribution of } + \text{ over } *)$$

$$\{0\}^1 \subseteq \{<=\} * \{1\}^1 \quad (DA141)(\text{Disjunction with } \{=\})$$

$$\{0\}^1 = (\{<=\} * \{1\}^1). \{0\}^1 \quad (AMX44)(\subseteq \text{Equality Form})(\text{Eqn4})$$

$$(\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1 = (\{<=\} * \{1\}^1).(\{>\} * \{0\}^1) + (\{<=\} * \{1\}^1). \{0\}^1$$

($AMX19$)(Substitution of Eqn4 into Eqn3)

$$(\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1 = (\{<=\} * \{1\}^1).(\{>\} * \{0\}^1 + \{0\}^1)$$

($TM407$)(Distribution of $.$ over $+$)

$$(\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1 = (\{<=\} * \{1\}^1).(\{>\} * \{0\}^1 + \{=\} * \{0\}^1)$$

($DA144$)(Definition of the Reflexive Relationship Set)

$$(\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1 = (\{<=\} * \{1\}^1).(\{>\} + \{=\}) * \{0\}^1$$

($AS65$)(Distribution of $+$ over $*$)

$$(\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1 = (\{<=\} * \{1\}^1).(\{>=\} * \{0\}^1)$$

($DA141$)(Disjunction with $\{=\}$)

$$(\{x\} * ((\{<\} * \{1\}^1).(\{>\} * \{0\}^1))) * ((\{<\} * \{1\}^1).(\{>\} * \{0\}^1)) = (\{<\} * \{1\}^1).(\{>\} * \{0\}^1) \quad (T_{S959})$$

Proof of (T_{S959})

$$(\{x\} * ((\{<\} * \{1\}^1).(\{>\} * \{0\}^1))) * ((\{<\} * \{1\}^1).(\{>\} * \{0\}^1)) = (\{<\} * \{1\}^1).(\{>\} * \{0\}^1) \quad (T_{7270})(Eqn1)$$

$$\text{Let } (\{<\} * \{1\}^1).(\{>\} * \{0\}^1) = \{a\} \quad (Eqn2)$$

$$(\{x\} * \{a\}) * \{a\} = \{a\} \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})(Eqn3)$$

$$(\{x\} * (\{a\} + \{1\}^1 + \{0\}^1)) * (\{a\} + \{1\}^1 + \{0\}^1) = (\{x\} * \{a\}) * \{a\} + \{a\} + \{1\}^1 + \{0\}^1$$

$$(T_{S850})(Eqn4)$$

$$(\{x\} * (\{a\} + \{1\}^1 + \{0\}^1)) * (\{a\} + \{1\}^1 + \{0\}^1) = \{a\} + \{a\} + \{1\}^1 + \{0\}^1$$

$$(A_{MX19})(\text{Substitution of Eqn3 into Eqn4})$$

$$(\{x\} * (\{a\} + \{1\}^1 + \{0\}^1)) * (\{a\} + \{1\}^1 + \{0\}^1) = \{a\} + \{1\}^1 + \{0\}^1 \quad (A_{MX01})(\text{Disjunction Tautology})(Eqn5)$$

$$(\{x\} * ((\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1)) * ((\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1) =$$

$$(\{<\} * \{1\}^1).(\{>\} * \{0\}^1) + \{1\}^1 + \{0\}^1 \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn5})$$

$$(\{x\} * ((\{<\} * \{1\}^1).(\{>\} * \{0\}^1))) * ((\{<\} * \{1\}^1).(\{>\} * \{0\}^1)) = (\{<\} * \{1\}^1).(\{>\} * \{0\}^1)$$

$$(T_{S849})$$

$$(\{x\} * \{a\}) * \{b\} = (\{x\} * \{a\}) * \{b\} \quad (T_{S856})$$

$$(\{<\} * \{a\}) * \{b\} = (\{<\} * \{a\}) * \{b\} \quad (T_{S857})$$

Example Proof of (T_{S857})

$$(X = \{x\}, \{<\})(Y = V, \{=\})(Z = V, \{=\}) \subseteq ((X * Y * \{a\}) * Z * \{b\} = Y * Z * (X * \{a\}) * \{b\})$$

$$(A_{R47})(\text{Positive/Negative Axiom})$$

$$(\{<\} * V * \{a\}) * V * \{b\} = V * V * (\{<\} * \{a\}) * \{b\} \quad \text{Extraction}$$

$$(\{<\} * \{a\}) * \{b\} = V * V * (\{<\} * \{a\}) * \{b\} \quad (D_{R27})(\text{Simplification of the Relationship Symmetry Function})$$

$$(\{<\} * \{a\}) * \{b\} = (\{<\} * \{a\}) * \{b\} \quad (T_{838})$$

$$(\{a\}^1 \subseteq \{<\} * \{0\}^1) \subseteq ((\{<\} * \{a\}^1) * \{a\}^1) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1) \quad (T_{S954})$$

Proof of (T_{S954})

$$(\{a\}^1 \subseteq \{>\} * \{0\}^1) \subseteq ((\{<\} * \{a\}^1) * \{a\}^1) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1) \quad (T_{S955})(Eqn1)$$

$$\text{Let } \{a\}^1 = \{a\}^1 \quad (Eqn2)$$

$$(\{a\}^1 \subseteq \{>\} * \{0\}^1) \subseteq ((\{<\} * \{a\}^1) * \{a\}^1) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)$$

$$(A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(\{a\}^1 \subseteq \{>\} * \{0\}^1) \subseteq ((\{<\} * \{a\}^1) * \{a\}^1) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)$$

$$(T_{R022d})(\text{Relationship Opposite of a Proposition})$$

$$(\{a\}^1 \subseteq \{<\} * \{0\}^1) \subseteq ((\{<\} * \{a\}^1) * \{a\}^1) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)$$

$$(T_{7215})$$

$$(\{a\}^1 \subseteq \{<\} * \{0\}^1) \subseteq ((\{<\} * \{a\}^1) * \{a\}^1) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)$$

$$(T_{7217})$$

$$(\{a\}^1 \subseteq \{<\} * \{0\}^1) \subseteq ((\{<\} * \{a\}^1) * \{a\}^1) \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)$$

$$(T_{S857})$$

$$(\{<\} * ((\{<\} * \{1\}^1).(\{>\} * \{0\}^1))) * \{1\}^1 = (\{<\} * \{1\}^1).(\{>\} * \{0\}^1) \quad (T_{7269})$$

Proof of (T₇₂₆₉)

$$(\{a\} = (\{<\} * \{1\}^1).(\{>\} * \{0\}^1)) \subseteq ((\{<\} * \{a\}) * \{1\}^1 = \{a\}) \quad (A_{R58})(2^{\text{nd}} \text{ Axiom of Fractions})$$

$$(\{a\} = (\{<\} * \{1\}^1).(\{>\} * \{0\}^1)) \subseteq ((\{<\} * \{a\}) * \{1\}^1 = \{a\}) \quad (T_{7193})$$

$$(\{<\} * ((\{<\} * \{1\}^1).(\{>\} * \{0\}^1))) * \{1\}^1 = (\{<\} * \{1\}^1).(\{>\} * \{0\}^1) \quad (A_{MX19})(\text{Substitution})$$

$$(\{<\} * (\{a\} + \{1\}^1 + \{0\}^1)) * \{1\}^1 = (\{<\} * \{a\}) * \{1\}^1 + \{0\}^1 + \{1\}^1 \quad (T_{S848})$$

Proof of (T_{S848})

$$(\{<\} * (\{a\} + \{1\}^1 + \{0\}^1)) * \{1\}^1 = (\{<\} * (\{a\} + \{1\}^1 + \{0\}^1)) * \{1\}^1 \quad (D_{M00})(\text{Definition of } =)$$

$$(\{<\} * (\{a\} + \{1\}^1 + \{0\}^1)) * \{1\}^1 = (\{<\} * \{a\} + \{<\} * \{1\}^1 + \{<\} * \{0\}^1) * \{1\}^1 \quad (T_{R002b})(\text{Distribution of } * \text{ over } +)$$

$$(\{<\} * (\{a\} + \{1\}^1 + \{0\}^1)) * \{1\}^1 = (\{<\} * \{a\}) * \{1\}^1 + (\{<\} * \{1\}^1) * \{1\}^1 + (\{<\} * \{0\}^1) * \{1\}^1$$

$$\begin{aligned}
(\{-\} * (\{a\} + \{1\}^1 + \{0\}^1)) * \{1\}^1 &= (\{-\} * \{a\}) * \{1\}^1 + \{0\}^1 + (\{-\} * \{0\}^1) * \{1\}^1 & (T_{R002c}) (\text{Distribution of } + \text{ over } *) \\
(\{-\} * (\{a\} + \{1\}^1 + \{0\}^1)) * \{1\}^1 &= (\{-\} * \{a\}) * \{1\}^1 + \{0\}^1 + \{1\}^1 & (T_{S838}) \\
& & (T_{T467})
\end{aligned}$$

$$(\{-\} * ((\{\leq\} * \{1\}^1).(\{\geq\} * \{0\}^1))) * \{1\}^1 = (\{\leq\} * \{1\}^1).(\{\geq\} * \{0\}^1) \quad (T_{T264})$$

Proof of (T_{T264})

$$\begin{aligned}
(\{-\} * ((\{\leq\} * \{1\}^1).(\{\geq\} * \{0\}^1))) * \{1\}^1 &= (\{-\} * \{1\}^1).(\{\geq\} * \{0\}^1) & (T_{T269}) (\text{Eqn1}) \\
\text{Let } (\{\leq\} * \{1\}^1).(\{\geq\} * \{0\}^1) &= \{a\} & (\text{Eqn2}) \\
(\{-\} * \{a\}) * \{1\}^1 &= \{a\} & (A_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) (\text{Eqn3}) \\
(\{-\} * (\{a\} + \{1\}^1 + \{0\}^1)) * \{1\}^1 &= (\{-\} * \{a\}) * \{1\}^1 + \{0\}^1 + \{1\}^1 & (T_{S848}) (\text{Eqn4}) \\
(\{-\} * (\{a\} + \{1\}^1 + \{0\}^1)) * \{1\}^1 &= \{a\} + \{0\}^1 + \{1\}^1 & (A_{MX19}) (\text{Substitution of Eqn3 into Eqn4}) (\text{Eqn5}) \\
(\{-\} * ((\{\leq\} * \{1\}^1).(\{\geq\} * \{0\}^1)) + \{1\}^1 + \{0\}^1) * \{1\}^1 &= (\{\leq\} * \{1\}^1).(\{\geq\} * \{0\}^1) + \{0\}^1 + \{1\}^1 & (A_{MX19}) (\text{Substitution of Eqn2 into Eqn5}) \\
(\{-\} * ((\{\leq\} * \{1\}^1).(\{\geq\} * \{0\}^1))) * \{1\}^1 &= (\{\leq\} * \{1\}^1).(\{\geq\} * \{0\}^1) & (T_{S849})
\end{aligned}$$

$$\{a\} + \{b\}^1 = (\{+\} * \{b\}^1) * \{a\} \quad (T_{S825})$$

$$\{a\} \times \{b\}^1 = (\{x\} * \{b\}^1) * \{a\} \quad (T_{T706})$$

Example Proof of (T_{S825})

$$\begin{aligned}
\text{Let } \{a\} &= \{a_1\}^1 +++ \{a_n\}^1 & (\text{Eqn1}) \\
\{a\} + \{b\}^1 &= \{a\} + \{b\}^1 & (D_{M00}) (\text{Definition of } =) (\text{Eqn2}) \\
\{a\} + \{b\}^1 &= (\{a_1\}^1 +++ \{a_n\}^1) + \{b\}^1 & (A_{MX19}) (\text{Substitution of Eqn1 into Eqn2}) \\
\{a\} + \{b\}^1 &= (\{a_1\}^1 + \{b\}^1) +++ (\{a_n\}^1 + \{b\}^1) & (A_{S88}) (\text{Distribution of Arithmetical Operators over } +) \\
\{a\} + \{b\}^1 &= ((\{+\} * \{b\}^1) * \{a_1\}^1) +++ ((\{+\} * \{b\}^1) * \{a_n\}^1) & (T_{T471}) \\
\{a\} + \{b\}^1 &= (\{+\} * \{b\}^1) * (\{a_1\}^1 +++ \{a_n\}^1) & (A_{S64}) (\text{Distribution of } * \text{ over } +) (\text{Eqn3}) \\
\{a\} + \{b\}^1 &= (\{+\} * \{b\}^1) * \{a\} & (A_{MX19}) (\text{Substitution of Eqn1 into Eqn3})
\end{aligned}$$

$$\{\>\} * \{a\}^1 + \{b\}^1 \subseteq \{\>\} * (\{a\}^1 + \{b\}^1) \quad (T_{T317})$$

$$\{\<\} * \{a\}^1 + \{b\}^1 \subseteq \{\<\} * (\{a\}^1 + \{b\}^1) \quad (T_{T371})$$

$$\{\>\} * \{a\}^1 - \{b\}^1 \subseteq \{\>\} * (\{a\}^1 - \{b\}^1) \quad (T_{T376})$$

$$\{\<\} * \{a\}^1 - \{b\}^1 \subseteq \{\<\} * (\{a\}^1 - \{b\}^1) \quad (T_{T378})$$

Example Proof of (T_{T317})

$$\begin{aligned}
(X = \{\>\}, \{\<\}, \{=\}, \{\geq\}, \{\leq\}) (o = \{+\}, \{-\}) &\subseteq (X * \{a\}^1 \langle o \rangle \{b\}^1 \subseteq X * (\{a\}^1 \langle o \rangle \{b\}^1)) \\
& & (A_{S90}) (\text{Range Displacement Axiom}) \\
\{\>\} * \{a\}^1 + \{b\}^1 &\subseteq \{\>\} * (\{a\}^1 + \{b\}^1) & \text{Extraction}
\end{aligned}$$

$$(\{a\}^1 > \{b\}^1) (\{c\}^1 = \{d\}^1) \subseteq (\{a\}^1 + \{c\}^1 > \{b\}^1 + \{d\}^1) \quad (T_{S811})$$

Proof of (T_{S811})

$$\begin{aligned}
\text{Let } \{a\}^1 &> \{b\}^1 & (\text{Eqn1}) \\
\{a\}^1 &\subseteq \{\>\} * \{b\}^1 & (T_{T962}) \text{ since } \{a\}^1 \neq \emptyset \text{ and } \{b\}^1 \neq \emptyset \\
(\{+\} * \{c\}^1) * \{a\}^1 &\subseteq (\{+\} * \{c\}^1) * \{\>\} * \{b\}^1 & (T_{R196}) (\text{Relational Deduction}) (\text{Eqn2}) \\
\text{Let } \{c\}^1 &= \{d\}^1 & (\text{Eqn3}) \\
(\{+\} * \{c\}^1) * \{a\}^1 &\subseteq (\{+\} * \{d\}^1) * \{\>\} * \{b\}^1 & (A_{MX19}) (\text{Substitution of Eqn3 into Eqn2}) \\
\{a\}^1 + \{c\}^1 &\subseteq \{\>\} * \{b\}^1 + \{d\}^1 & (T_{S825}) \\
\{a\}^1 + \{c\}^1 &\subseteq \{\>\} * (\{b\}^1 + \{d\}^1) & (T_{T317}) \\
\{a\}^1 + \{c\}^1 &> \{b\}^1 + \{d\}^1 & (T_{T471}) \text{ since } (\{b\}^1 + \{d\}^1) = (\{b\}^1 + \{d\}^1)^1 \\
& & \text{and } (\{a\}^1 + \{c\}^1) = (\{a\}^1 + \{c\}^1)^1 \\
\text{Thus } (\{a\}^1 > \{b\}^1) (\{c\}^1 = \{d\}^1) &\subseteq (\{a\}^1 + \{c\}^1 > \{b\}^1 + \{d\}^1) & (A_{MX02}) (\text{Conditional Introduction})
\end{aligned}$$

Chapter 18 Proofs of Theorems of Probability

18.1 Proofs of Theorems of Null Probability

$$(a^1 \subseteq S_i) \subseteq ((P * c^1) * a^1 = \emptyset) \quad (T_{7291})$$

Proof of (T₇₂₉₁)

Let $a^1 \subseteq S_i$

$$\mathcal{N}a^1 = \emptyset \quad (A_{718})(\text{Quantity of an Infinite Set})(\text{Eqn1})$$

$$\mathcal{N}(ac)^1 = \mathcal{N}a^1 - \mathcal{N}(ac)^1 \quad (T_{797})$$

$$\mathcal{N}(ac)^1 = (\{-\} * \mathcal{N}(ac)^1) * \mathcal{N}a^1 \quad (T_{472})(\text{Eqn2})$$

$$\mathcal{N}(ac)^1 = (\{-\} * \mathcal{N}(ac)^1) * \emptyset \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$$

$$\mathcal{N}(ac)^1 = \emptyset \quad (T_{5181})(\text{Null Object Theorem})(\text{Eqn3})$$

$$(P * c^1) * a^1 = (\{-\} * \mathcal{N}c^1) * \mathcal{N}(ac)^1 \quad (D_{79})(\text{Definition of Numerical Probability})(\text{Eqn4})$$

$$(P * c^1) * a^1 = (\{-\} * \mathcal{N}c^1) * \emptyset \quad (A_{MX19})(\text{Substitution of Eqn3 into Eqn4})$$

$$(P * c^1) * a^1 = \emptyset \quad (T_{5181})(\text{Null Object Theorem})$$

$$\text{Thus } (a^1 \subseteq S_i) \subseteq ((P * c^1) * a^1 = \emptyset) \quad (A_{MX02})(\text{Conditional Introduction})$$

$$(c^1 \subseteq S_i) \subseteq ((P * c^1) * a^1 = \emptyset) \quad (T_{7292})$$

Proof of (T₇₂₉₂)

Let $c^1 \subseteq S_i$

$$\mathcal{N}c^1 = \emptyset \quad (A_{718})(\text{Quantity of an Infinite Set})(\text{Eqn1})$$

$$(P * c^1) * a^1 = (\{-\} * \mathcal{N}c^1) * \mathcal{N}(ac)^1 \quad (D_{79})(\text{Definition of Numerical Probability})(\text{Eqn2})$$

$$(P * c^1) * a^1 = (\{-\} * \emptyset) * \mathcal{N}(ac)^1 \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$$

$$(P * c^1) * a^1 = \emptyset * \mathcal{N}(ac)^1 \quad (T_{5181})(\text{Null Object Theorem})$$

$$(P * c^1) * a^1 = \emptyset \quad (T_{R072})(\text{Null Function Theorem})$$

$$\text{Thus } (c^1 \subseteq S_i) \subseteq ((P * c^1) * a^1 = \emptyset) \quad (A_{MX02})(\text{Conditional Introduction})$$

$$(P * c^1) * \emptyset = \emptyset \quad (T_{5821})$$

$$(P * \emptyset) * a^1 = \emptyset \quad (T_{5759})$$

Example Proof of (T₅₈₂₁)

$$(P * \emptyset) * a^1 = (P * \emptyset) * a^1$$

$$(P * \emptyset) * a^1 = \emptyset * a^1 \quad (T_{5181})(\text{Null Object Theorem})$$

$$(P * \emptyset) * a^1 = \emptyset \quad (T_{R072})(\text{Null Function Theorem})$$

$$(c = \emptyset^1) \subseteq ((P * c^1) * a^1 = \emptyset) \quad (T_{5783})$$

$$(c = E^1) \subseteq ((P * c^1) * a^1 = \emptyset) \quad (T_{5779})$$

Example Proof of (T₅₇₈₃)

$$(P * c^1) * a^1 = (\{-\} * \mathcal{N}c^1) * \mathcal{N}(ac)^1 \quad (D_{79})(\text{Definition of Numerical Probability})(\text{Eqn1})$$

Let $c = \emptyset^1$

$$\mathcal{N}c^1 = \emptyset \quad (T_{5809})(\text{Eqn2})$$

$$(P * c^1) * a^1 = (\{-\} * \emptyset) * \mathcal{N}(ac)^1 \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$(P * c^1) * a^1 = \emptyset * \mathcal{N}(ac)^1 \quad (T_{5181})(\text{Null Object Theorem})$$

$$(P * c^1) * a^1 = \emptyset \quad (T_{R072})(\text{Null Function Theorem})$$

$$\text{Thus } (c = \emptyset^1) \subseteq ((P * c^1) * a^1 = \emptyset) \quad (A_{MX02})(\text{Conditional Introduction})$$

$$(c = \emptyset) \subseteq ((P * c^1) * a^1 = \emptyset) \quad (T_{S785})$$

Proof of (T_{S785})

$$\begin{aligned} (P * c^1) * a^1 &= (\{\div\} * \mathcal{N}c^1) * \mathcal{N}(ac)^1 & (D_{R79})(\text{Definition of Numerical Probability})(\text{Eqn1}) \\ \text{Let } c &= \emptyset \\ \{\div\} * \mathcal{N}c^1 &= \emptyset & (T_{S799})(\text{Eqn2}) \\ (P * c^1) * a^1 &= \emptyset * \mathcal{N}(ac)^1 & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (P * c^1) * a^1 &= \emptyset & (T_{R072})(\text{Null Function Theorem}) \\ \text{Thus } (c = \emptyset) &\subseteq ((P * c^1) * a^1 = \emptyset) & (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(c = E) \subseteq ((P * c^1) * a^1 = \emptyset) \quad (T_{S774})$$

Proof of (T_{S774})

$$\begin{aligned} (P * c^1) * a^1 &= (\{\div\} * \mathcal{N}c^1) * \mathcal{N}(ac)^1 & (D_{R79})(\text{Definition of Numerical Probability})(\text{Eqn1}) \\ \text{Let } c &= E \\ \mathcal{N}c^1 &= \emptyset & (T_{S786})(\text{Eqn2}) \\ (P * c^1) * a^1 &= (\{\div\} * \emptyset) * \mathcal{N}(ac)^1 & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (P * c^1) * a^1 &= \emptyset * \mathcal{N}(ac)^1 & (T_{S181})(\text{Null Object Theorem}) \\ (P * c^1) * a^1 &= \emptyset & (T_{R072})(\text{Null Function Theorem}) \\ \text{Thus } (c = E) &\subseteq ((P * c^1) * a^1 = \emptyset) & (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(a = \emptyset^1)(\emptyset^1 \subseteq c) \subseteq ((P * c^1) * a^1 = \emptyset) \quad (T_{S787}) \text{ Probability of } \emptyset^1$$

$$(a = E^1)(E^1 \subseteq c) \subseteq ((P * c^1) * a^1 = \emptyset) \quad (T_{S773}) \text{ Probability of } E^1$$

Example Proof of (T_{S781})

$$\begin{aligned} (P * c^1) * a^1 &= (\{\div\} * \mathcal{N}c^1) * \mathcal{N}(ac)^1 & (D_{R79})(\text{Definition of Numerical Probability})(\text{Eqn1}) \\ \text{Let } a &= \emptyset^1 & (\text{Eqn2}) \\ \text{Let } \emptyset^1 &\subseteq c & (\text{Eqn3}) \\ a &\subseteq c & (A_{MX19})(\text{Substitution of Eqn3 into Eqn2}) \\ a &= ac & (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn4}) \\ (P * c^1) * a^1 &= (\{\div\} * \mathcal{N}c^1) * \mathcal{N}a^1 & (A_{MX19})(\text{Substitution of Eqn4 into Eqn1})(\text{Eqn5}) \\ (P * c^1) * a^1 &= (\{\div\} * \mathcal{N}c^1) * \emptyset & (T_{S809}) \\ (P * c^1) * a^1 &= \emptyset & (T_{S181})(\text{Null Object Theorem}) \\ \text{Thus } (a = \emptyset^1)(\emptyset^1 \subseteq c) &\subseteq ((P * c^1) * a^1 = \emptyset) & (A_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(c = \emptyset) + (c = \emptyset^1) + (c = E) + (c = E^1) + (c^1 \subseteq S_i) + (a^1 \subseteq S_i) + (a = \emptyset^1)(\emptyset^1 \subseteq c) + (a = E^1)(E^1 \subseteq c) \subseteq ((P * c^1) * a^1 = \emptyset) \quad (T_{S760})$$

$$(c = \emptyset) + (c = \emptyset^1) + (c = E) + (c = E^1) + (c^1 \subseteq S_i) + (a^1 \subseteq S_i) + (a = \emptyset^1)(\emptyset^1 \subseteq c) + (a = E^1)(E^1 \subseteq c) = ((P * c^1) * a^1 = \emptyset) \quad (T_{S770})$$

Proof of (T_{S760}) & (T_{S770})

$$\begin{aligned} (c = \emptyset) &\subseteq ((P * c^1) * a^1 = \emptyset) & (T_{S785})(\text{Eqn1}) \\ (c = \emptyset^1) &\subseteq ((P * c^1) * a^1 = \emptyset) & (T_{S783})(\text{Eqn2}) \\ (c = E) &\subseteq ((P * c^1) * a^1 = \emptyset) & (T_{S774})(\text{Eqn3}) \\ (c = E^1) &\subseteq ((P * c^1) * a^1 = \emptyset) & (T_{S779})(\text{Eqn4}) \\ (c^1 \subseteq S_i) &\subseteq ((P * c^1) * a^1 = \emptyset) & (T_{T292})(\text{Eqn5}) \\ (a^1 \subseteq S_i) &\subseteq ((P * c^1) * a^1 = \emptyset) & (T_{T291})(\text{Eqn6}) \\ (a = \emptyset^1)(\emptyset^1 \subseteq c) &\subseteq ((P * c^1) * a^1 = \emptyset) & (T_{S781})(\text{Probability of } \emptyset^1)(\text{Eqn7}) \\ (a = E^1)(E^1 \subseteq c) &\subseteq ((P * c^1) * a^1 = \emptyset) & (T_{S773})(\text{Probability of } E^1)(\text{Eqn8}) \\ (c = \emptyset) + (c = \emptyset^1) + (c = E) + (c = E^1) + (c^1 \subseteq S_i) + (a^1 \subseteq S_i) + (a = \emptyset^1)(\emptyset^1 \subseteq c) + (a = E^1)(E^1 \subseteq c) &\subseteq ((P * c^1) * a^1 = \emptyset) & (T_{M131b})(\text{Conjunction Sufficiencies Eqns1to8})(\text{Eqn9}) \end{aligned}$$

$$\begin{aligned}
& ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \emptyset) \subseteq \\
& (\mathbf{c} = \emptyset) + (\mathbf{c} = \emptyset^1) + (\mathbf{c} = \mathbf{E}) + (\mathbf{c} = \mathbf{E}^1) + (\mathbf{c}^1 \subseteq \mathbf{S}_i) + (\mathbf{a}^1 \subseteq \mathbf{S}_i) + (\mathbf{a} = \emptyset^1)(\emptyset^1 \subseteq \mathbf{c}) + (\mathbf{a} = \mathbf{E}^1)(\mathbf{E}^1 \subseteq \mathbf{c}) \\
& \quad (\mathcal{A}_{R93})(\text{Conditions for Null Probability})(\text{Eqn10}) \\
& (\mathbf{c} = \emptyset) + (\mathbf{c} = \emptyset^1) + (\mathbf{c} = \mathbf{E}) + (\mathbf{c} = \mathbf{E}^1) + (\mathbf{c}^1 \subseteq \mathbf{S}_i) + (\mathbf{a}^1 \subseteq \mathbf{S}_i) + (\mathbf{a} = \emptyset^1)(\emptyset^1 \subseteq \mathbf{c}) + (\mathbf{a} = \mathbf{E}^1)(\mathbf{E}^1 \subseteq \mathbf{c}) = \\
& ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \emptyset) \quad (\mathcal{T}_{M430})(\text{Biconditional Introduction Eqns9\&10})
\end{aligned}$$

$$(\mathbf{c} \neq \emptyset)(\mathbf{c} \neq \emptyset^1)(\mathbf{c} \neq \mathbf{E})(\mathbf{c} \neq \mathbf{E}^1)(\mathbf{c}^1 \not\subseteq \mathbf{S}_i)(\mathbf{a}^1 \not\subseteq \mathbf{S}_i)((\mathbf{a} \neq \emptyset^1) + (\emptyset^1 \not\subseteq \mathbf{c}))((\mathbf{a} \neq \mathbf{E}^1) + (\mathbf{E}^1 \not\subseteq \mathbf{c})) = ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 \neq \emptyset)$$

($\mathcal{T}_{S769}$) Implicit Conditions of Probability

Proof of ($\mathcal{T}_{S769}$)

$$\begin{aligned}
& (\mathbf{c} = \emptyset) + (\mathbf{c} = \emptyset^1) + (\mathbf{c} = \mathbf{E}) + (\mathbf{c} = \mathbf{E}^1) + (\mathbf{c}^1 \subseteq \mathbf{S}_i) + (\mathbf{a}^1 \subseteq \mathbf{S}_i) + (\mathbf{a} = \emptyset^1)(\emptyset^1 \subseteq \mathbf{c}) + (\mathbf{a} = \mathbf{E}^1)(\mathbf{E}^1 \subseteq \mathbf{c}) = \\
& ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \emptyset) \quad (\mathcal{T}_{S770}) \\
& \sim((\mathbf{c} = \emptyset) + (\mathbf{c} = \emptyset^1) + (\mathbf{c} = \mathbf{E}) + (\mathbf{c} = \mathbf{E}^1) + (\mathbf{c}^1 \subseteq \mathbf{S}_i) + (\mathbf{a}^1 \subseteq \mathbf{S}_i) + (\mathbf{a} = \emptyset^1)(\emptyset^1 \subseteq \mathbf{c}) + (\mathbf{a} = \mathbf{E}^1)(\mathbf{E}^1 \subseteq \mathbf{c})) = \\
& \sim(((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \emptyset)) \quad (\mathcal{T}_{M008})(\text{Inverse of an Equality}) \\
& (\sim(\mathbf{c} = \emptyset). \sim(\mathbf{c} = \emptyset^1). \sim(\mathbf{c} = \mathbf{E}). \sim(\mathbf{c} = \mathbf{E}^1). \sim(\mathbf{c}^1 \subseteq \mathbf{S}_i). \sim(\mathbf{a}^1 \subseteq \mathbf{S}_i). \sim((\mathbf{a} = \emptyset^1)(\emptyset^1 \subseteq \mathbf{c})). \sim((\mathbf{a} = \mathbf{E}^1)(\mathbf{E}^1 \subseteq \mathbf{c}))) = \\
& \sim(((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \emptyset)) \quad (\mathcal{T}_{M026})(\text{DeMorgan's Law}) \\
& (\sim(\mathbf{c} = \emptyset). \sim(\mathbf{c} = \emptyset^1). \sim(\mathbf{c} = \mathbf{E}). \sim(\mathbf{c} = \mathbf{E}^1). \sim(\mathbf{c}^1 \subseteq \mathbf{S}_i). \\
& \sim(\mathbf{a}^1 \subseteq \mathbf{S}_i). \sim(\mathbf{a} = \emptyset^1) + \sim(\emptyset^1 \subseteq \mathbf{c})). \sim(\mathbf{a} = \mathbf{E}^1) + \sim(\mathbf{E}^1 \subseteq \mathbf{c})) = \sim(((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \emptyset)) \\
& \quad (\mathcal{T}_{M025})(\text{DeMorgan's Law}) \\
& (\mathbf{c} \neq \emptyset)(\mathbf{c} \neq \emptyset^1)(\mathbf{c} \neq \mathbf{E})(\mathbf{c} \neq \mathbf{E}^1)(\mathbf{c}^1 \not\subseteq \mathbf{S}_i)(\mathbf{a}^1 \not\subseteq \mathbf{S}_i)((\mathbf{a} \neq \emptyset^1) + (\emptyset^1 \not\subseteq \mathbf{c}))((\mathbf{a} \neq \mathbf{E}^1) + (\mathbf{E}^1 \not\subseteq \mathbf{c})) = ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 \neq \emptyset) \\
& \quad (\mathcal{T}_{M260})(\text{Definition of } \neq) \text{ \& } (\mathcal{T}_{M259})(\text{Definition of } \not\subseteq)
\end{aligned}$$

$$(\mathbf{a} = \emptyset^1)(\emptyset^1 \not\subseteq \mathbf{c}) \subseteq ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \{0\}^1) \quad (\mathcal{T}_{S772})$$

$$(\mathbf{a} = \mathbf{E}^1)(\mathbf{E}^1 \not\subseteq \mathbf{c}) \subseteq ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \{0\}^1) \quad (\mathcal{T}_{S771})$$

Example Proof of ($\mathcal{T}_{S772}$)

$$\begin{aligned}
& (\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = (\{\div\} * \mathcal{N}\mathbf{c}^1) * \mathcal{N}(\mathbf{ac})^1 \quad (\mathcal{D}_{R79})(\text{Definition of Numerical Probability})(\text{Eqn1}) \\
& \text{Let } \mathbf{a} = \emptyset^1 \quad (\text{Eqn2}) \\
& \text{Let } \emptyset^1 \not\subseteq \mathbf{c} \quad (\text{Eqn3}) \\
& \mathbf{a} \not\subseteq \mathbf{c} \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn3 into Eqn2}) \\
& \mathbf{a} \subseteq \mathbf{c} \quad (\mathcal{T}_{M604}) \text{ because } \mathbf{a} = \emptyset^1 \text{ and is a unit set, therefore.} \\
& \mathbf{ac} = \emptyset \quad (\mathcal{T}_{M688}) \text{ because } \mathbf{a} = \emptyset^1 \text{ and is a unit set, therefore.} \\
& \mathcal{N}(\mathbf{ac})^1 = \{0\}^1 \quad (\mathcal{D}_{R23})(\text{Definition of Zero})(\text{Eqn4}) \\
& (\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = (\{\div\} * \mathcal{N}\mathbf{c}^1) * \{0\}^1 \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn4 into Eqn1})(\text{Eqn5}) \\
& (\mathcal{N}\mathbf{c}^1 \neq \{0\}^1) \subseteq (\{0\}^1 = (\{\div\} * \mathcal{N}\mathbf{c}^1) * \{0\}^1) \quad (\mathcal{T}_{T655})(\text{Eqn6}) \\
& (\mathbf{c} = \emptyset) = (\mathcal{N}\mathbf{c}^1 = \{0\}^1) \quad (\mathcal{D}_{R23})(\text{Definition of Zero}) \\
& (\mathbf{c} \neq \emptyset) = (\mathcal{N}\mathbf{c}^1 \neq \{0\}^1) \quad (\mathcal{T}_{M008})(\text{Inverse of an Equality})(\text{Eqn7}) \\
& (\mathbf{c} \neq \emptyset) \subseteq (\{0\}^1 = (\{\div\} * \mathcal{N}\mathbf{c}^1) * \{0\}^1) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn7 into Eqn6})(\text{Eqn8}) \\
& (\mathbf{c} \neq \emptyset) \subseteq ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \{0\}^1) \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn8 into Eqn5}) \\
& \text{Thus } ((\mathbf{a} = \emptyset^1)(\emptyset^1 \not\subseteq \mathbf{c}) = \mathbf{E}) \subseteq ((\mathbf{c} \neq \emptyset) \subseteq ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \{0\}^1)) \\
& \quad (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\
& (\mathbf{a} = \emptyset^1)(\emptyset^1 \not\subseteq \mathbf{c})(\mathbf{c} \neq \emptyset) \subseteq ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \{0\}^1) \\
& \quad (\mathcal{A}_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\
& ((\mathbf{c} \neq \emptyset) = \mathbf{E}) \subseteq ((\mathbf{a} = \emptyset^1)(\emptyset^1 \not\subseteq \mathbf{c})) \subseteq ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \{0\}^1) \\
& \quad (\mathcal{A}_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\
& (\mathbf{a} = \emptyset^1)(\emptyset^1 \not\subseteq \mathbf{c}) \subseteq ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \{0\}^1) \quad (\mathcal{T}_{S769})(\text{Implicit Conditions of Probability})
\end{aligned}$$

18.2 Proofs of Theorems of Non-Null Probability

$$(\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = \mathcal{N}(\mathbf{ac})^1 \div \mathcal{N}\mathbf{c}^1 \quad (\mathcal{T}_{R801}) \text{ Definition of Numerical Probability}$$

Proof of ($\mathcal{T}_{R801}$)

$$\begin{aligned} (\mathcal{P} * \mathcal{c}^1) * \mathbf{a}^1 &= (\{\div\} * \mathcal{N}\mathcal{c}^1) * \mathcal{N}(\mathbf{ac})^1 & (D_{R79}) (\text{Definition of Numerical Probability}) \\ (\mathcal{P} * \mathcal{c}^1) * \mathbf{a}^1 &= \mathcal{N}(\mathbf{ac})^1 \div \mathcal{N}\mathcal{c}^1 & (T_{474}) \end{aligned}$$

$$(\mathbf{a} = \mathbf{E}) \subseteq ((\mathcal{P} * \mathcal{c}^1) * \mathbf{a}^1 = \{1\}^1) \quad (T_{R804}) \text{ Second Kolmogorov Axiom}$$

Proof of (T_{R804})

$$\begin{aligned} (\mathcal{P} * \mathcal{c}^1) * \mathbf{a}^1 &= \mathcal{N}(\mathbf{ac})^1 \div \mathcal{N}\mathcal{c}^1 & (T_{R801}) (\text{Definition of Numerical Probability}) (\text{Eqn1}) \\ \text{Let } \mathbf{a} &= \mathbf{E} & (\text{Eqn2}) \\ (\mathcal{P} * \mathcal{c}^1) * \mathbf{a}^1 &= \mathcal{N}(\mathbf{Ec})^1 \div \mathcal{N}\mathcal{c}^1 & (A_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\ (\mathcal{P} * \mathcal{c}^1) * \mathbf{a}^1 &= \mathcal{N}\mathcal{c}^1 \div \mathcal{N}\mathcal{c}^1 & (A_{MX11}) (\text{Universal Set Conjunction Axiom}) \\ (\mathcal{P} * \mathcal{c}^1) * \mathbf{a}^1 &= \{1\}^1 & (T_{R806}) \\ \text{Thus } (\mathbf{a} = \mathbf{E}) &\subseteq ((\mathcal{P} * \mathcal{c}^1) * \mathbf{a}^1 = \{1\}^1) & (A_{MX02}) (\text{Conditional Introduction}) \end{aligned}$$

$$(\mathcal{P} * \mathcal{c}^1) * \mathcal{c}^1 = \{1\}^1 \quad (T_{R834})$$

Proof of (T_{R834})

$$\begin{aligned} (\mathcal{P} * \mathcal{c}^1) * \mathcal{c}^1 &= \mathcal{N}(\mathbf{cc})^1 \div \mathcal{N}\mathcal{c}^1 & (T_{R801}) (\text{Definition of Numerical Probability}) \\ (\mathcal{P} * \mathcal{c}^1) * \mathcal{c}^1 &= \mathcal{N}\mathcal{c}^1 \div \mathcal{N}\mathcal{c}^1 & (T_{M667}) (\text{Conjunction Tautology}) \\ (\mathcal{P} * \mathcal{c}^1) * \mathcal{c}^1 &= \{1\}^1 & (T_{R806}) \end{aligned}$$

$$(\mathcal{P} * \mathcal{c}^1) * \mathbf{a}^1 + (\mathcal{P} * \mathcal{c}^1) * (\underline{\mathbf{a}})^1 = \{1\}^1 \quad (T_{R816})$$

$$(\{+\} * (\mathcal{P} * \mathcal{c}^1) * (\underline{\mathbf{a}})^1) * (\mathcal{P} * \mathcal{c}^1) * \mathbf{a}^1 = \{1\}^1 \quad (T_{R815})$$

$$\mathcal{N}(\mathbf{ac})^1 \div \mathcal{N}\mathcal{c}^1 + \mathcal{N}(\underline{\mathbf{ac}})^1 \div \mathcal{N}\mathcal{c}^1 = \{1\}^1 \quad (T_{R817})$$

Proof of (T_{R816}), (T_{R815}) & (T_{R817})

$$\begin{aligned} \mathcal{N}\mathcal{c}^1 \div \mathcal{N}\mathcal{c}^1 &= \{1\}^1 & (T_{R806}) \\ \mathcal{N}(\mathbf{c.E})^1 \div \mathcal{N}\mathcal{c}^1 &= \{1\}^1 & (A_{MX11}) (\text{Universal Set Conjunction Axiom}) \\ \mathcal{N}(\mathbf{c.(a + \underline{a})})^1 \div \mathcal{N}\mathcal{c}^1 &= \{1\}^1 & (T_{M671}) (\text{Tertium non Datur}) \\ \mathcal{N}(\mathbf{ca + c\underline{a}})^1 \div \mathcal{N}\mathcal{c}^1 &= \{1\}^1 & (T_{M407}) (\text{Distribution of } \cdot \text{ over } +) \\ (\mathbf{ca.c\underline{a}} = \emptyset) &\subseteq ((\mathcal{N}(\mathbf{ca})^1 + \mathcal{N}(\mathbf{c\underline{a}})^1) \div \mathcal{N}\mathcal{c}^1 = \{1\}^1) & (A_{S37}) (\text{Distribution of } \mathcal{N} \text{ over Disjoint Sets}) \\ (\mathbf{cc}\emptyset = \emptyset) &\subseteq ((\mathcal{N}(\mathbf{ca})^1 + \mathcal{N}(\mathbf{c\underline{a}})^1) \div \mathcal{N}\mathcal{c}^1 = \{1\}^1) & (T_{M621}) (\text{Complement Conjunction}) \\ (\emptyset = \emptyset) &\subseteq ((\mathcal{N}(\mathbf{ca})^1 + \mathcal{N}(\mathbf{c\underline{a}})^1) \div \mathcal{N}\mathcal{c}^1 = \{1\}^1) & (T_{M050}) (\text{Null Set Conjunction}) \\ \mathbf{E} &\subseteq ((\mathcal{N}(\mathbf{ca})^1 + \mathcal{N}(\mathbf{c\underline{a}})^1) \div \mathcal{N}\mathcal{c}^1 = \{1\}^1) & (T_{M343}) \\ \mathbf{E} &= ((\mathcal{N}(\mathbf{ca})^1 + \mathcal{N}(\mathbf{c\underline{a}})^1) \div \mathcal{N}\mathcal{c}^1 = \{1\}^1) & (T_{M275}) (\text{Supersets of the Universal Set}) \\ (\mathcal{N}(\mathbf{ca})^1 + \mathcal{N}(\mathbf{c\underline{a}})^1) \div \mathcal{N}\mathcal{c}^1 &= \{1\}^1 & (A_{MX16}) (\text{Order Axiom}) \\ \mathcal{N}(\mathbf{ca})^1 \div \mathcal{N}\mathcal{c}^1 + \mathcal{N}(\mathbf{c\underline{a}})^1 \div \mathcal{N}\mathcal{c}^1 &= \{1\}^1 & (T_{486}) (\text{Distribution of } \div \text{ over } +) \\ (\mathcal{P} * \mathcal{c}^1) * \mathbf{a}^1 + (\mathcal{P} * \mathcal{c}^1) * (\underline{\mathbf{a}})^1 &= \{1\}^1 & (T_{R801}) (\text{Definition of Numerical Probability}) (\text{Eqn1}) \\ (\{+\} * (\mathcal{P} * \mathcal{c}^1) * (\underline{\mathbf{a}})^1) * (\mathcal{P} * \mathcal{c}^1) * \mathbf{a}^1 &= \{1\}^1 & (T_{471}) \end{aligned}$$

$$(\mathcal{P} * \mathcal{c}^1) * (\underline{\mathbf{a}})^1 = \{1\}^1 - (\mathcal{P} * \mathcal{c}^1) * (\mathbf{a})^1 \quad (T_{R813})$$

$$(\mathcal{P} * \mathcal{c}^1) * (\underline{\mathbf{a}})^1 = (\{+\} * (\mathcal{P} * \mathcal{c}^1) * (\mathbf{a})^1) * \{1\}^1 \quad (T_{R812})$$

$$\mathcal{N}(\underline{\mathbf{ac}})^1 \div \mathcal{N}\mathcal{c}^1 = \{1\}^1 - \mathcal{N}(\mathbf{ac})^1 \div \mathcal{N}\mathcal{c}^1 \quad (T_{R814})$$

Proof of (T_{R813}), (T_{R812}) & (T_{R814})

$$\begin{aligned} (\mathbf{b}^1 \neq \emptyset)(\mathbf{c} * \mathbf{b}^1 \neq \emptyset) &\subseteq (\mathbf{b}^1 \subseteq \mathbf{Qc}) & (T_{S410}) (\text{Eqn1}) \\ \text{Let } \mathbf{b}^1 &= (\mathcal{P} * \mathcal{c}^1) * (\underline{\mathbf{a}})^1 & (\text{Eqn2}) \\ \text{Let } \mathbf{c} &= (\{+\} * (\mathcal{P} * \mathcal{c}^1) * (\mathbf{a})^1) & (\text{Eqn3}) \\ ((\mathcal{P} * \mathcal{c}^1) * (\underline{\mathbf{a}})^1 \neq \emptyset) &(((\{+\} * (\mathcal{P} * \mathcal{c}^1) * (\mathbf{a})^1) * (\mathcal{P} * \mathcal{c}^1) * (\underline{\mathbf{a}})^1 \neq \emptyset) \subseteq ((\mathcal{P} * \mathcal{c}^1) * (\underline{\mathbf{a}})^1 \subseteq \mathbf{Q}(\{+\} * (\mathcal{P} * \mathcal{c}^1) * (\mathbf{a})^1)) & (A_{MX19}) (\text{Substitution of Eqns2\&3 into Eqn1}) \\ (\mathcal{P} * \mathcal{c}^1) * (\underline{\mathbf{a}})^1 &\subseteq \mathbf{Q}(\{+\} * (\mathcal{P} * \mathcal{c}^1) * (\mathbf{a})^1) & (T_{S769}) (\text{Implicit Conditions of Probability}) (\text{Eqn4}) \\ (\mathbf{X} = \{+\}, \{-\}, \{x\}, \{\div\}) &\subseteq ((\mathbf{X} * \{\mathbf{c}\}^1))^1 \subseteq \mathbf{R}_0 & (T_{478}) \end{aligned}$$

$(\forall \{+\} * \{c^1\})^1 \subseteq R_0$	Extraction (Eqn5)
Let $\{c\}^1 = (P * c^1) * (a)^1$	(Eqn6)
$(\forall \{+\} * (P * c^1) * (a)^1)^1 \subseteq R_0$	($AMX19$)(Substitution of Eqn6 into Eqn5)(Eqn7)
$(\{+\} * (P * c^1) * (a)^1) * (P * c^1) * a^1 = \{1\}^1$	($TR815$)
$(\{+\} * (P * c^1) * (a)^1) * (P * c^1) * (a)^1 = \{1\}^1$	($TR853$)(Commutativity of Algebraic +)(Eqn8)
$\{1\}^1 \neq \emptyset$	($TR606$)(Eqn9)
$(\{+\} * (P * c^1) * (a)^1) * (P * c^1) * (a)^1 \neq \emptyset$	($AMX19$)(Substitution of Eqn9 into Eqn8)
$\forall \{+\} * (P * c^1) * (a)^1 * (\{+\} * (P * c^1) * (a)^1) * (P * c^1) * (a)^1 = \forall \{+\} * (P * c^1) * (a)^1 * \{1\}^1$	($TR522$)(Relational Deduction on Eqn1)(Eqn10)
$((\forall c)^1 \subseteq R_0)(b^1 \subseteq Qc) \subseteq (b^1 = \forall c * c * b^1)$	($TR519$)(Eqn11)
Let $c = (\{+\} * (P * c^1) * (a)^1)$	(Eqn12)
Let $b^1 = (P * c^1) * (a)^1$	(Eqn13)
$((\forall \{+\} * (P * c^1) * (a)^1)^1 \subseteq R_0)((P * c^1) * (a)^1 \subseteq Q(\{+\} * (P * c^1) * (a)^1)) \subseteq$	
$((P * c^1) * (a)^1 = \forall (\{+\} * (P * c^1) * (a)^1) * (\{+\} * (P * c^1) * (a)^1) * (P * c^1) * (a)^1)$	($AMX19$)(Substitution of Eqns12&13 into Eqn11)(Eqn14)
$EE \subseteq ((P * c^1) * (a)^1 = \forall (\{+\} * (P * c^1) * (a)^1) * (\{+\} * (P * c^1) * (a)^1) * (P * c^1) * (a)^1)$	($AMX19$)(Substitution of Eqns7&4 into Eqn14)
$E \subseteq ((P * c^1) * (a)^1 = \forall (\{+\} * (P * c^1) * (a)^1) * (\{+\} * (P * c^1) * (a)^1) * (P * c^1) * (a)^1)$	($TR667$)(Conjunction Tautology)
$E = ((P * c^1) * (a)^1 = \forall (\{+\} * (P * c^1) * (a)^1) * (\{+\} * (P * c^1) * (a)^1) * (P * c^1) * (a)^1)$	($TR275$)(Supersets of the Universal Set)
$(P * c^1) * (a)^1 = \forall (\{+\} * (P * c^1) * (a)^1) * (\{+\} * (P * c^1) * (a)^1) * (P * c^1) * (a)^1$	($AMX16$)(Order Axiom)(Eqn15)
$(P * c^1) * (a)^1 = \forall \{+\} * (P * c^1) * (a)^1 * \{1\}^1$	($AMX19$)(Substitution of Eqn15 into Eqn10)
$(P * c^1) * (a)^1 = (\{-\} * (P * c^1) * (a)^1) * \{1\}^1$	($TR248$)
$(P * c^1) * (a)^1 = \{1\}^1 - (P * c^1) * (a)^1$	($TR472$)(Eqn16)
$\mathcal{N}(ac) \div \mathcal{N}c^1 = \{1\}^1 - \mathcal{N}(ac) \div \mathcal{N}c^1$	($TR801$)(Definition of Numerical Probability)

$$(b = E) \subseteq ((P * c^1) * (a)^1 = (\{-\} * (P * c^1) * (a)^1) * ((P * c^1) * b^1)) \quad (TR811)$$

Proof of ($TR811$)

$(P * c^1) * (a)^1 = (\{-\} * (P * c^1) * (a)^1) * \{1\}^1$	($TR812$)(Eqn1)
$(b = E) \subseteq ((P * c^1) * b^1 = \{1\}^1)$	($TR804$)(Second Kolmogorov Axiom)(Eqn2)
$(b = E) \subseteq ((P * c^1) * (a)^1 = (\{-\} * (P * c^1) * (a)^1) * ((P * c^1) * b^1))$	($AMX19$)(Substitution of Eqn2 into Eqn1)

$$(c \subseteq a) = ((P * c^1) * a^1 = \{1\}^1) \quad (TR767)$$

$$(c \subseteq a) = ((\{+\} * \mathcal{N}c^1) * \mathcal{N}(ac)^1 = \{1\}^1) \quad (TR759)$$

Proof of ($TR767$) & ($TR759$)

$\mathcal{N}c^1 \div \mathcal{N}c^1 = \{1\}^1$	($TR806$)(Eqn1)
Let $c \subseteq a$	
$c = ca$	($AMX44$)($\subseteq$ Equality Form)(Eqn2)
$\mathcal{N}(ac)^1 \div \mathcal{N}c^1 = \{1\}^1$	($AMX19$)(Substitution of Eqn2 into Eqn1)
Thus $(c \subseteq a) = (\mathcal{N}(ac)^1 \div \mathcal{N}c^1 = \{1\}^1)$	($AMX02$)(Conditional Introduction) since this proof is reversible
$(c \subseteq a) = ((\{+\} * \mathcal{N}c^1) * \mathcal{N}(ac)^1 = \{1\}^1)$	($TR474$)
$(c \subseteq a) = ((P * c^1) * a^1 = \{1\}^1)$	($DR79$)(Definition of Numerical Probability)

$$(c \subseteq a) = ((P * c^1) * a^1 = \{0\}^1) \quad (TR769)$$

$$(c \subseteq a) = ((\{+\} * \mathcal{N}c^1) * \mathcal{N}(ac)^1 = \{0\}^1) \quad (TR761)$$

Proof of ($TR769$) & ($TR761$)

$\mathcal{N}c^1 \div \mathcal{N}c^1 = \{1\}^1$	($TR806$)(Eqn1)

Let $\mathbf{c} \subseteq \mathbf{a}$

$$\mathbf{c} = \mathbf{c}\mathbf{a}$$

$$\mathcal{N}(\mathbf{a}\mathbf{c})^1 \div \mathcal{N}\mathbf{c}^1 = \{1\}^1$$

$$\text{Thus } (\mathbf{c} \subseteq \mathbf{a}) = (\mathcal{N}(\mathbf{a}\mathbf{c})^1 \div \mathcal{N}\mathbf{c}^1 = \{1\}^1)$$

$$(\mathbf{c} \subseteq \mathbf{a}) = ((\mathcal{P}*\mathbf{c}^1)*(\mathbf{a})^1 = \{1\}^1)$$

$$(\mathcal{P}*\mathbf{c}^1)*\mathbf{a}^1 + (\mathcal{P}*\mathbf{c}^1)*(\mathbf{a})^1 = \{1\}^1$$

$$(\mathbf{c} \subseteq \mathbf{a}) = ((\mathcal{P}*\mathbf{c}^1)*\mathbf{a}^1 + \{1\}^1 = \{1\}^1)$$

$$(\mathbf{c} \subseteq \mathbf{a}) = ((\mathcal{P}*\mathbf{c}^1)*\mathbf{a}^1 + \{1\}^1 - \{1\}^1 = \{1\}^1 - \{1\}^1) \quad (T_{R860})$$

$$(\mathbf{c} \subseteq \mathbf{a}) = ((\mathcal{P}*\mathbf{c}^1)*\mathbf{a}^1 + \{0\}^1 = \{0\}^1) \quad (T_{T498})$$

$$(\mathbf{c} \subseteq \mathbf{a}) = ((\mathcal{P}*\mathbf{c}^1)*\mathbf{a}^1 = \{0\}^1) \quad (T_{T492})$$

$$(\mathbf{c} \subseteq \mathbf{a}) = ((\{ \div \} * \mathcal{N}\mathbf{c}^1) * \mathcal{N}(\mathbf{a}\mathbf{c})^1 = \{0\}^1) \quad (D_{R79})(\text{Definition of Numerical Probability})$$

$$(\mathcal{P}*\mathbf{c}^1)*\mathbf{a}^1 \subseteq (\{ \leq \} * \{1\}^1)(\{ \geq \} * \{0\}^1) \quad (T_{T659})$$

Proof of (T_{T659})

$$\mathbf{a}^1 \subseteq \mathbf{E}$$

$$(\mathcal{P}*\mathbf{c}^1)*\mathbf{a}^1 \subseteq (\mathcal{P}*\mathbf{c}^1)*\mathbf{E}$$

$$(\mathcal{P}*\mathbf{c}^1)*\mathbf{a}^1 \subseteq (\{ \leq \} * \{1\}^1)(\{ \geq \} * \{0\}^1)$$

(T_{M744})(Universal Set Theorem)

(T_{R196})(Relational Deduction)

(D_{S06})(Definition of Probability Range)

$$\mathcal{B}((\mathcal{P}*\mathbf{c}^1)*\mathbf{E})^1 \neq \emptyset \quad (T_{S824})$$

$$\mathcal{C}((\mathcal{P}*\mathbf{c}^1)*\mathbf{E})^1 \neq \emptyset \quad (T_{S823})$$

Example Proof of (T_{S824})

$$(\mathbf{X} = \mathbf{B}, \mathbf{C}) \subseteq (\mathbf{X} * ((\mathcal{P}*\mathbf{c}^1)*\mathbf{E})^1 \neq \emptyset)$$

$$\mathcal{B}((\mathcal{P}*\mathbf{c}^1)*\mathbf{E})^1 \neq \emptyset$$

(A_{R15})(Probability is Bounded)

Extraction

$$\mathcal{B}((\mathcal{P}*\mathbf{c}^1)*\mathbf{E})^1 = \{0\}^1 \quad (T_{T649})$$

$$\mathcal{C}((\mathcal{P}*\mathbf{c}^1)*\mathbf{E})^1 = \{1\}^1 \quad (T_{T650})$$

Example Proof of (T_{T649})

$$(\mathcal{P}*\mathbf{c}^1)*\mathbf{E} = (\{ \leq \} * \{1\}^1)(\{ \geq \} * \{0\}^1) \quad (D_{S06})(\text{Definition of Probability Range})(\text{Eqn1})$$

$$\mathcal{B}((\mathcal{P}*\mathbf{c}^1)*\mathbf{a})^1 = ((\mathcal{P}*\mathbf{c}^1)*\mathbf{a}).(\{ \geq \} * (\mathcal{P}*\mathbf{c}^1)*\mathbf{a}) \quad (T_{T209})(\text{Conditional Version})(\text{Eqn2})$$

$$\text{Let } \mathbf{a} = \mathbf{E} \quad (\text{Eqn3})$$

$$\mathcal{B}((\mathcal{P}*\mathbf{c}^1)*\mathbf{E})^1 = ((\mathcal{P}*\mathbf{c}^1)*\mathbf{E}).(\{ \geq \} * (\mathcal{P}*\mathbf{c}^1)*\mathbf{E}) \quad (A_{MX19})(\text{Substitution of Eqn3 into Eqn2})(\text{Eqn4})$$

$$\mathcal{B}((\mathcal{P}*\mathbf{c}^1)*\mathbf{E})^1 = (\{ \leq \} * \{1\}^1)(\{ \geq \} * \{0\}^1). \{ \geq \} * ((\{ \leq \} * \{1\}^1)(\{ \geq \} * \{0\}^1))$$

(A_{MX19})(Substitution of Eqn1 into Eqn4)(Eqn5)

$$\{1\}^1 \subseteq \{ \geq \} * \{0\}^1 \quad (T_{T582})$$

$$\{0\}^1 \subseteq \{ \geq \} * \{1\}^1 \quad (T_{R519})(\text{Active/Passive Theorem})$$

$$\{0\}^1 \subseteq \{ \leq \} * \{1\}^1 \quad (D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set})$$

$$\{0\}^1 \subseteq \{ \leq \} * \{1\}^1 + \{ = \} * \{1\}^1 \quad (T_{M150})(\text{Single Premise Syllogism})$$

$$\{0\}^1 \subseteq (\{ \leq \} + \{ = \}) * \{1\}^1 \quad (T_{R002c})(\text{Distribution of } + \text{ over } *)$$

$$\{0\}^1 \subseteq \{ \leq \} * \{1\}^1 \quad (D_{A141})(\text{Disjunction with } \{ = \})$$

$$\{0\}^1 = \{0\}^1. \{ \leq \} * \{1\}^1 \quad (A_{MX44})(\subseteq \text{ Equality Form})(\text{Eqn6})$$

$$\{ \leq \} * \{0\}^1 \subseteq \{ \leq \} * \{ \leq \} * \{1\}^1 \quad (T_{R196})(\text{Relational Deduction})$$

$$\{ \leq \} * \{0\}^1 \subseteq \{ \leq \} * \{1\}^1 \quad (T_{A711})(\text{Transitive Function Theorem})$$

$$(\{ \leq \} * \{0\}^1)(\{ \geq \} * \{0\}^1) \subseteq (\{ \leq \} * \{1\}^1)(\{ \geq \} * \{0\}^1) \quad (T_{M029})(\text{Deduction})(\text{Eqn7})$$

$$(\{ \leq \} . \{ \geq \}) * \{0\}^1 \subseteq (\{ \leq \} * \{0\}^1)(\{ \geq \} * \{0\}^1) \quad (T_{R220})(\text{Conditional Distribution of } . \text{ over } *) (\text{Eqn8})$$

$$(\{ \leq \} . \{ \geq \}) * \{0\}^1 \subseteq (\{ \leq \} * \{1\}^1)(\{ \geq \} * \{0\}^1) \quad (T_{M139})(\text{Hypothetical Syllogism Eqns7\&8})(\text{Eqn9})$$

$$\{ \leq \} . \{ \geq \} = \{ \leq \} . \{ \geq \} \quad (D_{M00})(\text{Definition of } =)$$

$$\{ \leq \} . \{ \geq \} = (\{ \geq \} + \{ = \})(\{ \leq \} + \{ = \}) \quad (D_{A141})(\text{Disjunction with } \{ = \})$$

$$\{ \leq \} . \{ \geq \} = (\{ \geq \} \{ \leq \} + \{ \geq \} \{ = \} + \{ = \} \{ \leq \} + \{ = \} \{ = \}) \quad (T_{M407})(\text{Distribution of } . \text{ over } +)$$

$$\{ \leq \} . \{ \geq \} = (\emptyset + \emptyset + \emptyset + \{ = \} \{ = \}) \quad (T_{S696}), (T_{S697}) \& (T_{S693})$$

$$\{ \leq \} . \{ \geq \} = \{ = \} \{ = \} \quad (T_{M668})(\text{Null Set Disjunction})$$

$$\begin{aligned}
\{\leq\}.\{\geq\} &= \{=\} & (T_{M667})(\text{Conjunction Tautology})(\text{Eqn10}) \\
\{=\}*\{0\}^1 &\subseteq (\{\leq\}*\{1\}^1)(\{\geq\}*\{0\}^1) & (A_{MX19})(\text{Substitution of Eqn10 into Eqn9}) \\
\{0\}^1 &\subseteq (\{\leq\}*\{1\}^1)(\{\geq\}*\{0\}^1) & (D_{A144})(\text{Definition of the Reflexive Relationship Set}) \\
\{>\}*\{0\}^1 &\subseteq \{>\}*((\{\leq\}*\{1\}^1)(\{\geq\}*\{0\}^1)) & (T_{R196})(\text{Relational Deduction}) \\
\{>\}*((\{\leq\}*\{1\}^1)(\{\geq\}*\{0\}^1)) &\subseteq \{>\}*\{0\}^1 & (T_{M012})(\text{Transposition}) \\
\{>\}*((\{\leq\}*\{1\}^1)(\{\geq\}*\{0\}^1)) &= (\{>\}*\{0\}^1).(\{>\}*((\{\leq\}*\{1\}^1)(\{\geq\}*\{0\}^1))) & (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn11}) \\
B((P*c^1)*E)^1 &= (\{\leq\}*\{1\}^1)(\{\geq\}*\{0\}^1).(\{>\}*\{0\}^1).(\{>\}*((\{\leq\}*\{1\}^1)(\{\geq\}*\{0\}^1))) & (A_{MX19})(\text{Substitution of Eqn11 into Eqn5})(\text{Eqn12}) \\
(\{\geq\}*\{0\}^1).(\{>\}*\{0\}^1) &= (\{\geq\}*\{0\}^1).(\{>\}*\{0\}^1) & (D_{M00})(\text{Definition of } =) \\
((\{>\} + \{=\})*\{0\}^1).(\{>\}*\{0\}^1) &= (\{\geq\}*\{0\}^1).(\{>\}*\{0\}^1) & (D_{A141})(\text{Disjunction with } \{=\}) \\
(\{>\}*\{0\}^1 + \{=\})*\{0\}^1 &= (\{\geq\}*\{0\}^1).(\{>\}*\{0\}^1) & (T_{R002c})(\text{Distribution of } + \text{ over } *) \\
(\{>\}*\{0\}^1).(\{>\}*\{0\}^1) + (\{=\})*\{0\}^1 &= (\{\geq\}*\{0\}^1).(\{>\}*\{0\}^1) & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
\emptyset + (\{=\})*\{0\}^1 &= (\{\geq\}*\{0\}^1).(\{>\}*\{0\}^1) & (T_{M621})(\text{Complement Conjunction}) \\
(\{=\})*\{0\}^1.(\{>\}*\{0\}^1) &= (\{\geq\}*\{0\}^1).(\{>\}*\{0\}^1) & (T_{M668})(\text{Null Set Disjunction}) \\
\{0\}^1.(\{>\}*\{0\}^1) &= (\{\geq\}*\{0\}^1).(\{>\}*\{0\}^1) & (D_{A144})(\text{Definition of the Reflexive Relationship Set})(\text{Eqn13}) \\
\{0\}^1 &\subseteq \{>\}*\{0\}^1 & (T_{T123}) \\
\{0\}^1 &= \{0\}^1.(\{>\}*\{0\}^1) & (A_{MX44})(\subseteq \text{Equality Form})(\text{Eqn14}) \\
\{0\}^1 &= (\{\geq\}*\{0\}^1).(\{>\}*\{0\}^1) & (A_{MX19})(\text{Substitution of Eqn14 into Eqn13})(\text{Eqn15}) \\
B((P*c^1)*E)^1 &= (\{\leq\}*\{1\}^1).\{0\}^1.(\{>\}*((\{\leq\}*\{1\}^1)(\{\geq\}*\{0\}^1))) & (A_{MX19})(\text{Substitution of Eqn15 into Eqn12})(\text{Eqn16}) \\
B((P*c^1)*E)^1 &= \{0\}^1.(\{>\}*((\{\leq\}*\{1\}^1)(\{\geq\}*\{0\}^1))) & (A_{MX19})(\text{Substitution of Eqn6 into Eqn16}) \\
B((P*c^1)*E)^1 &\subseteq \{0\}^1 & (T_{M073b})(\text{Single Premise Syllogism}) \\
(B((P*c^1)*E)^1 \subseteq \{0\}^1) + (B((P*c^1)*E)^1 = \{0\}^1) & & (T_{M266})(\text{Definition of } \subseteq) \\
(B((P*c^1)*E)^1 = \emptyset)(\{0\}^1 \neq \emptyset) + (B((P*c^1)*E)^1 = \{0\}^1) & & (A_{MX40})(\text{Proper Subset of Unit Set Axiom}) \\
(B((P*c^1)*E)^1 = \emptyset)E + (B((P*c^1)*E)^1 = \{0\}^1) & & (T_{M001})(\text{Universality of } (T_{T426})) \\
(B((P*c^1)*E)^1 = \emptyset) + (B((P*c^1)*E)^1 = \{0\}^1) & & (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn17}) \\
B((P*c^1)*E)^1 &\neq \emptyset & (T_{S824}) \\
(B((P*c^1)*E)^1 \neq \emptyset) &= E & (A_{MX16})(\text{Order Axiom}) \\
(B((P*c^1)*E)^1 = \emptyset) &= \sim E & (T_{M008})(\text{Inverse of an Equality}) \\
(B((P*c^1)*E)^1 = \emptyset) &= \emptyset & (D_{M02})(\text{Definition of } \emptyset)(\text{Eqn18}) \\
\emptyset + (B((P*c^1)*E)^1 = \{0\}^1) & & (A_{MX19})(\text{Substitution of Eqn18 into Eqn17}) \\
B((P*c^1)*E)^1 &= \{0\}^1 & (T_{M668})(\text{Null Set Disjunction})
\end{aligned}$$

$$(c \not\subseteq a) \subseteq ((P*c^1)*a^1 \subseteq \{<\}*\{1\}^1) \quad (T_{R768})$$

$$(c \not\subseteq a) \subseteq ((P*c^1)*a^1 \subseteq \{>\}*\{0\}^1) \quad (T_{R770})$$

$$(c \not\subseteq a) \subseteq ((\{ \div \}*\mathcal{N}c^1)*\mathcal{N}(ac)^1 \subseteq \{<\}*\{1\}^1) \quad (T_{R760})$$

$$(c \not\subseteq a) \subseteq ((\{ \div \}*\mathcal{N}c^1)*\mathcal{N}(ac)^1 \subseteq \{>\}*\{0\}^1) \quad (T_{R762})$$

Example Proof of (T_{R768}) & (T_{R760})

$$\begin{aligned}
(P*c^1)*a^1 &\subseteq (\{\leq\}*\{1\}^1)(\{\geq\}*\{0\}^1) & (T_{T659}) \\
(P*c^1)*a^1 &\subseteq \{\leq\}*\{1\}^1 & (T_{M073})(\text{Single Premise Syllogism}) \\
(P*c^1)*a^1 &\subseteq (\{<\} + \{=\})*\{1\}^1 & (D_{A141})(\text{Disjunction with } \{=\}) \\
(P*c^1)*a^1 &\subseteq (\{<\}*\{1\}^1) + (\{=\}*\{1\}^1) & (T_{R002c})(\text{Distribution of } + \text{ over } *) \\
((P*c^1)*a^1 \subseteq \{<\}*\{1\}^1) + ((P*c^1)*a^1 \subseteq \{=\}*\{1\}^1) & & (T_{M526}) \\
((P*c^1)*a^1 \subseteq \{<\}*\{1\}^1) + ((P*c^1)*a^1 \subseteq \{=\}*\{1\}^1) + ((P*c^1)*a^1 = \{=\}*\{1\}^1) & & (T_{M266})(\text{Definition of } \subseteq)
\end{aligned}$$

$$\begin{aligned}
& ((P * c^1) * a^1 \subseteq \{<\} * \{1\}^1) + ((P * c^1) * a^1 \subseteq \{=\} * \{1\}^1) + (c \subseteq a) \\
& \quad (T_{R767}) \\
& ((P * c^1) * a^1 \subseteq \{<\} * \{1\}^1) + ((P * c^1) * a^1 \subseteq \{1\}^1) + (c \subseteq a) \\
& \quad (D_{A144})(\text{Definition of the Reflexive Relationship Set})(\text{Eqn1}) \\
& ((P * c^1) * a^1 \subseteq \{1\}^1) = (\{1\}^1 \neq \emptyset)((P * c^1) * a^1 = \emptyset) (A_{MX40})(\text{Proper Subset of Unit Set Axiom}) \\
& ((P * c^1) * a^1 \subseteq \{1\}^1) = E((P * c^1) * a^1 = \emptyset) \quad (T_{M007})(\text{Universality of } (T_{T606})) \\
& ((P * c^1) * a^1 \subseteq \{1\}^1) = ((P * c^1) * a^1 = \emptyset) \quad (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn2}) \\
& ((P * c^1) * a^1 \subseteq \{<\} * \{1\}^1) + ((P * c^1) * a^1 = \emptyset) + (c \subseteq a) \\
& \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
& E = ((P * c^1) * a^1 \subseteq \{<\} * \{1\}^1) + ((P * c^1) * a^1 = \emptyset) + (c \subseteq a) \\
& \quad (A_{MX16})(\text{Order Axiom}) \\
& E \subseteq ((P * c^1) * a^1 \subseteq \{<\} * \{1\}^1) + ((P * c^1) * a^1 = \emptyset) + (c \subseteq a) \\
& \quad (T_{M275})(\text{Supersets of the Universal Set}) \\
& E((P * c^1) * a^1 \neq \emptyset)(c \not\subseteq a) \subseteq ((P * c^1) * a^1) \subseteq \{<\} * \{1\}^1 \\
& \quad (T_{M468}) \\
& ((P * c^1) * a^1 \neq \emptyset)(c \not\subseteq a) \subseteq ((P * c^1) * a^1) \subseteq \{<\} * \{1\}^1 \\
& \quad (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn3}) \\
& (c \not\subseteq a) \subseteq ((P * c^1) * a^1) \subseteq \{<\} * \{1\}^1 \quad (T_{S769}) \text{ because } ((P * c^1) * a^1 \neq \emptyset) \text{ is equal to the implied} \\
& \quad \text{conditions of probability}
\end{aligned}$$

$$(c \not\subseteq a) \subseteq ((P * c^1) * a^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \quad (T_{R772})$$

$$(c \not\subseteq a) \subseteq ((P * c^1) * a^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \quad (T_{S758})$$

Example Proof of (T_{R772})

$$\begin{aligned}
& (c \not\subseteq a) \subseteq ((P * c^1) * a^1 \subseteq \{<\} * \{1\}^1) \quad (T_{R768}) \\
& (c \not\subseteq a) \subseteq ((P * c^1) * a^1 \subseteq \{<\} * \{1\}^1)E \\
& a^1 \subseteq E \quad (A_{MX11})(\text{Universal Set Conjunction Axiom})(\text{Eqn1}) \\
& (P * c^1) * a^1 \subseteq (P * c^1) * E \quad (T_{M744})(\text{Universal Set Theorem}) \\
& (P * c^1) * E = (\{<\} * \{1\}^1)(\{>\} * \{0\}^1) \quad (T_{R196})(\text{Relational Deduction})(\text{Eqn2}) \\
& (P * c^1) * a^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1) \quad (D_{S06})(\text{Definition of Probability Range})(\text{Eqn3}) \\
& ((P * c^1) * a^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) = E \quad (A_{MX19})(\text{Substitution of Eqn3 into Eqn2}) \\
& (c \not\subseteq a) \subseteq ((P * c^1) * a^1 \subseteq \{<\} * \{1\}^1).((P * c^1) * a^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \\
& \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn4}) \\
& (c \not\subseteq a) \subseteq ((P * c^1) * a^1 \subseteq (\{<\} * \{1\}^1)(\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \quad (A_{MX19})(\text{Substitution of Eqn4 into Eqn1}) \\
& (c \not\subseteq a) \subseteq ((P * c^1) * a^1 \subseteq (\{<\} * \{1\}^1)(\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \quad (T_{M132})(\text{Composition}) \\
& (c \not\subseteq a) \subseteq ((P * c^1) * a^1 \subseteq (\{<\} * \{1\}^1)(\{<\} + \{=\}) * \{1\}^1)(\{>\} * \{0\}^1)) \\
& \quad (D_{A141})(\text{Disjunction with } \{=\}) \\
& (c \not\subseteq a) \subseteq ((P * c^1) * a^1 \subseteq (\{<\} * \{1\}^1)(\{<\} + \{=\}) * \{1\}^1)(\{>\} * \{0\}^1)) \quad (T_{R239})(\text{Distribution of } . \text{ over } * \text{ for Entities}) \\
& (c \not\subseteq a) \subseteq ((P * c^1) * a^1 \subseteq ((\{<\} * \{1\}^1) + \{<\} * \{=\}) * \{1\}^1)(\{>\} * \{0\}^1)) \\
& \quad (T_{M407})(\text{Distribution of } . \text{ over } +) \\
& (c \not\subseteq a) \subseteq ((P * c^1) * a^1 \subseteq ((\{<\} + \{<\} * \{=\}) * \{1\}^1)(\{>\} * \{0\}^1)) \quad (T_{M667})(\text{Conjunction Tautology}) \\
& (c \not\subseteq a) \subseteq ((P * c^1) * a^1 \subseteq ((\{<\} + \emptyset) * \{1\}^1)(\{>\} * \{0\}^1)) \quad (T_{S693}) \\
& (c \not\subseteq a) \subseteq ((P * c^1) * a^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \quad (T_{M668})(\text{Null Set Disjunction})
\end{aligned}$$

$$\{0\}^1 \subseteq (P * c^1) * E \quad (T_{S816})$$

$$\{0\}^1 \subseteq S(P * c^1) \quad (T_{S817})$$

Proof of (T_{S817})

$$\begin{aligned}
& (P * c^1) * E = (\{<\} * \{1\}^1)(\{>\} * \{0\}^1) \quad (D_{S06})(\text{Definition of Probability Range})(\text{Eqn1}) \\
& \{1\}^1 \subseteq \{>\} * \{0\}^1 \quad (T_{S582}) \\
& \{0\}^1 \subseteq \{>\} * \{1\}^1 \quad (T_{R519})(\text{Active/Passive Theorem}) \\
& \{0\}^1 \subseteq \{<\} * \{1\}^1 \quad (D_{A143})(\text{Symmetrical Opposite of an Asymmetrical Relationship Set}) \\
& \{0\}^1 \subseteq \{<\} * \{1\}^1 + \{=\} * \{1\}^1 \quad (T_{M150})(\text{Single Premise Syllogism}) \\
& \{0\}^1 \subseteq (\{<\} + \{=\}) * \{1\}^1 \quad (T_{R002c})(\text{Distribution of } + \text{ over } *)
\end{aligned}$$

$\{0\}^1 \subseteq \{\leq\} * \{1\}^1$	(D_{A141})(Disjunction with $\{=\}$)
$\{0\}^1 = \{0\}^1 . \{\leq\} * \{1\}^1$	($AMX44$)($\subseteq$ Equality Form)(Eqn2)
$\{\leq\} * \{0\}^1 \subseteq \{\leq\} * \{\leq\} * \{1\}^1$	($TR196$)(Relational Deduction)
$\{\leq\} * \{0\}^1 \subseteq \{\leq\} * \{1\}^1$	($TA711$)(Transitive Function Theorem)
$(\{\leq\} * \{0\}^1)(\{\geq\} * \{0\}^1) \subseteq (\{\leq\} * \{1\}^1)(\{\geq\} * \{0\}^1)$	($TM029$)(Deduction)(Eqn3)
$(\{\leq\} . \{\geq\}) * \{0\}^1 \subseteq (\{\leq\} * \{0\}^1)(\{\geq\} * \{0\}^1)$	($TR220$)(Conditional Distribution of . over *) (Eqn4)
$(\{\leq\} . \{\geq\}) * \{0\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{\geq\} * \{0\}^1)$	($TM139$)(Hypothetical Syllogism Eqns3&4)(Eqn5)
$\{\leq\} . \{\geq\} = \{\leq\} . \{\geq\}$	($DM00$)(Definition of $=$)
$\{\leq\} . \{\geq\} = (\{>\} + \{=\})(\{<\} + \{=\})$	($DA141$)(Disjunction with $\{=\}$)
$\{\leq\} . \{\geq\} = (\{>\}\{<\} + \{>\}\{=\} + \{=\}\{<\} + \{=\}\{=\})$	($TM407$)(Distribution of . over +)
$\{\leq\} . \{\geq\} = (\emptyset + \emptyset + \emptyset + \{=\}\{=\})$	($TS696$), ($TS697$) & ($TS693$)
$\{\leq\} . \{\geq\} = \{=\}\{=\}$	($TM668$)(Null Set Disjunction)
$\{\leq\} . \{\geq\} = \{=\}$	($TM667$)(Conjunction Tautology)(Eqn6)
$\{=\} * \{0\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{\geq\} * \{0\}^1)$	($AMX19$)(Substitution of Eqn6 into Eqn5)
$\{0\}^1 \subseteq (\{\leq\} * \{1\}^1)(\{\geq\} * \{0\}^1)$	($DA144$)(Definition of the Reflexive Relationship Set)(Eqn7)
$\{0\}^1 \subseteq (P * c^1) * E$	($AMX19$)(Substitution of Eqn1 into Eqn7)
$\{0\}^1 \subseteq S(P * c^1)$	($TR210$)

$$(a = \emptyset) \subseteq (a^1 = {}^v(P * c^1) * \{0\}^1) \quad (TS820)$$

Proof of ($TS820$)

$((c^1 \subseteq R_0)(a \subseteq S c) = E) \subseteq ((b = {}^v c * a) = (a = c * b))$	($TS522$)(Eqn1)
Let $c = P * c^1$	(Eqn2)
Let $a = \{0\}^1$	(Eqn3)
Let $b = a^1$ (Eqn4)	
$((P * c^1)^1 \subseteq R_0)(\{0\}^1 \subseteq S(P * c^1)) = E \subseteq ((a^1 = {}^v(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1))$	($AMX19$)(Substitution of Eqns2,3&4 into Eqn1)
$(E(\{0\}^1 \subseteq S(P * c^1)) = E) \subseteq ((a^1 = {}^v(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1))$	($TM001$)(Universality of ($TS719P$))
$(EE = E) \subseteq ((a^1 = {}^v(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1))$	($TM001$)(Universality of ($TS817$))
$(E = E) \subseteq ((a^1 = {}^v(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1))$	($TM667$)(Conjunction Tautology)
$E \subseteq ((a^1 = {}^v(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1))$	($TM342$)
$E = ((a^1 = {}^v(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1))$	($TM275$)(Supersets of the Universal Set)
$(a^1 = {}^v(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1)$	($AMX16$)(Order Axiom)(Eqn5)
Let $a = \emptyset$	(Eqn6)
$(c \subseteq a) = ((P * c^1) * a^1 = \{0\}^1)$	($TR769$)(Eqn7)
$(c \subseteq \sim \emptyset) = ((P * c^1) * a^1 = \{0\}^1)$	($AMX19$)(Substitution of Eqns6 into Eqn7)
$(c \subseteq E) = ((P * c^1) * a^1 = \{0\}^1)$	($TM054$)(Negative of Null Set)
$E = ((P * c^1) * a^1 = \{0\}^1)$	($TM001$)(Universality of ($TM744$))(Eqn8)
$(a^1 = {}^v(P * c^1) * \{0\}^1) = E$	($AMX19$)(Substitution of Eqns8 into Eqn5)
$a^1 = {}^v(P * c^1) * \{0\}^1$	($AMX16$)(Order Axiom)
Thus $(a = \emptyset) \subseteq (a^1 = {}^v(P * c^1) * \{0\}^1)$	($AMX02$)(Conditional Introduction)

$$(a^1 = {}^v(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1) \quad (TS810)$$

Proof of ($TS810$)

$((c^1 \subseteq R_0)(a \subseteq S c) = E) \subseteq ((b = {}^v c * a) = (a = c * b))$	($TS522$)(Eqn1)
Let $a = \{0\}^1$	(Eqn2)
Let $b = a^1$	(Eqn3)
Let $c = P * c^1$	(Eqn4)
$((P * c^1)^1 \subseteq R_0)(\{0\}^1 \subseteq S(P * c^1)) = E \subseteq ((a^1 = {}^v(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1))$	($AMX19$)(Substitution of Eqns2,3&4 into Eqn1)
$(E(\{0\}^1 \subseteq S(P * c^1)) = E) \subseteq ((a^1 = {}^v(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1))$	($TM001$)(Universality of ($TS719P$))
$(EE = E) \subseteq ((a^1 = {}^v(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1))$	($TM001$)(Universality of ($TS817$))

$$\begin{aligned}
(E = E) &\subseteq ((a^1 = {}^V(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1)) & (T_{M667})(\text{Conjunction Tautology}) \\
E &\subseteq ((a^1 = {}^V(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1)) & (T_{M342}) \\
E &= ((a^1 = {}^V(P * c^1) * \{0\}^1) = (\{0\}^1 = (P * c^1) * a^1)) & (T_{M275})(\text{Supersets of the Universal Set}) \\
(a^1 = {}^V(P * c^1) * \{0\}^1) &= (\{0\}^1 = (P * c^1) * a^1) & (A_{MX16})(\text{Order Axiom})
\end{aligned}$$

$$(P * c^1) * a^1 \subseteq (\{ \geq \} * \{0\}^1) \quad (T_{R802}) \text{ First Kolmogorov Axiom}$$

$$\mathcal{N}(ac)^1 \div \mathcal{N}c^1 \subseteq \{ \geq \} * \{0\}^1 \quad (T_{R803}) \text{ First Kolmogorov Axiom}$$

Proof of (T_{R802}) & (T_{R803})

$$\begin{aligned}
(P * c^1) * a^1 &\subseteq (\{ \leq \} * \{1\}^1)(\{ \geq \} * \{0\}^1) & (T_{T659}) \\
(P * c^1) * a^1 &\subseteq (\{ \geq \} * \{0\}^1) & (T_{M073})(\text{Single Premise Syllogism}) \\
\mathcal{N}(ac)^1 \div \mathcal{N}c^1 &\subseteq \{ \geq \} * \{0\}^1 & (D_{R79})(\text{Definition of Numerical Probability})
\end{aligned}$$

$$(ab = \emptyset) \subseteq ((P * c^1) * (a+b)^1 = (\{+\} * (P * c^1) * b^1) * (P * c^1) * a^1) \quad (T_{R807}) \text{ Third Kolmogorov Axiom}$$

$$(ab = \emptyset) \subseteq (\mathcal{N}(c.(a+b))^1 \div \mathcal{N}c^1 = (\mathcal{N}(ca)^1 \div \mathcal{N}c^1) + (\mathcal{N}(cb)^1 \div \mathcal{N}c^1)) \quad (T_{R808}) \text{ Third Kolmogorov Axiom}$$

Proof of (T_{R807}) & (T_{R808})

$$\begin{aligned}
(cacb = \emptyset) &\subseteq (\mathcal{N}(ca + cb)^1 = (\mathcal{N}(ca)^1 + \mathcal{N}(cb)^1)) & (A_{S37})(\text{Distribution of } \mathcal{N} \text{ over disjoint Sets})(Eqn1) \\
cacb &\subseteq ab & (T_{M083})(\text{Conjunction Elimination}) \\
(cacb \subseteq ab) &= E & (A_{MX16})(\text{Order Axiom}) \\
(ab = \emptyset) &\subseteq (cacb = \emptyset) & (T_{M097})(\text{Probabilistic Rule})(Eqn2) \\
(ab = \emptyset) &\subseteq (\mathcal{N}(ca + cb)^1 = (\mathcal{N}(ca)^1 + \mathcal{N}(cb)^1)) & (T_{M139})(\text{Hypothetical Syllogism Eqns1\&2}) \\
(ab = \emptyset) &\subseteq (\mathcal{N}(c.a + c.b)^1 \div \mathcal{N}c^1 = (\mathcal{N}(ca)^1 + \mathcal{N}(cb)^1) \div \mathcal{N}c^1) & (T_{R864}) \\
(ab = \emptyset) &\subseteq (\mathcal{N}(c.a + c.b)^1 \div \mathcal{N}c^1 = (\mathcal{N}(ca)^1 \div \mathcal{N}c^1) + (\mathcal{N}(cb)^1 \div \mathcal{N}c^1)) & (T_{T486}) \& (T_{T487})(\text{Distribution of } \div \text{ over } + \& -) \\
(ab = \emptyset) &\subseteq (\mathcal{N}(c.(a+b))^1 \div \mathcal{N}c^1 = (\mathcal{N}(ca)^1 \div \mathcal{N}c^1) + (\mathcal{N}(cb)^1 \div \mathcal{N}c^1)) & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
(ab = \emptyset) &\subseteq ((P * c^1) * (a+b)^1 = (P * c^1) * a^1 + (P * c^1) * b^1) & (T_{R801})(\text{Definition of Numerical Probability}) \\
(ab = \emptyset) &\subseteq ((P * c^1) * (a+b)^1 = (\{+\} * (P * c^1) * b^1) * (P * c^1) * a^1) & (T_{T471})
\end{aligned}$$

$$(P * c^1) * (a+b)^1 = (\{-\} * (P * c^1) * (a.b)^1) * (\{+\} * (P * c^1) * b^1) * ((P * c^1) * a^1) \quad (T_{R818})$$

$$(P * c^1) * (a+b)^1 = ((P * c^1) * a^1) + ((P * c^1) * b^1) - ((P * c^1) * (a.b)^1) \quad (T_{R819})$$

$$\mathcal{N}(ac + bc)^1 \div \mathcal{N}c^1 = \mathcal{N}(ac)^1 \div \mathcal{N}c^1 + \mathcal{N}(bc)^1 \div \mathcal{N}c^1 - \mathcal{N}(abc)^1 \div \mathcal{N}c^1 \quad (T_{R820})$$

Proof of (T_{R818}), (T_{R819}) & (T_{R820})

$$\begin{aligned}
\mathcal{N}(ca + cb)^1 &= \mathcal{N}(ca)^1 + \mathcal{N}(cb)^1 - \mathcal{N}(cacb)^1 & (T_{R798}) \\
\mathcal{N}(c.(a+b))^1 &= (\mathcal{N}(ca)^1 + \mathcal{N}(cb)^1 - \mathcal{N}(cab)^1) & (T_{M407})(\text{Distribution of } . \text{ over } +) \\
\mathcal{N}(c.(a+b))^1 \div \mathcal{N}c^1 &= (\mathcal{N}(ca)^1 + \mathcal{N}(cb)^1 - \mathcal{N}(cab)^1) \div \mathcal{N}c^1 & (T_{R864}) \\
\mathcal{N}(c.(a+b))^1 \div \mathcal{N}c^1 &= (\mathcal{N}(ca)^1 \div \mathcal{N}c^1) + (\mathcal{N}(cb)^1 \div \mathcal{N}c^1) - (\mathcal{N}(cab)^1 \div \mathcal{N}c^1) & (T_{T486}) \& (T_{T487})(\text{Distribution of } \div \text{ over } + \& -) \\
(P * c^1) * (a+b)^1 &= ((P * c^1) * a^1) + ((P * c^1) * b^1) - ((P * c^1) * (a.b)^1) & (T_{R801})(\text{Definition of Numerical Probability}) \\
(P * c^1) * (a+b)^1 &= (\{-\} * (P * c^1) * (a.b)^1) * (\{+\} * (P * c^1) * b^1) * ((P * c^1) * a^1) & (T_{T471}) \& (T_{T472}) \\
\mathcal{N}((a+b)c)^1 \div \mathcal{N}c^1 &= \mathcal{N}(ac)^1 \div \mathcal{N}c^1 + \mathcal{N}(bc)^1 \div \mathcal{N}c^1 - \mathcal{N}(abc)^1 \div \mathcal{N}c^1 & (T_{R801})(\text{Definition of Numerical Probability}) \\
\mathcal{N}(ac + bc)^1 \div \mathcal{N}c^1 &= \mathcal{N}(ac)^1 \div \mathcal{N}c^1 + \mathcal{N}(bc)^1 \div \mathcal{N}c^1 - \mathcal{N}(abc)^1 \div \mathcal{N}c^1
\end{aligned}$$

(T_{M407})(Distribution of . over +)

$$(\mathcal{P} * \mathbf{c}^1) * (\mathbf{ab})^1 = (\{x\} * (\mathcal{P} * (\mathbf{cb})^1) * \mathbf{a}^1) * (\mathcal{P} * \mathbf{c}^1) * \mathbf{b}^1 \quad (T_{R837})$$

$$(\mathcal{P} * \mathbf{c}^1) * (\mathbf{ab})^1 = ((\mathcal{P} * \mathbf{c}^1) * \mathbf{b}^1) \times ((\mathcal{P} * (\mathbf{cb})^1) * \mathbf{a}^1) \quad (T_{R838})$$

$$\mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}\mathbf{c}^1 = \mathcal{N}(\mathbf{bc})^1 \div \mathcal{N}\mathbf{c}^1 \times \mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1 \quad (T_{R839})$$

Proof of (T_{R837}), (T_{R838}) & (T_{R839})

$$\mathcal{N}(\mathbf{cab})^1 = \mathcal{N}(\mathbf{cab})^1 \times \{1\}^1 \quad (T_{T494})(\text{Eqn1})$$

$$(\mathcal{N}(\mathbf{cb})^1 \neq \{0\}^1) \subseteq (\mathcal{N}(\mathbf{cb})^1 \div \mathcal{N}(\mathbf{cb})^1 = \{1\}^1) \quad (T_{T499})$$

$$\mathcal{N}(\mathbf{cb})^1 \div \mathcal{N}(\mathbf{cb})^1 = \{1\}^1 \quad (T_{S761})(\text{Implied Conditions of Quantity})(\text{Eqn2})$$

$$\mathcal{N}(\mathbf{cab})^1 = \mathcal{N}(\mathbf{cab})^1 \times (\mathcal{N}(\mathbf{cb})^1 \div \mathcal{N}(\mathbf{cb})^1) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1})$$

$$\mathcal{N}(\mathbf{cab})^1 = (\mathcal{N}(\mathbf{cab})^1 \times \mathcal{N}(\mathbf{cb})^1) \div \mathcal{N}(\mathbf{cb})^1 \quad (T_{T660})$$

$$\mathcal{N}(\mathbf{cab})^1 = (\mathcal{N}(\mathbf{cb})^1 \times \mathcal{N}(\mathbf{cab})^1) \div \mathcal{N}(\mathbf{cb})^1 \quad (T_{T483})(\text{Commutativity of Algebraic } \times)$$

$$\mathcal{N}(\mathbf{cab})^1 \div \mathcal{N}\mathbf{c}^1 = ((\mathcal{N}(\mathbf{cb})^1 \times \mathcal{N}(\mathbf{cab})^1) \div \mathcal{N}(\mathbf{cb})^1) \div \mathcal{N}\mathbf{c}^1 \quad (T_{R864})$$

$$\mathcal{N}(\mathbf{cab})^1 \div \mathcal{N}\mathbf{c}^1 = (\mathcal{N}(\mathbf{cb})^1 \times \mathcal{N}(\mathbf{cab})^1) \div (\mathcal{N}\mathbf{c}^1 \times \mathcal{N}(\mathbf{cb})^1) \quad (T_{T662})$$

$$\mathcal{N}(\mathbf{cab})^1 \div \mathcal{N}\mathbf{c}^1 = (\mathcal{N}(\mathbf{cb})^1 \div \mathcal{N}\mathbf{c}^1) \times (\mathcal{N}(\mathbf{cab})^1 \div \mathcal{N}(\mathbf{cb})^1) \quad (T_{T663})$$

$$(\mathcal{P} * \mathbf{c}^1) * (\mathbf{ab})^1 = ((\mathcal{P} * \mathbf{c}^1) * \mathbf{b}^1) \times ((\mathcal{P} * (\mathbf{cb})^1) * \mathbf{a}^1) \quad (T_{R801})(\text{Definition of Numerical Probability})$$

$$(\mathcal{P} * \mathbf{c}^1) * (\mathbf{ab})^1 = (\{x\} * (\mathcal{P} * (\mathbf{cb})^1) * \mathbf{a}^1) * (\mathcal{P} * \mathbf{c}^1) * \mathbf{b}^1 \quad (T_{T473})$$

$$(\{x\} * (\mathcal{P} * (\mathbf{ca})^1) * \mathbf{b}^1) * ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1) = (\{x\} * (\mathcal{P} * (\mathbf{cb})^1) * \mathbf{a}^1) * ((\mathcal{P} * \mathbf{c}^1) * \mathbf{b}^1) \quad (T_{R840}) \text{ Bayes Theorem}$$

$$((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1) \times ((\mathcal{P} * (\mathbf{ca})^1) * \mathbf{b}^1) = ((\mathcal{P} * \mathbf{c}^1) * \mathbf{b}^1) \times ((\mathcal{P} * (\mathbf{cb})^1) * \mathbf{a}^1) \quad (T_{R841}) \text{ Bayes Theorem}$$

$$(\mathcal{N}(\mathbf{ac})^1 \div \mathcal{N}\mathbf{c}^1) \times (\mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{ac})^1) = (\mathcal{N}(\mathbf{bc})^1 \div \mathcal{N}\mathbf{c}^1) \times (\mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1) \quad (T_{T395}) \text{ Bayes Theorem}$$

Proof of (T_{R840}), (T_{R841}) & (T_{T395})

$$(\mathcal{P} * \mathbf{c}^1) * (\mathbf{ab})^1 = (\{x\} * (\mathcal{P} * (\mathbf{cb})^1) * \mathbf{a}^1) * ((\mathcal{P} * \mathbf{c}^1) * \mathbf{b}^1) \quad (T_{R837})(\text{Eqn1})$$

$$(\mathcal{P} * \mathbf{c}^1) * (\mathbf{ab})^1 = (\{x\} * (\mathcal{P} * (\mathbf{ca})^1) * \mathbf{b}^1) * ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1) \quad \text{Resymbolising (Eqn2)}$$

$$(\{x\} * (\mathcal{P} * (\mathbf{ca})^1) * \mathbf{b}^1) * ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1) = (\{x\} * (\mathcal{P} * (\mathbf{cb})^1) * \mathbf{a}^1) * ((\mathcal{P} * \mathbf{c}^1) * \mathbf{b}^1) \quad (A_{MX19})(\text{Substitution of Eqn1 into Eqn2})$$

$$((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1) \times ((\mathcal{P} * (\mathbf{ca})^1) * \mathbf{b}^1) = ((\mathcal{P} * \mathbf{c}^1) * \mathbf{b}^1) \times ((\mathcal{P} * (\mathbf{cb})^1) * \mathbf{a}^1)$$

(T_{T473})

$$(\mathcal{N}(\mathbf{ac})^1 \div \mathcal{N}\mathbf{c}^1) \times (\mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{ac})^1) = (\mathcal{N}(\mathbf{bc})^1 \div \mathcal{N}\mathbf{c}^1) \times (\mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1)$$

(T_{R801})(Definition of Numerical Probability)

$$(((\mathcal{P} * (\mathbf{bc})^1) * \mathbf{a}^1) - ((\mathcal{P} * (\mathbf{bc})^1) * \mathbf{a}^1) = \{0\}^1) \subseteq ((\mathcal{P} * \mathbf{c}^1) * \mathbf{a}^1 = (\mathcal{P} * (\mathbf{bc})^1) * \mathbf{a}^1) \quad (T_{T678})$$

Proof of (T_{T678})

$$(\{a\}^1 \div \{c\}^1 = \{b\}^1 \div \{d\}^1) \subseteq ((\{a\}^1 + \{b\}^1) \div (\{c\}^1 + \{d\}^1) = \{a\}^1 \div \{c\}^1) \quad (T_{T677})(\text{Eqn1})$$

$$\text{Let } \{a\}^1 = \mathcal{N}(\mathbf{abc})^1 \quad (\text{Eqn2})$$

$$\text{Let } \{b\}^1 = \mathcal{N}(\mathbf{abc})^1 \quad (\text{Eqn3})$$

$$\text{Let } \{c\}^1 = \mathcal{N}(\mathbf{bc})^1 \quad (\text{Eqn4})$$

$$\text{Let } \{d\}^1 = \mathcal{N}(\mathbf{bc})^1 \quad (\text{Eqn5})$$

$$(\mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1 = \mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1) \subseteq$$

$$((\mathcal{N}(\mathbf{abc})^1 + \mathcal{N}(\mathbf{abc})^1) \div (\mathcal{N}(\mathbf{bc})^1 + \mathcal{N}(\mathbf{bc})^1) = \mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1)$$

(A_{MX19})(Substitution of Eqns2,3,4&5 into Eqn1)

$$(\mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1 = \mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1) \subseteq (\mathcal{N}(\mathbf{abc} + \mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc} + \mathbf{bc})^1) = \mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1$$

(A_{S37})(Distribution of $\mathcal{N}$ over Disjoint Sets)

$$(\mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1 = \mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1) \subseteq (\mathcal{N}(\mathbf{ac}(\mathbf{b} + \mathbf{b}))^1 \div \mathcal{N}(\mathbf{c}(\mathbf{b} + \mathbf{b}))^1) = \mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1$$

(T_{M407})(Distribution of . over +)

$$(\mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1 = \mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1) \subseteq (\mathcal{N}(\mathbf{acE})^1 \div \mathcal{N}(\mathbf{cE})^1) = \mathcal{N}(\mathbf{abc})^1 \div \mathcal{N}(\mathbf{bc})^1$$

(T_{M671})(Tertium non Datur)

$$\begin{aligned}
(\mathcal{N}(abc)^1 \div \mathcal{N}(bc)^1 = \mathcal{N}(abc)^1 \div \mathcal{N}(bc)^1) &\subseteq (\mathcal{N}(ac)^1 \div \mathcal{N}(c)^1 = \mathcal{N}(abc)^1 \div \mathcal{N}(bc)^1) \\
&\quad (\mathbf{AMX11})(\text{Universal Set Conjunction Axiom}) \\
(((\mathcal{P}*(bc)^1)*a^1) = ((\mathcal{P}*(bc)^1)*a^1)) &\subseteq (((\mathcal{P}*c^1)*a^1) = ((\mathcal{P}*(bc)^1)*a^1)) \\
&\quad (\mathbf{TR801})(\text{Definition of Numerical Probability}) \\
(((\mathcal{P}*(bc)^1)*a^1) - ((\mathcal{P}*(bc)^1)*a^1) = ((\mathcal{P}*(bc)^1)*a^1) - ((\mathcal{P}*(bc)^1)*a^1)) &\subseteq (((\mathcal{P}*c^1)*a^1) = ((\mathcal{P}*(bc)^1)*a^1)) \\
&\quad (\mathbf{TR860}) \\
(((\mathcal{P}*(bc)^1)*a^1) - ((\mathcal{P}*(bc)^1)*a^1) = \{0\}^1) &\subseteq (((\mathcal{P}*c^1)*a^1) = ((\mathcal{P}*(bc)^1)*a^1)) \\
&\quad (\mathbf{TR498}) \\
(\mathcal{N}(abc)^1 \div \mathcal{N}(bc)^1 - \mathcal{N}(abc)^1 \div \mathcal{N}(bc)^1 = \{0\}^1) &\subseteq (\mathcal{N}(ac)^1 \div \mathcal{N}c^1 = (\mathcal{N}(abc)^1 \div \mathcal{N}(bc)^1)) \\
&\quad (\mathbf{TR801})(\text{Definition of Numerical Probability})
\end{aligned}$$

$$\begin{aligned}
(((\mathcal{P}*(bc)^1)*a^1) - ((\mathcal{P}*(bc)^1)*a^1) = \{0\}^1) &\subseteq ((\mathcal{P}*c^1)*(ab)^1 = ((\mathcal{P}*c^1)*a^1) \times ((\mathcal{P}*c^1)*b^1)) \quad (\mathbf{TR844}) \\
(((\mathcal{P}*(bc)^1)*a^1) - ((\mathcal{P}*(bc)^1)*a^1) = \{0\}^1) &\subseteq ((\mathcal{P}*c^1)*(ab)^1 = (\{x\}*(\mathcal{P}*c^1)*b^1)*((\mathcal{P}*c^1)*a^1)) \quad (\mathbf{TR845}) \\
((\mathcal{N}(abc)^1 \div \mathcal{N}(bc)^1) - (\mathcal{N}(abc)^1 \div \mathcal{N}(bc)^1) = \{0\}^1) &\subseteq (\mathcal{N}(abc)^1 \div \mathcal{N}c^1 = (\mathcal{N}(ac)^1 \div \mathcal{N}c^1) \times (\mathcal{N}(bc)^1 \div \mathcal{N}c^1)) \\
&\quad (\mathbf{TR846})
\end{aligned}$$

Proof of (TR845), (TR844) & (TR846)

$$\begin{aligned}
(\mathcal{P}*c^1)*(ab)^1 &= ((\mathcal{P}*c^1)*b^1) \times ((\mathcal{P}*(cb)^1)*a^1) \quad (\mathbf{TR838})(\text{Eqn1}) \\
(((\mathcal{P}*(bc)^1)*a^1) - ((\mathcal{P}*(bc)^1)*a^1) = \{0\}^1) &\subseteq (((\mathcal{P}*c^1)*a^1) = ((\mathcal{P}*(bc)^1)*a^1)) \\
&\quad (\mathbf{TR678})(\text{Eqn2}) \\
(((\mathcal{P}*(bc)^1)*a^1) - ((\mathcal{P}*(bc)^1)*a^1) = \{0\}^1) &\subseteq ((\mathcal{P}*c^1)*(ab)^1 = ((\mathcal{P}*c^1)*b^1) \times ((\mathcal{P}*c^1)*a^1)) \\
&\quad (\mathbf{AMX19})(\text{Substitution of Eqn2 into Eqn1}) \\
(((\mathcal{P}*(bc)^1)*a^1) - ((\mathcal{P}*(bc)^1)*a^1) = \{0\}^1) &\subseteq ((\mathcal{P}*c^1)*(ab)^1 = ((\mathcal{P}*c^1)*a^1) \times ((\mathcal{P}*c^1)*b^1)) \\
&\quad (\mathbf{TR483})(\text{Commutativity of Algebraic } x) \\
(((\mathcal{P}*(bc)^1)*a^1) - ((\mathcal{P}*(bc)^1)*a^1) = \{0\}^1) &\subseteq ((\mathcal{P}*c^1)*(ab)^1 = (\{x\}*(\mathcal{P}*c^1)*b^1)*((\mathcal{P}*c^1)*a^1)) \\
&\quad (\mathbf{TR473}) \\
((\mathcal{N}(abc)^1 \div \mathcal{N}(bc)^1) - (\mathcal{N}(abc)^1 \div \mathcal{N}(bc)^1) = \{0\}^1) &\subseteq (\mathcal{N}(abc)^1 \div \mathcal{N}c^1 = (\mathcal{N}(ac)^1 \div \mathcal{N}c^1) \times (\mathcal{N}(bc)^1 \div \mathcal{N}c^1)) \\
&\quad (\mathbf{TR801})(\text{Definition of Numerical Probability})
\end{aligned}$$

$$\begin{aligned}
((\mathcal{P}*c^1)*a^1 = \{n\}^1)((\mathcal{P}*d^1)*a^1 = \{n\}^1) &\subseteq ((\mathcal{P}*(c+d)^1)*a^1 = \{n\}^1) \quad (\mathbf{TR847}) \\
(\mathcal{N}(ca)^1 \div \mathcal{N}c^1 = \{n\}^1).(\mathcal{N}(da)^1 \div \mathcal{N}d^1 = \{n\}^1) &\subseteq (\mathcal{N}((c+d)a)^1 \div \mathcal{N}(c+d)^1 = \{n\}^1) \quad (\mathbf{TR848})
\end{aligned}$$

Proof of (TR847) & (TR848)

$$\begin{aligned}
\text{Let } (\mathcal{P}*c^1)*a^1 &= \{n\}^1 \\
\mathcal{N}(ca)^1 \div \mathcal{N}c^1 &= \{n\}^1 \quad (\mathbf{TR801})(\text{Definition of Numerical Probability}) \\
\mathcal{N}(ca)^1 &= \{n\}^1 \times \mathcal{N}c^1 \quad (\mathbf{TR680}) \\
(\mathcal{N}(ca)^1 = \{n\}^1 \times \mathcal{N}c^1) &= E \quad (\mathbf{AMX16})(\text{Order Axiom})(\text{Eqn1}) \\
\text{Let } (\mathcal{P}*d^1)*a^1 &= \{n\}^1 \\
\mathcal{N}(da)^1 \div \mathcal{N}d^1 &= \{n\}^1 \quad (\mathbf{TR801})(\text{Definition of Numerical Probability}) \\
\mathcal{N}(da)^1 &= \{n\}^1 \times \mathcal{N}d^1 \quad (\mathbf{TR680}) \\
(\mathcal{N}(da)^1 = \{n\}^1 \times \mathcal{N}d^1) &= E \quad (\mathbf{AMX16})(\text{Order Axiom})(\text{Eqn2}) \\
(\mathcal{N}(ca)^1 = \{n\}^1 \times \mathcal{N}c^1)(\mathcal{N}(da)^1 = \{n\}^1 \times \mathcal{N}d^1) &\subseteq (\mathcal{N}(ca)^1 + \mathcal{N}(da)^1 = \{n\}^1 \times \mathcal{N}c^1 + \{n\}^1 \times \mathcal{N}d^1) \\
&\quad (\mathbf{TR553})(\text{Arithmetical Combination})(\text{Eqn3}) \\
EE &\subseteq (\mathcal{N}(ca)^1 + \mathcal{N}(da)^1 = \{n\}^1 \times \mathcal{N}c^1 + \{n\}^1 \times \mathcal{N}d^1) \\
&\quad (\mathbf{AMX19})(\text{Substitution of Eqns1\&2 into Eqn3}) \\
E &\subseteq (\mathcal{N}(ca)^1 + \mathcal{N}(da)^1 = \{n\}^1 \times \mathcal{N}c^1 + \{n\}^1 \times \mathcal{N}d^1) \\
&\quad (\mathbf{TR667})(\text{Conjunction Tautology}) \\
E &= (\mathcal{N}(ca)^1 + \mathcal{N}(da)^1 = \{n\}^1 \times \mathcal{N}c^1 + \{n\}^1 \times \mathcal{N}d^1) \\
&\quad (\mathbf{TR275})(\text{Supersets of the Universal Set}) \\
\mathcal{N}(ca)^1 + \mathcal{N}(da)^1 &= \{n\}^1 \times \mathcal{N}c^1 + \{n\}^1 \times \mathcal{N}d^1 \quad (\mathbf{AMX16})(\text{Order Axiom}) \\
\mathcal{N}(ca)^1 + \mathcal{N}(da)^1 &= \{n\}^1 \times (\mathcal{N}c^1 + \mathcal{N}d^1) \quad (\mathbf{TR484})(\text{Distribution of } x \text{ over } +)(\text{Eqn4}) \\
\text{Let } cd &= \emptyset
\end{aligned}$$

$$\begin{aligned}
(\{+\} * \mathcal{N}d^1) * \mathcal{N}c^1 &= \mathcal{N}(c + d)^1 & (T_{R800}) (\text{Distribution of } \mathcal{N} \text{ over Disjoint Sets}) \\
\mathcal{N}c^1 + \mathcal{N}d^1 &= \mathcal{N}(c+d)^1 & (T_{T471}) (\text{Eqn5}) \\
cda &= \emptyset & (T_{M350}) \\
(\{+\} * \mathcal{N}(da)^1) * \mathcal{N}(ca)^1 &= \mathcal{N}(ca + da)^1 & (T_{R800}) (\text{Distribution of } \mathcal{N} \text{ over Disjoint Sets}) \\
\mathcal{N}(ca)^1 + \mathcal{N}(da)^1 &= \mathcal{N}(ca+da)^1 & (T_{T471}) \\
\mathcal{N}(ca)^1 + \mathcal{N}(da)^1 &= \mathcal{N}((c+d)a)^1 & (T_{M407}) (\text{Distribution of } . \text{ over } +) (\text{Eqn6}) \\
\mathcal{N}((c+d)a)^1 &= \{n\}^1 \times \mathcal{N}(c+d)^1 & (A_{MX19}) (\text{Substitution of Eqns5\&6 into Eqn4}) \\
\mathcal{N}((c+d)a)^1 \div \mathcal{N}(c+d)^1 &= \{n\}^1 & (T_{T680}) \\
(P * (c+d)^1) * a^1 &= \{n\}^1 & (T_{R801}) (\text{Definition of Numerical Probability}) \\
\text{Thus } ((P * c^1) * a^1 = \{n\}^1) & ((P * d^1) * a^1 = \{n\}^1) \subseteq ((P * (c+d)^1) * a^1 = \{n\}^1) & (A_{MX02}) (\text{Conditional Introduction}) \\
(\mathcal{N}(ca)^1 \div \mathcal{N}c^1 = \{n\}^1) \cdot (\mathcal{N}(da)^1 \div \mathcal{N}d^1 = \{n\}^1) &\subseteq (\mathcal{N}((c+d)a)^1 \div \mathcal{N}(c+d)^1 = \{n\}^1) & (T_{R801}) (\text{Definition of Numerical Probability})
\end{aligned}$$

$$\begin{aligned}
(((P * (bc)^1) * a^1) - ((P * (\underline{bc})^1) * a^1) = \{0\}^1) &= E \subseteq (((P * (bc)^1) * a^1 = \{n\}^1) \subseteq ((P * c^1) * a^1 = \{n\}^1)) & (T_{R849}) \text{ Deduction/Induction Rule} \\
((\mathcal{N}(abc)^1 \div \mathcal{N}(bc)^1 - \mathcal{N}(\underline{abc})^1 \div \mathcal{N}(\underline{bc})^1 = \{0\}^1) &= E) \subseteq ((\mathcal{N}(abc)^1 \div \mathcal{N}c^1 = \{n\}^1) \subseteq (\mathcal{N}(ac)^1 \div \mathcal{N}c^1 = \{n\}^1)) & (T_{R850}) \text{ Deduction/Induction Rule}
\end{aligned}$$

Proof of (T_{R849}) & (T_{R850})

$$\begin{aligned}
(((P * (bc)^1) * a^1) - ((P * (\underline{bc})^1) * a^1) &= \{0\}^1) \subseteq ((P * c^1) * a^1 = (P * (bc)^1) * a^1) & (T_{T678}) (\text{Eqn1}) \\
\text{Let } (P * (bc)^1) * a^1 &= \{n\}^1 & (\text{Eqn2}) \\
(((P * (bc)^1) * a^1) - ((P * (\underline{bc})^1) * a^1) &= \{0\}^1) \subseteq ((P * c^1) * a^1 = \{n\}^1) & (A_{MX19}) (\text{Substitution of Eqn2 into Eqn1}) \\
\text{Thus } (((P * (bc)^1) * a^1) - ((P * (\underline{bc})^1) * a^1) &= \{0\}^1) ((P * (bc)^1) * a^1 = \{n\}^1) \subseteq ((P * c^1) * a^1 = \{n\}^1) & (A_{MX02}) (\text{Conditional Introduction}) \\
(((P * (bc)^1) * a^1) - ((P * (\underline{bc})^1) * a^1) &= \{0\}^1) = E \subseteq (((P * (bc)^1) * a^1 = \{n\}^1) \subseteq ((P * c^1) * a^1 = \{n\}^1)) & (A_{MX35}) (\text{Exportation/Importation}) \text{ for truth sets} \\
((\mathcal{N}(abc)^1 \div \mathcal{N}(bc)^1 - \mathcal{N}(\underline{abc})^1 \div \mathcal{N}(\underline{bc})^1 &= \{0\}^1) = E) \subseteq ((\mathcal{N}(abc)^1 \div \mathcal{N}c^1 = \{n\}^1) \subseteq (\mathcal{N}(ac)^1 \div \mathcal{N}c^1 = \{n\}^1)) & (T_{R801}) (\text{Definition of Numerical Probability})
\end{aligned}$$

$$\begin{aligned}
((P * c^1) * a^1 \neq \{1\}^1) &= ((P * c^1) * a^1 \subseteq (\{<\} * \{1\}^1) (\{>\} * \{0\}^1)) & (T_{S748}) \\
((P * c^1) * a^1 \neq \{0\}^1) &= ((P * c^1) * a^1 \subseteq (\{<=\} * \{1\}^1) (\{>\} * \{0\}^1)) & (T_{S750}) \\
((P * c^1) * a^1 = \{1\}^1) &= ((P * c^1) * a^1 \not\subseteq (\{<\} * \{1\}^1) (\{>\} * \{0\}^1)) & (T_{S752}) \\
((P * c^1) * a^1 = \{0\}^1) &= ((P * c^1) * a^1 \not\subseteq (\{<=\} * \{1\}^1) (\{>\} * \{0\}^1)) & (T_{S751})
\end{aligned}$$

Example Proof of (T_{S748})

$$\begin{aligned}
\text{Let } (P * c^1) * a^1 &\neq \{1\}^1 \\
(P * c^1) * a^1 &\subseteq \{\neq\} * \{1\}^1 & (T_{T736}) (\text{Definition of a Relationship}) (\text{Eqn1}) \\
(P * c^1) * a^1 &\subseteq (\{<=\} * \{1\}^1) (\{>\} * \{0\}^1) & (T_{T659}) (\text{Eqn2}) \\
(P * c^1) * a^1 &\subseteq (\{\neq\} * \{1\}^1) (\{<=\} * \{1\}^1) (\{>\} * \{0\}^1) & (T_{M132}) (\text{Composition of Eqns1\&2}) \\
(P * c^1) * a^1 &\subseteq (\{\neq\} * \{1\}^1) (\{=\} + \{<\} * \{1\}^1) (\{>\} * \{0\}^1) & (D_{A141}) (\text{Disjunction with } \{=\}) \\
(P * c^1) * a^1 &\subseteq (\{\neq\} \cdot \{=\} + \{<\} * \{1\}^1) (\{>\} * \{0\}^1) & (T_{R239}) (\text{Distribution of } . \text{ over } * \text{ for Entities}) \\
(P * c^1) * a^1 &\subseteq ((\{\neq\} \cdot \{=\} + \{\neq\} \cdot \{<\} * \{1\}^1) (\{>\} * \{0\}^1) & (T_{M407}) (\text{Distribution of } . \text{ over } +) \\
(P * c^1) * a^1 &\subseteq ((\emptyset + \{\neq\} \cdot \{<\} * \{1\}^1) (\{>\} * \{0\}^1) & (T_{M621}) (\text{Complement Conjunction}) \\
(P * c^1) * a^1 &\subseteq ((\{\neq\} \cdot \{<\} * \{1\}^1) (\{>\} * \{0\}^1) & (T_{M668}) (\text{Null Set Disjunction}) \\
(P * c^1) * a^1 &\subseteq (\{<\} * \{1\}^1) (\{>\} * \{0\}^1) & (T_{S685})
\end{aligned}$$

$$((P * c^1) * a^1 \not\leq (P * c^1) * b^1) = ((P * c^1) * a^1 > (P * c^1) * b^1) \quad (T_{T323})$$

$$((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \not\leq (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) = ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 < (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) \quad (\mathcal{T}_{7318})$$

Example Proof of ($\mathcal{T}_{7323}$)

$$\begin{aligned} & \forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} \subseteq \{>\} + \{<\} + \{=\} & (\mathcal{T}_{454} \mathcal{N}) \\ & (\forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} = \mathcal{E}) \subseteq (\mathcal{E} \subseteq \{>\} + \{<\} + \{=\}) & (\mathcal{A}_{MX35})(\text{Exportation/Importation}) \text{ for truth sets} \\ & (\forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} = \mathcal{E}) \subseteq (\mathcal{E} = \{>\} + \{<\} + \{=\}) & (\mathcal{T}_{M275})(\text{Supersets of the Universal Set}) \\ & (\forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} = \mathcal{E}) \subseteq (\mathcal{E}. \{>\} = \{>\}. (\{>\} + \{<\} + \{=\})) & (\mathcal{T}_{M029})(\text{Deduction}) \\ & (\forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} = \mathcal{E}) \subseteq (\{>\} = \{>\}. (\{>\} + \{<\} + \{=\})) & (\mathcal{A}_{MX11})(\text{Universal Set Conjunction Axiom}) \\ & (\forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} = \mathcal{E}) \subseteq (\{>\} = \{>\} \{>\} + \{>\} \{<\} + \{>\} \{=\}) & (\mathcal{T}_{M407})(\text{Distribution of } . \text{ over } +) \\ & (\forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} = \mathcal{E}) \subseteq (\{>\} = \{>\} \{>\} + \{<\} + \{>\} \{=\}) & (\mathcal{T}_{S692})(\text{Eqn1}) \\ & \{>\} \subseteq \{>\} & (\mathcal{T}_{S642}) \\ & \{=\} \subseteq \{>\} & (\mathcal{T}_{M012})(\text{Transposition}) \\ & \{=\} = \{>\} \{=\} & (\mathcal{A}_{MX44})(\subseteq \text{Equality Form})(\text{Eqn2}) \\ & (\forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} = \mathcal{E}) \subseteq (\{>\} = \{>\} \{>\} + \{<\} + \{=\}) & (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ & (\forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} = \mathcal{E}) \subseteq (\{>\} = \emptyset + \{<\} + \{=\}) & (\mathcal{T}_{M621})(\text{Complement Conjunction}) \\ & (\forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} = \mathcal{E}) \subseteq (\{>\} = \{<\} + \{=\}) & (\mathcal{T}_{M668})(\text{Null Set Disjunction}) \\ & (\forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} = \mathcal{E}) \subseteq (\{>\} = \{\leq\}) & (\mathcal{T}_{887})(\text{Definition of } \leq) \\ & (\forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} = \mathcal{E}) \subseteq (\{>\} = \{\neq\}) & (\mathcal{T}_{M008})(\text{Inverse of an Equality})(\text{Eqn3}) \\ & \text{Let } \{a\}^1 \subseteq \mathcal{N} \mathcal{E} \\ & \forall \mathcal{S} \{a\}^1 \subseteq \forall \mathcal{S} \mathcal{N} \mathcal{E} & (\mathcal{T}_{R196})(\text{Relational Deduction})(\text{Eqn4}) \\ & \text{Let } \{b\}^1 \subseteq \mathcal{N} \mathcal{E} \\ & \forall \mathcal{Q} \{b\}^1 \subseteq \forall \mathcal{Q} \mathcal{N} \mathcal{E} & (\mathcal{T}_{R196})(\text{Relational Deduction})(\text{Eqn5}) \\ & \forall \mathcal{S} \{a\}^1. \forall \mathcal{Q} \{b\}^1 \subseteq \forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} & (\mathcal{T}_{M669})(\text{Combination of Eqns4\&5}) \\ & (\forall \mathcal{S} \{a\}^1. \forall \mathcal{Q} \{b\}^1 \subseteq \forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E}) = \mathcal{E} & (\mathcal{A}_{MX16})(\text{Order Axiom}) \\ & (\forall \mathcal{S} \{a\}^1. \forall \mathcal{Q} \{b\}^1 = \mathcal{E}) \subseteq (\forall \mathcal{S} \mathcal{N} \mathcal{E}. \forall \mathcal{Q} \mathcal{N} \mathcal{E} = \mathcal{E}) & (\mathcal{T}_{M094})(\text{Probabilistic Rule})(\text{Eqn6}) \\ & (\forall \mathcal{S} \{a\}^1. \forall \mathcal{Q} \{b\}^1 \subseteq \mathcal{E}) \subseteq (\{>\} = \{\neq\}) & (\mathcal{T}_{M139})(\text{Hypothetical Syllogism Eqns6\&3}) \\ & \mathcal{E} \subseteq (\{>\} = \{\neq\}) & (\mathcal{T}_{M001})(\text{Universality of } (\mathcal{T}_{M744}) \text{ Universal Set Theorem}) \\ & \mathcal{E} = (\{>\} = \{\neq\}) & (\mathcal{T}_{M275})(\text{Supersets of the Universal Set}) \\ & \{>\} = \{\neq\} & (\mathcal{A}_{MX16})(\text{Order Axiom}) \\ & \forall \mathcal{S} \{a\}^1 \{>\} \forall \mathcal{Q} \{b\}^1 = \forall \mathcal{S} \{a\}^1 \{\neq\} \forall \mathcal{Q} \{b\}^1 & (\mathcal{T}_{M029})(\text{Deduction}) \\ & (\forall \mathcal{S} \{a\}^1 \{>\} \forall \mathcal{Q} \{b\}^1 = \forall \mathcal{S} \{a\}^1 \{\neq\} \forall \mathcal{Q} \{b\}^1) = \mathcal{E} & (\mathcal{A}_{MX16})(\text{Order Axiom}) \\ & ((\forall \mathcal{S} \{a\}^1 \{>\} \forall \mathcal{Q} \{b\}^1) \neq \emptyset) = ((\forall \mathcal{S} \{a\}^1 \{\neq\} \forall \mathcal{Q} \{b\}^1) \neq \emptyset) & (\mathcal{T}_{M096})(\text{Probabilistic Rule}) \\ & (\{a\}^1 \subseteq \{>\} * \{b\}^1) = (\{a\}^1 \subseteq \{\neq\} * \{b\}^1) & (\mathcal{T}_{R608}) \\ & (\{a\}^1 > \{b\}^1) = (\{a\}^1 \neq \{b\}^1) & (\mathcal{T}_{890})(\text{Definition of an Ordering Equation})(\text{Eqn7}) \\ & \text{Let } \{a\}^1 = (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 & (\text{Eqn8}) \\ & \text{Let } \{b\}^1 = (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 & (\text{Eqn9}) \\ & ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) = ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \neq (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) & (\mathcal{A}_{MX19})(\text{Substitution of Eqns8\&9 into Eqn7}) \end{aligned}$$

$$(\mathbf{a} = \mathbf{b}) \subseteq ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 = (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) \quad (\mathcal{T}_{7381})$$

Proof of ($\mathcal{T}_{7381}$)

$$\begin{aligned} & \text{Let } \mathbf{a} = \mathbf{b} \\ & \mathbf{ca} = \mathbf{cb} & (\mathcal{T}_{M029})(\text{Deduction}) \\ & \mathcal{N}(\mathbf{ca})^1 = \mathcal{N}(\mathbf{cb})^1 & (\mathcal{T}_{545} \mathcal{N}) \\ & (\{\div\} * \mathcal{N} \mathcal{C}^1) * \mathcal{N}(\mathbf{ca})^1 = (\{\div\} * \mathcal{N} \mathcal{C}^1) * \mathcal{N}(\mathbf{cb})^1 & (\mathcal{T}_{R196})(\text{Relational Deduction}) \\ & (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 = (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 & (\mathcal{T}_{R801})(\text{Definition of Numerical Probability}) \\ & \text{Thus } (\mathbf{a} = \mathbf{b}) \subseteq ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 = (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) & (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \end{aligned}$$

$$(\mathcal{P} * \mathcal{C}^1) * (\mathbf{ca})^1 = (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \quad (\mathcal{T}_{S747})$$

Proof of ($\mathcal{T}_{S747}$)

$$\begin{aligned} & (\mathcal{P} * \mathcal{C}^1) * (\mathbf{ca})^1 = (\mathcal{P} * \mathcal{C}^1) * (\mathbf{ca})^1 & (\mathcal{D}_{M00})(\text{Definition of } =) \\ & (\mathcal{P} * \mathcal{C}^1) * (\mathbf{ca})^1 = (\{\div\} * \mathcal{N} \mathcal{C}^1) * \mathcal{N}(\mathbf{cca})^1 & (\mathcal{T}_{R801})(\text{Definition of Numerical Probability}) \end{aligned}$$

$$\begin{aligned} (\mathcal{P}^*c^1)*(ca)^1 &= (\{\div\}*\mathcal{N}c^1)*\mathcal{N}(ca)^1 \\ (\mathcal{P}^*c^1)*(ca)^1 &= (\mathcal{P}^*c^1)*a^1 \end{aligned}$$

(T_{M667})(Conjunction Tautology)(T_{R801})(Definition of Numerical Probability)

$$(ca = cb) \subseteq ((\mathcal{P}^*c^1)*a^1 = (\mathcal{P}^*c^1)*b^1) \quad (T_{S739})$$

$$(ca \subset cb) \subseteq ((\mathcal{P}^*c^1)*a^1 < (\mathcal{P}^*c^1)*b^1) \quad (T_{T379})$$

$$(ca \supset cb) \subseteq ((\mathcal{P}^*c^1)*a^1 > (\mathcal{P}^*c^1)*b^1) \quad (T_{S741})$$

$$(ca \subseteq cb) \subseteq ((\mathcal{P}^*c^1)*a^1 \leq (\mathcal{P}^*c^1)*b^1) \quad (T_{S742})$$

$$(ca \supseteq cb) \subseteq ((\mathcal{P}^*c^1)*a^1 \geq (\mathcal{P}^*c^1)*b^1) \quad (T_{S744})$$

Example Proof of (T_{T379})

$$(((\mathcal{X}E)^1 \subseteq D_0) = E) \subseteq ((a^1 \subseteq \{\subset\}b^1) \subseteq (\mathcal{X}a^1 \subseteq \{\subset\}\mathcal{X}b^1))$$

(A_{S33})(Axiom of Physical Proper Sub-entities for Discrete Dimensions)(Eqn1)

$$\text{Let } \mathcal{X} = (\mathcal{P}^*c^1)$$

(Eqn2)

$$\text{Let } a = ca$$

(Eqn3)

$$\text{Let } b = cb$$

(Eqn4)

$$(((\mathcal{P}^*c^1)*E)^1 \subseteq D_0) = E \subseteq (((ca)^1 \subseteq \{\subset\}(cb)^1) \subseteq ((\mathcal{P}^*c^1)*(ca)^1 \subseteq \{\subset\}(\mathcal{P}^*c^1)*(cb)^1))$$

(A_{MX19})(Substitution of Eqns2,3&4 into Eqn1)(Eqn5)

$$(((\mathcal{P}^*c^1)*E)^1 \subseteq D_0) = E \subseteq (((ca)^1 \subseteq \{\subset\}(cb)^1) \subseteq ((\mathcal{P}^*c^1)*a^1 \subseteq \{\subset\}(\mathcal{P}^*c^1)*b^1))$$

(T_{S747})

$$((\mathcal{P}^*c^1)*E)^1 \subseteq D_0$$

(A_{S50})(Probability is a Discrete Dimension)

$$(((\mathcal{P}^*c^1)*E)^1 \subseteq D_0) = E$$

(A_{MX16})(Order Axiom)(Eqn6)

$$(E = E) \subseteq (((ca)^1 \subseteq \{\subset\}(cb)^1) \subseteq ((\mathcal{P}^*c^1)*a^1 \subseteq \{\subset\}(\mathcal{P}^*c^1)*b^1))$$

(A_{MX19})(Substitution of Eqn6 into Eqn5)

$$E \subseteq (((ca)^1 \subseteq \{\subset\}(cb)^1) \subseteq ((\mathcal{P}^*c^1)*a^1 \subseteq \{\subset\}(\mathcal{P}^*c^1)*b^1))$$

(T_{M342})

$$E = (((ca)^1 \subseteq \{\subset\}(cb)^1) \subseteq ((\mathcal{P}^*c^1)*a^1 \subseteq \{\subset\}(\mathcal{P}^*c^1)*b^1))$$

(T_{M275})(Supersets of the Universal Set)

$$((ca)^1 \subseteq \{\subset\}(cb)^1) \subseteq ((\mathcal{P}^*c^1)*a^1 \subseteq \{\subset\}(\mathcal{P}^*c^1)*b^1)$$

(A_{MX16})(Order Axiom)

$$((ca)^1 \subseteq \{\subset\}(cb)^1) \subseteq ((\mathcal{P}^*c^1)*a^1 < (\mathcal{P}^*c^1)*b^1)$$

(T_{T890})(Definition of an Ordering Equation)(Eqn7)

$$((ca) \subset (cb)) = ((ca)^1 \subseteq \{\subset\}*(cb)^1)$$

(T_{R653})(Definition of a Relationship)(Eqn8)

$$((ca) \subset (cb)) \subseteq ((\mathcal{P}^*c^1)*a^1 < (\mathcal{P}^*c^1)*b^1)$$

(A_{MX19})(Substitution of Eqn8 into Eqn7)

$$((\mathcal{P}^*c^1)*a^1 > (\mathcal{P}^*c^1)*d^1).((\mathcal{P}^*c^1)*d^1 > (\mathcal{P}^*c^1)*b^1) \subseteq ((\mathcal{P}^*c^1)*a^1 > (\mathcal{P}^*c^1)*b^1) \quad (T_{T305})$$

Proof of (T_{T305})

$$(a \subseteq \{\subset\}^*c)(c \subseteq \{\subset\}^*b) \subseteq (a \subseteq \{\subset\}^*b) \quad (T_{A319})(\text{Transitive Asymmetrical Syllogism})(\text{Eqn1})$$

$$\text{Let } \{\subset\} = \{\supset\}$$

(Eqn2)

$$\text{Let } a = (\mathcal{P}^*c^1)*a^1$$

(Eqn3)

$$\text{Let } c = (\mathcal{P}^*c^1)*d^1$$

(Eqn4)

$$\text{Let } b = (\mathcal{P}^*c^1)*b^1$$

(Eqn5)

$$((\mathcal{P}^*c^1)*a^1 \subseteq \{\supset\}^*((\mathcal{P}^*c^1)*d^1)).((\mathcal{P}^*c^1)*d^1 \subseteq \{\supset\}^*((\mathcal{P}^*c^1)*b^1)) \subseteq ((\mathcal{P}^*c^1)*a^1 \subseteq \{\supset\}^*((\mathcal{P}^*c^1)*b^1))$$

(A_{MX19})(Substitution of Eqns2,3,4&5 into Eqn1)

$$((\mathcal{P}^*c^1)*a^1 > (\mathcal{P}^*c^1)*d^1)((\mathcal{P}^*c^1)*d^1 > (\mathcal{P}^*c^1)*b^1) \subseteq ((\mathcal{P}^*c^1)*a^1 > (\mathcal{P}^*c^1)*b^1)$$

(T_{T890})(Definition of an Ordering Equation)

$$((\mathcal{P}^*c^1)*a^1 \geq (\mathcal{P}^*c^1)*d^1).((\mathcal{P}^*c^1)*d^1 > (\mathcal{P}^*c^1)*b^1) \subseteq ((\mathcal{P}^*c^1)*a^1 > (\mathcal{P}^*c^1)*b^1) \quad (T_{T299})$$

$$((\mathcal{P}^*c^1)*a^1 > (\mathcal{P}^*c^1)*d^1)((\mathcal{P}^*c^1)*d^1 \geq (\mathcal{P}^*c^1)*b^1) \subseteq ((\mathcal{P}^*c^1)*a^1 > (\mathcal{P}^*c^1)*b^1) \quad (T_{T296})$$

Example Proof of (T_{T299})

$$\text{Let } ((\mathcal{P}^*c^1)*a^1 \geq (\mathcal{P}^*c^1)*d^1).((\mathcal{P}^*c^1)*d^1 > (\mathcal{P}^*c^1)*b^1)$$

$$(((\mathcal{P}^*c^1)*a^1 > (\mathcal{P}^*c^1)*d^1) + ((\mathcal{P}^*c^1)*a^1 = (\mathcal{P}^*c^1)*d^1)).((\mathcal{P}^*c^1)*d^1 > (\mathcal{P}^*c^1)*b^1)$$

$$\begin{aligned}
& ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1).((\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) + ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 = (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1).((\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) \\
& \quad (T_{788})(\text{Definition of } \geq) \\
& ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) + ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 = (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1).((\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) \\
& \quad (T_{M407})(\text{Distribution of } . \text{ over } +) \\
& ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) \\
& \quad (T_{7305}) \\
& ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) + ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) \\
& \quad (A_{MX19})(\text{Substitution}) \\
& ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) \\
& \quad (A_{MX01})(\text{Disjunction Tautology}) \\
& \text{Thus } ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \geq (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1).((\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) \subseteq ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1) \\
& \quad (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$((\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1) \subseteq ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1) \quad (T_{R773})$$

Proof of (T_{R773})

$$\begin{aligned}
& \text{Let } (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 \\
& (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 \subseteq \{>\} * (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 \quad (T_{7962}) \text{ since } (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 \neq \emptyset \text{ and } (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 \neq \emptyset \\
& (\{x\} * (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1) * (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 \subseteq (\{x\} * (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1) * \{>\} * (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 \\
& \quad (T_{R196})(\text{Relational Deduction}) \\
& (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \subseteq \{>\} * (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \\
& (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \subseteq \{>\} * ((\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1) \quad (T_{7706}) \\
& (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \quad (T_{7471}) \\
& \quad \text{since } ((\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1) = ((\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1)^1 \\
& \quad \text{and } ((\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1) = ((\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1)^1 \\
& (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1 \quad (T_{7483})(\text{Commutativity of Algebraic } \times) \\
& \text{Thus } ((\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1) \subseteq ((\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{b}^1 > (\mathcal{P} * \mathcal{C}^1) * \mathbf{a}^1 \times (\mathcal{P} * \mathcal{C}^1) * \mathbf{d}^1) \\
& \quad (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

18.3 Proofs of Theorems of Epistemic Probability

$$\{p\} = (J^1 \subseteq H_S * \{\{p\}\}) \quad (T_{7279})$$

Proof of (T₇₂₇₉)

$$\begin{aligned}
& \{p\} = (\{\{p\}\} \subseteq {}^V H_S * J^1) \quad (D_{R02})(\text{Definition of Knowing that}) \\
& \{p\} = (J^1 \subseteq H_S * \{\{p\}\}) \quad (T_{R519})(\text{Active/Passive Theorem})
\end{aligned}$$

$$\{p\} = (J \subseteq H_S * \{\{p\}\}) \quad (T_{7284})$$

Proof of (T₇₂₈₄)

$$\begin{aligned}
& \{p\} = (\{\{p\}\} \subseteq {}^V H_S * J^1) \quad (D_{R02})(\text{Definition of Knowing that}) \\
& \{p\} = (\{\{p\}\} \subseteq {}^V H_S \diamond J) \quad (A_{S15})(\text{Axiom of Alternative Mental Models}) \\
& \{p\} = (J \subseteq H_S * \{\{p\}\}) \quad (T_{R521})(\text{Active/Passive Theorem})
\end{aligned}$$

$$(\{p\} = \emptyset) = (\sim\{p\} \subseteq {}^V H_S * J^1) \quad (T_{S999})$$

$$(\{p\} = \emptyset) = (J^1 \subseteq H_S * \sim\{p\}) \quad (T_{7265})$$

Proof of (T_{S999}) & (T₇₂₆₅)

$$\begin{aligned}
& \{p\} = (\{\{p\}\} \subseteq {}^V H_S * J^1) \quad (D_{R02})(\text{Definition of Knowing that}) \\
& (\{p\} = E) = (\{\{p\}\} \subseteq {}^V H_S * J^1) \quad (A_{MX16})(\text{Order Axiom})(\text{Eqn1}) \\
& \text{Let } \{p\} = \sim\{p\} \quad (\text{Eqn2}) \\
& (\sim\{p\} = E) = (\sim\{p\} \subseteq {}^V H_S * J^1) \quad (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
& (\{p\} = \sim E) = (\sim\{p\} \subseteq {}^V H_S * J^1) \quad (T_{M008})(\text{Inverse of an Equality}) \\
& (\{p\} = \emptyset) = (\sim\{p\} \subseteq {}^V H_S * J^1) \quad (D_{M02})(\text{Definition of } \emptyset)
\end{aligned}$$

$$(\{p\} = \emptyset) = (J^1 \subseteq H_s * \sim\{p\}) \quad (T_{R519})(\text{Active/Passive Theorem})$$

$$(\{p\} = \emptyset) = (J \subseteq H_s * \sim\{p\}) \quad (T_{S753})$$

Proof of (T_{S753})

$$\begin{aligned} \{p\} &= (J \subseteq H_s * \sim\{p\}) & (T_{T284}) \\ (\{p\} = E) &= (J \subseteq H_s * \sim\{p\}) & (A_{MX16})(\text{Order Axiom})(\text{Eqn1}) \\ \text{Let } \{p\} &= \sim\{p\} & (\text{Eqn2}) \\ (\sim\{p\} = E) &= (J \subseteq H_s * \sim\{p\}) & (A_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\ (\{p\} = \sim E) &= (J \subseteq H_s * \sim\{p\}) & (T_{M008})(\text{Inverse of an Equality}) \\ (\{p\} = \emptyset) &= (J \subseteq H_s * \sim\{p\}) & (D_{M02})(\text{Definition of } \emptyset) \end{aligned}$$

$$(\{p\} = E) = ((P * J^1) * (H_s * \sim\{p\})^1 = \{1\}^1) \quad (T_{S754})$$

Proof of (T_{S754})

$$\begin{aligned} (c \subseteq a) &= ((P * c^1) * a^1 = \{1\}^1) & (T_{R767})(\text{Eqn1}) \\ \text{Let } a &= J^1 & (\text{Eqn2}) \\ \text{Let } c &= H_s * \sim\{p\} & (\text{Eqn3}) \\ (J^1 \subseteq H_s * \sim\{p\}) &= ((P * J^1) * (\vee H_s * \sim\{p\})^1 = \{1\}^1) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\ \{p\} &= ((P * J^1) * (\vee H_s * \sim\{p\})^1 = \{1\}^1) & (T_{T279}) \\ (\{p\} = E) &= ((P * J^1) * (\vee H_s * \sim\{p\})^1 = \{1\}^1) & (A_{MX16})(\text{Order Axiom}) \end{aligned}$$

$$(\{p\} = \emptyset) = ((P * J^1) * (H_s * \sim\{p\})^1 = \{0\}^1) \quad (T_{S757})$$

Proof of (T_{S757})

$$\begin{aligned} (c \subseteq a) &= ((P * c^1) * a^1 = \{1\}^1) & (T_{R767})(\text{Eqn1}) \\ \text{Let } c &= J^1 & (\text{Eqn2}) \\ \text{Let } a &= H_s * \sim\{p\} & (\text{Eqn3}) \\ (J^1 \subseteq H_s * \sim\{p\}) &= ((P * J^1) * (H_s * \sim\{p\})^1 = \{1\}^1) & (A_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\ (\{p\} = \emptyset) &= ((P * J^1) * (H_s * \sim\{p\})^1 = \{1\}^1) & (T_{T265})(\text{Eqn5}) \\ (P * c^1) * (a)^1 &= \{1\}^1 - (P * c^1) * (a)^1 & (T_{R813})(\text{Eqn6}) \\ \text{Let } c &= J^1 & (\text{Eqn7}) \\ \text{Let } a &= H_s * \sim\{p\} & (\text{Eqn8}) \\ a &= H_s * \sim\{p\} & (T_{T259})(\text{Eqn9}) \\ (P * J^1) * (H_s * \sim\{p\})^1 &= \{1\}^1 - (P * J^1) * (H_s * \sim\{p\})^1 & (A_{MX19})(\text{Substitution of Eqns7,8\&9 into Eqn1})(\text{Eqn10}) \\ (\{p\} = \emptyset) &= (\{1\}^1 - (P * J^1) * (H_s * \sim\{p\})^1 = \{1\}^1) & (A_{MX19})(\text{Substitution of Eqn10 into Eqn5}) \\ (\{p\} = \emptyset) &= (\{1\}^1 - \{1\}^1 = (P * J^1) * (H_s * \sim\{p\})^1) & (T_{T681}) \\ (\{p\} = \emptyset) &= (\{0\}^1 = (P * J^1) * (H_s * \sim\{p\})^1) & (T_{T498}) \\ (\{p\} = \emptyset) &= ((P * J^1) * (H_s * \sim\{p\})^1 = \{0\}^1) & (T_{M197})(\text{Commutativity of } =) \end{aligned}$$

$$(\{p\} \neq E) = ((P * J^1) * (H_s * \sim\{p\})^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) \quad (T_{S755})$$

$$(\{p\} \neq \emptyset) = ((P * J^1) * (H_s * \sim\{p\})^1 \subseteq (\{<=\} * \{1\}^1)(\{>\} * \{0\}^1)) \quad (T_{S756})$$

Example Proof of (T_{S755})

$$\begin{aligned} (\{p\} = E) &= ((P * J^1) * (H_s * \sim\{p\})^1 = \{1\}^1) & (T_{S754}) \\ (\{p\} \neq E) &= ((P * J^1) * (H_s * \sim\{p\})^1 \neq \{1\}^1) & (T_{M008})(\text{Inverse of an Equality}) \\ (\{p\} \neq E) &= ((P * J^1) * (H_s * \sim\{p\})^1 \subseteq (\{<\} * \{1\}^1)(\{>\} * \{0\}^1)) & (T_{S748}) \end{aligned}$$

$$((P * J^1) * (H_s * \sim\{p\})^1 \neq \{1\}^1) \cdot ((P * J^1) * (H_s * \sim\{p\})^1 = \{1\}^1) \subseteq ((P * J^1) * (H_s * \sim\{p\})^1 \neq \{0\}^1)$$

(U_{R001}) Induction Theorem (Probability Form)

Proof of (U_{R001})

$$\begin{aligned}
((\mathbf{ac} \neq \emptyset)(\mathbf{ac} \subseteq \mathbf{b}) = \mathbf{E}) &\subseteq ((\mathbf{a} \subseteq \mathbf{b}) \neq \emptyset) & (\mathbf{U}_{M017})(\text{Induction Theorem}) \\
((\mathbf{ac} \neq \emptyset) = \mathbf{E}).((\mathbf{ac} \subseteq \mathbf{b}) = \mathbf{E}) &\subseteq ((\mathbf{a} \subseteq \mathbf{b}) \neq \emptyset) & (\mathbf{T}_{M108})(\text{Conjunction Introduction})(\text{Eqn1}) \\
((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \neq \emptyset))^1 = \{1\}^1. &((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}))^1 = \{1\}^1 \subseteq ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{a} \subseteq \mathbf{b}))^1 \neq \{0\}^1 & (\mathbf{T}_{S754}), (\mathbf{T}_{S757}) \& (\mathbf{T}_{M008})
\end{aligned}$$

$$((\mathbf{a} \subseteq \mathbf{b}) = \mathbf{E}) = ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{a} \subseteq \mathbf{b}))^1 = \{1\}^1 \quad (\mathbf{T}_{T658})$$

$$((\mathbf{a} \subseteq \mathbf{b}) = \emptyset) = ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{a} \subseteq \mathbf{b}))^1 = \{0\}^1 \quad (\mathbf{T}_{T648})$$

Example Proof of ($\mathbf{T}_{T658}$)

$$\begin{aligned}
(\{p\} = \mathbf{E}) &= ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}))^1 = \{1\}^1 & (\mathbf{T}_{S754})(\text{Eqn1}) \\
\text{Let } \{p\} &= (\mathbf{a} \subseteq \mathbf{b}) & (\text{Eqn2}) \\
((\mathbf{a} \subseteq \mathbf{b}) = \mathbf{E}) &= ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{a} \subseteq \mathbf{b}))^1 = \{1\}^1 & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$((\mathbf{a} \subseteq \mathbf{b}) \neq \mathbf{E}) \subseteq ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{a} \subseteq \mathbf{b}))^1 < \{1\}^1 \quad (\mathbf{T}_{T653})$$

$$((\mathbf{a} \subseteq \mathbf{b}) \neq \emptyset) \subseteq ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{a} \subseteq \mathbf{b}))^1 > \{0\}^1 \quad (\mathbf{T}_{T651})$$

Example Proof of ($\mathbf{T}_{T653}$)

$$\begin{aligned}
(\{p\} \neq \mathbf{E}) &= ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}))^1 \subseteq (\{<\}*\{1\}^1)(\{>\}*\{0\}^1) & (\mathbf{T}_{S755}) \\
(\{p\} \neq \mathbf{E}) &\subseteq ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}))^1 \subseteq (\{<\}*\{1\}^1) & (\mathbf{T}_{M073})(\text{Single Premise Syllogism}) \\
(\{p\} \neq \mathbf{E}) &\subseteq ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}))^1 < \{1\}^1 & (\mathbf{T}_{T963})(\text{Eqn1}) \text{ since } (\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 \neq \emptyset \text{ and } \{1\}^1 \neq \emptyset \\
\text{Let } \{p\} &= (\mathbf{a} \subseteq \mathbf{b}) & (\text{Eqn2}) \\
((\mathbf{a} \subseteq \mathbf{b}) \neq \mathbf{E}) &\subseteq ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{a} \subseteq \mathbf{b}))^1 < \{1\}^1 & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1})
\end{aligned}$$

$$(\{p\} = \{q\}) \subseteq ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}))^1 = (\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 \quad (\mathbf{T}_{T290})$$

Proof of ($\mathbf{T}_{T290}$)

$$\begin{aligned}
\text{Let } \{p\} &= \{q\} & (\text{Eqn1}) \\
(\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 &= (\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 & (\mathbf{D}_{M00})(\text{Definition of } =) (\text{Eqn2}) \\
(\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 &= (\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 & (\mathbf{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
\text{Thus } (\{p\} = \{q\}) &\subseteq ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}))^1 = (\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 & (\mathbf{A}_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$(\mathbf{J} \subseteq \mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}) = ((\{<\}*\mathbf{J}^1)*\mathbf{J}^1)(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 = \{1\}^1 \quad (\mathbf{T}_{R759b})$$

$$(\mathbf{J} \not\subseteq \mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}) \subseteq ((\{<\}*\mathbf{J}^1)*\mathbf{J}^1)(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 \subseteq \{<\}*\{1\}^1 \quad (\mathbf{T}_{R760b}) \quad \text{or } \neq \{1\}^1 \quad \text{or } < \{1\}^1$$

$$(\mathbf{J} \not\subseteq \mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}) \subseteq ((\{<\}*\mathbf{J}^1)*\mathbf{J}^1)(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 \subseteq \{>\}*\{0\}^1 \quad (\mathbf{T}_{R762b}) \quad \text{or } \neq \{0\}^1 \quad \text{or } > \{0\}^1$$

$$(\mathbf{J} \subseteq \mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}) = ((\{<\}*\mathbf{J}^1)*\mathbf{J}^1)(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 = \{0\}^1 \quad (\mathbf{T}_{R761b})$$

Example Proof of ($\mathbf{T}_{R759b}$)

$$\begin{aligned}
(\mathbf{c} \subseteq \mathbf{a}) &= ((\{<\}*\mathbf{J}^1)*\mathbf{J}^1)(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 = \{1\}^1 & (\mathbf{T}_{R759})(\text{Eqn1}) \\
\text{Let } \mathbf{c} &= \mathbf{J} & (\text{Eqn2}) \\
\text{Let } \mathbf{a} &= \mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b} & (\text{Eqn3}) \\
(\mathbf{J} \subseteq \mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}) &= ((\{<\}*\mathbf{J}^1)*\mathbf{J}^1)(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 = \{1\}^1 & (\mathbf{A}_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1})
\end{aligned}$$

$$(((\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})^1 \neq \emptyset) = \mathbf{E}) \subseteq ((\{p\} \subseteq \{q\}) \subseteq ((\mathbf{P}^*\mathbf{J}^1)(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}))^1 \leq (\mathbf{P}^*\mathbf{J}^1)(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}))^1) \quad (\mathbf{T}_{T397})$$

Proof of ($\mathbf{T}_{T397}$)

$$\begin{aligned}
\text{Let } \{p\} &\subseteq \{q\} \\
(\{p\} \subseteq \{q\}) &\subseteq ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}))^1 \subseteq ((\mathbf{P}^*\mathbf{J}^1)*(\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}))^1 & (\mathbf{D}_{R02})(\text{Definition of Knowing that}) \\
(\mathbf{J}^1 \subseteq \mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}) &\subseteq (\mathbf{J}^1 \subseteq \mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}) & (\mathbf{T}_{R519})(\text{Active/Passive Theorem}) \\
\text{Thus } (\{p\} \subseteq \{q\}) &= ((\mathbf{J}^1 \subseteq \mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}) \subseteq (\mathbf{J}^1 \subseteq \mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b})) & (\mathbf{A}_{MX02})(\text{Conditional Introduction}) \\
&\text{Since this proof is reversible} \\
(\{p\} \subseteq \{q\}) &= ((\mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}) \subseteq \mathbf{H}_s*\mathbf{ac} \subseteq \mathbf{b}) = \mathbf{E} & (\mathbf{T}_{M750})(\text{Deductive Rule}) \text{ since } \mathbf{J}^1 \neq \emptyset
\end{aligned}$$

$$\begin{aligned}
&(\{p\} \subseteq \{q\}) = (Hs^* \{p\} \subseteq Hs^* \{q\}) & (A_{MX16})(\text{Order Axiom}) \\
&\text{Let } (Hs^* \{q\})^1 \neq \emptyset \\
&(\{p\} \subseteq \{q\}) = ((Hs^* \{p\})^1 \subseteq \{\subseteq\}(Hs^* \{q\})^1) & (T_{S412})(\text{Eqn1}) \\
&(a^1 \subseteq \{\subseteq\}b^1) \subseteq ((Pc^1)a^1 \subseteq \{\subseteq\}(Pc^1)b^1) & (T_{T615P})(\text{Eqn2}) \\
&\text{Let } a^1 = (Hs^* \{p\})^1 & (\text{Eqn3}) \\
&\text{Let } b^1 = (Hs^* \{q\})^1 & (\text{Eqn4}) \\
&\text{Let } c^1 = J^1 & (\text{Eqn5}) \\
&((Hs^* \{p\})^1 \subseteq \{\subseteq\}(Hs^* \{q\})^1) \subseteq ((PJ^1)(Hs^* \{p\})^1 \subseteq \{\subseteq\}(PJ^1)(Hs^* \{q\})^1) \\
& & (A_{MX19})(\text{Substitution of Eqns3,4\&5 into Eqn2}) \\
& & (\text{Eqn6}) \\
&(\{p\} \subseteq \{q\}) \subseteq ((PJ^1)(Hs^* \{p\})^1 \subseteq \{\subseteq\}(PJ^1)(Hs^* \{q\})^1) & (A_{MX19})(\text{Substitution of Eqn1 into Eqn6}) \\
&(\{p\} \subseteq \{q\}) \subseteq ((PJ^1)(Hs^* \{p\})^1 \leq (PJ^1)(Hs^* \{q\})^1) & (T_{T963}) \text{ NB } (PJ^1)(Hs^* \{p\})^1 \neq \emptyset \\
& & \text{and } (PJ^1)(Hs^* \{q\})^1 \neq \emptyset \\
&\text{Thus } (((Hs^* \{q\})^1 \neq \emptyset) = E) \subseteq ((\{p\} \subseteq \{q\}) \subseteq ((PJ^1)(Hs^* \{p\})^1 \leq (PJ^1)(Hs^* \{q\})^1)) \\
& & (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$(((Hs^* \{q\})^1 \neq \emptyset) = E) \subseteq ((\{p\} \subseteq \{q\}) \subseteq ((PJ^1)(Hs^* \{p\})^1 \leq (PJ^1)(Hs^* \{q\})^1)) \quad (T_{S835})$$

Proof of (T_{S835})

$$\begin{aligned}
&\text{Let } (Hs^* \{q\})^1 \neq \emptyset \\
&(\{p\} \subseteq \{q\}) \subseteq ((PJ^1)(Hs^* \{p\})^1 \leq (PJ^1)(Hs^* \{q\})^1) & (T_{T397}) \\
&(\{p\} \subseteq \{q\}) + (\{p\} = \{q\}) \subseteq ((PJ^1)(Hs^* \{p\})^1 \leq (PJ^1)(Hs^* \{q\})^1) & (T_{M266})(\text{Definition of } \subseteq) \\
&(\{p\} \subseteq \{q\}) \subseteq ((PJ^1)(Hs^* \{p\})^1 \leq (PJ^1)(Hs^* \{q\})^1) & (T_{M072})(\text{Single Premise Syllogism}) \\
&\text{Thus } (((Hs^* \{q\})^1 \neq \emptyset) = E) \subseteq ((\{p\} \subseteq \{q\}) \subseteq ((PJ^1)(Hs^* \{p\})^1 \leq (PJ^1)(Hs^* \{q\})^1)) \\
& & (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$(Hs^* \{p\}) + (Hs^* \{q\}) \subseteq (Hs^* \sim(\{p\} + \{q\})) \quad (T_{T313})$$

Proof of (T_{T313})

$$\begin{aligned}
&(Hs^* \{p\}) \subseteq Hs^* \sim\{p\} & (T_{T259})(\text{Eqn1}) \\
&(Hs^* \{q\}) \subseteq Hs^* \sim\{q\} & (T_{T259})(\text{Eqn2}) \\
&(Hs^* \{p\}) + (Hs^* \{q\}) \subseteq (Hs^* \sim\{p\}) + (Hs^* \sim\{q\}) & (A_{MX21})(\text{Combination of Eqns1\&2}) \\
&(Hs^* \{p\}) + (Hs^* \{q\}) \subseteq \sim((Hs^* \sim\{p\}) \cdot (Hs^* \sim\{q\})) & (T_{M025})(\text{DeMorgan's Law}) \\
&(Hs^* \{p\}) + (Hs^* \{q\}) \subseteq (Hs^* \sim\{p\} \cdot \sim\{q\}) & (A_{S46})(\text{Conjunction of Information}) \\
&(Hs^* \{p\}) + (Hs^* \{q\}) \subseteq (Hs^* \sim(\{p\} + \{q\})) & (T_{M026})(\text{DeMorgan's Law})
\end{aligned}$$

$$(J \subseteq Hs^* \{p\}) \cdot (J \subseteq Hs^* \{q\}) = (J \subseteq Hs^* (\{p\} \cdot \{q\})) \quad (T_{T283})$$

Proof of (T_{T283})

$$\begin{aligned}
&\text{Let } (J \subseteq Hs^* \{p\}) \cdot (J \subseteq Hs^* \{q\}) \\
&J \subseteq (Hs^* \{p\}) \cdot (Hs^* \{q\}) & (T_{M132})(\text{Composition}) \\
&J \subseteq Hs^* (\{p\} \cdot \{q\}) & (A_{S46})(\text{Conjunction of Information}) \\
&\text{Thus } (J \subseteq Hs^* \{p\}) \cdot (J \subseteq Hs^* \{q\}) = (J \subseteq Hs^* (\{p\} \cdot \{q\})) & (A_{MX02})(\text{Conditional Introduction}) \\
&\text{Since this proof is reversible}
\end{aligned}$$

$$(J \subseteq Hs^* \{p\}) + (J \subseteq Hs^* \{q\}) \subseteq (J \subseteq Hs^* \sim(\{p\} + \{q\})) \quad (T_{T282})$$

Proof of (T_{T282})

$$\begin{aligned}
&\text{Let } (J \subseteq Hs^* \{p\}) + (J \subseteq Hs^* \{q\}) \\
&J \subseteq Hs^* \{p\} + Hs^* \{q\} & (T_{M170})(\text{Disjunction of Necessities}) \\
&J \subseteq (Hs^* \sim(\{p\} + \{q\})) & (T_{T313}) \\
&\text{Thus } (J \subseteq Hs^* \{p\}) + (J \subseteq Hs^* \{q\}) \subseteq (J \subseteq Hs^* \sim(\{p\} + \{q\})) & (A_{MX02})(\text{Conditional Introduction})
\end{aligned}$$

$$((P^*(J \cdot (Hs^* \{q\}))^1) \cdot (Hs^* \{p\})^1 = (P^*J^1) \cdot (Hs^* \{p\})^1):$$

$$((P^*J^1) \cdot (Hs^* \{p\} \cdot \{q\})^1 = ((P^*J^1) \cdot (Hs^* \{q\})^1) \times ((P^*J^1) \cdot (Hs^* \{p\})^1)) \quad (T_{S645})$$

N.B. the conditions mean that there is no correlation between $\{p\}$ and $\{q\}$, i.e., the probability that the one is true does not vary with the probability that the other is true.

Proof of (Ts645)

$$(\mathcal{P}^*c^1) * (ab)^1 = ((\mathcal{P}^*c^1) * b^1) \times ((\mathcal{P}^*(cb)^1) * a^1) \quad (\mathcal{T}_{R838})(\text{Eqn1})$$

$$\text{Let } a = (\mathcal{H}_s * \{p\}) \quad (\text{Eqn2})$$

$$\text{Let } b = (\mathcal{H}_s * \{q\}) \quad (\text{Eqn3})$$

$$\text{Let } c = J \quad (\text{Eqn4})$$

$$(\mathcal{P}^*J^1) * ((\mathcal{H}_s * \{p\}) * (\mathcal{H}_s * \{q\}))^1 = ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{q\}))^1 \times ((\mathcal{P}^*(J * (\mathcal{H}_s * \{q\})))^1 * (\mathcal{H}_s * \{p\}))^1 \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqns2,3\&4 into Eqn1})(\text{Eqn5})$$

$$\text{Let } (\mathcal{P}^*(J * (\mathcal{H}_s * \{q\})))^1 * (\mathcal{H}_s * \{p\})^1 = (\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\})^1 \quad (\text{Eqn6})$$

$$(\mathcal{P}^*J^1) * ((\mathcal{H}_s * \{p\}) * (\mathcal{H}_s * \{q\}))^1 = ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{q\}))^1 \times ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\}))^1 \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqn6 into Eqn5})$$

$$(\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\} * \{q\})^1 = ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{q\}))^1 \times ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\}))^1 \quad (\mathcal{A}_{S46})(\text{Conjunction of Information})$$

$$\text{Thus } ((\mathcal{P}^*(J * (\mathcal{H}_s * \{q\})))^1 * (\mathcal{H}_s * \{p\}))^1 = (\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\})^1:$$

$$((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\} * \{q\}))^1 = ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{q\}))^1 \times ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\}))^1 \quad (\mathcal{A}_{MX02})(\text{Conditional Introduction})$$

$$(\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\} + \{q\})^1 = (\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\})^1 + (\mathcal{P}^*J^1) * (\mathcal{H}_s * \{q\})^1 - (\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\} * \{q\})^1 \quad (\mathcal{T}_{S812})$$

Proof of (Ts812)

$$(\mathcal{P}^*c^1) * (a+b)^1 = ((\mathcal{P}^*c^1) * a^1) + ((\mathcal{P}^*c^1) * b^1) - ((\mathcal{P}^*c^1) * (a.b)^1) \quad (\mathcal{T}_{R819})(\text{Eqn1})$$

$$\text{Let } a = (\mathcal{H}_s * \{p\}) \quad (\text{Eqn2})$$

$$\text{Let } b = (\mathcal{H}_s * \{q\}) \quad (\text{Eqn3})$$

$$\text{Let } c = J \quad (\text{Eqn4})$$

$$(\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\} + \mathcal{H}_s * \{q\})^1 = ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\}))^1 + ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{q\}))^1 - ((\mathcal{P}^*J^1) * ((\mathcal{H}_s * \{p\}) * (\mathcal{H}_s * \{q\})))^1 \quad (\mathcal{A}_{MX19})(\text{Substitution of Eqns2,3\&4 into Eqn1})(\text{Eqn5})$$

$$(\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\} + \mathcal{H}_s * \{q\})^1 = ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\}))^1 + ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{q\}))^1 - ((\mathcal{P}^*J^1) * ((\mathcal{H}_s * \{p\} * \{q\})))^1 \quad (\mathcal{A}_{S46})(\text{Conjunction of Information})(\text{Eqn6})$$

$$(\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\} + \{q\})^1 = ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\}))^1 + ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{q\}))^1 - ((\mathcal{P}^*J^1) * ((\mathcal{H}_s * \{p\} * \{q\})))^1 \quad (\mathcal{A}_{S72})(\text{Disjunction of Information})$$

$$(\{c\}^1 > \{0\}^1): (\{a\}^1 > \{b\}^1) = ((\{+\} * \{c\}^1) * \{a\}^1 > (\{+\} * \{c\}^1) * \{b\}^1) \quad (\mathcal{T}_{S636})$$

$$(\{c\}^1 > \{0\}^1): (\{a\}^1 > \{b\}^1) = (\{a\}^1 + \{c\}^1 = \{b\}^1 + \{c\}^1) \quad (\mathcal{T}_{S635})$$

$$(\{c\}^1 > \{0\}^1): (\{a\}^1 > \{b\}^1) = ((\{-\} * \{c\}^1) * \{a\}^1 > (\{-\} * \{c\}^1) * \{b\}^1) \quad (\mathcal{T}_{S634})$$

$$(\{c\}^1 > \{0\}^1): (\{a\}^1 > \{b\}^1) = (\{a\}^1 - \{c\}^1 > \{b\}^1 - \{c\}^1) \quad (\mathcal{T}_{S633})$$

$$(\{c\}^1 > \{0\}^1): (\{a\}^1 > \{b\}^1) = ((\{x\} * \{c\}^1) * \{a\}^1 > (\{x\} * \{c\}^1) * \{b\}^1) \quad (\mathcal{T}_{S632})$$

$$(\{c\}^1 > \{0\}^1): (\{a\}^1 > \{b\}^1) = (\{a\}^1 \times \{c\}^1 > \{b\}^1 \times \{c\}^1) \quad (\mathcal{T}_{S646})$$

$$(\{c\}^1 > \{0\}^1): (\{a\}^1 > \{b\}^1) = ((\{\div\} * \{c\}^1) * \{a\}^1 > (\{\div\} * \{c\}^1) * \{b\}^1) \quad (\mathcal{T}_{S631})$$

$$(\{c\}^1 > \{0\}^1): (\{a\}^1 > \{b\}^1) = (\{a\}^1 \div \{c\}^1 > \{b\}^1 \div \{c\}^1) \quad (\mathcal{T}_{S630})$$

Example Proof of (Ts632) & (Ts646)

$$(X = \{+, \{-\}, \{x\}, \{\div\}) (\{c\}^1 > \{0\}^1): ((\{a\}^1 > \{b\}^1) = ((X\{c\}^1) * \{a\}^1 > (X\{c\}^1) * \{b\}^1)) \quad (\mathcal{A}_{S83})(\text{Relative Deduction})$$

$$(\{c\}^1 > \{0\}^1): ((\{a\}^1 > \{b\}^1) = ((\{x\} * \{c\}^1) * \{a\}^1 > (\{x\} * \{c\}^1) * \{b\}^1)) \quad \text{Extraction}$$

$$(\{c\}^1 > \{0\}^1): (\{a\}^1 > \{b\}^1) = (\{a\}^1 \times \{c\}^1 > \{b\}^1 \times \{c\}^1) \quad (\mathcal{T}_{7473})$$

$$((\mathcal{P}^*(J * (\mathcal{H}_s * \{q\})))^1 * (\mathcal{H}_s * \{p\}))^1 = (\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\})^1:$$

$$(((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{p\}))^1 > (\mathcal{P}^*J^1) * (\mathcal{H}_s * \{q\}))^1 \subseteq ((\mathcal{P}^*J^1) * (\mathcal{H}_s * \{r\} * \{p\}))^1 > (\mathcal{P}^*J^1) * (\mathcal{H}_s * \{r\} * \{q\}))^1 \quad (\mathcal{T}_{7280})$$

N.B. the conditions mean that there is no correlation between $\{p\}$ and $\{q\}$, i.e., the probability that the one is true does not vary with the probability that the other is true.

Proof of (T₂₈₀)

$$\begin{aligned}
 & \text{Let } (\mathcal{P}*(\mathcal{J}.(\mathcal{H}_s*\{\mathcal{q}\})))^1*(\mathcal{H}_s*\{\mathcal{p}\})^1 = (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1 \\
 & \text{Let } (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1 > (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{q}\})^1 \\
 & (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\})^1 \times (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1 > (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\})^1 \times (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{q}\})^1 \\
 & \hspace{15em} (\mathcal{T}_{S646}) \\
 & (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\}\{\mathcal{p}\})^1 > (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\}\{\mathcal{q}\})^1 \hspace{15em} (\mathcal{T}_{S645}) \\
 & \text{Thus } ((\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1 > (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{q}\})^1) \subseteq ((\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\}\{\mathcal{p}\})^1 > (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\}\{\mathcal{q}\})^1) \\
 & \hspace{15em} (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\
 & \text{Thus } ((\mathcal{P}*(\mathcal{J}.(\mathcal{H}_s*\{\mathcal{q}\})))^1*(\mathcal{H}_s*\{\mathcal{p}\})^1 = (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1): \\
 & (((\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1 > (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{q}\})^1) \subseteq ((\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\}\{\mathcal{p}\})^1 > (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\}\{\mathcal{q}\})^1)) \\
 & \hspace{15em} (\mathcal{A}_{MX02})(\text{Conditional Introduction})
 \end{aligned}$$

$$(\{\mathcal{p}\} \neq \{\mathcal{q}\}) = (\{\mathcal{p}\}.\{\mathcal{q}\} = \emptyset) \hspace{10em} (\mathcal{T}_{S637})$$

Proof of (T_{S637})

$$\begin{aligned}
 & ((\mathcal{a}^1 \neq \emptyset) = \mathcal{E}) \subseteq ((\mathcal{a}^1 \neq \mathcal{b}^1) = (\mathcal{a}^1.\mathcal{b}^1 = \emptyset)) \hspace{10em} (\mathcal{T}_{M205})(\text{Unit Set Conjunction Rule})(\text{Eqn1}) \\
 & \text{Let } \mathcal{a}^1 = \{\mathcal{p}\} \hspace{15em} (\text{Eqn2}) \\
 & \text{Let } \mathcal{b}^1 = \{\mathcal{q}\} \hspace{15em} (\text{Eqn3}) \\
 & ((\{\mathcal{p}\} \neq \emptyset) = \mathcal{E}) \subseteq ((\{\mathcal{p}\} \neq \{\mathcal{q}\}) = (\{\mathcal{p}\}.\{\mathcal{q}\} = \emptyset)) \\
 & \hspace{15em} (\mathcal{A}_{MX19})(\text{Substitution of Eqns2\&3 into Eqn1}) \\
 & (\mathcal{E} = \mathcal{E}) \subseteq ((\{\mathcal{p}\} \neq \{\mathcal{q}\}) = (\{\mathcal{p}\}.\{\mathcal{q}\} = \emptyset)) \hspace{10em} (\mathcal{T}_{M001})(\text{Universality of } (\mathcal{T}_{398}) \text{ Knowledge exists}) \\
 & \mathcal{E} \subseteq ((\{\mathcal{p}\} \neq \{\mathcal{q}\}) = (\{\mathcal{p}\}.\{\mathcal{q}\} = \emptyset)) \hspace{15em} (\mathcal{T}_{M342}) \\
 & \mathcal{E} = ((\{\mathcal{p}\} \neq \{\mathcal{q}\}) = (\{\mathcal{p}\}.\{\mathcal{q}\} = \emptyset)) \hspace{15em} (\mathcal{T}_{M275})(\text{Supersets of the Universal Set}) \\
 & (\{\mathcal{p}\} \neq \{\mathcal{q}\}) = (\{\mathcal{p}\}.\{\mathcal{q}\} = \emptyset) \hspace{15em} (\mathcal{A}_{MX16})(\text{Order Axiom})
 \end{aligned}$$

$$((\mathcal{P}*(\mathcal{J}.(\mathcal{H}_s*\{\mathcal{q}\})))^1*(\mathcal{H}_s*\{\mathcal{p}\})^1 = (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1):$$

$$(\{\mathcal{p}\}.\{\mathcal{q}\} = \{\mathcal{r}\}) = ((\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1 \times (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{q}\})^1 = (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\})^1) \hspace{10em} (\mathcal{T}_{390})$$

N.B. the conditions mean that there is no correlation between $\{p\}$ and $\{q\}$, i.e., the probability that the one is true does not vary with the probability that the other is true.

Proof of (T₃₉₀)

$$\begin{aligned}
 & \text{Let } ((\mathcal{P}*(\mathcal{J}.(\mathcal{H}_s*\{\mathcal{q}\})))^1*(\mathcal{H}_s*\{\mathcal{p}\})^1 = (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1) \\
 & (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\})^1 = (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\})^1 \hspace{10em} (\mathcal{D}_{M00})(\text{Definition of } =)(\text{Eqn1}) \\
 & \text{Let } \{\mathcal{p}\}.\{\mathcal{q}\} = \{\mathcal{r}\} \hspace{15em} (\text{Eqn2}) \\
 & (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\}.\{\mathcal{q}\})^1 = (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\})^1 \hspace{10em} (\mathcal{A}_{MX19})(\text{Substitution of Eqn2 into Eqn1}) \\
 & ((\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{q}\})^1) \times ((\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1) = (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\})^1 \hspace{10em} (\mathcal{T}_{S645}) \\
 & \text{Thus } (\{\mathcal{p}\}.\{\mathcal{q}\} = \{\mathcal{r}\}) = ((\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1 \times (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{q}\})^1 = (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\})^1) \\
 & \hspace{15em} (\mathcal{A}_{MX02})(\text{Conditional Introduction}) \\
 & \hspace{15em} \text{Since this proof is reversible} \\
 & \text{Thus } ((\mathcal{P}*(\mathcal{J}.(\mathcal{H}_s*\{\mathcal{q}\})))^1*(\mathcal{H}_s*\{\mathcal{p}\})^1 = (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1): \\
 & ((\{\mathcal{p}\}.\{\mathcal{q}\} = \{\mathcal{r}\}) = ((\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1 \times (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{q}\})^1 = (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\})^1)) \\
 & \hspace{15em} (\mathcal{A}_{MX02})(\text{Conditional Introduction})
 \end{aligned}$$

$$((\mathcal{P}*(\mathcal{J}.(\mathcal{H}_s*\{\mathcal{q}\})))^1*(\mathcal{H}_s*\{\mathcal{p}\})^1 = (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1):$$

$$(\{\mathcal{p}\} + \{\mathcal{q}\} = \{\mathcal{r}\}) =$$

$$((\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1 + (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{q}\})^1 - ((\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{p}\})^1 \times (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{q}\})^1) = (\mathcal{P}*\mathcal{J}^1)*(\mathcal{H}_s*\{\mathcal{r}\})^1)$$

(T₃₉₆)

N.B. the conditions mean that there is no correlation between $\{p\}$ and $\{q\}$, i.e., the probability that the one is true does not vary with the probability that the other is true.

Proof of (T₇₃₉₆)

$$(\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{r}\})^1 = (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{r}\})^1$$

(D_{M00})(Definition of =)(Eqn1)

$$\text{Let } \{\mathcal{p}\} + \{\mathcal{q}\} = \{\mathcal{r}\}$$

(Eqn2)

$$(\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\} + \{\mathcal{q}\})^1 = (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{r}\})^1$$

(A_{MX19})(Substitution of Eqn2 into Eqn1)

$$(\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\})^1 + (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{q}\})^1 - (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\}\{\mathcal{q}\})^1 = (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{r}\})^1$$

(T_{S812})

$$\text{Thus } (\{\mathcal{p}\} + \{\mathcal{q}\} = \{\mathcal{r}\}) =$$

$$((\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\})^1 + (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{q}\})^1 - (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\}\{\mathcal{q}\})^1 = (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{r}\})^1)$$

(A_{MX02})(Conditional Introduction)

Since this proof is reversible

$$\text{Let } \mathcal{P} * (\mathcal{J} * (\mathcal{H}_s * \{\mathcal{q}\}))^1 * (\mathcal{H}_s * \{\mathcal{p}\})^1 = (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\})^1$$

$$(\{\mathcal{p}\} + \{\mathcal{q}\} = \{\mathcal{r}\}) =$$

$$((\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\})^1 + (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{q}\})^1 - ((\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\})^1 \times (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{q}\})^1) =$$

$$(\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{r}\})^1$$

(T_{S645})

$$\text{Thus } ((\mathcal{P} * (\mathcal{J} * (\mathcal{H}_s * \{\mathcal{q}\}))^1 * (\mathcal{H}_s * \{\mathcal{p}\})^1 = (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\})^1) : (\{\mathcal{p}\} + \{\mathcal{q}\} = \{\mathcal{r}\}) =$$

$$((\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\})^1 + (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{q}\})^1 - ((\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\})^1 \times (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{q}\})^1) =$$

$$(\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{r}\})^1$$

(A_{MX02})(Conditional Introduction)

$$((\mathcal{a}^1 \neq \emptyset) = \mathcal{E}) \subseteq ((\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \mathcal{a}^1 \subseteq \mathcal{b}))^1 = (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \mathcal{a}^1 \subseteq \mathcal{bc})^1 + (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \mathcal{a}^1 \subseteq \mathcal{bc})^1 \quad (\text{T}_{S813})$$

Proof of (T_{S813})

$$(\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\} + \{\mathcal{q}\})^1 = (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\})^1 + (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{q}\})^1 - (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\}\{\mathcal{q}\})^1$$

(T_{S812})(Eqn1)

$$\text{Let } \{\mathcal{p}\} = (\mathcal{a}^1 \subseteq \mathcal{bc})$$

(Eqn2)

$$\text{Let } \{\mathcal{q}\} = (\mathcal{a}^1 \subseteq \mathcal{bc})$$

(Eqn3)

$$\text{Let } \mathcal{a}^1 \neq \emptyset$$

$$(\mathcal{a}^1 \neq \emptyset) = \mathcal{E}$$

(A_{MX16})(Order Axiom)

$$(\mathcal{a}^1 = \emptyset) = \sim \mathcal{E}$$

(T_{M008})(Inverse of an Equality)

$$(\mathcal{a}^1 = \emptyset) = \emptyset$$

(D_{M02})(Definition of $\emptyset$)(Eqn4)

$$\{\mathcal{p}\} + \{\mathcal{q}\} = (\mathcal{a}^1 \subseteq \mathcal{bc}) + (\mathcal{a}^1 \subseteq \mathcal{bc})$$

(T_{M230})(Combination of Eqns2&3)

$$\{\mathcal{p}\} + \{\mathcal{q}\} = (\mathcal{a}^1 \subseteq \mathcal{bc} + \mathcal{bc})$$

(T_{M170})(Disjunction of Necessities)

$$\{\mathcal{p}\} + \{\mathcal{q}\} = (\mathcal{a}^1 \subseteq \mathcal{b}(\mathcal{c} + \mathcal{c}))$$

(T_{M407})(Distribution of $\cdot$ over $+$)

$$\{\mathcal{p}\} + \{\mathcal{q}\} = (\mathcal{a}^1 \subseteq \mathcal{b}\mathcal{E})$$

(T_{M671})(Tertium non Datur)

$$\{\mathcal{p}\} + \{\mathcal{q}\} = (\mathcal{a}^1 \subseteq \mathcal{b})$$

(A_{MX11})(Universal Set Conjunction Axiom)(Eqn5)

$$\{\mathcal{p}\} \cdot \{\mathcal{q}\} = (\mathcal{a}^1 \subseteq \mathcal{bc}) \cdot (\mathcal{a}^1 \subseteq \mathcal{bc})$$

(T_{M231})(Combination of Eqns2&3)

$$\{\mathcal{p}\} \cdot \{\mathcal{q}\} = (\mathcal{a}^1 \subseteq \mathcal{bc}\mathcal{bc})$$

(T_{M132})(Composition)

$$\{\mathcal{p}\} \cdot \{\mathcal{q}\} = (\mathcal{a}^1 \subseteq \mathcal{bb}\emptyset)$$

(T_{M621})(Complement Conjunction)

$$\{\mathcal{p}\} \cdot \{\mathcal{q}\} = (\mathcal{a}^1 \subseteq \emptyset)$$

(T_{M050})(Null Set Conjunction)

$$\{\mathcal{p}\} \cdot \{\mathcal{q}\} = (\mathcal{a}^1 = \emptyset)$$

(T_{M052})(Subsets of the Null Set)(Eqn6)

$$\{\mathcal{p}\} \cdot \{\mathcal{q}\} = \emptyset$$

(A_{MX19})(Substitution of Eqn4 into Eqn6)

$$(\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \{\mathcal{p}\}\{\mathcal{q}\})^1 = \{0\}^1$$

(T_{S757})(Eqn7)

$$(\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \mathcal{a}^1 \subseteq \mathcal{b})^1 = (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \mathcal{a}^1 \subseteq \mathcal{bc})^1 + (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \mathcal{a}^1 \subseteq \mathcal{bc})^1 - \{0\}^1$$

(A_{MX19})(Substitution of Eqns2,3,5&7 into Eqn1)

$$(\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \mathcal{a}^1 \subseteq \mathcal{b})^1 = (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \mathcal{a}^1 \subseteq \mathcal{bc})^1 + (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \mathcal{a}^1 \subseteq \mathcal{bc})^1$$

(T₇₄₉₃)

$$\text{Thus } ((\mathcal{a}^1 \neq \emptyset) = \mathcal{E}) \subseteq ((\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \mathcal{a}^1 \subseteq \mathcal{b}))^1 = (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \mathcal{a}^1 \subseteq \mathcal{bc})^1 + (\mathcal{P} * \mathcal{J}^1) * (\mathcal{H}_s * \mathcal{a}^1 \subseteq \mathcal{bc})^1$$

(A_{MX02})(Conditional Introduction)

Chapter 19 Proofs of Theorems of Conventional Set Theory

19.1 Proofs

$$(a \subseteq b)(b \subseteq a) = (a^1 = b^1)$$

(T_{7314}) Axiom of Extensionality

Proof of (T_{7314})

$$(a \subseteq b)(b \subseteq a) = (a = b)$$

(T_{M430}) (Biconditional Introduction)

$$(a \subseteq b)(b \subseteq a) = (a^1 = b^1)$$

(T_{M479})

$$(a^1 \neq \emptyset)(b^1 \neq \emptyset) \subseteq ((a^1 + b^1)^1 \neq \emptyset)$$

(T_{7399}) Axiom of Pairs

Proof of (T_{7399})

Let $a^1 \neq \emptyset$

$$(a^1 \neq \emptyset) = E$$

(A_{MX16}) (Order Axiom) (Eqn1)

Let $b^1 \neq \emptyset$

(Eqn2)

$$(a^1 + b^1 \neq \emptyset) \subseteq (a^1 \neq \emptyset)$$

(T_{M086})

$$(a^1 \neq \emptyset) \subseteq (a^1 + b^1 \neq \emptyset)$$

(T_{M012}) (Transposition) (Eqn3)

$$E \subseteq (a^1 + b^1 \neq \emptyset)$$

(A_{MX19}) (Substitution of Eqn1 into Eqn3)

$$E = (a^1 + b^1 \neq \emptyset)$$

(T_{M275}) (Supersets of the Universal Set)

$$a^1 + b^1 \neq \emptyset$$

(A_{MX16}) (Order Axiom)

$$(a^1 + b^1)^1 \neq \emptyset$$

(T_{M503})

$$\text{Thus } (a^1 \neq \emptyset)(b^1 \neq \emptyset) \subseteq ((a^1 + b^1)^1 \neq \emptyset)$$

(A_{MX02}) (Conditional Introduction)

$$((a_1^1 \neq \emptyset^1) \dots (a_n^1 \neq \emptyset^1)(a_1^{1+++} a_n \neq \emptyset^1)(a_1^{1+++} a_n^1 \neq \emptyset^1) = E) \subseteq (((a_1^{1+++} a_n)^1 \neq \emptyset) = ((a_1^{1+++} a_n^1)^1 \neq \emptyset))$$

(T_{7401}) Axiom of the Union

Proof of (T_{7401})

$$(a_r = \emptyset) \subseteq (a_r^1 = \emptyset) + (a_r^1 = \emptyset^1)$$

(T_{M496})

$$(a_r = \emptyset)(a_r^1 \neq \emptyset^1) \subseteq (a_r^1 = \emptyset)$$

(T_{M468})

$$((a_r^1 \neq \emptyset^1) = E) \subseteq ((a_r = \emptyset) \subseteq (a_r^1 = \emptyset))$$

(A_{MX35}) (Exportation/Importation) for truth sets (Eqn1)

$$(a_r^1 = \emptyset) \subseteq (a_r = \emptyset)$$

(A_{MX50}) (Components of a Null Entity)

$$E = ((a_r^1 = \emptyset) \subseteq (a_r = \emptyset))$$

(A_{MX16}) (Order Axiom) (Eqn2)

$$((a_r^1 \neq \emptyset^1) = E).E \subseteq ((a_r = \emptyset) \subseteq (a_r^1 = \emptyset))((a_r^1 = \emptyset) \subseteq (a_r = \emptyset))$$

(T_{M669}) (Combination of Eqns1&2)

$$((a_r^1 \neq \emptyset^1) = E) \subseteq ((a_r = \emptyset) \subseteq (a_r^1 = \emptyset))((a_r^1 = \emptyset) \subseteq (a_r = \emptyset))$$

(A_{MX17}) (Universal Set Conjunction Axiom)

$$((a_r^1 \neq \emptyset^1) = E) \subseteq ((a_r^1 = \emptyset) = (a_r = \emptyset))$$

(T_{M430}) (Biconditional Introduction)

$$((a_r^1 \neq \emptyset^1) = E) \subseteq ((a_r \neq \emptyset) = (a_r^1 \neq \emptyset))$$

(T_{M008}) (Inverse of an Equality) (Eqn3)

$$((a_1^{1+++} a_n \neq \emptyset^1) = E) \subseteq ((a_1^{1+++} a_n \neq \emptyset) = ((a_1^{1+++} a_n)^1 \neq \emptyset))$$

Resymbolising (Eqn4)

$$((a_1^{1+++} a_n^1 \neq \emptyset^1) = E) \subseteq ((a_1^{1+++} a_n^1 \neq \emptyset) = ((a_1^{1+++} a_n^1)^1 \neq \emptyset))$$

Resymbolising (Eqn5)

$$((a_1^1 \neq \emptyset^1) \dots (a_n^1 \neq \emptyset^1) = E) \subseteq ((a_1 \neq \emptyset) = (a_1^1 \neq \emptyset)) \dots ((a_n \neq \emptyset) = (a_n^1 \neq \emptyset))$$

(T_{M669}) (Combination of Eqn3 for $r = 1$ to n)

$$((a_1^1 \neq \emptyset^1) \dots (a_n^1 \neq \emptyset^1) = E) \subseteq ((a_1 \neq \emptyset) +++ (a_n \neq \emptyset) = (a_1^1 \neq \emptyset) +++ (a_n^1 \neq \emptyset))$$

(A_{MX27}) (Combination for $r = 1$ to n) (Eqn6)

$$(a_1 \neq \emptyset) +++ (a_n \neq \emptyset) = (a_1^{1+++} a_n \neq \emptyset)$$

(T_{M238}) (Probabilistic Disjunction) (Eqn7)

$$(a_1^1 \neq \emptyset) +++ (a_n^1 \neq \emptyset) = (a_1^{1+++} a_n^1 \neq \emptyset)$$

(T_{M238}) (Probabilistic Disjunction) (Eqn8)

$$((a_1^1 \neq \emptyset^1) \dots (a_n^1 \neq \emptyset^1) = E) \subseteq ((a_1^{1+++} a_n \neq \emptyset) = (a_1^{1+++} a_n^1 \neq \emptyset))$$

(A_{MX19}) (Substitution of Eqns7&8 into Eqn6) (Eqn9)

$$((a_1^1 \neq \emptyset^1) \dots (a_n^1 \neq \emptyset^1)(a_1^{1+++} a_n \neq \emptyset^1)(a_1^{1+++} a_n^1 \neq \emptyset^1) = E) \subseteq (((a_1^{1+++} a_n)^1 \neq \emptyset) = ((a_1^{1+++} a_n^1)^1 \neq \emptyset))$$

(A_{MX19}) (Substitution of Eqns4&5 into Eqn9)

$$((Qc = E)(c^1 \subseteq Ro) = E) \subseteq ((x^1 \neq \emptyset) \subseteq ((c * x^1)^1 \neq \emptyset)) \quad (T_{7402})$$

Proof of (T_{7402})

Let $Qc = E$

(Eqn1)

Let $x^1 \neq \emptyset$

$(x^1 \neq \emptyset) = E$

($AMX16$)(Order Axiom)(Eqn2)

$x^1 \subseteq E$

($TM744$)(Universal Set Theorem)(Eqn3)

$x^1 \subseteq Qc$

($AMX19$)(Substitution of Eqn1 into Eqn3)

$(x^1 \subseteq Qc) = E$

($AMX16$)(Order Axiom)(Eqn4)

$(x^1 \neq \emptyset)(c * x^1 \neq \emptyset) = (x^1 \subseteq Qc)$

($TS410$)(Eqn5)

$E(c * x^1 \neq \emptyset) = E$

($AMX19$)(Substitution of Eqn2&4 into Eqn5)

$(c * x^1 \neq \emptyset) = E$

($AMX11$)(Universal Set Conjunction Axiom)(Eqn6)

$((c * x^1)^1 = \emptyset) \subseteq (c * x^1 = \emptyset)$

($AMX50$)(Components of a Null Entity)

$(c * x^1 \neq \emptyset) \subseteq ((c * x^1)^1 \neq \emptyset)$

($TM012$)(Transposition)(Eqn7)

$E \subseteq ((c * x^1)^1 \neq \emptyset)$

($AMX19$)(Substitution of Eqn6 into Eqn7)

$E = ((c * x^1)^1 \neq \emptyset)$

($TM275$)(Supersets of the Universal Set)

$(c * x^1)^1 \neq \emptyset$

($AMX16$)(Order Axiom)

Thus $(Qc = E)(x^1 \neq \emptyset) \subseteq ((c * x^1)^1 \neq \emptyset)$

($AMX02$)(Conditional Introduction)(Eqn8)

$(Qc = E)(c^1 \subseteq Ro)(x^1 \neq \emptyset) \subseteq (Qc = E)(x^1 \neq \emptyset)$

($TM083$)(Conjunction Elimination)(Eqn9)

$(Qc = E)(c^1 \subseteq Ro)(x^1 \neq \emptyset) \subseteq ((c * x^1)^1 \neq \emptyset)$

($TM139$)(Hypothetical Syllogism Eqns8&9)

$((Qc = E)(c^1 \subseteq Ro) = E) \subseteq ((x^1 \neq \emptyset) \subseteq ((c * x^1)^1 \neq \emptyset))$ ($AMX35$)(Exportation/Importation) for truth sets